Possibility & Impossibility of Liquidity Adaptation in Prediction Markets

Rafael Frongillo, Harvard University

Based on joint works with Jacob Abernethy, Miroslav Dudík, Xiaolong Li, and Jennifer Wortman Vaughan

November 14, 2014

Liquidity

Prediction market:
offers securities contingent on some future outcome.

Liquidity

Prediction market:
offers securities contingent on some future outcome.

Liquidity:
extent to which traders can profit from knowledge about the future outcome.
I.e., magnitude of market incentives

Liquidity

Prediction market:
offers securities contingent on some future outcome.

Liquidity:
extent to which traders can profit from knowledge about the future outcome.
I.e., magnitude of market incentives

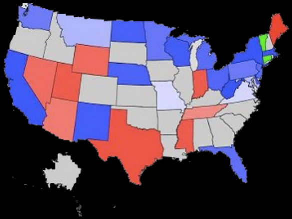
Our goal:
extend current prediction market frameworks to allow liquidity levels to change over time.

This Talk

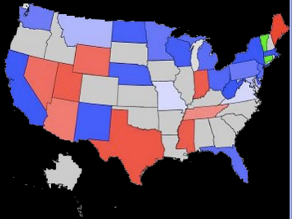
1. Increasing liquidity
   as market activity increases
2. Decreasing liquidity
   when information becomes less valuable

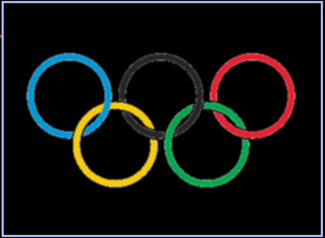
Prolog: Fundamentals

Setting: Complex Markets



Setting: Complex Markets





events: pay $1 iff Bob wins gold @ Men's Downhill Skiing

counts: pay $1 iff Norway wins at least 3 gold medals

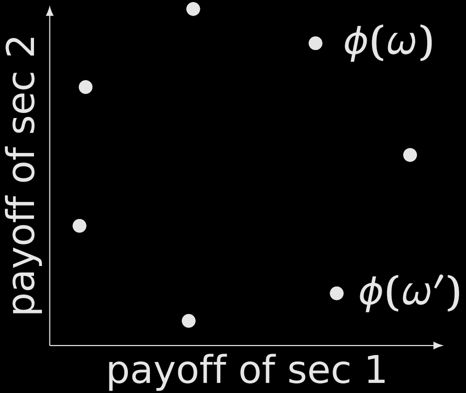
General Securities

- Outcome $\omega \in \Omega$

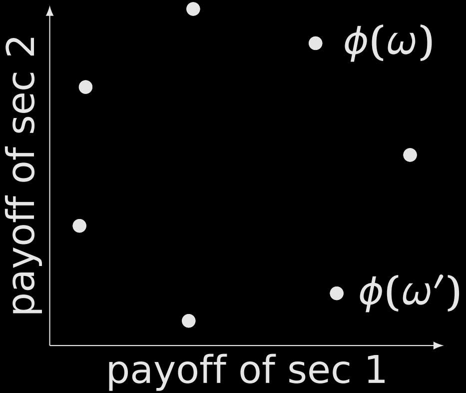
  E.g. $\Omega = \{$*all assignments of medals to athletes*$\}$
- k securities
- Payoffs encoded by $\phi : \Omega \to \mathbb{R}^k$

$$\phi(\omega) = \begin{bmatrix} \text{payoff of security 1 given } \omega \\ \vdots \\ \text{payoff of security k given } \omega \end{bmatrix}$$

Payoff Space

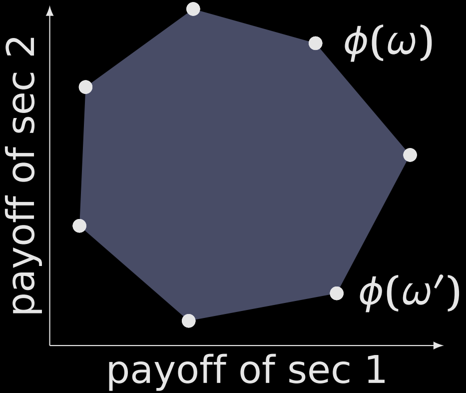


Payoff Space



Payoff space := conv $\phi(\Omega)$

Payoff Space



Payoff space := conv $\phi(\Omega)$

I. Increasing Liquidity

General Market Making

- Market maker offers to buy or sell any bundle $\boldsymbol{r} \in \mathbb{R}^k$ for a cost

General Market Making

- Market maker offers to buy or sell any bundle $\boldsymbol{r} \in \mathbb{R}^k$ for a cost

$$N(\boldsymbol{r};\ \boldsymbol{r}_1, \ldots, \boldsymbol{r}_T\ ) \in \mathbb{R}$$

General Market Making

- Market maker offers to buy or sell any bundle $\boldsymbol{r} \in \mathbb{R}^k$ for a cost

$$N(\boldsymbol{r};\ [\text{hist}]) \in \mathbb{R}$$

Net profit: $\boldsymbol{\phi}(\omega) \cdot \boldsymbol{r} - N(\boldsymbol{r};\ [\text{hist}])$

