CONSIDERATIONS

SUR LA

THÉORIE MATHÉMATIQUE

DU JEU*

André-Marie Ampère

Year II. — 1802

1. Many writers, among whom one must distinguish the celebrated Dussaulx,[1] have had recourse to experience in order to prove that the passion of the game leads those who indulge in it to an inevitable ruin. The set of facts that they have reunited, suffice without doubt, to convince each impartial man; but the players pay little attention, because they are accustomed to see only the effect of chance in the events most proper to make known to them all the extent of the dangers where they rush forward. These events would make perhaps more impression on their spirit, if one demonstrated to them that they must consider them as a necessary consequence of the combination of chances, and that they are able to avoid the same misfortunes only by ceasing to expose themselves. Such was, without doubt, the motive which engaged the illustrious Buffon, this author of whom the errors themselves bear the imprint of genius, to examine this question under a point of view purely mathematical in his essay on moral arithmetic.

2. One finds in this work some ideas which should have led the author to the true principles of the general theory of the game, that one must not at all confound with the theory of the different games considered each in particular. This has been the object of the researches of a great number of Mathematicians, who have given to it all the perfection of which it was susceptible: the first appears to me to have been suspected only by Buffon. I believe indispensable to cite here some passages, where he puts the first foundations of this new theory, in the most clear and most precise manner. "One knows in general that the game is an avid passion of which the practice is ruin, but this truth has perhaps never been demonstrated but by a sad experience, on which one has not reflected enough in order to be corrected by the conviction. A player of whom the fortune exposed each day to the trials of chance, is consumed little by little, and is found finally necessarily destroyed, attributes his losses only to this same chance that he accuses of injustice . . . in his despair he lays the blame on his unlucky star; he does

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. December 18, 2011

[1]Jean Dusaulx (1728 – 1799) wrote two books on games. The are *Lettre et Réflexions sur la fureur du jeu*, 1775 and *De la Passion du Jeu, depuis les temps anciens jusqu'a nos jours*, 1779. RP

not imagine that this blind power, the fortune of the game, marches to the truth by a path indifferent and uncertain, but that at each step it tends nevertheless to an end, and draws to a certain term, which is the ruin of those who try it. He does not see that the apparent indifference that it has for the good and for the bad, produces with the times the necessity of the bad: that a long sequence of chances is a fatal chain of which the prolongation leads to misfortune."[2]

3. It is impossible to make an exposé more eloquent and more exact of the principles which serve as base to the theory that we examine; and if the Author of it had, by aid of the calculus, developed all the consequences, the memoir that I present to the public would have object no longer. But soon he abandoned his first ideas in order to cast himself into some hypotheses which are strange to them, and delivering himself immediately to new considerations, he seeks only to prove that two equally rich players, who play the half of their fortune, each diminish this fortune by a twelfth. I swear that the sum that one hazards in the game, produces in general less advantage to the one who wins it, than privations to the who who loses it; but I do not believe that this difference establishes between the real value of the sum lost, and that of the sum won, which is to him numerically equal, the ratio of the half to the third of the fortune of each player, rather than every other ratio. As if it were possible to evaluate that which depends on the needs of each player, of his state, of the rank that he holds in society, and of the circumstances where he is found.

4. At the same time as one would be able to determine exactly this difference, one could take no account of it in a calculation where it would be a concern to explicate how a long sequence of chances is a fatal chain which carries necessarily to misfortune, since the sums lost do not approach more the ruin of the player than the sums won prolong it, and that the effects which result from it are mutually destroyed when these sums are equals.

5. I myself am therefore decided to make enter into this calculation only the absolute values of the sums played, as one does constantly in the ordinary theory of probabilities: I have found in this manner some results rather different from those of Buffon, but on which I do not believe that the following demonstrations are able to leave the least doubt. I have banished from these demonstrations the methods of induction, of which one makes, to that which it seems, too much usage in the theory of probabilities, and in that of the series; the desire to employ only direct proofs, has obliged me to have recourse to some formulas that I believe new, and that one will find in this memoir. These formulas will be able to become very useful for different researches of the calculus; they appear especially proper to furnish the most simple and most direct means that one is able to employ in order to demonstrate many important theorems, which have not at all yet been completion.[3]

6. Here are the principle results to which I have been led, and of which the demonstration is the object of this memoir: 1° by setting aside the moral considerations which

[2] *Essai d'arithm. morale*, art. XII.

[3] See the appendices at the end of this memoir.

make the value of money vary, according to the circumstances where the players are found, he would not know how to have any disadvantage to play in an equal game against an adversary equally rich, since one is able to lose nothing that the other not win, and that each is equal on both sides; 2° the same thing holds between two players, of unequal fortunes, if they have decided to make only a determined number of games, and small enough in order that neither the one nor the other is able to be in the case of losing all that which he possessed; 3° it is not likewise when the concern is of an indefinite number of games: the possibility to hold the game a long time, gives to the richest of the two players an advantage so much greater as there is more difference between their fortunes; 4° this advantage would become infinite, if the one of the fortunes would be able to be it, the player least rich would be then sure of his ruin, and it is for this that is to run to a certain ruin, that to play indifferently against all those who are encountered in society: one must in fact, in the theory, to consider them as a single adversary of whom the fortune would be infinite. But as there would be able to result from it some obscurity, I am going to begin by treating this case independently of the one where one supposes that these are the same two players who always play one against the other; and in order to leave nothing to desire in this regard, I will examine first that which one must understand in the theory of probabilities by moral certitude, the only one of which there is here question.

7. In representing, as one does ordinarily, by unity absolute certitude, that for example which results from a rigorous demonstration, one will be able to regard as a moral certitude each variable fraction which, without ever becoming equal to unity, is able to approach near enough in order to surpass each determined fraction. It is thus that a man is morally certain to bring forth a sonnet by playing all his life at tric-trac, although the probability of this event is only $\frac{1}{36}$ at the first coup, $\frac{1}{36} + \frac{35}{36.36}$ in the first two coups, $\frac{1}{36} + \frac{35}{36.36} + \frac{35.35}{36.36.36}$ in the first three, and so forth: it is easy to see that these different sums of probabilities, are never able to become equal to unity to which they approach more and more, until differing only by a quantity less than every given fraction.[4]

8. Every time that nothing limits the number of coups where an event is able to arrive, the probability of this event increases necessarily with the number of coups: but according to that which we just said, one must especially be interested in distinguishing the case where this increase tends toward a determined limit, from the one where it has not at all a limit other than certitude; this which renders the event morally certain, by supposing always the number of coups indeterminate.

9. The subject that we treat is able to furnish some examples of the one and of the other case: we have just indicated (7) one of them of the one where the sum of the probabilities is able to approach certitude as near as one wishes; in order to give one of the cases where this sum is able to increase only by remaining constantly below a certain limit, it suffices to consider the one where two players, equally rich, play in an equal game against one another, until one of them is ruined.

[4] This is demonstrated immediately by aid of the formula that we will give below (41), by supposing $q = \frac{1}{35}$, so that one has $\frac{1}{1+q} = \frac{35}{36}$, and $\frac{q}{1+q} = \frac{1}{36}$.

10. It is easy to see that nothing then determines the number of games that the two players will make, and that the probability that one of them will be ruined, will increase with the number of games, without being able however to surpass ever the limit $\frac{1}{2}$, since this player is not able to be ruined only if there arrives on the contrary that he ruins his adversary, an event as probable as the other, when all, as one supposes it here, is equal between the two players.[5]

11. The man who indulges in the love of the game, supposes certainly no limit to the number of games that he will play; he knows that he is able to be ruined, and that the probability of this event will become so much greater as he will play more games; he regards however this probability as rather small, in order to have to inspire in him only weak anxieties; so that he believes to be, in this regard, in the first of the two cases of which we just spoke, and of which he has a confused sentiment, similar to the one which all the players have of the principal points of the theory of probabilities. What would be his astonishment, if he knew that it is to the contrary in the second, and that this probability, quite far from being as small as he imagines, becomes rather great, after a sufficient number of games, in order to surpass every given probability; the demonstration that one will find here of the truth of this assertion, reposes on one of the fundamental propositions of the theory of series, namely: *That in summing a convergent series, under the assumption that the number of its terms is infinite, one finds always a limit of which the sums formed of the consecutive terms of the same series, is able to approach in a manner by differing from it only by a quantity less than every given quantity.* I would not be able to occupy myself here in the examination of this proposition, admitted by all the mathematicians, without leaving the limits of my subject; but as it seems to me that there is lacking yet something to the demonstrations that one has given of them to the present, I will return in this regard to a work on series, on which the professor of mathematics of the central school of the department of Ain and myself, we are working in concert, and which probably will be published soon.[6] One will find in this work new researches on different points of the theory of series, and some direct and general demonstrations of the theorems which depend on it, particularly of those which have yet been demonstrated only in a vague manner, or by induction.

12. In order to determine the limit of the probabilities contrary to the player, in the case that we examine, it is necessary first to find the general term of the series which comprehends them all, that is, the probability that the player will be ruined at the end of any number of games. We suppose, in order to simplify the calculation, that the sum played is the same at each game, and that it is an exact fraction of the fortune that the player has in entering into the game. These two assumptions are certainly not at all in accord with that which the players ordinarily do; but as the calculation, if one were not to admit them, would be too complicated in order that one could draw from it some satisfying result, it is so much more à propos to adopt them as one is always able to an exact fraction of the fortune of the player, less than the different sums that he risks in

[5]By applying to this particular case the formulas demonstrated in this memoir, we will show (76) that $\frac{1}{2}$ is in fact the limit of this probability.

