Research concerning the probabilities of the errors which happen in making observations, &c.

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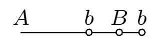
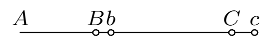
The question which I propose to resolve is this: Supposing 

AB to be the true value of any quantity, of which the measure by observation or experiment is Ab, the error being Bb; what is the expression of the probability that the error Bb happens in measuring AB?

Let AB, BC, &c. be several successive distances of 

which the values by measure are Ab, bc, &c. the whole error being Cc: now supposing the measures Ab, bc, to be given and also the whole error Cc, we assume as an evident principle that the most probable distances AB, BC are proportional to the measures Ab, bc; and therefore the errors belonging to AB, BC are proportional to their lengths, or to their measured values Ab, bc. If therefore we represent the values of AB, BC, or of their measures Ab, bc, by a, b, the whole error Cc by E, and the errors of the measures Ab, bc, by x, y, we must, for the greatest probability, have the equation $\frac{x}{a} = \frac{y}{b}$.

Let X and Y be similar functions of a, x, and of b, y, expressing the probabilities that the errors x, y, happen in the distances a, b; and, by the fundamental principle of the doctrine of chance, the probability that both these errors happen together will be expressed by the product XY. If now we were to determine the values of x and y from the equations $x + y = E$, and $XY =$maximum, we ought evidently to arrive at the equation $\frac{x}{a} = \frac{y}{b}$: and since x and y are rational functions of the simplest order possible of a, b and E, we ought to arrive at the equation $\frac{x}{a} = \frac{y}{b}$ without the intervention of roots, in other words by simple equations; or, which amounts to the same thing in effect, if there be several forms of X and Y that will fulfill the required condition, we must choose the simplest possible, as having the greatest possible degree of probability.

Let X', Y', be the logarithms of X and Y, to any base or modulus e: and when $XY = \max.$ its logarithm $X' + Y' = \max.$ and therefore $\dot{X}' + \dot{Y}' = 0$; which fluxional equation we may express by $X''\dot{x} + Y''\dot{y} = 0$; for as X' involves only the variable quantity x, its fluxion $\dot{X}'$ will evidently involve only the fluxion of x; in like manner the fluxion of Y' may be expressed by $Y''\dot{y}$; and from the equation $X''\dot{x} + Y''\dot{y} = 0$ we have $X''\dot{x} = -Y''\dot{y}$: but since $x + y = E$ we have also $\dot{x} + \dot{y} = 0$, and $\dot{x} = -\dot{y}$ by which dividing the equation $X''\dot{x} = -Y''\dot{y}$, we obtain $X'' = Y''$.

Now this equation ought to be equivalent to $\frac{x}{a}=\frac{y}{b}$; and this circumstance is effected in the simplest manner possible, by assuming $X''=\frac{mx}{a}$, and $Y''=\frac{my}{b}$; m being any fixed number which the question may require.

Since therefore $X''=\frac{mx}{a}$, we have $X''\dot{x}=\dot{X}'=\frac{mx\dot{x}}{a}$, and taking the fluent, we have $X'=a'+\frac{mx^2}{2a}$.

The constant quantity a' being either absolute, or some function of the distance a.

We have discovered therefore, that the logarithm of the probability that the error x happens in the distance a is expressed by $a'+\frac{mx^2}{2a}=X'$, and consequently the probability itself is

$$X=e^{X'}=e^{\left(a'+\frac{mx^2}{2a}\right)}.$$

Such is the formula by which the probabilities of different errors may be compared, when the values of the determinate quantities e, a', and m are properly adjusted. If this probability of the error x be denoted by u, the ordinate of a curve to the abscissa x, we shall have $u=e^{\left(a'+\frac{mx^2}{2a}\right)}$, which is the general equation of *the curve of probability.*

When only the maximum of probability is required, we have no need of the values of e, a' and m; it is proper however to observe that m must be negative. This is easily shown. the probability that the errors x, y, z, &c. happen in the distances a, b, c, &c. is

$$e^{\left(a'+\frac{mx^2}{2a}\right)}\times e^{\left(b'+\frac{my^2}{2b}\right)}\times e^{\left(c'+\frac{mz^2}{2c}\right)},\ \text{\&c.}$$

which is equal to

$$e^{\left(a'+b'+c'+\frac{mx^2}{2a}+\frac{my^2}{2b}+\frac{mz^2}{2c}\right)},$$

and this quantity will evidently be a maximum or minimum as its index or logarithm