General Market Making

- Market maker offers to buy or sell any bundle $\boldsymbol{r} \in \mathbb{R}^k$ for a cost

$$N(\boldsymbol{r};\ [\text{hist}]\ ) \in \mathbb{R}$$

Net profit: $\boldsymbol{\phi}(\omega) \cdot \boldsymbol{r} - N(\boldsymbol{r}; [\text{hist}])$

Our focus: the design of N

Desiderata of N

- **WCL** – bounded worst-case loss

Desiderata of *N*

- **WCL** – bounded worst-case loss
- **ARB** – no arbitrage

Desiderata of N

- **WCL** – bounded worst-case loss
- **ARB** – no arbitrage
- **II** – information incorporation

Desiderata of *N*

- **WCL** – bounded worst-case loss
- **ARB** – no arbitrage
- **II** – information incorporation
- **L** – increasing liquidity

Desiderata of *N*

- **WCL** – bounded worst-case loss
- **ARB** – no arbitrage
- **II** – information incorporation
- **L** – increasing liquidity
- **SS** – shrinking bid-ask spread

Desiderata of *N*

- **WCL** – bounded worst-case loss
- **ARB** – no arbitrage
- **II** – information incorporation
- **L** – increasing liquidity
- **SS** – shrinking bid-ask spread

Desiderata of N

- **WCL** – bounded worst-case loss
- **ARB** – no arbitrage
- **II** – information incorporation
- **L** – increasing liquidity
- **SS** – shrinking bid-ask spread

II: $N(\boldsymbol{r}; [\text{hist}] \oplus \boldsymbol{r}) \geq N(\boldsymbol{r}; [\text{hist}])$

SS: $N(\boldsymbol{r}; [\text{hist}]) + N(-\boldsymbol{r}; [\text{hist}]) \xrightarrow{[\text{hist}] \to \infty} 0$

Scoreboard

Market maker	WCL	ARB	II	L	SS
Fixed Price	✗	✓	✓	✓	✓

$$N(\boldsymbol{r};[\,\text{hist}\,]) = \boldsymbol{\pi}\cdot\boldsymbol{r}$$

(for fixed price vector $\boldsymbol{\pi}$)

Scoreboard

Market maker	**WCL**	**ARB**	**II**	**L**	**SS**
Fixed Price	✗	✓	✓	✓	✓
Potential-based	✓	✓	✓	✗	✓

$$N(\boldsymbol{r};\boldsymbol{q}) = C(\boldsymbol{q}+\boldsymbol{r}) - C(\boldsymbol{q})$$

Scoreboard

Market maker	**WCL**	**ARB**	**II**	**L**	**SS**
Fixed Price	✗	✓	✓	✓	✓
Potential-based	✓	✓	✓	✗	✓
Profit-charging	✓	✓	✗	✓	✓

[Othman-Sandholm 2012]

Scoreboard

Market maker	WCL	ARB	II	L	SS
Fixed Price	✗	✓	✓	✓	✓
Potential-based	✓	✓	✓	✗	✓
Profit-charging	✓	✓	✗	✓	✓
Buy-only	✓	✓	✓	✓	✗

[Li-Vaughan 2013]

Scoreboard

Market maker	WCL	ARB	II	L	SS
Fixed Price	✗	✓	✓	✓	✓
Potential-based	✓	✓	✓	✗	✓
Profit-charging	✓	✓	✗	✓	✓
Buy-only	✓	✓	✓	✓	✗

Q: What other combinations can we achieve?

Scoreboard

Market maker	WCL	ARB	II	L	SS
Fixed Price	✗	✓	✓	✓	✓
Potential-based	✓	✓	✓	✗	✓
Profit-charging	✓	✓	✗	✓	✓
Buy-only	✓	✓	✓	✓	✗

Q: What other combinations can we achieve?

Q: Can we achieve all five??

Impossibility

Theorem

*No market $(\boldsymbol{\phi}, N)$ with at least two securities satisfies **WCL**, **ARB**, **II**, **L**, & **SS**.*