[6]I find no evidence that such a work ever appeared. RP

each game, and that if one demonstrates then that he must necessarily be ruined, one will be able to conclude, the more so, that he will be ruined by hazarding in each game some more considerable sums.

13. We represent by m the number of times that this fraction is contained in the original fortune of the player: since he risks under this hypothesis only $\frac{1}{m}$ of his fortune in each game, it is evident that he will not be able to find himself ruined before the game of which the rank is designated by m: in order that he was it in fact at this game, it would be necessary that he lost it after having lost all the preceding; if he won one of them and if he lost all the others, he will be found ruined only after $m+2$ games; if he wins a second of them, he will be able no longer to be that by losing $m+2$ of them, this which supposes necessarily $m+4$ games; and it is easy to see that in general p designating any number whatsoever, it will be necessary in order that there remains nothing to the player that the number of all the games be $m+2p$, the number of games that he wins p, and the one of the games that he loses $m+p$.

14. Let $q:1$ be the ratio which is found in each game between the chances which are favorable to the player and those which are contrary to him, so that $q=1$ when he plays at par, and if one has for example $q=\frac{8}{3}$, if according to the nature and the conditions of the game, he must win in general 8 games out of 11. Certitude being ordinarily represented by unity, the probability that the player will win one game, will be by the fraction $\frac{q}{q+1}$, and the probability that he will lose it by $\frac{1}{1+q}$. If one wishes to have the probability that p games won, and $m+p$ games lost are successively in a determined order, it will be necessary to make the product of p factors equal to $\frac{q}{1+q}$, and of $m+p$ factors equal to $\frac{1}{1+q}$, this which will give $\frac{q^p}{(1+q)^{m+2p}}$.

15. This probability is the same for all the arrangements that one is able to imagine among these games won and lost, and as they are absolutely independent from one another, it is evident that the probability that we just found must be multiplied by the number of these arrangements, by observing to set aside from those which would not have permitted the player to arrive to the game that we are considering, by depriving him of all his fortune as of the preceding games. Let $m+2r$ be the rank of one of these games, r being smaller than p, it will be necessary to reject all the arrangements of p games won, and of $m+p$ games lost, of which the $m+2r$ first games would contain r games won, and $m+r$ games lost, because these are precisely those arrangements which would have ruined the player after $m+2r$ games.

16. Without this condition the number of arrangements would be

$$\frac{m+2p}{1}\cdot\frac{m+2p-1}{2}\cdot\frac{m+2p-2}{3}\cdots\frac{m+p+1}{p};$$

in order to know that which it becomes in the present case, we express in general by $A^{(t)}$ the number of the arrangements of any number whatsoever t of games, which bring forth the ruin of the player at the last of these t games, without having it brought forth at any of the preceding, the parentheses which accompany the number t serving to designate that this number must be considered as an index and not as an exponent.

According to this notation the number of which we seek the value will be expressed by $A^{(m+2p)}$, and $A^{(m+2r)}$ will represent the number of the arrangements of r games won, and of $m+r$ games lost, which would have ruined the player at one of the preceding games, of which the rank is in general designated by $m+2r$, r being always smaller than p.

17. If one joins $p-r$ games won, and as many games lost, to each of these last arrangements, one will form from them p games won, and $m+p$ games lost, which must be subtracted from the number

$$\frac{m+2p}{1}\cdot\frac{m+2p-1}{2}\cdot\frac{m+2p-2}{3}\cdots\frac{m+p+1}{p}.$$

so that after having given to r all the possible values, in whole numbers, from $r=0$ to $r=p-1$, there remains only the arrangements of which the number is designated by $A^{(m+2p)}$.

18. Each of the arrangements of which we just spoke will give in this manner a number expressed by

$$\frac{2p-2r}{1}\cdot\frac{2p-2r-1}{2}\cdot\frac{2p-2r-2}{3}\cdots\frac{p-r+1}{p-r}$$

because of the $2p-2r$ games that it is necessary to partition into two groups of $p-r$ games each. One will have therefore

$$\frac{2p-2r}{1}\cdot\frac{2p-2r-1}{2}\cdot\frac{2p-2r-2}{3}\cdots\frac{p-r+1}{p-r}A^{(m+2r)},$$

for the number of the arrangements to subtract.

19. Making successively $r=p-1$, $r=p-2$, $r=p-3$, etc. One will find for the different values of the preceding expression,

$$\frac{2}{1}A^{(m+2p-2}, \quad \frac{4}{1}\cdot\frac{3}{2}A^{(m+2p-4)}, \quad \frac{6}{1}\cdot\frac{5}{2}\cdot\frac{4}{3}A^{(m+2p-6)}, \quad \text{etc.}$$

whence it will be easy to conclude that

$$\begin{aligned} & A^{(m+2p)} = \frac{m+2p}{1}\cdot\frac{m+2p-1}{2}\cdot\frac{m+2p-2}{3}\cdots\frac{m+p+1}{p} \\ - \quad & \frac{2}{1}A^{(m+2p-2)} - \frac{4}{1}\cdot\frac{3}{2}\cdot A^{(m+2p-4)} - \frac{6}{1}\cdot\frac{5}{2}\cdot\frac{4}{3}A^{(m+2p-6)} - \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ - \quad & \frac{2p-2r}{1}\cdot\frac{2p-2r-1}{2}\cdot\frac{2p-2r-2}{3}\cdots\frac{p-r+1}{p-r}A^{(m+2r)} - \text{etc.} \end{aligned}$$

One is able to divide above and below by $p-r$ the term

$$-\frac{2p-2r}{1}\cdot\frac{2p-2r-1}{2}\cdot\frac{2p-2r-2}{3}\cdots\frac{p-r+1}{p-r}A^{(m+2r)},$$

and to make an analogous reduction in the preceding terms, which are of the same form. One will change thus the preceding equation into

$$\begin{aligned} & A^{(m+2p)} = \frac{m+2p}{1}\cdot\frac{m+2p-1}{2}\cdot\frac{m+2p-2}{3}\cdots\frac{m+p+1}{p} \\ - \quad & 2A^{(m+2p-2)} - 2\frac{3}{1}\cdot A^{(m+2p-4)} - 2\frac{3}{1}\cdot\frac{4}{2}A^{(m+2p-6)} - \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ - \quad & 2\frac{2p-2r-1}{1}\cdot\frac{2p-2r-2}{2}\cdot\frac{2p-2r-3}{3}\cdots\frac{p-r+1}{p-r-1}A^{(m+2r)} - \text{etc.} \end{aligned}$$

20. In order to have a value of $A^{(m+2p)}$ independent of the quantities $A^{(m+2p-2)}$, $A^{(m+2p-4)}$, $A^{(m+2p-6)} \dots A^{(m+2r)}$, etc. One will observe that the player is not able to be ruined at the game of which the rank is designated by $m+2p$, unless the $m+2p-1$ preceding games had reduced him to having no more than $\frac{1}{m}$ of his original fortune, since we have expressed by this fraction the sum that he plays in each game. It is necessary for this that out of these $m+2p-1$ games, he has p of them won, and $m+p-1$ lost. One sees besides that the number of the different arrangements that one is able to give to these games, without supposing that any one of them has ruined the player, must be equal to the one of the arrangements of p games won, and $m+p$ games lost, of which the number is represented by $A^{(m+2p)}$, since each of these here is formed from one of the first, by adding one lost game. We draw from this consideration a value of $A^{(m+2p)}$ that we are able to compare with the preceding.

21. The number of all the arrangements that one is able to make with $m+2p-1$ games, by supposing them partitioned into two groups, the one of p games won, and the other of $m+p-1$ games lost, is in general equal to

$$\frac{m+2p-1}{1}\cdot\frac{m+2p-2}{2}\cdot\frac{m+2p-3}{3}\cdots\frac{m+p}{p}.$$

or that which reverts to the same to

$$\frac{m+p}{1}\cdot\frac{m+2p-1}{2}\cdot\frac{m+2p-2}{3}\cdots\frac{m+p+1}{p}.$$

The concern is therefore no longer to have the value of $A^{(m+2p)}$, but to subtract from the number expressed by that formula, the number of arrangements which would have ruined the player since the preceding games. Those are formed evidently from the arrangements of r games won, and $m+r$ games lost, of which the number is represented by $A^{(m+2r)}$, by adding $2p-2r-1$ games, of which $p-r$ won, and $p-r-1$ lost, this which is able to be made in

$$\frac{2p-2r-1}{1}\cdot\frac{2p-2r-2}{2}\cdot\frac{2p-2r-3}{3}\cdots\frac{p-r+1}{p-r-1},$$

different ways.