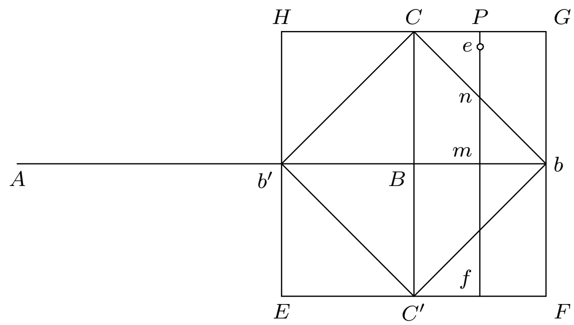
$$a'+b'+c'+\frac{mx^2}{2a}+\frac{my^2}{2b}+\frac{mz^2}{2c}$$

is a maximum or minimum, that is when $\frac{m}{2}\times\left\{\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\ \text{\&c.}\right\}=$ a maximum or minimum. Now when $x+y+z+\text{\&c.}=E$, we know that $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}$ &c. $=\min.$ when $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, &c. and therefore $-\left\{\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}+\text{\&c.}\right\}=$ maximum when $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=$&c. it is evident therefore that m must be negative, and as we may for the case of maxima use any value of it we please we may put $m=-2$, and the probability of x in a is $u=e^{\left(a'-\frac{x^2}{a}\right)}$.

If we put $\frac{m}{2a}=-1$ and $a'=f^2$, we have $u=e^{(f^2-x^2)}$ for the equation of the curve of probability; but if we suppose $f^2=0$, the ordinates u will still be proportional to their former values, and we shall have $u=e^{-x^2}$, or $u=\frac{1}{e^{x^2}}$, which is the simplest form of the equation expressing the nature of the curve of probability.

We shall now confirm what has been said by a different method of investigation.

Suppose that the length and bearing of AB are to be measured; and that the little equal straight lines Bb, BC, are the equally probable errors, the one $Bb = Bb'$ of the length of AB, the other $BC = BC'$ (perpendicular to the former) of the angle at A, when measured on a circular arc to the radius AB: and let the question be to find such a curve passing through the four points b, C, b', C', which are equally distant from B, that, supposing the measurement to commence at A, the probability of terminating on any point of the curve may be the same as the probability of terminating on any one of the four points b, C, b', C'.



Describe the squares $bCb'C$, $EFGH$. I say the curve sought must pass within the greater square $EFGH$, but without the less square $bCb'C'$.

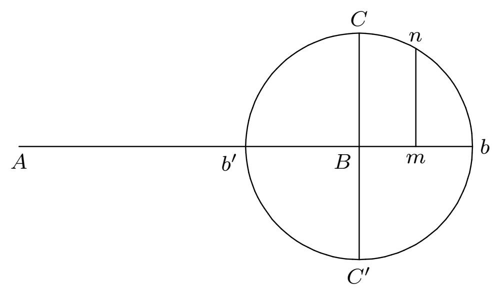
Let $mneP$ be drawn parallel to BC; and since the probabilities of the indefinitely little equal errors BC, mP, are ultimately in the ratio of equality; but the probability of the error Bm in the distance is less than the probability of the error 0 at B, (for it is self evident that the greater the error is, the less is its probability) therefore, by the laws of chance, the probability of terminating on P is less than that of terminating on C,and therefore the point P is without the curve sought.

By the same argument we prove that bG is without the curve.

Again, since the sum of the two errors Bm, mn, in distance and bearing, is together equal to the error Bb, it follows that the probability of terminating on n is greater than that on b: for it certainly is more reasonable to suppose that each of two equal sources of error should produce a part of the whole error $Bm + mn = Bb$, than that the whole error Bb should be produced from one of these sources alone, without any assistance from the other.

The same thing may also be shown thus, the probability of mn is the same as if it were reckoned on BC from B, and therefore the probability of mn is greater than that of mb, because any particle of error in Bb or BC is always less probable as that particle is farther removed from the point B, and of course the point n is within the curve; and therefore the curve must fall without the square $bCb'C'$. This curve therefore passes through the four points b, C, b', C' equally distant from B, and lies in the four triangles, bGC, CHb', $b'EC'$, and $C'Fb$.

Farther, the curve in question ought evidently to be continuous, and have its four portions similar which lie in the four triangles bGC, CHb', &c. Its arcs proceeding from b to C or from C to b ought to be similar to each other, and to each of those proceeding from C, b', and C'. It must have two and only two ordinates me, mf to the same abscissa Bm; and those ordinates must be equal, the one positive, the other negative. The value of the ordinate must be the same whether the error Bm be positive or negative, that is in excess, or defect. The equation of the curve must therefore have two equal values of the ordinate $y = ne = nf$ to the same abscissa $= x$; and the abscissa x must have two equal values to the same value of the ordinate y. Lastly, the curve must be the simplest possible having all the preceding conditions, and must consequently be the circumference of a circle having its centre in B.



Now let us investigate the probability of the error $Bm = x$, and of $mn = y$.

Let X and Y be two similar functions of x and y denoting those probabilities, X', Y' their logarithms, then $X \times Y =$ constant, or $X' + Y', =$ constant, and therefore $\dot{X}' + \dot{Y}' = 0$, or $X''\dot{x} + Y''\dot{y} = 0$, whence $X''\dot{x} = -Y''\dot{y}$.