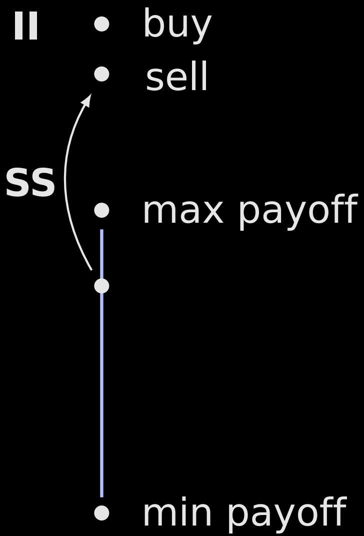
Proof Intuition

- buy
- max payoff
- sell
- min payoff

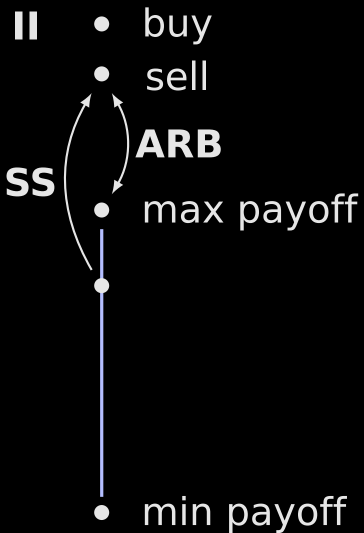
Proof Intuition

- buy
- max payoff
- sell
- min payoff

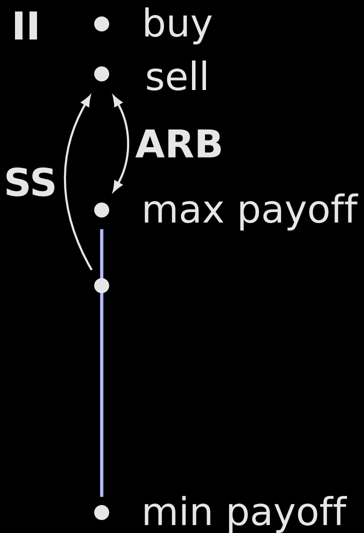
Proof Intuition

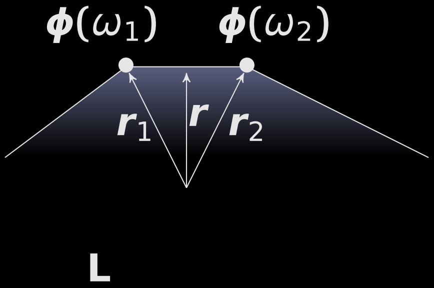


Proof Intuition

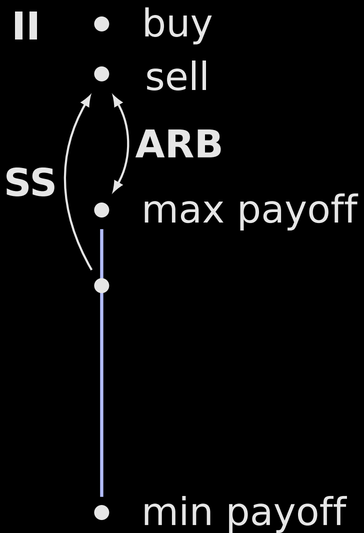


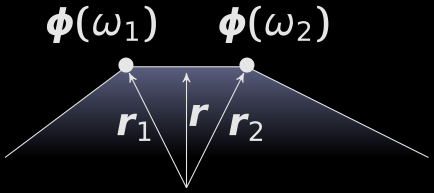
Proof Intuition





Proof Intuition





$$\mathbf{L} \implies \neg\mathbf{WCL}$$

Theorem

No market $(\boldsymbol{\phi}, N)$ *with at least two securities satisfies* ***WCL***, ***ARB***, ***II***, ***L***, *&* ***SS***.

Theorem

No market $(\boldsymbol{\phi}, N)$ *with at least two securities satisfies* ***WCL****,* ***ARB****,* ***II****,* ***L****, &* ***SS****.*

Q: What other combinations can we achieve?

Q: Can we achieve all five??

Theorem

No market $(\boldsymbol{\phi}, N)$ *with at least two securities satisfies* ***WCL***, ***ARB***, ***II***, ***L***, & ***SS***.

Q: What other combinations can we achieve? **?**

Q: Can we achieve all five?? ✗

Theorem

No market $(\boldsymbol{\phi}, N)$ *with at least two securities satisfies* ***WCL***, ***ARB***, ***II***, ***L***, & ***SS***.

Q: What other combinations can we achieve? **?**
Q: Can we achieve all five?? ✗

Introducing the Volume-Parameterized Market:

$$N(\boldsymbol{r}; \boldsymbol{q}, v) = C(\boldsymbol{q} + \boldsymbol{r}, v + \|\boldsymbol{r}\|) - C(\boldsymbol{q}, v)$$

Generalizes previous work, still tractable

Theorem

No market $(\boldsymbol{\phi}, N)$ with at least two securities satisfies ***WCL***, ***ARB***, ***II***, ***L***, & ***SS***.

Q: What other combinations can we achieve? **?**
Q: Can we achieve all five?? ✗

Introducing the Volume-Parameterized Market:

$$N(\boldsymbol{r}; \boldsymbol{q}, v) = C(\boldsymbol{q} + \boldsymbol{r}, v + \|\boldsymbol{r}\|) - C(\boldsymbol{q}, v)$$

Generalizes previous work, still tractable

Special case: Perspective Market (new)

Perspective Market

- **CPLX** – beyond $\boldsymbol{\phi}(\omega)_i = \mathbf{1}[\omega = \omega_i]$

Market maker	**WCL**	**ARB**	**II**	**L**	**SS**	**CPLX**
Fixed Price	✗	✓	✓	✓	✓	
Potential-based	✓	✓	✓	✗	✓	
Profit-charging	✓	✓	✗	✓	✓	
Buy-only	✓	✓	✓	✓	✗	

Perspective Market

- **CPLX** – beyond $\boldsymbol{\phi}(\omega)_i = \mathbf{1}[\omega = \omega_i]$

Market maker	**WCL**	**ARB**	**II**	**L**	**SS**	**CPLX**
Fixed Price	✗	✓	✓	✓	✓	✓
Potential-based	✓	✓	✓	✗	✓	✓
Profit-charging	✓	✓	✗	✓	✓	✗
Buy-only	✓	✓	✓	✓	✗	✗

Perspective Market

- **CPLX** – beyond $\boldsymbol{\phi}(\omega)_i = \mathbf{1}[\omega = \omega_i]$

Market maker	**WCL**	**ARB**	**II**	**L**	**SS**	**CPLX**
Fixed Price	✗	✓	✓	✓	✓	✓
Potential-based	✓	✓	✓	✗	✓	✓
Profit-charging	✓	✓	✗	✓	✓	✗
Buy-only	✓	✓	✓	✓	✗	✗
Perspective	✓	✓	✗	✓	✓	✓