22. By reasoning here as in the preceding calculation, one will see that the total number of the arrangements to subtract, will be found by giving successively to r all the possible values in whole numbers, from $r = p - 1$, to $r = 0$, in the formula

$$\frac{2p-2r-1}{1} \cdot \frac{2p-2r-2}{2} \cdot \frac{2p-2r-3}{3} \cdots \frac{p-r+1}{p-r-1} A^{(m+2r)}$$

If one reunites next all the results thus obtained, namely:

$$A^{(m+2p-2)}, \quad \frac{3}{1}A^{(m+2p-4)}, \quad \frac{5}{1} \cdot \frac{4}{2}A^{(m+2p-6)}, \quad \text{etc.}$$

one will have

$$\begin{aligned} A^{(m+2p)} \quad = \quad & \frac{m+p}{1} \cdot \frac{m+2p-1}{2} \cdot \frac{m+2p-2}{3} \cdots \frac{m+p+1}{p} \\ & -A^{(m+2p-2)} - \frac{3}{1}A^{(m+2p-4)} - \frac{5}{1}\frac{4}{2}A^{(m+2p-6)} - \cdots \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & -\frac{2p-2r-1}{1} \cdot \frac{2p-2r-2}{2} \cdot \frac{2p-2r-3}{3} \cdots \frac{p-r+1}{p-r-1} A^{(m+2r)} - \text{etc.} \end{aligned}$$

By doubling all the terms of this equation, one finds

$$\begin{aligned} 2A^{(m+2p)} \quad = \quad & \frac{2m+2p}{1} \cdot \frac{m+2p-1}{2} \cdot \frac{m+2p-2}{3} \cdots \frac{m+p+1}{p} \\ & -2A^{(m+2p-2)} - 2\frac{3}{1}A^{(m+2p-4)} - 2\frac{5}{1}\frac{4}{2}A^{(m+2p-6)} - \cdots \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & -2\frac{2p-2r-1}{1} \cdot \frac{2p-2r-2}{2} \cdot \frac{2p-2r-3}{3} \cdots \frac{p-r+1}{p-r-1} A^{(m+2r)} - \text{etc.} \end{aligned}$$

and by subtracting from this last equation that which we have obtained previously

$$\begin{aligned} A^{(m+2p)} \quad = \quad & \frac{m+2p}{1} \cdot \frac{m+2p-1}{2} \cdot \frac{m+2p-2}{3} \cdots \frac{m+p+1}{p} \\ & -2A^{(m+2p-2)} - 2\frac{3}{1}A^{(m+2p-4)} - 2\frac{5}{1}\frac{4}{2}A^{(m+2p-6)} - \cdots \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & -2\frac{2p-2r-1}{1} \cdot \frac{2p-2r-2}{2} \cdot \frac{2p-2r-3}{3} \cdots \frac{p+r+1}{p-r-1} A^{(m+2r)} - \text{etc.} \end{aligned}$$

there remains

$$A^{(m+2p)} = \frac{m}{1} \cdot \frac{m+2p-1}{2} \cdot \frac{m+2p-2}{3} \cdots \frac{m+p+1}{p}.$$

This value of $A^{(m+2p)}$, remarkable by its simplicity and its elegance, would have been easy to find by induction, but the preceding analysis has the advantage of giving it in a direct and general manner.

23. The formula that we just found holds not only in regard to the diverse arrangements that one is able to give to $m+2p$ games, partitioned into two groups, conformably to the conditions of the present question: it would be able to have an infinity of other applications. It is this, for example, which would give the number of different products of p letters, that one could make with $m+2p$ letters, by confining oneself to arrange them according to alphabetical order, and to choose the first letter of each product, among the first m, the second among the first $m+2$ letters, the third among the first $m+4$, and so forth. I will not stop myself to demonstrate this proposition of which one will perceive easily the liaison with that which precedes, if one pays attention that it is necessary in order that the player not be ruined before the game of which the rank is $m+2p$, that he wins at least one time out of the first m games, two times out of the first $m+2$, three times out of the first $m+4$, and in general r times out of the first $m+2r-2$ games; because if he won only $r-1$ games, he would lose $m+r-1$, and would find himself ruined after the $m+2r-2$ games.

24. The number $\frac{m}{1}\cdot\frac{m+2p-1}{2}\cdot\frac{m+2p-2}{3}\cdots\frac{m+p+1}{p}$, of the products of p letters which satisfy the conditions of which we just spoke, differ from the total number of the same products

$$\frac{m+2p}{1}\cdot\frac{m+2p-1}{2}\cdot\frac{m+2p-2}{3}\cdots\frac{m+p+1}{p},$$

only in regard to the first factor, where the term $+2p$ is lacking; these conditions restrict therefore the number of these products in the ratio of $m+2p$ to m. There results from it a new specie of combinations of which the consideration will be able to become very useful in the progress of the theory of probabilities.

25. The series of numbers that one obtains by supposing successively $p=0$, $p=1$, $p=2$, $p=3$ etc., and that one is able to represent by

$$A^{(m)},\quad A^{(m+2)},\quad A^{(m+4)},\quad \ldots A^{(m+2p-4)},\quad A^{(m+2p-2)},\quad A^{(m+2p)},$$

enjoy some remarkable properties, which depend on a general formula on which we are going to occupy ourselves. This formula will serve us in the following of this work, to give to the demonstrations a rigor and a generality that it would be perhaps difficult to obtain otherwise.

26. One has first by transposing the terms

$$-\frac{2}{1}A^{(m+2p-2)},\quad -\frac{4}{1}\cdot\frac{3}{2}A^{(m+2p-4)},\quad -\frac{6}{1}\cdot\frac{5}{2}\cdot\frac{4}{3}A^{(m+2p-6)},\ldots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$-\frac{2p-2r}{1}\cdot\frac{2p-2r-1}{2}\cdot\frac{2p-2r-2}{3}\cdots\frac{p-r+1}{p-r}A^{(m+2r)},\ \text{etc.}$$

of the first value that we have found for $A^{(m+2p)}$, the equation

$$A^{(m+2p)} + \frac{2}{1}A^{(m+2p-2)} + \frac{4}{1}\cdot\frac{3}{2}A^{(m+2p-4)} + \frac{6}{1}\cdot\frac{5}{2}\cdot\frac{4}{3}A^{(m+2p-6)} +$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$+\frac{2p-2r}{1}\cdot\frac{2p-2r-1}{2}\cdot\frac{2p-2r-2}{3}\cdots\frac{p-r+1}{p-r}A^{(m+2r)} + \text{etc.} =$$
$$\frac{m+2p}{1}\cdot\frac{m+2p-1}{2}\cdot\frac{m+2p-2}{3}\cdots\frac{m+p+1}{p} \qquad [1]$$

which is only one particular case of the general formula of which we occupy ourselves.

27. In order to obtain this formula, one will substitute into $A^{(m+2p)}$ its value

$$\frac{m}{1}\cdot\frac{m+2p-1}{2}\cdot\frac{m+2p-2}{3}\cdots\frac{m+p+1}{p},$$

and one will have by permitting to pass into the second member the term which will result from it

$$\frac{2}{1}A^{(m+2p-2)} + \frac{4}{1}\cdot\frac{3}{2}A^{(m+2p-4)} + \frac{6}{1}\cdot\frac{5}{2}\cdot\frac{4}{3}A^{(m+2p-6)} + \cdots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$+\frac{2p-2r}{1}\cdot\frac{2p-2r-1}{2}\cdot\frac{2p-2r-2}{3}\cdots\frac{p-r+1}{p-r}A^{(m+2r)} + \text{etc.} =$$
$$\frac{m+2p}{1}\cdot\frac{m+2p-1}{2}\cdot\frac{m+2p-2}{3}\cdots\frac{m+p+1}{p} - \frac{m}{1}\cdot\frac{m+2p-1}{2}\cdot\frac{m+2p-2}{3}\cdot\frac{m+p+1}{p}$$
$$= \frac{2p}{1}\cdot\frac{m+2p-1}{2}\cdot\frac{m+2p-2}{3}\cdots\frac{m+p+1}{p}$$

28. If one recalls that

$$\frac{2p-2r}{1}\cdot\frac{2p-2r-1}{2}\cdot\frac{2p-2r-2}{3}\cdots\frac{p-r+1}{p-r} =$$
$$2\frac{2p-2r-1}{1}\cdot\frac{2p-2r-2}{2}\cdot\frac{2p-2r-3}{3}\cdots\frac{p-r+1}{p-r-1},$$

it will be easy to see that one is able by dividing by two all the terms of the preceding equation, to reduce it to

$$A^{(m+2p-2)} + \frac{3}{1}A^{(m+2p-4)} + \frac{5}{1}\cdot\frac{4}{2}A^{(m+2p-6)} + \cdots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$+\frac{2p-2r-1}{1}\cdot\frac{2p-2r-2}{2}\cdot\frac{2p-2r-3}{3}\cdots\frac{p-r+1}{p-r-1}A^{(m+2r)}+\text{etc.}$$
$$=\frac{p}{1}\cdot\frac{m+2p-1}{2}\cdot\frac{m+2p-2}{3}\cdots\frac{m+p+1}{p}=$$
$$\frac{m+2p-1}{1}\cdot\frac{m+2p-2}{2}\cdot\frac{m+2p-3}{3}\cdots\frac{m+p+1}{p-1},$$

this one holding for all the values of p, still will be true, if one substitutes $p+1$ in p, this which gives

$$A^{(m+2p)}+\frac{3}{1}A^{(m+2p-2)}+\frac{5}{1}\cdot\frac{4}{2}A^{(m+2p-4)}+\frac{7}{1}\cdot\frac{6}{2}\cdot\frac{5}{3}A^{(m+2p-6)}+$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+\frac{2p-2r+1}{1}\cdot\frac{2p-2r}{2}\cdot\frac{2p-2r-1}{3}\cdots\frac{p-r+2}{p-r}A^{(m+2r)}+\text{etc.}=$$
$$\frac{m+2p+1}{1}\cdot\frac{m+2p}{2}\cdot\frac{m+2p-1}{3}\cdots\frac{m+p+2}{p};\quad [2]$$

which is a second particular case of the formula of which the equation

$$A^{(m+2p)}+\frac{2}{1}A^{(m+2p-2)}+\frac{4}{1}\cdot\frac{3}{2}A^{(m+2p-4)}+\frac{6}{1}\cdot\frac{5}{2}\cdot\frac{4}{3}A^{(m+2p-6)}+$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+\frac{2p-2r}{1}\cdot\frac{2p-2r-1}{2}\cdot\frac{2p-2r-2}{3}\cdots\frac{p-r+1}{p-r}A^{(m+2r)}+\text{etc.}=$$
$$\frac{m+2p}{1}\cdot\frac{m+2p-1}{2}\cdot\frac{m+2p-2}{3}\cdots\frac{m+p+1}{p},$$

has offered to us the first case.