But $x^2 + y^2 = r^2 = Bb^2$ therefore $x\dot{x} = -y\dot{y}$, by which dividing $X''\dot{x} = -Y''\dot{y}$, we have $\frac{X''}{x} = \frac{Y''}{y}$; and therefore, by a fundamental principle of similar functions, the similar functions $\frac{X''}{x}$ and $\frac{Y''}{y}$ must be each a constant quantity: put then $\frac{X''}{x} = n$, and we have $X''\dot{x} = nx\dot{x}$, that is $\dot{X}' = nx\dot{x}$, and the fluent is $X' = C + \frac{nx^2}{2}$; in like manner we find $Y' = C + \frac{ny^2}{2}$, and therefore the probabilities themselves are $e^{C+\frac{nx^2}{2}}$, and $e^{C+\frac{ny^2}{2}}$, in which n ought to be negative, for the probability of x grows less as x grows greater.

If now we put the constant quantities C and n equal to a', and $\frac{m}{2a}$ we have $u = e^{\left(a' + \frac{m}{2a}x^2\right)}$ as before.

As the application of this formula to maxima and minima does not require the value of a' or m we shall suppose $a' = o$, and $m = -2$, in which case we have $u = e^{-\frac{x^2}{a}}$.

If there be only one quantity to which the errors relate, we may put $C = 0$, and $\frac{n}{2} = -1$, in which case $u = e^{-x^2}$, in which u is the probability of the error x, or the ordinate of the curve of probability to the abscissa x.

Suppose now that the equally probable errors Bb, Bc are in any proposed ratio of 1 to p; let Bb and Bc be expressed by x and X, and supposing that $e^{-\frac{x^2}{a}}$ is the probability of x, I say the probability of X will be $e^{-\frac{X^2}{pa}}$.

For since $1 : p :: x : X$, therefore $x = \frac{X}{p}$, $x^2 = \frac{X^2}{p^2}$, and $e^{-\frac{x^2}{a}} = e^{-\frac{X^2}{p^2 a}}$.

In this case the curve of *equal probability* is an ellipse. We shall now show the use of this theory in the solution of the following problems.

PROBLEM I.

Supposing a, b, c, d, &c. to be the observed measures of any quantity x, the most probable value of x is required.

The several errors are $x - a$, $x - b$, $x - c$, $x - d$, &c. and the logarithms of their probabilities are, by what has just been shown, $-(x - a)^2$, $-(x - b)^2$, $-(x - c)^2$, &c. therefore $(x - a)^2 + (x - b)^2 + (x - c)^+ (x - d)^2 + \text{\&c.} = \text{min.}$

The fluxion of this divided by $2\dot{x}$ gives us $x - a + x - b + x - c + x - d, \text{\&c.} = 0$: let n be the number of terms and we have $nx = a + b + c + d, \text{\&c.}$ to n terms, therefore $\frac{a+b+c+d,\ \text{\&c.}}{n} = \frac{s}{n}$ putting s for the sum of a, b, c, &c.

Hence the following rule:

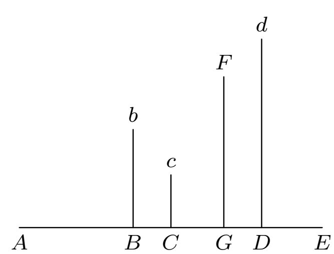
Divide the sum of all the observed values by their number, and the quotient will be the most probable value required.

This rule coincides exactly with that commonly practised by astronomers, navigators, &c.

It is worthy of notice, that, according to the solution given above, the sum of all the errors in excess is precisely equal to the sum of all the errors in defect; in other words, the sum of all the errors is precisely equal to 0, each error being taken with its proper sign: this is evident from the equation $(x - a) + (x - b) + (x - c) + (x - d) + \text{\&c.} = 0$.

PROBLEM II.

Given the observed positions of a point in space, to find the most probable position of the point.



On any fixed plane let fall perpendiculars from all the points of position, and also from the point sought, meeting the plane in b, c, d, &c. and in F, and on the straight line AE given in position in this plane let fall the perpendiculars bB, cC, &c. Take A any fixed point in AE, and let AB, AC, AD, &c. be denoted by a, b, c, &c. Bb, Cc, Dd, &c. by a', b', c', &c. the altitudes at b, c, d, &c. by a'', b'', c'', &c. also let the sums of a, b, &c. a', b', &c. a'', b'', &c. be denoted by s, s', s'', and the number of given points by n; finally, let the three co-ordinates of the points sought, viz. AG, GF, and the altitude of the point above the plane AGF, be denoted by x, y, and z.

Now the square of the distance from F to b is $(x - a)^2 + (y - b)^2$, and the difference of the altitudes at F and b is $z - a''$, therefore the square of the first error in distance is

$$(x - a)^2 + (y - a')^2 + (z - a'')^2$$

The square of the 2d is $(x - b)^2 + (y - b')^2 + (z - b'')^2$

of the 3d $(x - c)^2 + (y - c')^2 + (z - c'')^2$

&c. &c.