Future Directions

Market maker	WCL	ARB	II	L	SS	CPLX
Fixed Price	✗	✓	✓	✓	✓	✓
Potential-based	✓	✓	✓	✗	✓	✓
Profit-charging	✓	✓	✗	✓	✓	✗
Buy-only	✓	✓	✓	✓	✗	✗
Perspective	✓	✓	✗	✓	✓	✓

Future Directions

Market maker	WCL	ARB	II	L	SS	CPLX
Fixed Price	✗	✓	✓	✓	✓	✓
Potential-based	✓	✓	✓	✗	✓	✓
Profit-charging	✓	✓	✗	✓	✓	✗
Buy-only	✓	✓	✓	✓	✗	✗
Perspective	✓	✓	✗	✓	✓	✓

Future Directions

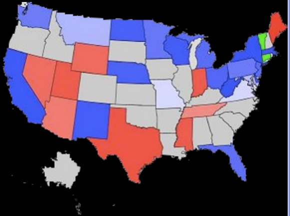
Market maker	WCL	ARB	II	L	SS	CPLX
Fixed Price	✗	✓	✓	✓	✓	✓
Potential-based	✓	✓	✓	✗	✓	✓
Profit-charging	✓	✓	✗	✓	✓	✗
Buy-only	✓	✓	✓	✓	✗	✗
Perspective	✓	✓	✗	✓	✓	✓
???	✓	✗	✓	✓	✓	✓

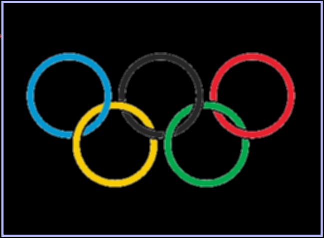
Future Directions

Market maker	**WCL**	**ARB**	**II**	**L**	**SS**	**CPLX**
Fixed Price	✗	✓	✓	✓	✓	✓
Potential-based	✓	✓	✓	✗	✓	✓
Profit-charging	✓	✓	✗	✓	✓	✗
Buy-only	✓	✓	✓	✓	✗	✗
Perspective	✓	✓	✗	✓	✓	✓
???	✓	✗	✓	✓	✓	✓

II. Decreasing Liquidity

Setting: Complex Markets





events: pay $1 iff Bob wins gold @ Men's Downhill Skiing

counts: pay $1 iff Norway wins at least 3 gold medals

Most Market Models:

Most Market Models:

1. Market opens

   Trading begins

Most Market Models:

1. Market opens

   Trading begins

2. Market closes

Most Market Models:

1. Market opens

   Trading begins

2. Market closes

3. Outcome revealed

   All security payoffs given to traders

Most Market Models:

1 Market opens

Trading begins

2 Market closes

3 Outcome revealed

All security payoffs given to traders

PROBLEM:
Winner of Men’s Downhill announced before Women’s Downhill takes place!

Just-in-Time Arbitrage

pay $1 iff Bob wins Men’s Downhill

price: $0.4

Just-in-Time Arbitrage

pay $1 iff Bob wins Men’s Downhill

price: $0.4

Bob wins Men’s Downhill!

Just-in-Time Arbitrage

pay $1 iff Bob wins Men’s Downhill

price: $0.4

Bob wins Men’s Downhill!

Buy buy buy buy buy buy buy...

Price ⟶ $1, *trader makes a huge profit*

Just-in-Time Arbitrage

pay $1 iff Bob wins Men's Downhill

price: $0.4

Bob wins Men's Downhill!

Buy buy buy buy buy buy buy...

Price ⟶ $1, *trader makes a huge profit*

Inefficient allocation of wealth!

Solutions?

- Close the market?

Solutions?

- Close the market?
  - *Other events to trade on!*

Solutions?

- Close the market?
  - *Other events to trade on!*
- Close the submarket?

Solutions?

- Close the market?
  Other events to trade on!
- Close the submarket?
  Counts and other related securities!

Solutions?

- Close the market?
  Other events to trade on!
- Close the submarket?
  Counts and other related securities!

Need new tools!

Cost Func Market Makers

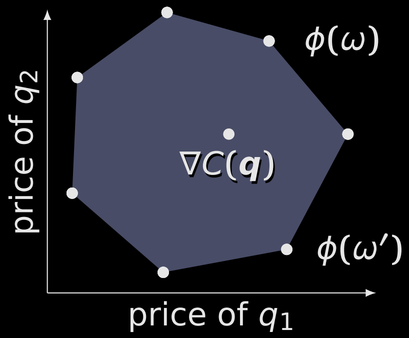
[Abernethy, Chen, Vaughan '11]

- Cost of bundle $\boldsymbol{r}$ is $C(\boldsymbol{q}+\boldsymbol{r}) - C(\boldsymbol{q})$
- Instantaneous price of security i: $\frac{\partial}{\partial q_i} C(\boldsymbol{q})$

Cost Func Market Makers

[Abernethy, Chen, Vaughan '11]

- Cost of bundle $\boldsymbol{r}$ is $C(\boldsymbol{q}+\boldsymbol{r}) - C(\boldsymbol{q})$
- Instantaneous price of security i: $\frac{\partial}{\partial q_i} C(\boldsymbol{q})$



