The Principle of Coherence

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Abstract

Standard finance theory uses a probabilistic principle, the principle of no arbitrage, to formalize the assumption that investors cannot make money for sure. Shafer and Vovk (2001) use instead a simpler and purely game-theoretic principle, which we call the principle of coherence. This note compares the two principles.

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1 Introduction

The modern theory of option pricing is based on the law of one price: if we can replicate an option by trading in the underlying asset and other securities, then the price of the option must equal the cost of the replication. The law of one price, in turn, is usually presented as a consequence of the principle that the market will not allow an investor to make money for sure.

On their face, these ideas have nothing to do with probability theory. The law of one price says that two ways of getting the same outcome will always cost the same, not merely that they will probably cost the same. It is widely believed, however, that option pricing must be based on probability, and it is customary to formalize the notion that the market will not allow an investor to make money for sure in terms of probability. One assumes that market prices are governed by some probability distribution, say P. An *arbitrage opportunity* is a strategy for an investor that begins with zero capital, has zero probability under P of producing a net loss, and has nonzero probability under P of producing a net gain. The *principle of no arbitrage* says that no arbitrage opportunity exists [4, 8].

One striking aspect of the definition of an arbitrage opportunity is that it looks only at whether P assigns events zero or nonzero probability. Other details about P do not matter. This reflects how probability models are used to price options. The usual models, Itô processes, are defined by their volatility and their drift. The volatility, which determines the possible paths the price of the underlying asset may follow, affects the replication of the option and therefore its price, whereas the drift, which determines the probabilities for the possible paths, is irrelevant. All that matters is which paths are impossible (the entire set of these is assigned probability zero) and which are possible.

Although the points just reviewed are well understood, conceptual and practical questions and difficulties remain. If the non-extreme probabilities given by the model do not matter, then is it necessary to assume that market prices are stochastic in some sense? Is it possible to reformulate the theory in a way that simply states what is possible without using probability theory? Most confusing, perhaps, is the fact that the picture is so firmly tied to continuous time, whereas real trading is discrete. If only probabilities of zero and one are meaningful, then how can we analyze the approximate replication that is possible in discrete time [2]?

These questions have recently been addressed by Shafer and Vovk [10], who propose methods of option pricing that do not make stochastic assumptions. Instead of assuming that asset prices are stochastic, they assume only that changes in asset prices are approximately of the order $\sqrt{dt}$—an assumption that is supported both empirically and by efficient-market arguments [5, 14]. They then bound the success of discrete replication in terms of the accuracy of the $\sqrt{dt}$ assumption—not in terms of probabilities. In order to make this analysis mathematically rigorous, they work in terms of explicit games between two players, Market and Investor. The $\sqrt{dt}$ assumption appears as a constraint on the moves by Market.

Unfortunately, the existing stochastic approach to option pricing is so deeply engrained that students and even experienced researchers often have difficulty grasping the thoroughly non-probabilistic nature of the game-theoretic framework. One symptom of this difficulty is this frequently asked question:

> Do Shafer and Vovk use the principle of no arbitrage?

The questioner is usually genuinely puzzled. He or she knows that the principle of no arbitrage involves probabilities. If Shafer and Vovk use it, then they are using probabilities. If they do not use it, on the other hand, they must be foregoing the most important insights of modern finance theory.

The short answer to the frequently asked question is that Shafer and Vovk use a form of the principle of no arbitrage that avoids appealing to the doubtful assumption that markets are stochastic. In this note, which is merely a more extended answer, we call this non-stochastic principle the *principle of coherence.*

The principle of coherence can be stated at a very abstract level, in terms of a general game between Market and Investor that may continue for many rounds. An *incoherency* is a strategy for Investor that begins with zero capital and produces a strictly positive gain for Investor no matter how Market moves. Such a strategy may be available to Investor at the beginning of the game, or it may arise only later, because of moves Market makes. The *principle of coherence* says that (1) no incoherency is present at the beginning of the game, and (2) Market will move so that no incoherency arises.