29. By comparing these two equations, one sees that they differ only by the numerators of the coefficients which multiply the quantities

$$A^{(m+2p)},\quad A^{(m+2p-1)},\quad A^{(m+2p-4)},\quad A^{(m+2p-6)},\quad \ldots$$
$$\cdots\cdots\cdots\cdots\cdots A^{(m+2r)},\ \text{etc.}$$

and that all the factors of these numerators have each increased by one unit, by the operations which have led from equation [1] to equation [2]. By subtracting the first from the second, and paying attention that one has, whatever be the values of m, of p, and of r,

$$\frac{2p-2r+1}{1}\cdot\frac{2p-2r}{2}\cdot\frac{2p-2r-1}{3}\cdots\frac{p-r+2}{p-r}-\frac{2p-2r}{1}\cdot\frac{2p-2r-1}{2}\cdot\frac{2p-2r-2}{3}$$
$$\cdots\frac{p-r+1}{p-r}=\frac{2p-2r}{1}\cdot\frac{2p-2r-1}{2}\cdot\frac{2p-2r-2}{3}\cdots$$

$$\frac{p-r+2}{p-r-1}\cdot\frac{p-r}{p-r}=\frac{2p-2r}{1}\cdot\frac{2p-2r-1}{2}\cdot\frac{2p-2r-2}{3}\cdots\frac{p-r+2}{p-r-1},$$

and

$$\frac{m+2p+1}{1}\cdot\frac{m+2p}{2}\cdot\frac{m+2p-1}{3}\cdots\frac{m+p+2}{p}-\frac{m+2p}{1}\cdot\frac{m+2p-1}{2}\cdot$$
$$\frac{m+2p-2}{3}\cdots\frac{m+p+1}{p}=\frac{m+2p}{1}\cdot\frac{m+2p-1}{2}\cdot\frac{m+2p-2}{3}\cdots\frac{m+p+2}{p-1},$$

one will find

$$A^{(m+2p-2)}+\frac{4}{1}A^{(m+2p-4)}+\frac{6}{1}\frac{5}{2}A^{(m+2p-6)}+\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+\frac{2p-2r}{1}\cdot\frac{2p-2r-1}{2}\cdot\frac{2p-2r-2}{3}\cdots\frac{p-r+2}{p-r-1}A^{(m+2r)}+\text{etc.}$$
$$=\frac{m+2p}{1}\cdot\frac{m+2p-1}{2}\cdot\frac{m+2p-2}{3}\cdots\frac{m+p+2}{p-1}.$$

30. This equation must also hold for all the values of p, one will write $p+1$ instead of p, and there will come

$$A^{(m+2p)}+\frac{4}{1}A^{(m+2r-2)}+\frac{6}{1}\cdot\frac{5}{2}A^{(m+2r-4)}+\frac{8}{1}\cdot\frac{7}{2}\cdot\frac{6}{3}A^{(m+2r-6)}+$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+\frac{2p-2r+2}{1}\cdot\frac{2p-2r+1}{2}\cdot\frac{2p-2r}{3}\cdots\frac{p-r+3}{p-r}A^{(m+2r)}+\text{ etc.}=$$
$$\frac{m+2p+2}{1}\cdot\frac{m+2p+1}{2}\cdot\frac{m+2p}{3}\cdots\frac{m+p+3}{p},\quad [3]$$

which is again of the same form as equations [1] and [2], and differs from them only by the increase of a new unit, that the operations by which one has passed from equation [2] to equation [3], have produced in the numerators of the coefficients. One will perceive easily, by considering the form of these equations, that this increase results from it necessarily all the time that one subtracts one equation of this form from that which follows it, and that one writes next $p+1$ instead of p in the remaining equation. By executing these operations on equations [2] and [3], one obtains

$$A^{(m+2p)}+\frac{5}{1}A^{(m+2p-2)}+\frac{7}{1}\cdot\frac{6}{2}\cdot\frac{5}{3}A^{(m+2p-4)}+\frac{9}{1}\cdot\frac{8}{2}\cdot\frac{7}{3}\cdot\frac{6}{4}A^{(m+2p-6)}+$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+\frac{2p-2r+3}{1}\cdot\frac{2p-2r+2}{2}\cdot\frac{2p-2r+1}{3}\cdots\frac{p-r+4}{p-r}A^{(m+2r)}+\text{ etc.}$$

$$= \frac{m+2p+3}{1} \cdot \frac{m+2p+2}{2} \cdot \frac{m+2p+1}{3} \cdots \frac{m+p+4}{p},$$

and so forth.

31. This increase by one unit in the numerators taking place in each successive transformation, if one represents by u the number of these transformations, departing from equation [1]; each numerator will be increased by u, and the last transformed will be

$$A^{(m+2p)} + \frac{u+2}{1} A^{(m+2p-2)} + \frac{u+4}{1} \cdot \frac{u+3}{2} A^{(m+2p-4)} + \frac{u+6}{1} \cdot \frac{u+5}{2} \cdot \frac{u+4}{3} A^{(m+2r-6)} + \cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+ \frac{u+2p-2r}{1} \cdot \frac{u+2p-2r-1}{2} \cdot \frac{u+2p-2r-2}{3} \cdots \frac{u+p-r+1}{p-r} A^{(m+2r)} + \text{etc.} =$$
$$= \frac{u+m+2p}{1} \cdot \frac{u+m+2p-1}{2} \cdot \frac{u+m+2p-2}{3} \cdots \frac{u+m+p+1}{p}. \quad [4]$$

u being absolutely arbitrary in this equation, one must consider it as a general formula which comprehends all the equations of like form as we have just found.

32. By setting in place of

$$A^{(m+2p)}, \quad A^{(m+2p-2)}, \quad A^{(m+2p-4)}, \quad A^{(m+2p-6)}, \ldots A^{(m+2r)}, \text{ etc.}$$

the values represented by these characters, namely:

$$\begin{aligned}
A^{(m+2p)} &= \frac{m}{1} \cdot \frac{m+2p-1}{2} \cdot \frac{m+2p-2}{3} \cdots \frac{m+p+1}{p}, \\
A^{(m+2p-2)} &= \frac{m}{1} \cdot \frac{m+2p-3}{2} \cdot \frac{m+2p-4}{3} \cdots \frac{m+p}{p-1}, \\
A^{(m+2p-4)} &= \frac{m}{1} \cdot \frac{m+2p-5}{2} \cdot \frac{m+2p-6}{3} \cdots \frac{m+p-1}{p-2}, \\
A^{(m+2p-6)} &= \frac{m}{1} \cdot \frac{m+2p-7}{2} \cdot \frac{m+2p-8}{3} \cdots \frac{m+p-2}{p-3}, \\
\cdots\cdots & \quad \cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots & \quad \cdots\cdots\cdots\cdots\cdots\cdots \\
A^{(m+2r)} &= \frac{m}{1} \cdot \frac{m+2r-1}{2} \cdot \frac{m+2r-2}{3} \cdots \frac{m+r+1}{r},
\end{aligned}$$

the preceding formula would become

$$\frac{m}{1} \cdot \frac{m+2p-1}{2} \cdot \frac{m+2p-2}{3} \cdots \frac{m+p+1}{p} +$$
$$\frac{u+2}{1} \cdot \frac{m}{1} \cdot \frac{m+2p-3}{2} \cdot \frac{m+2p-4}{3} \cdots \frac{m+p}{p-1} +$$
$$\frac{u+4}{1} \cdot \frac{u+3}{2} \cdot \frac{m}{1} \cdot \frac{m+2p-5}{2} \cdot \frac{m+2p-6}{3} \cdots \frac{m+p-1}{p-2} +$$

$$\frac{u+6}{1}\cdot\frac{u+5}{2}\cdot\frac{u+4}{3}\cdot\frac{m}{1}\cdot\frac{m+2p-7}{2}\cdot\frac{m+2p-8}{3}\cdots\frac{m+p-2}{p-3}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\frac{u+2p-2r}{1}\cdot\frac{u+2p-2r-1}{2}\cdot\frac{u+2p-2r-2}{3}\cdots$$

$$\frac{u+p-r+1}{p-r}\cdot\frac{m}{1}\cdot\frac{u+2r-1}{2}\cdot\frac{u+2r-2}{3}\cdots\frac{m+r+1}{r}+$$

$$\text{etc.}=\frac{u+m+2p}{1}\cdot\frac{u+m+2p-1}{2}\cdot\frac{u+m+2p-2}{3}\cdots\frac{u+m+p+1}{p}; \quad [5]$$

but as this transformation renders it much more complicated, we will leave it in the different applications as we do in that formula, under the form where we have first found it, and we will consider

$$A^{(m+2p)},\quad A^{(m+2p-2)},\quad A^{(m+2p-4)},\quad A^{(m+2p-6)},\ldots A^{(m+2r)},$$

as some symbols destined to designate in a brief manner the quantities which they represent.

33. One could believe that the preceding demonstration leaving the liberty to assign to u the value that one wishes among the positive whole numbers, does not permit to give to it some negative or fractional values, but one will be convinced easily that the value of u, is absolutely indeterminate, if one pays attention that the preceding equation is not able to take place for all the whole and positive values of u unless executing the indicated operations, reducing the two members to the same denominator, and ordering with respect to u, one does not find for coefficients of one same power of u in the two members, two functions of p and of m absolutely identical; whence there results necessarily that the equation is yet identical, when n is fractional or negative.