Implicit Submarket Closing

- Let $\Omega = \Omega_1 \cup \Omega_2 \cup \; \Omega_3 \; \cup \cdots \cup \Omega_N$ *partition*

  $\Omega_x = \{\mathit{assignments\ where\ x\ wins\ Men's\ Downhill}\}$

Implicit Submarket Closing

- Let $\Omega = \Omega_1 \cup \Omega_2 \cup \boxed{\Omega_3} \cup \cdots \cup \Omega_N$ *partition*

  $\Omega_x = \{$*assignments where x wins Men's Downhill*$\}$

- At time t, traders learn x $(\omega \in \Omega_x)$

  x = winner of Men's Downhill

Implicit Submarket Closing

- Let $\Omega = \Omega_1 \cup \Omega_2 \cup \boxed{\Omega_3} \cup \cdots \cup \Omega_N$ *partition*

  $\Omega_x = \{$*assignments where x wins Men's Downhill*$\}$

- At time t, traders learn x $(\omega \in \Omega_x)$

  $x =$ *winner of Men's Downhill*

- Market maker knows partition and time t, but ***not*** Ω_x

Implicit Submarket Closing

GOAL:

At time t, swap cost function $C \rightarrow \tilde{C}$ so that:

Implicit Submarket Closing

GOAL:

At time t, swap cost function $C \rightarrow \tilde{C}$ so that:

1 Traders cannot profit from knowing Ω_x

Implicit Submarket Closing

GOAL:

At time t, swap cost function $C \rightarrow \tilde{C}$ so that:

1. Traders cannot profit from knowing Ω_x
2. Traders rewarded as before for all other info

Implicit Submarket Closing

GOAL:

At time t, swap cost function $C \to \tilde{C}$ so that:

1. Traders cannot profit from knowing Ω_x
2. Traders rewarded as before for all other info
3. Information already gathered is preserved

Implicit Submarket Closing

GOAL:

At time t, swap cost function $C \rightarrow \tilde{C}$ so that:

1. Traders cannot profit from knowing Ω_x
2. Traders rewarded as before for all other info
3. Information already gathered is preserved

$\mathtt{Util}_C(I; \boldsymbol{q})$: max profit a trader could make knowing information I at state $\boldsymbol{q}$

Single Security

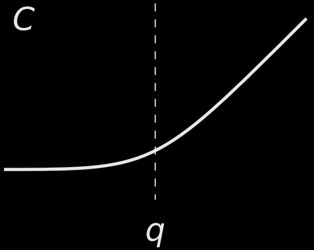
$\Omega_1 = \{$  $\}$

$\Omega_0 = \{$  $\}$

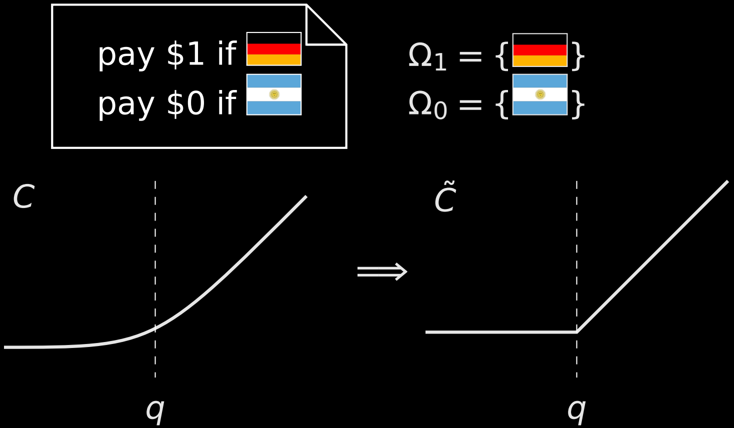
Single Security

pay \$1 if 
pay \$0 if 

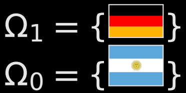
$\Omega_1 = \{$  $\}$
$\Omega_0 = \{$  $\}$