Although the principle of coherence and the principle of no arbitrage both formalize the idea that an investor cannot make money for sure, the two principles are quite different. Neither implies the other.

In order to discuss the relation between the two principles more precisely, we must make some assumptions about the game between Market and Investor. We do this in §2. In §3, we discuss the two principles, their relation with each other, and their relation with the law of one price. In §4, we look at examples where the principle of coherence comes into play in different ways. In Appendix A, we provide formal notation for strategies and their capital processes.

2 A general market game

Shafer and Vovk consider a variety of games between Market and Investor. Here we consider only perfect-information games of the following form, in which Market is divided into two players, Opening Market and Closing Market, and each move by Investor affects only the change in his capital on that round:[1]

[1]In order to deal with non-European options we need games in which a move by Investor can affect the change of his capital on a later round. See, for example, p. 321 of [10].

General Market Game
Parameters: $\alpha \in \mathbb{R}$, sets $\mathbf{O}$, $\mathbf{I}$, $\mathbf{C}$, and a mapping $\lambda : \mathbf{O} \times \mathbf{I} \times \mathbf{C} \to \mathbb{R}$
Players: Opening Market, Investor, Closing Market
Protocol:
  $\mathcal{K}_0 := \alpha$.
  For $t = 1, 2, \dots, T$:
    Opening Market announces $\mathbf{o}_t \in \mathbf{O}$.
    Investor announces $\mathbf{i}_t \in \mathbf{I}$.
    Closing Market announces $\mathbf{c}_t \in \mathbf{C}$.
    $\mathcal{K}_t := \mathcal{K}_{t-1} + \lambda(\mathbf{o}_t, \mathbf{i}_t, \mathbf{c}_t)$.

Here α is Investor's initial capital, and $\mathcal{K}_t$ is his capital after the tth round of play. Intuitively, $\mathbf{o}_t$ represents opening prices on the tth round, $\mathbf{i}_t$ represents investments made by Investor at the beginning of the round, and $\mathbf{c}_t$ are closing prices. We suppose that Investor liquidates his investments at the end of the round, and $\lambda(\mathbf{o}_t, \mathbf{i}_t, \mathbf{c}_t)$ is his net gain when he does so.

In the protocol, each player has the same move space on each round: Opening Market always selects from $\mathbf{O}$, Investor always selects from $\mathbf{I}$, and Closing Market always selects from $\mathbf{C}$. The possibilities do not change, and in particular, they are not affected by previous moves. We do, however, sometimes impose side restrictions on the sequence of moves by Opening and Closing Market, and this may make the moves permitted to these players on a given round depend on the moves they made earlier.

Let us call any game of this general form a *market game*. We get different market games by

- different specifications of the move spaces $\mathbf{O}$, $\mathbf{I}$, and $\mathbf{C}$,
- different specifications of the payoff function λ, and
- different side restrictions on the sequence of moves Market may make.

We look at some examples in §4.

2.1 Voluntary and linear market games

We say that a market game is *voluntary for Investor* if there is an element $\mathbf{i} \in \mathbf{I}$ such that $\lambda(\mathbf{o}, \mathbf{i}, \mathbf{c}) = 0$ for all $\mathbf{o} \in \mathbf{O}$ and $\mathbf{c} \in \mathbf{C}$. This means that Investor always has the option of preserving his current capital; intuitively, he has the option of making no investment. Not all market games that interest us are voluntary for Investor. In some games, Investor starts with positive capital and must invest it in some fashion or other on each round.