34. One will be able therefore to suppose $u=-x$, x being positive, and one will give thus to the preceding equation the form

$$A^{(m+2p)}-\frac{x-2}{1}A^{(m+2p-2)}+\frac{x-4}{1}\cdot\frac{x-3}{2}A^{(m+2p-4)}-$$

$$\frac{x-6}{1}\cdot\frac{x-5}{2}\cdot\frac{x-4}{3}A^{(m+2p-6)}+\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\pm\frac{x-2p+2r}{1}\cdot\frac{x-2p+2r+1}{2}\cdot\frac{x-2p+2r+2}{3}\cdots\frac{x-p+r-1}{p-r}A^{(m+2r)}\mp\text{etc.}$$

$$=\frac{m+2p-x}{1}\cdot\frac{m+2p-x-1}{2}\cdot\frac{m+2p-x-2}{3}\cdots\frac{m+p+1-x}{p}; \quad [6]$$

where it is necessary to employ the upper sign when the indeterminate number r is such that $p-r$ is even, and the lower sign when $p-r$ is odd, this which depends on the rank which occupies in the first member, the term of which one wishes to calculate the value by aid of the general term

$$\pm\frac{x-2p+2r}{1}\cdot\frac{x-2p+2r+1}{2}\cdot\frac{x-2p+2r+2}{3}\cdots\frac{x-p+r-1}{p-r}A^{(m+2r)},$$

which gives immediately all the others by supposing successively $r=p-1$, $r=p-2$, $r=p-3$, etc.

35. By giving to x a value comprehended between these two limits inclusively:

$$x=m+2p, \qquad x=m+p+1,$$

one of the factors of the second member vanishing, this second member is reduced to zero, and the first becomes consequently also equal to zero. If one supposed in the formula $x=m$ it would be much simplified, and would give

$$A^{(m+2p)}-\frac{m-2}{1}A^{(m+2p-2)}+\frac{m-4}{1}\cdot\frac{m-3}{2}A^{(m+2p-4)}-$$
$$\frac{m-6}{1}\cdot\frac{m-5}{2}\cdot\frac{m-4}{3}A^{(m+2p-6)}+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\pm\frac{m-2p+2r}{1}\cdot\frac{m-2p+2r+1}{2}\cdot\frac{m-2p+2r+2}{3}\cdots\frac{m-p+r-1}{p-r}A^{(m+2r)}\mp\text{ etc.}$$
$$=\frac{2p}{1}\cdot\frac{2p-1}{2}\cdot\frac{2p-2}{3}\cdots\frac{p+1}{p}=2\frac{2p-1}{1}\cdot\frac{2p-2}{2}\cdot\frac{2p-3}{3}\cdots\frac{p+1}{p-1}. \quad [7]$$

36. We return to the problem that we had proposed, and we substitute in the place of $A^{(m+2p)}$, its value in the expression

$$A^{(m+2p)}\frac{q^p}{(1+q)^{m+2p}}$$

of the probability that we wish to calculate; it will become

$$\frac{m}{1}\cdot\frac{m+2p-1}{2}\cdot\frac{m+2p-2}{3}\cdots\frac{m+p+1}{p}\cdot\frac{q^p}{(1+q)^{m+2p}}.$$

By making successively $p=0$, $p=1$, $p=2$, $p=3$, etc., one will have the following probabilities, that the player will be ruined
in the game of which the rank is designated by m . . . . $\frac{1}{(1+q)^m}$,
in the game of which the rank is designated by $m+1$ . . . . $\frac{m}{1}\cdot\frac{q}{(1+q)^{m+2}}$,
in the game of which the rank is designated by $m+2$ . . . . $\frac{m}{1}\cdot\frac{m+3}{2}\cdot\frac{q^2}{(1+q)^{m+4}}$,
in the game of which the rank is designated by $m+3$ . . . . $\frac{m}{1}\cdot\frac{m+5}{2}\cdot\frac{m+4}{3}\cdot\frac{q^3}{(1+q)^{m+6}}$,
and so forth.

37. Before seeking the limit of the series formed by the reunion of the probabilities that we just found, it is necessary to demonstrate that this limit exists, by showing that if this series is not convergent in all its extent, it becomes it at least necessarily after a certain number of terms. For this let us divide the general term

$$\frac{m}{1} \cdot \frac{m+2p-1}{2} \cdot \frac{m+2p-2}{3} \cdots \frac{m+p+1}{p} \cdot \frac{q^p}{(1+q)^{m+2p}}$$

by the preceding term

$$\frac{m}{1} \cdot \frac{m+2p-3}{2} \cdot \frac{m+2p-4}{3} \cdots \frac{m+p}{p-1} \cdot \frac{q^{p-1}}{(1+q)^{m+2p-2}}$$

we will have for the quotient

$$\frac{(m+2p-1)(m+2p-2)}{p(m+p)} \cdot \frac{q}{(1+q)^2}$$

and the series will be convergent all the time that this quantity will be smaller than unity. We examine separately the two factors of which it is composed.

38. The fraction $\frac{q}{(1+q)^2}$ has the same value for all the terms of one same series, in order to find the case where it is the greatest possible one will equate its differential to zero, and one will have in order to determine q the equation

$$\frac{(1+q)^2dq - 2q(1+q)dq}{(1+q)^4} = 0,$$

which will give $q = 1$ and the maximum sought $\frac{q}{(1+q)^2} = \frac{1}{4}$, whence it follows that the series will be convergent all the time that the other factor

$$\frac{(m+2p-1)(m+2p-2)}{p(m+p)}$$

will not surpass four. The value of this factor depends on the number p of the terms which are found in the series before the general term, but it is easy to see that after having executed the multiplications indicated, one is able to set it under the form

$$4 + \frac{m^2 - 3m - 6p + 2}{pm + p^2}$$

which is less than 4 all the time that p is greater than $\frac{m^2-3m+2}{6}$, the series becomes therefore necessarily convergent as soon as one arrives to the terms for which p surpasses this last quantity.

39. Nothing is easier now than to find the limit of the proposed series

$$\frac{1}{(1+q)^m} + \frac{m}{1} \cdot \frac{q}{(1+q)^{m+2}} + \frac{m}{1} \cdot \frac{m+3}{2} \cdot \frac{q^2}{(1+q)^{m+4}} +$$

$$+\frac{m}{1}\cdot\frac{m+2p-1}{2}\cdot\frac{m+2p-2}{3}\cdots\frac{m+p+1}{p}\cdot\frac{q^p}{(1+q)^{m+2p}}+\text{etc.}$$

or that which returns to the same

$$\frac{1}{(1+q)^m}+A^{(m+2)}\frac{q}{(1+q)^{m+2}}+A^{(m+4)}\frac{q^2}{(1+q)^{m+4}}+$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots A^{(m+2p)}\frac{q^p}{(1+q)^{m+2p}}+\text{etc.}$$

it suffices for this to change, in each term, the denominators into fractional powers, and to develop them by the formula of Newton, in a manner that the series which result from them are convergent, that which requires that they proceed according to the ascending powers of q, when this quantity is smaller than 1, and according to the descending powers when it is greater. One will have thus in the first case

$$
\begin{aligned}
&(1+q)^{-m}+A^{(m+2)}q(1+q)^{-m-2}+A^{(m+4)}q^2(1+q)^{-m-4}+\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+A^{(m+2p)}q^p(1+q)^{-m-2p}+\text{etc.}=\\
&1-\frac{m}{1}q+\frac{m}{1}\cdot\frac{m+1}{2}q-\cdots\pm\frac{m}{1}\cdot\frac{m+1}{2}\cdot\frac{m+2}{3}\cdots\frac{m+p-1}{p}q^p\mp\text{etc.}\\
&+A^{(m+2)}q-\frac{m+2}{1}A^{(m+2)}q^2+\cdots\mp\frac{m+2}{1}\cdot\frac{m+3}{2}\cdot\frac{m+4}{3}\cdots\frac{m+p}{p-1}A^{(m+2)}q^p\pm\text{etc.}\\
&+A^{(m+4)}q^2-\cdots\pm\frac{m+4}{1}\cdot\frac{m+5}{2}\cdot\frac{m+6}{3}\cdots\frac{m+p+1}{p-2}A^{(m+4)}q^p\mp\text{etc.}\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&+\frac{m+2p-4}{1}\cdot\frac{m+2p-3}{2}A^{(m+2p-4)}q^p-\text{etc.}\\
&-\frac{m+2p-2}{1}A^{(m+2p-2)}q^p+\text{etc.}\\
&+A^{(m+2p)}q^p-\text{etc.}\\
&+\text{etc.}\qquad[8]
\end{aligned}
$$

and in the second

$$
\begin{aligned}
&(q+1)^{-m}+A^{(m+2)}q(q+1)^{-m-2}+A^{(m+4)}q^2(q+1)^{-m-4}+\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+A^{(m+2p)}q^p(q+1)^{-m-2p}+\text{etc.}=\\
&q^{-m}-\frac{m}{1}q^{-m-1}+\frac{m}{1}\cdot\frac{m+1}{2}q^{-m-2}-\cdots\pm\frac{m}{1}\cdot\frac{m+1}{2}\cdot\frac{m+2}{3}\cdots\frac{m+p-1}{p}q^{-m-p}\mp\text{etc.}\\
&+A^{(m+2)}q^{-m-1}-\frac{m+2}{1}A^{(m+2)}q^{-m-2}+\cdots\mp\frac{m+2}{1}\cdot\frac{m+3}{2}\cdot\frac{m+4}{3}\cdots\frac{m+p}{p-1}A^{(m+2)}q^{-m-p}\pm\text{etc.}\\
&+A^{(m+4)}q^{-m-2}-\cdots\pm\frac{m+4}{1}\cdot\frac{m+5}{2}\cdot\frac{m+6}{3}\cdots\frac{m+p+1}{p-2}A^{(m+4)}q^{-m-p}\mp\text{etc.}
\end{aligned}
$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\begin{aligned} &+\frac{m+2p-4}{1}\cdot\frac{m+2p-3}{2}A^{(m+2p-4)}q^{-m-p} - \text{etc.}\\ &-\frac{m+2p-2}{1}A^{(m+2p-2)}q^{-m-p} + \text{etc.}\\ &+A^{(m+2p)}q^{-m-p} - \text{etc.}\\ &+\text{etc.} \quad [9] \end{aligned}$$

These two developments which differ only by the exponents of which q is affected, are able to serve equally in the case where $q = 1$, they become then evidently identical.