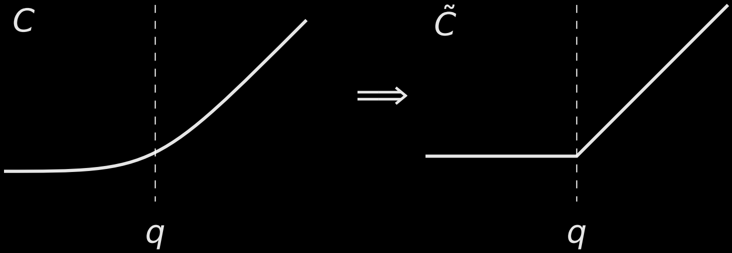


Single Security



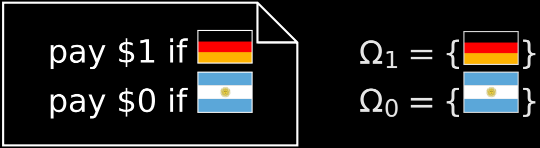
Single Security

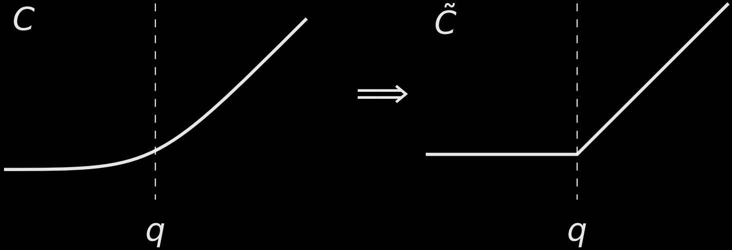




1 No profit for Ω_x

Single Security





1 No profit for Ω_x

*(**implicitly** closed)*

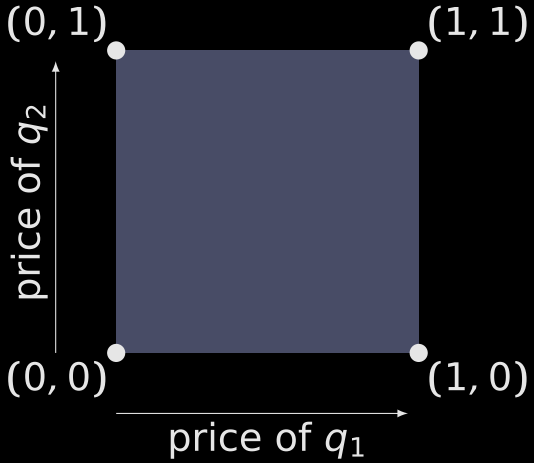
Two binary securities, learn 1 before 2

Two binary securities, learn 1 before 2

$$C(\boldsymbol{q}) = C_1(q_1) + C_2(q_2)$$

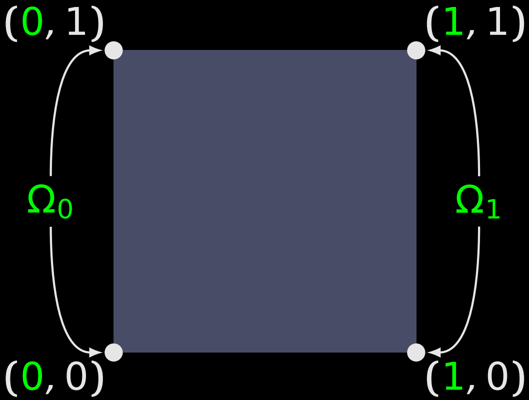
Two binary securities, learn 1 before 2

$$C(\boldsymbol{q}) = C_1(q_1) + C_2(q_2)$$



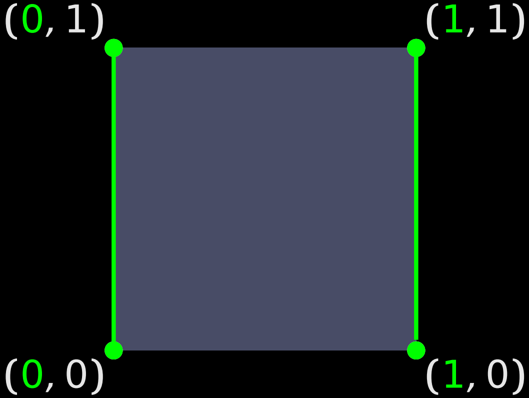
Two binary securities, learn 1 before 2

$$C(\boldsymbol{q}) = C_1(q_1) + C_2(q_2)$$



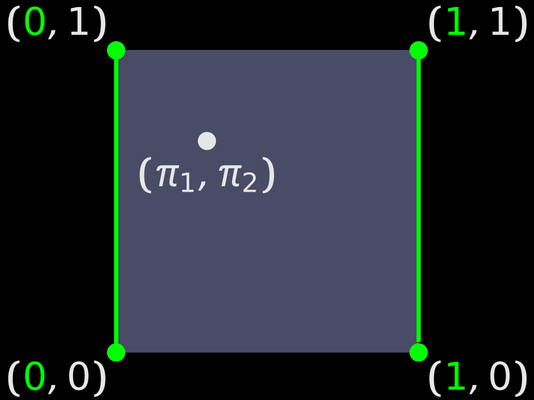
Two binary securities, learn 1 before 2

$$C(\boldsymbol{q}) = C_1(q_1) + C_2(q_2)$$



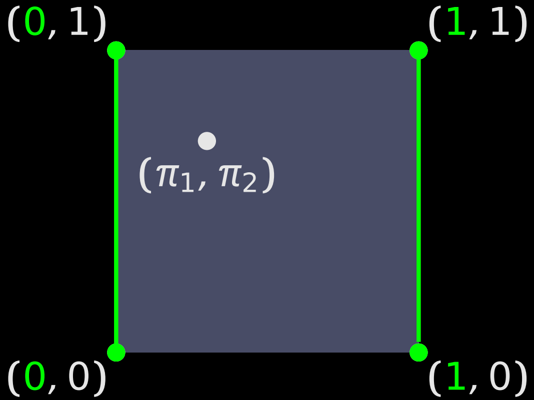
Two binary securities, learn 1 before 2

$$C(\boldsymbol{q}) = C_1(q_1) + C_2(q_2)$$



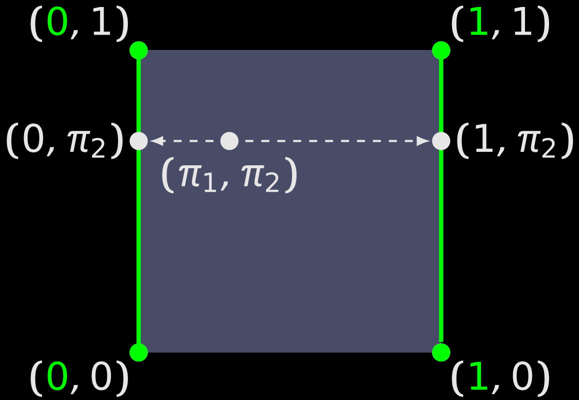
Two binary securities, learn 1 before 2

$$\tilde{C}(\boldsymbol{q}) = \tilde{C}_1(q_1) + C_2(q_2)$$