We say that a market game is *linear for Investor* if S is a linear space and λ is linear in its second argument: $\lambda(\mathbf{o}, a_1\mathbf{i}_1 + a_2\mathbf{i}_2, \mathbf{c}) = a_1\lambda(\mathbf{o}, \mathbf{i}_1, \mathbf{c}) + a_2\lambda(f, \mathbf{i}_2, \mathbf{c})$ for any real numbers a_1 and a_2. The intuitive meaning of this condition is that Investor can buy in any amount or go short in any amount in the assets being offered. If the game is linear for Investor, then it is voluntary for Investor. On the other hand, a market game that is voluntary for Investor is not necessarily linear for Investor. Intuitively, a market game will not be linear for Investor if unlimited short selling is not always allowed.

2.2 Information and strategies for Investor

We have said that a market game is a game of *perfect information*: each player sees the other players' moves as they are made. We leave open the possibility that the players also receive other information, but we do not specify what that information might be. This reflects the way market games actually work; often no one can say in advance what information players will receive.

Our unwillingness to specify what information will be available means that we need to be careful about the concept of a strategy. In general, a strategy for a player in a game is a rule that tells the player how to move on each round depending on the information he has received. When the possibilities for the information are not specified in advance, a strategy is not a well-defined mathematical concept.

We can, however, define the concept of an internal strategy for one of the players in a market game. An *internal strategy*, let us say, is a rule that tells the player how to move on each round depending just on previous moves by other players. This is a well-defined mathematical concept.

We are particularly interested in internal strategies for Investor. An internal strategy for Investor can specify his moves for the entire game, or it can specify only his moves starting in a situation resulting from a sequence of previous moves $\mathbf{o}_1, \mathbf{c}_1, \ldots, \mathbf{o}_t, \mathbf{c}_t, \mathbf{o}_{t+1}$ by Market. In either case the internal strategy together with Investor's initial capital (at the beginning of the game or in the later situation, as the case may be) determines a capital process for Investor, which specifies his subsequent capital as a function of his and Market's subsequent moves. (For a mathematical notation anchoring these concepts, see Appendix A.)

In the rest of this note, we will be concerned only with internal strategies for Investor. We will interpret both the principle of no arbitrage and our own principle of coherence as ruling out only opportunities to make money for sure based on market information—i.e., opportunities to make money for sure using internal strategies. A broader interpretation might be possible and useful for some purposes, but this narrow interpretation seems to be appropriate for a discussion of option pricing and consistent with the established literature on option pricing.

3 Coherence and arbitrage

We now use the concept on an internal strategy in a market game to give the principles of coherence and no arbitrage precise mathematical meaning, so that we can say something precise about how they are related.

3.1 Coherence and the law of one price

An *incoherency* is an internal strategy for Investor that begins with zero capital and produces a strictly positive gain for Investor at the end of the game no matter how Market moves. Such a strategy may be available to Investor at the beginning of the game, or it may arise only later, because of moves Market makes. Shafer and Vovk (p. 186) call a market game *coherent* if no incoherency can arise, no matter how Market moves.[2]

The *principle of coherence* says that (1) no incoherency is present at the beginning of the game, and (2) Market will move so that no incoherency arises. If the market game itself is coherent, then (1) is true, and (2) is automatically satisfied; it puts no restrictions on Market's moves. In some games, however, Market's moves can create incoherencies, and in these games the principle of coherence is a nonempty prediction about Market's moves.

Given an internal strategy $\mathcal{P}$ for Investor and a value α for his initial capital, let us write $\mathcal{K}^{\mathcal{P},\alpha}$ for the resulting capital process. The *law of one price* says that if $\mathcal{P}_1$ and $\mathcal{P}_2$ are internal strategies for Investor, α_1 and α_2 are real numbers, and the two capital processes $\mathcal{K}^{\mathcal{P}_1,\alpha_1}$ and $\mathcal{K}^{\mathcal{P}_2,\alpha_2}$ are equal in all final situations, then $\alpha_1 = \alpha_2$. This is true both at the beginning of the game and in all later situations created by Market's moves.[3] Like the principle of coherence, the law of one price is in part a prediction concerning how Market will move. If there are situations where two internal strategies for Investor produce identical results with different initial capital, Market is not supposed to move into those situations.