By the preceding theory, the probability of all these errors will be the maximum when the sum of their squares is a minimum, therefore, since each of the three quantities x, y, and z, is independent, the three following expressions must each be a minimum, viz.

$$\begin{aligned}
(x-a)^2+(x-b)^2+(x-c)^2+\&c. &= \min. \\
(y-a')^2+(y-b')^2+(y-c')^2+\&c. &= \min. \\
(z-a'')^2+(z-b'')^2+(z-c'')^2+\&c. &= \min.
\end{aligned}$$

These three equations put into fluxions and divided by $2\dot{x}$, $2\dot{y}$, and $2\dot{z}$ respectively, become

$$\begin{aligned}
x-a+x-b+x-c+\&c. = 0, \\
y-a'+y-b'+y-c'+\&c. = 0, \\
z-a''+z-b''+z-c''+\&c. = 0,
\end{aligned}$$

Whence,

$$x=\frac{a+b+c,\ \&c.}{n},\quad y=\frac{a'+b'+c',\ \&c.}{n},\quad z=\frac{a''+b''+c'',\ \&c.}{n}$$

that is,

$$x=\frac{s}{n},\quad y=\frac{s'}{n},\quad z=\frac{s''}{n}.$$

Whence this rule: divide the sum of each system of ordinates by the number of given points, and the three quotients will be the three orthosonal co-ordinates of the most probable point required.

From this solution we may deduce the following remarkable consequences.

I. The point sought is so situated that the sum of the errors estimated in any direction whatever is precisely equal to 0.

II. The point sought is precisely in the centre of gravity of all the given points, those points being supposed all equal: this is easily shown.

Let p be the mass at any point, and $np = M$ will be the whole mass at all the points, then we have

$$\begin{aligned}
x &= \frac{a+b+c\ \&c.}{n} &= \frac{ap+bp+cp+\&c.}{np} &= \frac{ap+bp+cp\ \&c.}{M} \\
y &= \frac{a'+b'+c'\ \&c.}{n} &= \frac{a'p+b'p+c'p+\&c.}{np} &= \frac{a'p+b'p+c'p\ \&c.}{M} \\
z &= \frac{a''+b''+c''\ \&c.}{n} &= \frac{a''p+b''p+c''p+\&c.}{np} &= \frac{a''p+b''p+c''p\ \&c.}{M}
\end{aligned}$$

and these last values of x, y, z, are well known to be the three proper expressions for the three rectangular co-ordinates of the centre of gravity of all the equal quantities of matter p, p', p'', &c. Hence it follows that if a point be sought such that the sum of the squares of its distances from any number of fixed points may be a minimum, the point required will be the center of gravity of all the fixed points.

Hence also, if a point be sought such that when the squares of its distances from any number of fixed points are multiplied by the fixed numbers p, p', p'', &c. respectively, the sum may be a minimum; the required point will be precisely in the centre of gravity of all the fixed points, the quantities of matter at those points being supposed p, p', p'', respectively.

III. The data of prob. ii. being still supposed, if the *locus* of a point be required, such, that a point situated any where on it may have an equal degree of probability to be the point sought, we may determine the *locus* from the original formulas of the preceding solution.

We must evidently have the following equation,

$$\left\{\begin{array}{l}(x-a)^2+(x-b)^2+(x-c)^2+\&\text{c.}\\(y-a')^2+(y-b')^2+(y-c')^2+\&\text{c.}\\(z-a'')^2+(z-b'')^2+(z-c'')^2+\&\text{c.}\end{array}\right\}=nD^2=\text{const.}$$

that is

$$\left\{\begin{array}{l}nx^2-2sx+a^2+b^2+c^2=\&\text{c.}\\ny^2-2s'y+a'^2+b'^2+c'^2=\&\text{c.}\\nz^2-2s''z+a''^2+b''^2+c''^2=\&\text{c.}\end{array}\right\}=nD^2,$$

which, by putting a^2+b^2 &c. $+\,a'^2+b'^2$ &c. $+\,a''^2-$ &c. $=n'D^2$, dividing by n, and putting x', y', z' for the values of $\frac{s}{n}$, $\frac{s'}{n}$, $\frac{s''}{n}$, becomes

$$\left\{\begin{array}{l}x^2-2x'x\\y^2-2y'y+D'^2\\z^2-2z'z\end{array}\right\}=D^2,$$

and this last by completing the squares, transposing D'^2, and putting $D^2-D'^2+x'+y'+z'=r$, is converted into

$$(x'-x)^2+(y'-y)^2+(z'-z)^2=r^2,$$

which is manifestly the equation of a spherical surface having its radius r, and its centre in the point of greatest probability.

If therefore the locus of a point be required, such, that the sum of the squares of its distances from any number of fixed points may be a constant quantity, the locus sought will be a spherical surface having its centre in the centre of gravity of all the fixed points considered as equal to one another.