Two binary securities, learn 1 before 2

$$\tilde{C}(\boldsymbol{q}) = \tilde{C}_1(q_1) + C_2(q_2)$$

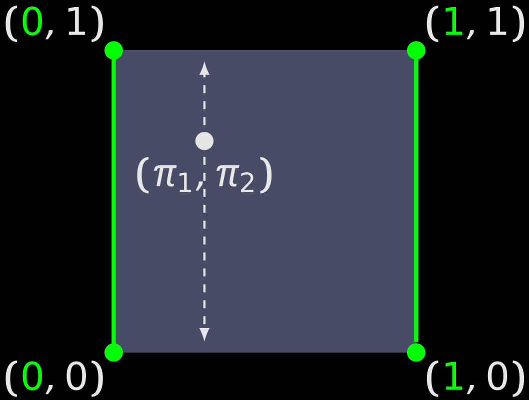


1 No profit for Ω_x

3 Information preserved

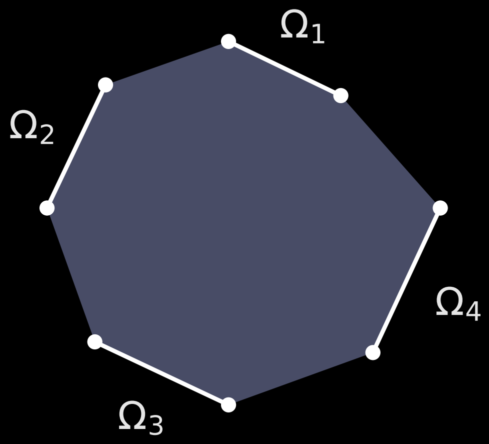
Two binary securities, learn 1 before 2

$$\tilde{C}(\boldsymbol{q}) = \tilde{C}_1(q_1) + C_2(q_2)$$



2 Same rewards for q_2

More Complicated Example (?)



Theorem

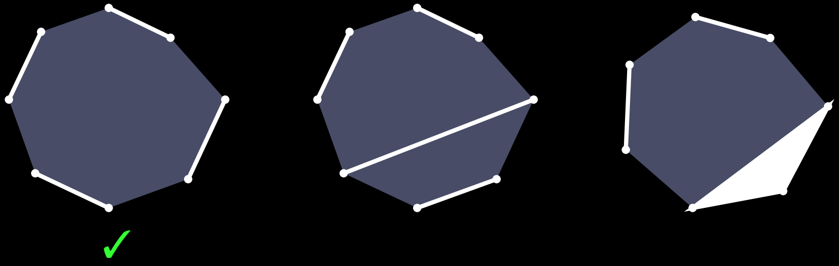
Implicit submarket closing is possible if:

> Conditional price spaces are *faces* of the price space

Theorem

Implicit submarket closing is possible if:

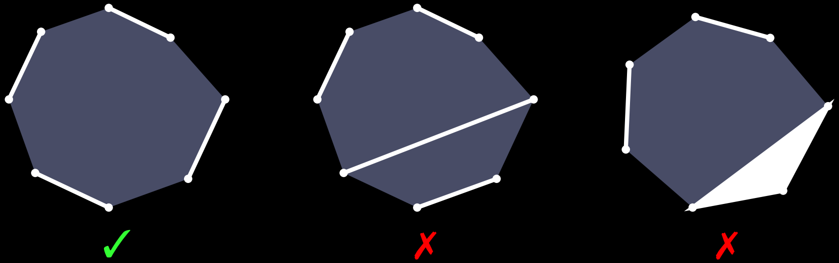
Conditional price spaces are *faces* of the price space



Theorem

Implicit submarket closing is possible if:

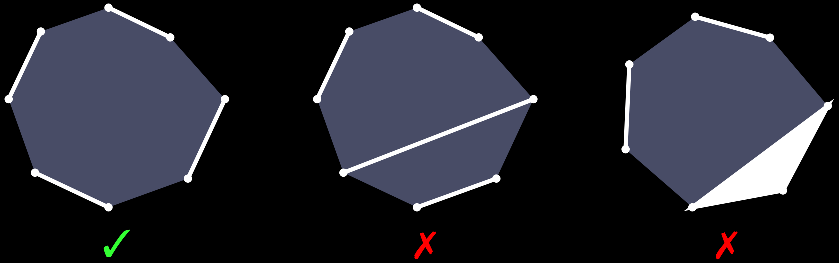
Conditional price spaces are *faces* of the price space



Theorem

Implicit submarket closing is possible if:

Conditional price spaces are *faces* of the price space

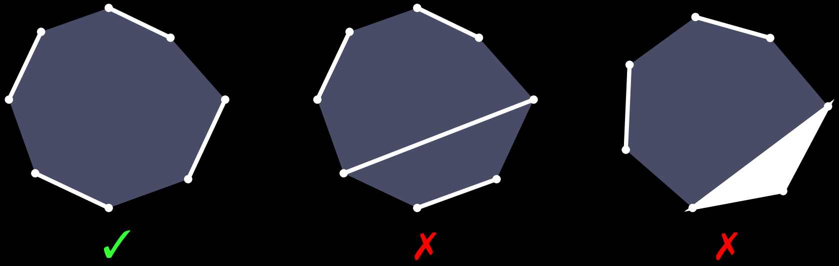


Holds for any C

Theorem

Implicit submarket closing is possible if:

Conditional price spaces are *faces* of the price space



Holds for any C

Can always add securities to satisfy

Lemma: $\texttt{Util}$ = Breg Divergence

Let $R = C^*$,

$$D_R(\pi, \pi') = R(\pi) - R(\pi') - \nabla R(\pi') \cdot (\pi - \pi')$$

1 $$\texttt{Util}_C(X = x; \boldsymbol{q}) = \min_{\boldsymbol{\mu}' \in \operatorname{conv} \phi(\Omega_x)} D_R(\boldsymbol{\mu}', \nabla C(\boldsymbol{q}))$$

2 $$\texttt{Util}_C(\mathbb{E}[\phi] = \boldsymbol{\mu}; \boldsymbol{q}) = D_R(\boldsymbol{\mu}, \nabla C(\boldsymbol{q}))$$

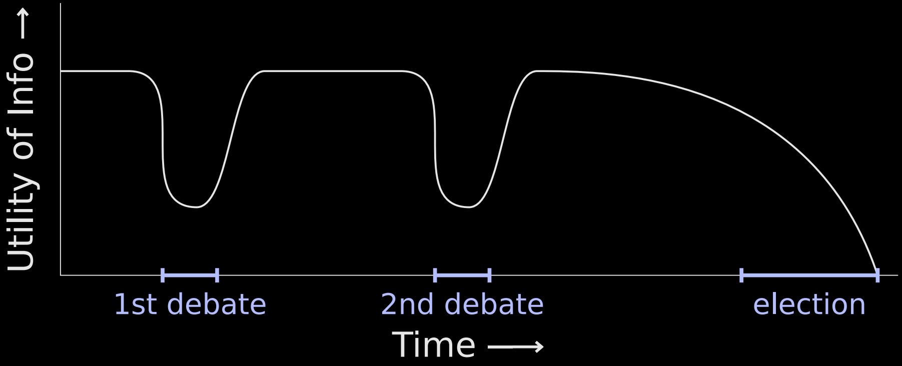
3 $$\pi_C(X = x; \boldsymbol{q}) = \operatorname*{argmin}_{\boldsymbol{\mu}' \in \operatorname{conv} \phi(\Omega_x)} D_R(\boldsymbol{\mu}', \nabla C(\boldsymbol{q}))$$

Gradual Setting

- Implicit submarket closing = sudden drop in utility of info
- Also consider *gradual* decrease
  - *E.g. unemployment statistics for 2014*

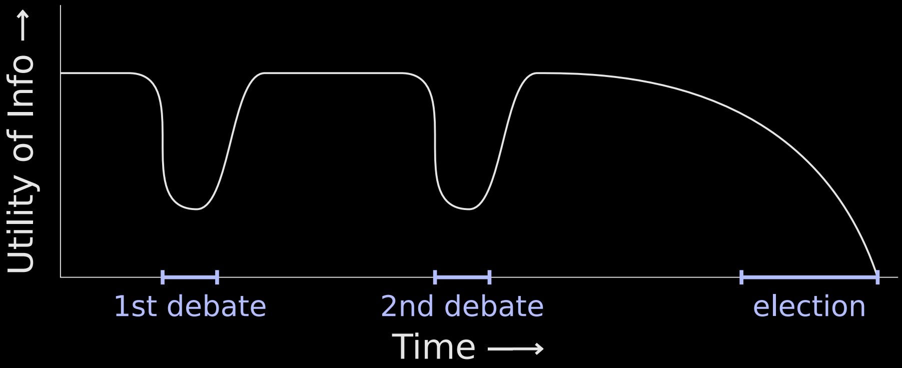
Future

1 Responding to information shocks



Future

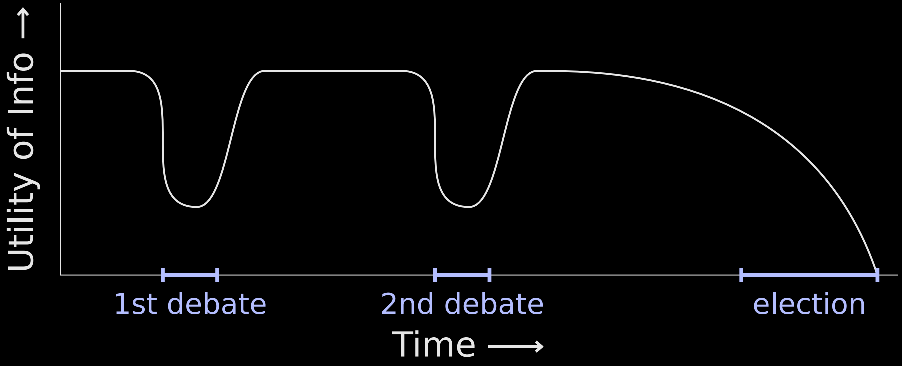
1 Responding to information shocks



2 Compare to real market making data

Thanks!!

1 Responding to information shocks



2 Compare to real market making data