It is easy to see that if the market game is linear for Investor, and the principle of coherence holds, then the law of one price holds. Indeed, if starting at α_1 and following $\mathcal{P}_1$ gives the same result as starting at α_2 and following $\mathcal{P}_2$, and $\alpha_1 < \alpha_2$, say, then starting at zero and following $\mathcal{P}_2 - \mathcal{P}_1$ (this is the strategy that always moves $\mathbf{i}_2 - \mathbf{i}_1$, where $\mathbf{i}_i$ is the move recommended by $\mathcal{P}_i$) produces $\alpha_2 - \alpha_1$ at the end of the game no matter how Market moves.

3.2 No arbitrage and the probabilistic law of one price

As we have already explained, the notion of no arbitrage is based on the assumption that Market's moves are governed by a probability distribution P. An *arbitrage opportunity* is an internal strategy for Investor that has zero probability under P of producing a net loss, and has nonzero probability under P of producing a net gain. The *principle of no arbitrage* says that no arbitrage opportunity exists.

[2]Bruno de Finetti was perhaps the first to call betting offers "coherent" when they do not allow an adversary to make money for sure. De Finetti was interested in hypothetical betting offers that represent a person's degrees of belief [3, 9].

[3]If $\mathcal{P}$ starts at the beginning of the game, then the capital process $\mathcal{K}^{\mathcal{P},\alpha}$ gives Investor's capital in all possible situations, including all situations at the end of the game. If it starts in a situation $\mathbf{o}_1, \mathbf{c}_1, \ldots, \mathbf{o}_t, \mathbf{c}_t, \mathbf{o}_{t+1}$, and α is the capital in that situation, then $\mathcal{K}^{\mathcal{P},\alpha}$ gives his capital in all possible situations subsequent to $\mathbf{o}_1, \mathbf{c}_1, \ldots, \mathbf{o}_t, \mathbf{c}_t, \mathbf{o}_{t+1}$.

So far as we are aware, the definition of arbitrage opportunity and the principle of no arbitrage are always formulated, in the existing literature, in terms of the probability distribution P that governs Market's moves at the outset of the game; nothing is said about situations later in the game (see, e.g., [1, 4, 8, 12]). Whenever multiple investment periods are considered, however, it is assumed that the probability space on which P is defined, say $(\Omega, \mathcal{F}, \mathsf{P})$ comes with a filtration, to which the moves by Market are adapted ([12], p. 411), and it is natural to further assume that regular conditional probabilities with respect to this filtration are specified ([11], p. 226). Under these assumptions, we can define the concept of an arbitrage opportunity later in the game. This is an internal strategy for Investor that has, under the conditional probability distribution in that later situation, zero probability of producing a net loss and positive probability of producing a net gain. And if the market game is voluntary, then the assumption that no arbitrage opportunity exists at the beginning of the game implies that *with probability one*, Market will not move into a later situation where one exists. (If there were a positive probability that an arbitrage opportunity will arise later in the game, then Investor could construct an arbitrage opportunity at the beginning of the game by waiting to invest until such an opportunity arises.)

We could, if we wanted, formulate a stronger principle of no arbitrage, one that states that Market definitely will not move into a situation where an arbitrage opportunity exists, not merely that he will not do so with probability one. We are not aware, however, of such a principle being discussed in the existing literature.

In the case where the game is linear for Investor, the principle of no arbitrage implies a probabilistic law of one price: if $\mathcal{P}_1$ and $\mathcal{P}_2$ are internal strategies for Investor, α_1 and α_2 are real numbers, and the two capital processes $\mathcal{K}^{\mathcal{P}_1,\alpha_1}$ and $\mathcal{K}^{\mathcal{P}_2,\alpha_2}$ are equal in all final situations with probability one, then $\alpha_1 = \alpha_2$. This will be true at the beginning of the game, and with probability one, it will be true in all later situations into which Market moves.