Hence also, if the *locus* of a point be required, such, that when the squres of its distances from any number of fixed points are respectively multiplied by the fixed numbers p, p', p'', &c. the sum of all the products may be a constant quantity, the *locus* sought will be the surface of a sphere having its centre in the common centre of gravity of all the points; the quqantities of matter in the several points being respectively p, p', p'', &c.

IV. If the *locus* of equal probability (still retaining the data of prob. ii.) be restricted to a given surface, it is clear that the *locus* sought will be the line which is the common intersection of the given surface, and of a spherical surface having its centre in the point of greatest probability: and the equations of the given surface, and of the spherical surface, when referred to the same system of rectangular co-ordiantes will be the two equations of the *locus* required. If the given surface were plane, or spherical, the *locus* of equal probability would be the circumference of a circle.

And therefore, if the *locus* of a point moving on a given surface be required, such, that when the squares of its distances from any number of fixed points are multiplied respectively by the numbers p, p', p'', &c. the sum of all the products may be a constant quantity; the *locus* sought will be the common intersection of the given surface, and of a spherical surface having its centre in the common centre of gravity of all the points, their quantities of matter being supposed to be expressed by the fixed number p, p', p'', &c. respectively.

V. The same data still remaining, we may also determine the points of greatest and least probability on any given line or surface.

Let $V = 0$ be the equation of the given line or surface, referred to the same system of co-ordinates x, y, z. In this case let the fluxion of two equations $V = 0$, and

$$\left\{ \begin{array}{l} (x-a)^2 + (x-b)^2 + (x-c)^2 + \&c. \\ (y-a')^2 + (y-b')^2 + (y-c')^2 + \&c. \\ (z-a'')^2 + (z-b'')^2 + (z-c'')^2 + \&c. \end{array} \right\} = \max. \text{ or } \min.$$

be taken; and exterminating any one of the three fluxions $\dot{x}$, $\dot{y}$, $\dot{z}$, let the coefficients of the other two fluxions in the resulting equation be put each $= 0$: this will give two equations, from which and the equation $V = 0$, the values of x, y, and z may be determined by the common rules of algebra.

We may also give a geometrical plan of solution as follows; let a straight line be drawn from the point of greatest probability perpendicular to the given line or surface, and the points of intersection will be those of the maxima or minima required.

In the very same manner we determine the position of a point on a given line or surface, such, that when the squares of its distances from any number of fixed points are respectively multiplied by p, p', p'', &c. the sum of all the products may be a maximum or minimum.

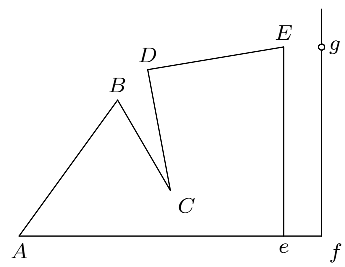
From the centre of gravity of all the fixed points having the quantities of matter p, p', p'', &c. let a straight line be drawn perpendicularly to the given figure, and the intersections will give the points of maxima and minima required.

PROBLEM III.

To correct the Dead Reckoning at sea, by an observation of the latitude.

SOLUTION.

Let $ABCDE$ be a traverse of which the difference of latitude, and departure, according to the dead reckoning are Ae, Ee; the true difference of latitude being Af; and let fg be parallel to eE.



Now the position of the points B, C, D, &c. must be changed in such a manner, that the last point E may fall somewhere on the true parallel fg, and the probability of all these changes must be a maximum.

Let a, b, c, &c. A, B, C, &c. be the lengths and bearings of AB, BC, &c. the radius being unity; x, y, z, &c. X, Y, Z, &c. the motions or translations of the angular points B, C, D, &c. in the directions AB, BC, &c. and in directions perpendicular to the former; D', D'', D''', &c. L', L'', L''', &c. the several changes in departure and latitude, and $L = ef$ the whole error in latitude.

The several departures are a, $\sin A$, b, $\sin B$, &c. if therefore a, and A were variable, the fluxion of the departure would evidently be

$$\sin A.\dot{a} + a\cos A.\dot{A},$$

which by putting x and X for $\dot{a}$ and $a.\dot{A}$, &c. gives us the equations

$$\begin{aligned} D' &= \sin A.x + \cos A.X, \\ D'' &= \sin B.y + \cos B.Y \\ D''' &= \sin C.z + \cos C.Z, \\ \text{\&c.} & \quad \text{\&c.} \end{aligned}$$

The several differences of latitude are $a\cos A$, $b\cos B$, &c. and because when a, and A are variable the fluxion of $a\cos A$ is

$$\cos A.\dot{a} - a\sin A.\dot{A},$$

therefore putting x and X for $\dot{a}$, and $a.\dot{A}$, &c. we have the following equations,

$$\begin{aligned} L' &= \cos A.x - \sin A.X, \\ L'' &= \cos B.y - \sin B.Y \\ L''' &= \cos C.z - \sin C.Z, \\ \text{\&c.} & \quad \text{\&c.} \end{aligned}$$