40. It will be easy to find by induction that the second members of equations [8] and [9] are reduced respectively to their first terms,[7] by substituting in the place of

$$A^{(m+2)},\quad A^{(m+4)},\quad \cdots A^{(m+2p-4)},\quad A^{(m+2p-2)},\quad A^{(m+2p)}, \text{etc.}$$

the values represented by these signs, and by reducing after having executed the indicated multiplications; but in order to arrive to the same end in a direct and general manner, it is worth more to have recourse to equation [6], and to suppose $x = m + 2p$, this which changes it into

$$\begin{aligned} &A^{(m+2p)} - \frac{m+2p-2}{1}A^{(m+2p-2)} + \frac{m+2p-4}{1}\cdot\frac{m+2p-3}{2}A^{(m+2p-4)} - \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &\pm\frac{m+2r}{1}\cdot\frac{m+2r+1}{2}\cdot\frac{m+2r+2}{3}\cdots\frac{m+p+r-1}{p-r}A^{(m+2r)} \mp \text{etc.} = 0, \end{aligned}$$

the last terms of its first member that one finds by making successively $r = 2$, $r = 1$, $r = 0$, and by recalling that $A^{(m)} = 1$, are

$$\begin{aligned} &\frac{m+4}{1}\cdot\frac{m+5}{2}\cdot\frac{m+6}{3}\cdots\frac{m+p+1}{p-2}A^{(m+4)} \mp \frac{m+2}{1}\cdot\frac{m+3}{2}\cdot\frac{m+4}{3}\cdots\\ &\qquad\frac{m+p}{p-1}A^{(m+2)} \pm \frac{m}{1}\cdot\frac{m+1}{2}\cdot\frac{m+2}{3}\cdots\frac{m+p-1}{p}, \end{aligned}$$

whence it follows that this first member is precisely the same thing as the coefficient of q^p in equation [8], or of q^{-m-p} in equation [9]; the terms affected of this coefficient are reduced therefore to zero, p being indeterminate it is necessary likewise of all the terms which are found in the second members of these two equations, after 1 in the one

[7] These first terms being 1 when q is smaller than 1, and $\frac{1}{q^m}$ when it is greater, the limit of the series which we examine is constant in the first case and variable in the second: by writing $\frac{a}{x}$ in the place of q, one would have a series of which the limit would be constant or variable, according as x would be greater or smaller than a. This series reunited to one function whatsoever of x, one would form therefore one of the kind of those that one has named discontinuous functions, and of which I know not if one is to be arrived to represent the value by any combination of algebraic characters; the expression which furnishes the preceding remark, shows the possibility of having at least some developments into always convergent series.

and after q^{-m} in the other: it suffices in fact to suppose successively $p = 1$, $p = 2$, etc. and one obtains

$$A^{(m+2)} - \frac{m}{1} = 0,$$
$$A^{(m+4)} - \frac{m+2}{1} A^{(m+2)} + \frac{m}{1} \cdot \frac{m+1}{2} = 0,$$
$$\text{etc.} \qquad\qquad \text{etc.}$$

equations of which the first members are nothing other than the coefficient of these terms.

41. When the number of chances favorable to the player outweighs in each game the one of the chances which are contrary to him q is greater than 1, and it is necessary to serve oneself with the second development which gives q^{-m} or $\frac{1}{q^m}$ for the sought limit, so that the probability of ruin of the player remains always finite whatever be the number of games, and it is able even to be less than the contrary probability if, $\frac{1}{q^m}$ is smaller than $\frac{1}{2}$, or that which reverts to the same if q is greater than $\sqrt[m]{2}$.[8] But it is necessary to observe well that this case or the game, if there is not a tax established by the Government, must be considered as a theft made to the public, and against which the laws should prevail with reason, is the sole one where the player is able to avoid a certain ruin. In fact, when q is smaller than 1, it is necessary to serve oneself with the first development, and one has 1 for the limit of the probabilities of ruin of the player: this event is therefore morally certain (7). It is likewise in the case where the chances are equally divided, and where q being equal to 1, the two developments agree to give 1 for the same limit. It is easy to sense that it is uniquely from the results given by the calculation in this last case that it is necessary to draw all the applications that one is able to make from the mathematical theory of the game to that which is passed habitually in society, for an unequal game being able to present no other side an advantage greater than the disadvantage that results to it from the other, there must be in the course of life of a player a necessary compensation between the case where the probability is found in his favor and the one where it is contrary to him. I do not speak of the players who are knavish enough or rather easily deceived in order to set themselves voluntarily and constantly into the one or into the other of these two cases, because the first must be repressed by the public authority, and that is so evident that the others must be ruined, that it must perhaps be useless to demonstrate it. I myself proposed especially in this work to prove that the certitude of ruin of the player is also complete, when likewise the probability is equal in each game between him and his adversary. This truth that one took in the first glance for a paradox, results evidently from this that the limit of the probabilities contrary to the player, is the same when one takes q equal to 1, or when one supposes that it is smaller. There is to note that one finds also the same result in a case wher the necessity of the ruin of the player is

[8]One is able also to conclude from this formula that a man who would make trade of a game where he would have a determined advantage, and who would not wish that the probability of his ruin ever be able to attain a probability known and represented by $\frac{1}{a}$, he would arrive easily by never playing that with the fraction $\frac{1}{m}$ of his fortune of which the denominator m was greater than $\frac{\ln a}{\ln q}$.

yet more evident, and where whatever be the value of q, the probability of this event has precisely the same limit. This case is the one where, commencing by putting into the game all his fortune as of the first game, the player would continue indefinitely to play to quit or double, so that, only one game lost would suffice always to ruin him completely.

42. If one continues, under this new hypothesis, to represent by $q:1$ the ratio which exists at each game between the chances favorable to the player, and those which are contrary to him: the probabilities that he will win or that he will lose a game, will be always represented respectively, by

$$\frac{q}{1+q} \text{ and } \frac{1}{1+q}.$$

Since under the actual supposition, the player is able to be ruined in the last in any number t of games, only in the case where he would lose this game after having won all the preceding, of which the number is expressed by $t-1$, it is evident that the probability of this event will be represented by the product of $t-1$ factors equal to $\frac{q}{1+q}$, and by a factor equal to $\frac{1}{1+q}$, that is, by $\frac{q^{t-1}}{(1+q)^t}$; making successively $t=1$, $t=2$, $t=3$, etc. one will find the following probabilities that the player will be ruined

in the first game . . . . $\frac{1}{(1+q)}$,

in the second . . . . $\frac{q}{(1+q)^2}$,

in the third . . . . $\frac{q^2}{(1+q)^3}$,

and so forth.

43. The series that one forms by reuniting the probabilities that we just determined

$$\frac{1}{1+q} + \frac{q}{(1+q)^2} + \frac{q^2}{(1+q)^3} + \cdots + \frac{q^{t-1}}{(1+q)^t} + \text{etc.}$$

is evidently a progression by quotients, of which the limit found by the known methods is reduced to one. This limit is therefore precisely the same under the hypothesis that we just examined, and under that where the player exposes at each game only a constant portion of his original fortune. The moral certitude of his ruin is therefore the same in these two cases, and the only difference which is able to exist between them, is only in the number of games which will give for the sum of the probabilities contrary to the player, some values which approach equally to certitude. This number must be so much greater as the sum played in each game is smaller. It could be small enough so that the ruin of the player requires more games than the ordinary limits of life permits him to play it, it is this which arrives in regard to those who expose themselves only to some losses incapable of diminishing sensibly their fortune; every other manner of play leads to a certain ruin. The witness of experience has a long time set this truth beyond doubt, being found confirmed in the most complete manner by the preceding calculations, the end of this memoir would be fulfilled, and I would have been able to end it here, if it were not necessary, in order to leave nothing in obscurity on this theory, to examine also the case where the same two players play constantly one against the other.

44. It is necessary to first calculate the probability that one of the two players will find himself ruined at the last of any number of games. We suppose, under the view of rendering the calculation most simple, that the sum played is the same at each game, and that it is an exact fraction of the fortune of each player, contained m times in that of player B, of whom we calculate the chances, and n times in the fortune of the other player C, $m : n$ expressing the ratio of the two fortunes. It is evident that under the supposition the first player will be able to be found ruined only after $m + 2p$ games, of which p won and $m + p$ lost, whence it follows that by representing always by $q : 1$, the ratio of the chances favorable to this player, and from those which are contrary to him, $\frac{q^p}{(1+q)^{m+2p}}$ will express the probability of each of the arrangement of these $m + 2p$ games, which will remove in the last game the rest of his fortune. This probability is precisely the same as in the problem that we have already resolved, (n^{o} 12 and the following); but the number of arrangements of the $m + 2p$ parts, by which it will be necessary to multiply this probability, will not be the same, because it will be necessary to exclude from the total number of the arrangements of p games won, and of $m+p$ games lost, not only the arrangements which would have ruined player B, before the game of which the rank is designated by $m + 2p$, but yet those who would have brought forth the ruin of his adversary before the same game, since the game ceasing necessarily as soon as one of the two players is ruined, it would not have been able to be continued, in this case, until the game for which we calculate the probability of the ruin of the first player.