3.3 Comparing the principles

The principle of no arbitrage says something about a probability distribution for Market's moves in a market game. The principle of coherence, on the other hand, makes an unqualified (nonprobabilistic) prediction about Market's behavior. So the two are not directly comparable.

We can, however, point to ways in which each principle seems stronger than the other:

- As we will verify shortly, it is easy to construct a coherent market game and a probability distribution for Market's moves with respect to which the principle of no arbitrage does not hold. This is because coherence only rules out strategies that ensure a strictly positive gain. The principle of no arbitrage rules out more—it rules out any strategy that ensure a probability of gain with no chance of loss, including strategies that are most likely to produce zero gain or loss. In this respect, the principle of no arbitrage seems stronger.

- If the probability distribution P for Market's moves satisfies the principle of no arbitrage, and Market is governed by P, then Market will obey the principle of coherence with probability one. But saying that something will happen with probability one is weaker than saying it will happen for certain. In this respect, the principle of coherence seems stronger.

Speaking roughly, we may say that neither principle implies the other.

Here is a market game in which Market necessarily obeys the principle of coherence but the principle of no arbitrage does not necessarily hold. The game has only one round, and we assume Opening Market's move does matter and so can be omitted.

$\mathcal{K}_0 := \alpha$.
Investor announces $s \in \{0, 1\}$.
Market announces $r \in \{0, 1\}$.
$\mathcal{K}_1 := \mathcal{K}_0 + rs$.

Here Market can always choose $r = 0$, preventing Investor from making any gain, and so Investor has no strategy for making money for sure at the beginning of the game. Since there is no later situation in which Investor moves, the principle of coherence holds. On the other hand, if P gives a positive probability to Market's choosing $r = 1$, then the move $s = 1$ by Investor constitutes an arbitrage opportunity, and so the principle of no arbitrage does not hold.

Here, on the other hand, is a market game and a probability measure P such that the principle of no arbitrage holds but Market is not obliged to obey the principle of coherence. The game again has one round, but this time it is Closing Market whose move does not matter and is omitted.

$\mathcal{K}_0 := \alpha$.
Market announces $r \in \{0, 1\}$.
Investor announces $s \in \{0, 1\}$.
$\mathcal{K}_1 := \mathcal{K}_0 + rs$.

We assume that P assigns probability one to the Market making the move $r = 0$. There is no strategy for Investor that gives him a positive probability of gain, and so the principle of no arbitrage holds. On the other hand, it is possible for Market to make the move $r = 1$, creating an incoherency, and so we cannot say that Market is obliged to follow the principle of coherence.

The law of one price and the probabilistic law of one price are similarly incomparable. The law of one price is stronger than the probabilistic law of one price in that it says Market will definitely not create a situation where Investor has a choice between two ways, with different initial investments, of getting the same result, whereas the probabilistic law only says that Market will probably not create such a situation. But the probabilistic law is stronger in that it rules out more: it rules out situations where the two different strategies with different initial capital produce the same result only with probability one.

4 Examples

We now turn to more serious examples, which confirm that the principle of coherence has a role to play in a purely game-theoretic treatment of option pricing.

First, we formulate a game in which the principle of coherence can be used to price a forward. Second, we formulate a game for multiple-period trading in a securities market, where the principle of coherence is automatically satisfied. Third, we discuss a version of Shafer and Vovk's game for pricing European options using the principle of coherence. In all these cases, the the principle of no arbitrage does not come into play because no probability distribution is assumed.

4.1 Pricing a forward

If an investor can take any long or short position in a forward contract or its underlying asset, and the asset provides no income and has no cost of storage, then we can expect that

$$F = S_0(1 + r), \tag{1}$$

where F is the forward price, r is the risk-free interest rate and S_0 is the spot price for the asset. Many authors (e.g. Hull [6]) give the following argument for this equation:

- If $F > S_0(1 + r)$, an investor can initially short the forward and borrow money to buy the asset. When the forward matures, he sells the asset for F, which exceeds the money required to repay the loan with interest, $S_0(1 + r)$. This produces a profit of $F - S_0(1 + r)$.
- On the other hand, if $F < S_0(1 + r)$, an investor can initially short the asset, put the money in the bank, and go long in a forward contract. At the end of the period, the bank account is worth $S_0(1 + r)$. To close out his position, he only needs to pay F. As a result, he makes a profit of $S_0(1 + r) - F$.