Now the sum of the translations of all the angular points B, C, D, &c. in the direction ef, must be equal to ef; if therefore we reckon all the bearing one way round from Af, we shall have

$$\cos A.x + \cos B.y + \text{\&c.} - \sin A.X - \sin B.Y + \text{\&c.} = L : \qquad \text{(I.)}$$

and by the preceding theory, if 1 to p be the ratio of the equally probable linear errors in any proposed distance, the former in the direction of that distance, the latter at right angles to the former, we have

$$\frac{x^2}{a} + \frac{y^2}{b} + \text{\&c.} + \frac{X^2}{p^2a} + \frac{Y^2}{p'^2b} + \text{\&c.} = \min. \qquad \text{(II.)}$$

Put now the equations I, and II, into fluxions, and having multiplied the former by m, and the latter by $-\frac{1}{2}$, we shall have, by addition,

$$(m\cos A - \frac{x}{a} \times \dot{x} + (m\cos B - \frac{y}{b}) \times \dot{y} + \text{\&c.}$$
$$- (m\sin A + \frac{X}{p^2a})\dot{X} - (m\sin B + \frac{Y}{p^2b}) \times \dot{Y}, \text{\&c.} = 0.$$

This equation is satisfied by making the co-efficients of the several fluxions each $= 0$, whence we obtain the following equations,

$$x = ma\cos A, \quad y = mb\cos B, \quad \text{\&c.}$$

$$X = -mp^2a\sin A, \quad Y = -mp^2b\sin B, \quad \text{\&c.}$$

These equations show us that the several motions of the angular points A, B, C, &c. in the directions of the lengths are directly as the several differences of latitude, and that their motions in directions perpendicular to the former are directly as the departures.

By substituting for x, y, X, Y, &c. their values as just determined we obtain the several corrections in latitude and departure as follows

$$\begin{array}{ll} L' = mp^2a\sin^2 A + ma\cos^2 A & D' = (1-p^2)ma\sin A\cos A \\ L'' = mp^2b\sin^2 B + mb\cos^2 B & D'' = (1-p^2)mb\sin B\cos B \\ L''' = mp^2c\sin^2 C + mc\cos^2 C & D''' = (1-p^2)mc\sin C\cos C \\ \text{\&c.} \qquad \text{\&c.} & \text{\&c.} \qquad \text{\&c.} \end{array}$$

These equations, by putting $p^2 = 1 + r$, become

$$\begin{array}{ll} L' = ma + mra\sin^2 A & D' = rma\sin A\cos A \\ L'' = mb + mrb\sin^2 B & D' = rmb\sin B\cos B \\ \text{\&c.} \qquad \text{\&c.} & \text{\&c.} \qquad \text{\&c.} \end{array}$$

Which we may also express more simply thus,

$$\begin{array}{ll} L' = ma + \frac{1}{2}mra\,\text{versin}\,2A & D' = \frac{1}{2}rma\sin 2A \\ L'' = mb + \frac{1}{2}mrb\,\text{versin}\,2B & D' = \frac{1}{2}rmb\sin 2B \\ \text{\&c.} \qquad \text{\&c.} & \text{\&c.} \qquad \text{\&c.} \end{array}$$

And the value of m is to be derived from the equation

$$m\{a+b+\&c.+\frac{1}{2}r(a\,\text{versin}\,2A+b\,\text{versin}\,2B\&c.\}=L$$

But when $p=1$ or $r=0$, which is at once the simplest and most probable value, we have

$$m=\frac{L}{a+b+c+\&c.}$$

also

$$L'=ma,\quad L''=mb,\quad L'''=mc,\quad \&c.$$

and

$$D'=0,\quad D''=0,\quad D'''=0,\quad \&c.$$

These equations furnish the following practical rule for correcting dead reckoning. *Say, as the sum of all the distances in the traverse is to each particular distance, so is the whole error in latitude to the correction of the latitude corresponding to said distance; those corrections in latitude being always applied in such a manner as to diminish the whole error in latitude; but no corrections whatever must be applied to the several departures by account; and the differences of longitude are to be deduced from the correct differences of latitude and the corresponding departures by dead reckoning.*

But if, as is commonly practised, the whole difference of longitude is to be obtained at once by finding the whole departure, the corrections for the particular differences of latitude are unnecessary; and the rule for obtaining the correct difference of longitude is simply this:

With the departure by account, and the correct difference of latitude find the correct difference of longitude.

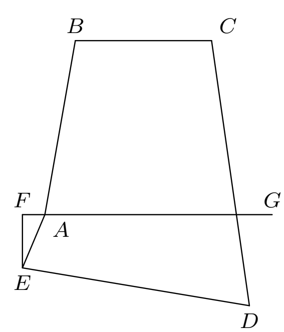
If the correct course and distance be required, they must, in like manner, be obtained from the departure by account, and the correct difference of latitude.

From this solution, it appears, that the various rules given by authors for correcting the dead reckoning are totally erroneous; and we hope they will be abandoned by all future writers on the subject.

PROBLEM IV.