45. It follows from this observation that the probability of ruin of one of the players is not able to be calculated independently from the probability of that from the other: now, the entire loss of the fortune of player C, supposes that player B has won n games more than he has lost of them. This event is able to arrive therefore only after $n + 2p$ games, p designating always any number; and by supposing that B has won $n + p$ of these games, and that he has lost p, this which gives $\frac{q^{n+p}}{(1+q)^{n+2p}}$ for the probability of each of the arrangements that one is able to give to $n + 2p$ games, in a manner to satisfy this condition. We represent in general by $\mathrm{B}^{(t)}$, the number of arrangements of any number t of games, which cause the ruin of player B at the last of these t games, and by $\mathrm{C}^{(t)}$, the number of arrangements, which bring forth the ruin of his adversary in the same game, by comprehending in these arrangements only those which have ruined neither the one nor the other player at any of the preceding games, we have the two series

$$\mathrm{B}^{(m)}\frac{1}{(1+q)m} + \mathrm{B}^{(m+2)}\frac{q}{(1+q)^{m+2}} + \mathrm{B}^{(m+4)}\frac{q^2}{(1+q)^{m+4}} +$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots + \mathrm{B}^{(m+2p)}\frac{q^p}{(1+q)^{m+2p}} + \text{etc. and}$$

$$\mathrm{C}^{(n)}\frac{q^n}{(1+q)^n} + \mathrm{C}^{(n+2)}\frac{q^{n+1}}{(1+q)^{n+2}} + \mathrm{C}^{(n+4)}\frac{q^{n+2}}{(1+q)^{n+4}} +$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots + \mathrm{C}^{(n+2p)}\frac{q^{n+p}}{(1+q)^{n+2p}} + \text{etc.}$$

of which each term will indicate the probability that the player to whom the series corresponds, will be ruined in the game of which the rank is designated by the index of B or of C in the same term.

46. In the two series the coefficient $\mathrm{B}^{(m)}$ or $\mathrm{C}^{(n)}$ of the first term is equal to unity, because there is only a single arrangement of m games, all losses by player B, which is able to ruin this player at the m^{th} game; and there is likewise only a single arrangement of n games, all won by the same player, which is able to ruin his adversary at the game of which the rank is designated by n.

47. In order to find the relations which exist between the coefficients of the different terms of these two series, one will observe that $\mathrm{B}^{(m+2p)}$ must be equal to the arrangements of p games won, and of $m+p$ games lost, which remain after one has taken off from the total number of these arrangements, namely:

$$\frac{m+2p}{1}\cdot\frac{m+2p-1}{2}\cdot\frac{m+2p-2}{3}\cdots\frac{m+p+1}{p},$$

1° the number of the arrangements which would suppose player B ruined at any of the preceding games. One will find, as in the first problem, that we have resolved, and for the same reasons, that this number is expressed by this sequence of terms

$$\begin{gathered}\frac{2}{1}\mathrm{B}^{(m+2p-2)} + \frac{4}{1}\cdot\frac{3}{2}\mathrm{B}^{(m+2p-4)} + \frac{6}{1}\cdot\frac{5}{2}\cdot\frac{4}{3}\mathrm{B}^{(m+2p-6)} + \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ + \frac{2p-2r}{1}\cdot\frac{2p-2r-1}{2}\cdot\frac{2p-2r-2}{3}\cdots\frac{p-r+1}{p-r}\mathrm{B}^{(m+2r)} + \text{etc.}\end{gathered}$$

or that which reverts to the same, by

$$\begin{gathered}2\mathrm{B}^{(m+2p-2)} + 2\frac{3}{1}\mathrm{B}^{(m+2p-4)} + 2\frac{5}{1}\cdot\frac{4}{2}\mathrm{B}^{(m+2p-6)} + \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ +2\frac{2p-2r-1}{1}\cdot\frac{2p-2r-2}{2}\cdots\frac{p-r+1}{p-r-1}\mathrm{B}^{(m+2r)} + \text{etc.}\end{gathered}$$ [9]

2° the number of arrangements which would have ruined player C in one of the preceding games. In order to find it one will represent in general by $n+2s$ the rank of this game. The arrangement of the $n+2s$ games that it terminates, being necessarily composed of $n+s$ games won by player B, and of the s games lost by the same player,

[9] The first of these two formulas gives the most regular, the second the most simple to calculate, thus as we have already seen in regard to the analogous formulas of the preceding problem; this is that which we will determine to employ sometimes the one and sometimes the other according to the requirement of the cases.

it will be necessary to join $p-n-s$ games won, and $m+p-s$ games lost, in order to form the arrangements of the p won games, and of the $m+p$ games lost, this which is able to be executed for each of the arrangements of which the number is represented by $\mathrm{C}^{(n+2s)}$, of

$$\frac{2p+m-n-2s}{1}\cdot\frac{2p+m-n-2s-1}{2}\cdots\frac{p+m-s+1}{p-n-s}$$

different ways, since there are $2p+m-n-2s$ games divided into two groups, the one of $p-n-s$, and the other of $m+p-s$ games. By multiplying the number that we just found by $\mathrm{C}^{(n+2s)}$, one has

$$\frac{2p+m-n-2s}{1}\cdot\frac{p+m-n-2s-1}{2}\cdots\frac{p+m-s+1}{p-n-s}\mathrm{C}^{(n+2s)}$$

48. The concern now is to give to s all the values in positive whole numbers which are able to agree with the state of the question, in order to reunite all the terms which result from it with those that we have found just now and by subtracting the sum from the total number of the arrangements

$$\frac{m+2p}{1}\cdot\frac{m+2p-1}{2}\cdot\frac{m+2p-2}{3}\cdots\frac{m-p+1}{p}$$

Now it is evident that the number $p-n-s$ of games won by player B, from the game of which the rank is expressed by $n+2s$, to that of which the rank is designated by $m+2p$, not being able to be negative, the greatest value that one is able to give to s, is $s=p-n$, making successively $s=p-n$, $s=p-n-1$, $s=p-n-2$, etc., this which gives

$$\begin{array}{c}
n+2s=2p-n,\ \text{and}\ 2p+m-n-2s=m+n,\\
n+2s=2p-n-2\ \text{and}\ 2p+m-n-2s=m+n+2\\
n+2s=2p-n-4\ \text{and}\ 2p+m-n-2s=m+n+4\\
\text{etc.}\qquad\qquad\qquad \text{etc.}
\end{array}$$

will have from it this series of terms

$$\begin{array}{l}
\mathrm{C}^{(2p-n)}+\frac{m+n+2}{1}\mathrm{C}^{(2p-n-2)}+\frac{m+n+4}{1}\cdot\frac{m+n+3}{2}\mathrm{C}^{(2p-n-4)}+\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
+\frac{2p+m-n-2s}{1}\cdot\frac{2p+m-n-2s-1}{2}\cdots\frac{p+m-s+1}{p-n-s}\mathrm{C}^{(n+2s)}
\end{array}$$

and one will conclude that

$$\begin{aligned}
\mathrm{B}^{(m+2p)}=&\frac{m+2p}{1}\cdot\frac{m+2p-1}{2}\cdot\frac{m+2p-2}{3}\cdots\frac{m+p+1}{p}\\
&-2\mathrm{B}^{(m+2p-2)}-2\frac{3}{1}\mathrm{B}^{(m+2p-4)}-2\frac{5}{1}\cdot\frac{4}{2}\mathrm{B}^{(m+2p-6)}-
\end{aligned}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots - 2\frac{2p-2r-1}{1}\cdot\frac{2p-2r-2}{2}\cdots\frac{p-r+1}{p-r-1}\mathrm{B}^{(m+2r)} - \text{ etc.}$$

$$-\,\mathrm{C}^{(2p-n)} - \frac{m+n+2}{1}\mathrm{C}^{(2p-n-2)} - \frac{m+n+4}{1}\cdot\frac{m+n+3}{2}\mathrm{C}^{(2p-n-4)} -$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$-\frac{2p+m-n-2s}{1}\cdot\frac{2p+m-n-2s-1}{2}\cdots\frac{p+m-s+1}{p+m-s}\mathrm{C}^{(n+2s)} - \text{ etc.} \qquad [10]$$

49. If one makes for brevity $m+n=k$, this which gives $m-n=k-2n$, and $2p+m-n-2s=k+2(p-n-s)$, one will obtain

$$\begin{aligned}\mathrm{B}^{(m+2p)} = {} & \frac{m+2p}{1}\cdot\frac{m+2p-1}{2}\cdot\frac{m+2p-2}{3}\cdots\frac{m+p+1}{p}\\ & - 2\mathrm{B}^{(m+2p-2)} - 2\frac{3}{1}\mathrm{B}^{(m+2p-4)} - 2\frac{5}{1}\cdot\frac{4}{2}\mathrm{B}^{(m+2p-6)} -\end{aligned}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$- 2\frac{2p-2r-1}{1}\cdot\frac{2p-2r-2}{2}\cdots\frac{p-r+1}{p-r-1}\mathrm{B}^{(m+2r)} - \text{ etc.}$$

$$-\,\mathrm{C}^{(2p-n)} - \frac{k+2}{1}\mathrm{C}^{(2p-n-2)} - \frac{k+4}{1}\cdot\frac{k+3}{2}\mathrm{C}^{(2p-n-4)} -$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$-\frac{k+2(p+n-s)}{1}\cdot\frac{k+2(p+n-s)-1}{2}\cdots\frac{k+p-n-s+1}{p-n-s}\mathrm{C}^{(n+2s)} - \text{ etc.} \qquad [11]$$