The conclusion is that the investor can make $|F - S_0(1 + r)|$ for sure, without taking any risk of loss, if $F \neq S_0(1 + r)$. Because we do not expect an investor to have such opportunities, we expect Equation (1) to hold.

This argument can be regarded as an application of the principle of coherence within the following one-round game between Investor and Market, where s is the amount of the asset the Investor buys and f is the number of forward contracts in which he goes long:

The Forward Game
Parameter: Interest rate $r \in [0, \infty)$
Players: Opening Market, Investor, Closing Market
Protocol:
$\mathcal{K}_0 := 0$.
Opening Market announces $S_0 \in (0, \infty)$ and $F \in (0, \infty)$.
Investor announces $s \in \mathbb{R}$ and $f \in \mathbb{R}$.
Closing Market announces $S_1 \in (0, \infty)$.
$\mathcal{K}_1 := \mathcal{K}_0 + s(S_1 - S_0) - srS_0 + f(S_1 - F)$.

Investor's net gain is his gain from holding the stock, $s(S_1 - S_0)$, less the interest he pays in order to finance buying the stock, srS_0, plus his gain from holding the forward contracts, $f(S_1 - F)$.

This game is linear for Investor. It is not a coherent game, because Opening Market can create an incoherency. But the principle of coherence says that Opening Market will not do so, and this implies, by the argument just given, that he will choose S_0 and F that satisfy (1).

We could also derive (1) from the law of one price. Textbook authors prefer the argument we have just discussed, however, because it makes clear why we should expect the law of one price to hold.

The principle of no arbitrage (or the probabilistic law of one price) implies only that Opening Market will choose S_0 and F to satisfy (1) with probability one. This conclusion is sufficient, perhaps, for we can hardly hope for stronger certainty in practical matters. But the argument for the conclusion is weak, inasmuch as it begins with the unsupported assumption that Market is behaving stochastically according to some probability distribution.

4.2 A securities market

The following game provides one way of thinking about a market in which K securities are traded in T successive periods (cf. [13]). We assume that Investor is required to distribute his current capital among the K securities during each period, and we describe Market's moves by giving the rate of return for each security over the period (this allows us to omit Opening Market).

A Securities Market
Parameter: $\alpha > 0$
Players: Investor, Market
Protocol:
$\mathcal{K}_0 := \alpha$.
For $t = 1, 2, \dots, T$:
Investor announces $g_t \in \mathbb{R}^K$ such that $\sum_{k=1}^K g_t^k = \mathcal{K}_{t-1}$.
Market announces $x_t \in (-1, \infty)^K$.
$\mathcal{K}_t := \sum_{k=1}^K g_t^k (1 + x_t^k) = \mathcal{K}_{t-1} + g_t \cdot x_t$.

Here $\cdot$ represents the dot product: $g \cdot x = \sum_{k=1}^{K} g^k x^k$. The number x_t^k is the rate of return of the kth security in the tth period, and $g_t \cdot x_t$ is the rate of return for Investor in the tth period.

Investor can go short in any particular security (we assume that $K > 1$), but because of the constraint $\sum_{k=1}^{K} g_t^k = \mathcal{K}_{t-1}$, the game is neither voluntary nor linear for Investor.

On the other hand, no matter how Investor distributes his capital on a given round, Market can choose the returns so that Investor loses money on that round. This implies that the game is coherent, and thus that Market necessarily obeys the principle of coherence. It is also clear that Market necessarily obeys the law of one price.

Whether the game obeys the principle of no arbitrage or the probabilistic law of one price depends on probability distribution P we choose.