To correct a survey.

SOLUTION.



Let $ABCDE$ be a survey accurately protracted according to the measured lengths and bearings of the sides AB, BC, &c. A the place of beginning, E of ending, AG a meridian, and AF, FE the errors in latitude and departure. Now the problem requires us to make such changes in the positions of the points B, C, &c. that we may remove the errors AF, FE; in other words, that E may coincide with A, and those changes must be made in the most probable manner. We have therefore to fulfil the three following conditions.

All the changes in departure must remove the error in dep. EF. All the changes in latitude must remove the error in lat. AF. The probability of all these changes must be a maximum.

Let a, b, c, &c. A, B, C, &. denote the lengths and bearings of the several sides AB, BC, &c. the radius being unity; x, y, z, &c. X, Y, Z, &c. the exceedingly small motions or tanslations of the angular points B, C, &c. in the directions of the sides AB, BC, &c. and in directions perpendicular to those lengths; D, L, the errors EF, FA, and D', D'', &c. L', L'', &c. the several corrections required in departure and latitude.

The several departures are $a \sin A$, $b \sin B$, &c. and the fluxion of $a \sin A$ being

$$\sin A.\dot{a} + a \cos A.\dot{A};$$

by putting x and X for $\dot{a}$ and $a.\dot{A}$, &c. we have

$$\begin{aligned} D' &= \sin A.x + \cos A.X \\ D'' &= \sin B.y + \cos B.Y \\ \&c. &\qquad \&c. \end{aligned}$$

And therefore

$$\sin A.x + \sin B.y + \text{\&c.} + \cos A.X + \cos B.Y + \text{\&c.} = D \tag{I}$$

Again, the differences of latitude are $a \cos A$, $b \cos B$, &c. and because the fluxion of $a \cos A$ is

$$\cos A.\dot{a} - a \sin A.\dot{A},$$

by putting x and X for $\dot{a}$ and $a.\dot{A}$, &c. we have

$$\begin{aligned} L' &= \cos A.x - \sin A.X \\ L'' &= \cos B.y - \sin B.Y \\ \&c. &\qquad \&c. \end{aligned}$$

and therefore, reckoning the bearings all one way round from the meridian AG, we have

$$\cos A.x + \cos B.y + \text{\&c.} - \sin A.X - \sin B.Y - \text{\&c.} = L \tag{II}$$

Also, retaining the same signification of the letter p, we have, by the preceding theory, for the greatest probability,

$$\frac{x^2}{a} + \frac{y^2}{b} + \text{\&c.} + \frac{X^2}{p^2 a} + \frac{Y^2}{p^2 b} + \text{\&c.} = \min. \tag{III}$$

Now let the fluxions of the three equations I, II, III, be multiplied respectively by m, n, and $-\frac{1}{2}$, and by additon we have

$$\left\{\begin{array}{c} m\sin A.\dot{x} + m\sin B.\dot{y} + \text{\&c.} + m\cos A.\dot{X} + m\cos B.\dot{Y} + \text{\&c.} \\ n\cos A.\dot{x} + n\cos B.\dot{y} + \text{\&c.} - n\sin A.\dot{X} - n\sin B.\dot{Y} - \text{\&c.} \\ -\frac{x}{a}.\dot{x} - \frac{y}{b}.\dot{y} - \text{\&c.} - \frac{X}{p^2a}.\dot{X} - \frac{Y}{p^2b}.\dot{Y} - \text{\&c.} \end{array}\right\} = 0.$$

This equation is satisfied by making the sum of the coefficients of each fluxion separately equal to 0, whence we obtain the following equations,

$$\begin{aligned} x &= ma\sin A + ma\cos A, \\ y &= mb\sin B + nb\cos B, \\ &\text{\&c.} \qquad \text{\&c.} \\ X &= mp^2a\cos A - np^2\sin A, \\ Y &= mp^2b\cos C - np^2\sin B, \\ &\text{\&c.} \qquad \text{\&c.} \end{aligned}$$

If the departures and differences of latitude of a, b, c, &c. be denoted by a', b', c', &c. a'', b'', c'', &c. we may express the several values of x, y, X, &c. thus,

$$\begin{aligned} x &= ma' + na'' & X &= p^2 \times \{ma'' - na'\} \\ y &= mb' + nb'' & Y &= p^2 \times \{mb'' - nb'\} \\ z &= mc' + nc'' & Z &= p^2 \times \{mc'' - nc'\} \\ &\text{\&c.} \quad \text{\&c.} & &\text{\&c.} \quad \text{\&c.} \end{aligned}$$

But the proper algebraic signs of $\sin A$, $\cos A$, $\sin B$, $\cos B$, &c. must also be transferred to those values of $a'b'$, &c. $a''b''$, &c.

Putting now for x, y, X, &c. their values, we obtain the several corrections in departure and latitude as follows;

$$\begin{aligned} D' &= ma\sin^2 A + mp^2a\cos^2 A + (na - np^2a)\sin A\cos A, \\ D'' &= mb\sin^2 B + mp^2b\cos^2 B + (nb - np^2b)\sin B\cos B, \\ D''' &= \qquad \text{\&c.} \qquad \text{\&c.}, \\ L' &= na\cos^2 A + np^2a\sin^2 A + (ma - mp^2a)\sin A\cos A, \\ L'' &= nb\cos^2 B + np^2b\sin^2 B + (mb - mp^2b)\sin B\cos B, \\ L''' &= \qquad \text{\&c.} \qquad \text{\&c.} \end{aligned}$$

Which expressions, by putting $p^2 = 1 + r$, become

$$
\begin{aligned}
D' &= ma + mra\cos^2 A - nra\sin A\cos A,\\
D'' &= mb + mrb\cos^2 B - nrb\sin B\cos B,\\
&\text{\&c.} \qquad\qquad \text{\&c.},\\
L' &= na + nra\sin^2 A - mra\sin A\cos A,\\
L'' &= nb + nrb\sin^2 B - mrb\sin B\cos B,\\
&\text{\&c.} \qquad\qquad \text{\&c.}
\end{aligned}
$$

and these may be transformed into the following,

$$
\begin{aligned}
D' &= ma + \frac{ra}{2}\{m + m\cos 2A - n\sin 2A\}\\
D'' &= mb + \frac{rb}{2}\{m + m\cos 2B - n\sin 2B\}\\
&\text{\&c.} \qquad\qquad \text{\&c.},\\
L' &= na + \frac{ra}{2}\{n - n\cos 2A - m\sin 2A\}\\
L'' &= nb + \frac{rb}{2}\{n - n\cos 2B - m\sin 2B\}\\
&\text{\&c.} \qquad\qquad \text{\&c.}
\end{aligned}
$$

The values of m and n are discovered by equating the sum of the values of D', D'', &c. to D, and the sum of those of L', L'', &c. to L; m and n will therefore be found by simple equations.

In these calculations the signs of all angles in the 3d and 4th quadrants must be taken negatively, as well as the cosines of those in the 2d and 3d quadrants; according to the common rule in such cases.

Also, when the errors EF, AF are on the same side of the beginning, A, with the first departure and difference of latitude, then D and L must be taken negatively.

The simplest case of the problem is, when $p = 1$, or $r = 0$, which is also the most probable supposition by Prob. I; besides, this seems to agee best with the imperfections of the common instruments used in surveying.

In this case, the values of m, n, and of the required corrections in departure and latitude are as follows;

$$
m = \frac{D}{a + b + c + \text{\&c.}}, \qquad n = \frac{L}{a + b + c + \text{\&c.}}
$$

$$
\begin{aligned}
D' &= ma, \quad D'' = mb, \quad D''' = mc, \quad \text{\&c.}\\
L' &= na, \quad L'' = nb, \quad L''' = nc, \quad \text{\&c.}
\end{aligned}
$$

Hence the following practical rule for correcting a survey:

Say, as the sum of all the distances is to each particular distance, so is the whole error in departure to the correction of the corresponding departure; each correction being so applied as to diminish the whole error in departure: proceed the same way for the corrections in latitude.

When $p = 1$, as in the practical rule, the motions of the angular points B, C, D, &c. are parallel to the whole linear error EA. This appears by imagining the meridian to coincide with EA; for in this case $D = 0$, and therefore

$$D' = 0, \quad D'' = 0, \quad D''' = 0, \quad \&c.$$

which equations show that the motions of B, C, D, &c. are parallel to EA. Also because

$$L' = na, \quad L'' = nb, \quad L''' = nc; \quad \&c.$$

it follows that the motions of B, C, D, &c. in the direction EA are proportional to the several distances a, b, c, &c.

But when p is not equal to unity, the motions of B, C, &c. are not parallel to EA: when $p = 0$, their motions are in the directions a, b, c, &c. because in this case the equations

$$X = p^2(ma'' - ma'), \qquad Y = p^2(mb'' - nb') \quad \&c.$$

become

$$X = 0, \qquad Y = 0, \qquad \&c.$$

When p is infinite, we have only to remove p^2 from the values of X, Y, &c. to those of x, y, &c. and make $p = 0$; in this case the equations

$$x = p^2(ma' + na''), \quad y = p^2(mb' + nb''), \quad \&c.$$

become

$$x = 0, \qquad y = 0, \qquad \&c.$$

and the remaining motions $X = ma'' - na'$, $Y = mb'' - nb'$, &c. are manifestly perpendicular to the distances a, b, c, &c.

From this investigation it appears, that the rules hitherto given by authors for correcting a survey, are altogether erroneous, and ought to be entirely rejected. The true method here given is exemplified by Mr. Bowditch, in his solution of Mr. Patterson's question of correcting a survey; his practical rule and mine being precisely the same.

I have applied the principle of this essay to the determination of the most probable value of the earth's ellipticity, &c. but want of room will not permit me to give the investigations at this time.