4.3 Pricing European options

The usual theory for pricing nonlinear options assumes that the price of the underlying asset follows geometric Brownian motion (or some other Itô process) and that an investor can trade in the underlying asset continuously. Under these assumptions, a nonlinear European option can be replicated exactly. A more realistic theory, which permits a nonlinear option to be replicated approximately, can be based on the assumption that it is possible to trade in the underlying asset at a very large number T of time points.

One approach, which is studied by Shafer and Vovk in §10.3 of [10], is to suppose that the investor can trade not only in the underlying security, whose price at the end of period t we designate by S_t, but also in a dividend-paying derivative security, whose price we designate by D_t. This derivative security pays $(\Delta S_t / S_{t-1})^2$ at the end of each period t and becomes worthless at the end of the game. Because the sum of the dividends in the course of the game is a measure of the actual variance of underlying security over T periods, the derivative's price at the beginning of the game, D_0, can be thought of as the market's evaluation of this variance. It plays the role played by $\sigma^2 T$ in the standard Black-Scholes theory.

Writing ΔS_t for $S_t - S_{t-1}$, we can describe Shafer and Vovk's discrete pricing game as follows:

Discrete Black-Scholes with Constrained Variation
Parameters: $\alpha > 0$, $\delta \in (0,1)$, $C > 0$
Players: Market, Investor
Protocol:
 $\mathcal{K}_0 := \alpha$.
 Market announces $S_0 > 0$ and $D_0 > 0$.
  FOR $t = 1, \dots, T$:
  Investor announces $M_t \in \mathbb{R}$ and $V_t \in \mathbb{R}$.
  Market announces $S_t > 0$ and $D_t \geq 0$.
  $\mathcal{K}_t := \mathcal{K}_{t-1} + M_t \Delta S_t + V_t((\Delta S_t / S_{t-1})^2 + \Delta D_t)$.

Additional Constraints on Market: Market's moves must satisfy $0 < S_t < C$ for $t = 1, \ldots, T$, $0 < D_t < C$ for $t = 1, \ldots, T-1$, $D_T = 0$,

$$\sum_{t=1}^{T} |\Delta S_t|^{2.5} < \delta, \qquad \text{and} \qquad \sum_{t=1}^{T} |\Delta D_t|^{1.5} < \delta. \tag{2}$$

The constraints (2) are consistent with the usual stochastic theory. According to that theory,

- the price process S_t should follow the $\sqrt{dt}$ effect, and so $\sum_{t=1}^{T} |\Delta S_t|^p$ should be very small whenever p is much greater than 2, and
- the price process D_t should satisfy $D_t = \sigma^2(T-t)$, and so $\sum_{t=1}^{T} |\Delta D_t|^p$ should be very small whenever p is much larger than 1.

Even with these constraints, there is no incoherency at the beginning of the game if T is sufficiently large (how large it needs to be depends on how small δ is), because Market can satisfy the constraints while preventing Investor from making any gains by setting $D_t = D_0(T-t)/T$ and $\Delta S_t / S_{t-1} = \pm\sqrt{D_0/T}$. Moreover, Market can prevent any incoherency from arising by choosing D_t and S_t so that these equations are approximately satisfied.

Shafer and Vovk show ([10], p. 249) that any European option with a payoff that is Lipschitzian can be replicated in this game, with error smaller than $8ce^{5C}\delta^{1/4}$, where c is the Lipschitz coefficient. The approximate cost of the replication is given by the Black-Scholes formula with D_0 in the place of $\sigma^2 T$. If we were to expand the game to allow trading in the European option as well as the underlying security and the dividend-paying derivative, then the principle of coherence would require Market to set the price of the European option equal to this cost of replication.

We can also think of this conclusion as a consequence of an approximate law of one price, which requires two strategies that give approximately the same final capital to start with approximately the same initial capital. Because the game is linear, such an approximate law of one price follows from the principle of coherence.

A Strategies and their capital processes

In this appendix, we lay out notation for internal strategies and capital processes for Investor in the general market game. This gives a concrete algebraic representation of these concepts, which may help readers who are not accustomed to thinking about strategies in perfect-information games.

A complete sequence of moves by Market in the general market game has the form

$$\mathbf{o}_1 \mathbf{c}_1 \ldots \mathbf{o}_T \mathbf{c}_T;$$

in other words, it is an element of the Cartesian product $(\mathbf{O} \times \mathbf{C})^T$. We set

$$\Theta := (\mathbf{O} \times \mathbf{C})^T,$$

and we call Θ the *sample space* for the game. We call an element of the sample space a *path*.

A partial sequence of moves by Market leading up to a move by Investor has the form

$$\mathbf{o}_1\mathbf{c}_1 \dots \mathbf{o}_{t-1}\mathbf{c}_{t-1}\mathbf{o}_t,$$

where $1 \le t < T$. We call a sequence of this form an *investing situation*. We write $\mathbf{S}$ for the set of investing situations. When a path θ has an investing situation $\mathbf{s}$ as an initial segment, say

$$\mathbf{s} = \mathbf{o}_1\mathbf{c}_1 \dots \mathbf{o}_{t-1}\mathbf{c}_{t-1}\mathbf{o}_t \quad \text{and} \quad \theta = \mathbf{s}\mathbf{c}_t \dots \mathbf{o}_T\mathbf{c}_T,$$

we say that θ *goes through* $\mathbf{s}$.

A partial sequence of moves leading up to a new value for Investor's capital has the form

$$\mathbf{o}_1\mathbf{c}_1 \dots \mathbf{o}_t\mathbf{c}_t,$$

where $0 \le t \le T$. (When $t = 0$, the sequence is empty; we write $\Box$ for the empty sequence.) We call a sequence of this form an *accounting situation*. We write $\mathbf{A}$ for the set of accounting situations. Notice that the sample space Θ is a subset of $\mathbf{A}$.

An *internal strategy* for Investor is defined by a mapping

$$\mathcal{P} : \mathbf{S} \to \mathbf{I};$$

$\mathcal{P}(\mathbf{s})$ is the move it recommends in investing situation $\mathbf{s}$.

An internal strategy $\mathcal{P}$ together with an initial value α for Investor's capital determines a *capital process* for Investor. This is the mapping

$$\mathcal{K}^{\mathcal{P},\alpha} : \mathbf{A} \to \mathbb{R}$$

defined by

$$\mathcal{K}^{\mathcal{P},\alpha}(\Box) = \alpha$$

and

$$\mathcal{K}^{\mathcal{P},\alpha}(\mathbf{o}_1\mathbf{c}_1 \dots \mathbf{o}_{t+1}\mathbf{c}_{t+1}) = \mathcal{K}^{\mathcal{P},\alpha}(\mathbf{o}_1\mathbf{c}_1 \dots \mathbf{o}_t\mathbf{c}_t)$$
$$+\lambda(\mathbf{o}_{t+1}, \mathcal{P}(\mathbf{o}_1\mathbf{c}_1 \dots \mathbf{o}_{t+1}), \mathbf{c}_{t+1})$$

Notice that

$$\mathcal{K}^{\mathcal{P},\alpha} = \alpha + \mathcal{K}^{\mathcal{P},0}.$$

If the market game is linear, Investor can form linear combinations of strategies, resulting in the corresponding linear combination of the capital processes:

$$\mathcal{K}^{a_1\mathcal{P}_1 + a_2\mathcal{P}_2, 0} = a_1\mathcal{K}^{\mathcal{P}_1,0} + a_2\mathcal{K}^{\mathcal{P}_2,0}$$

for any real numbers a_1 and a_2.

We leave it to the reader to give notation for an internal strategy and its capital process from an investing situation onward.

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Shashi Murthy called our attention to [5].

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