50. It is easy to find another value of $\mathrm{B}^{(m+2p)}$, by observing that player B is not able to be ruined in the game of which the rank is marked by $m+2p$, without having been reduced, the preceding game, to having no more than $\frac{1}{m}$ of that which he had on entering into the game; whence it follows that $\mathrm{B}^{(m+2p)}$ is also equal to the number of arrangement of p games won and of $m+p-1$ games lost, which have ruined neither the one nor the other of the players in any of the preceding games; without this condition the number of these arrangements were

$$\frac{m+2p-1}{1}\cdot\frac{m+2p-2}{2}\cdot\frac{m+2p-3}{3}\cdots\frac{m+p}{p},$$

from which it is necessary to subtract, 1° the number of those of these arrangements which have ruined player B before the $(m+2p)^{\text{th}}$ game, a number that one will find here as in the preceding problem, expressed by the series

$$\mathrm{B}^{(m+2p-2)} + \frac{3}{1}\mathrm{B}^{(m+2p-4)} + \frac{5}{1}\cdot\frac{4}{2}\mathrm{B}^{(m+2p-6)} +$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\frac{2p-2r-1}{1}\cdot\frac{2p-2r-2}{2}\cdots\frac{p-r+1}{p-r-1}\mathrm{B}^{(m+2r)}+\text{ etc.}$$

2° all the arrangements which suppose on the contrary player C ruined before the same game. In these here the $n+2s$ first games that we suppose susceptible of $\mathrm{C}^{(n+2s)}$ different arrangements, are composed of $n+s$ games won by player B, and of s games lost by the same player; it is necessary therefore to join $p-n-s$ games won, and $p+m-s$ games lost, in order to have the arrangements to subtract; these $2p+m-n-2s-1$ games are able to be partitioned thus, in

$$\frac{2p+m-n-2s-1}{1}\cdot\frac{2p+m-n-2s-2}{2}\cdots\frac{p+m-s}{p-n-s}$$

different ways, one will have the expression

$$\frac{2p+m-n-2s-1}{1}\cdot\frac{2p+m-n-2s-2}{2}\cdots\frac{p+m-s}{p-n-s}\mathrm{C}^{(n+2s)}$$

where it will be necessary to make successively

$$s=p-n,\quad s=p-n-1,\quad s=p-n-2,\quad \text{etc.}$$

this which will give for $n+2s$ and for $2p+m-n-2s$ the same values as above (48). One will conclude from it easily, by reuniting all the terms which will result from these diverse substitutions, that the number that we wish to calculate is represented by the series

$$\mathrm{C}^{(2p-n)}+\frac{m+n+1}{1}\mathrm{C}^{(2p-n-2)}+\frac{m+n+3}{1}\cdot\frac{m+n+2}{2}\mathrm{C}^{(2p-n-4)}+$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+\frac{2p+m-n-2s-1}{1}\cdot\frac{2p+m-n-2s-2}{2}\cdots\frac{p+m-s}{p-n-s}\mathrm{C}^{(n+2s)}+\text{ etc.}$$

or that which returns to the same (49) by

$$\mathrm{C}^{(2p-n)}+\frac{k+1}{1}\mathrm{C}^{(2p-n-2)}+\frac{k+3}{1}\cdot\frac{k+2}{2}\mathrm{C}^{(2p-n-4)}+$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots+\frac{k+2(p-n-s)-1}{1}\cdot\frac{k+2(p-n-s)-2}{2}\cdots\frac{k+p-n-s}{p-n-s}\mathrm{C}^{(n+2s)}+\text{ etc.}$$

it follows from all this that we just said, that

$$\mathrm{B}^{(m+2p)}=\frac{m+2p-1}{1}\cdot\frac{m+2p-2}{2}\cdots\frac{m+p}{p}-$$
$$\mathrm{B}^{(m+2p-2)}-\frac{3}{1}\mathrm{B}^{(m+2p-4)}-\frac{5}{1}\cdot\frac{4}{2}\mathrm{B}^{(m+2p-6)}-\cdots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$- \frac{2p-2r-1}{1} \cdot \frac{2p-2r-2}{2} \cdots \frac{p-r+1}{p-r-1} \mathrm{B}^{(m+2r)} - \text{ etc.}$$

$$- \mathrm{C}^{(2p-n)} - \frac{k+1}{1} \mathrm{C}^{(2p-n-2)} - \frac{k+3}{1} \cdot \frac{k+2}{2} \mathrm{C}^{(2p-n-4)} - \cdots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$- \frac{k+2(p-n-s)-1}{1} \cdot \frac{k+2(p-n-s)-2}{2} \cdots \frac{k+p-n-s}{p-n-s} \mathrm{C}^{(n+2s)} - \text{ etc.} \quad [12]$$

51. If one doubles this equation, and if one subtracts equation [11] from it, all the affected terms of

$$\mathrm{B}^{(m+2p-2)}, \quad \mathrm{B}^{(m+2p-4)}, \quad \mathrm{B}^{(m+2p-6)}, \cdots \mathrm{B}^{(m+2r)}, \text{etc.}$$

will vanish, and there will remain

$$\mathrm{B}^{(m+2p)} = \frac{2m+2p}{1} \cdot \frac{m+2p-1}{2} \cdot \frac{m+2p-2}{3} \cdots \frac{m+p+1}{p}$$

$$-2\mathrm{C}^{(2p-n)} - \frac{2k+2}{1} \mathrm{C}^{(2p-n-2)} - \frac{2k+4}{1} \cdot \frac{k+3}{2} \mathrm{C}^{(2p-n-4)} -$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$- \frac{2k+2(p-n-s)}{1} \cdot \frac{k+2(p-n-s)-1}{2} \cdots \frac{k+p-n-s+1}{p-n-s} \mathrm{C}^{(n+2s)} - \text{ etc.}$$

$$- \frac{m+2p}{1} \cdot \frac{m+2p-1}{2} \cdot \frac{m+2p-2}{3} \cdots \frac{m+p+1}{p}$$

$$+\mathrm{C}^{(2p-n)} + \frac{k+2}{1} \mathrm{C}^{(2p-n-2)} + \frac{k+4}{1} \cdot \frac{k+3}{2} \mathrm{C}^{(2p-n-4)} +$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ \frac{k+2(p-n-s)}{1} \cdot \frac{k+2(p-n-s)-1}{2} \cdots \frac{k+p-n-s+1}{p-n-s} \mathrm{C}^{(n+2s)} + \text{ etc.}$$

which is reduced to

$$\mathrm{B}^{(m+2p)} = \frac{m}{1} \cdot \frac{m+2p-1}{2} \cdot \frac{m+2p-2}{3} \cdots \frac{m+p+1}{p} -$$

$$\mathrm{C}^{(2p-n)} - \frac{k}{1} \mathrm{C}^{(2p-n-2)} - \frac{k}{1} \cdot \frac{k+3}{2} \mathrm{C}^{(2p-n-4)} -$$

$$\frac{k}{1} \cdot \frac{k+5}{2} \cdot \frac{k+4}{3} \mathrm{C}^{(2p-n-6)} - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$- \frac{k}{1} \cdot \frac{k+2(p-n-s)-1}{2} \cdot \frac{k+2(p-n-s)-2}{3} \cdots \frac{k+p-n-s+1}{p-n-s} \mathrm{C}^{(n+2s)} - \text{ etc.} \quad [13]$$

52. If one pays attention that $\mathrm{C}^{(n)} = 1$, and that every term of the series of C of which the index would be small, would equate to zero, one will see easily that as long as $2p - n$ is smaller than n, that is as long as p is smaller than n, one has simply

$$\mathrm{B}^{(m+2p)} = \frac{m}{1} \cdot \frac{m+2p-1}{2} \cdot \frac{m+2p-2}{3} \cdots \frac{m+p+1}{p},$$

which is precisely the value that we have found for $\mathrm{A}^{(m+2p)}$ in the preceding problem, whence it follows that the first terms of the series of B, are the same as those of the series of A. In order to determine the number of the terms common to these two series, it suffices to observe that $m + 2p$ represents always the index of any term, their first term corresponds to $p = 0$, and the last of those which are the same in the two series to $2p - n = n - 2$, or $p = n - 1$, this which gives n common terms; the one which comes after these n terms corresponds to $p = n$, and this term, which is represented by $\mathrm{B}^{(m+2n)}$, is found consequently equal to

$$\frac{m}{1} \cdot \frac{m+2n-1}{2} \cdot \frac{m+2n-2}{3} \cdots \frac{m+n+1}{n} - \mathrm{C}^{(n)} =$$
$$\frac{m}{1} \cdot \frac{m+2n-1}{2} \cdot \frac{m+2n-2}{3} \cdots \frac{m+n+1}{n} - 1,$$

a value less one unit than the corresponding term of the series of A,

$$\mathrm{A}^{(m+2n)} = \frac{m}{1} \cdot \frac{m+2n-1}{2} \cdot \frac{m+2n-2}{3} \cdots \frac{m+n+1}{n}.$$

53. If one subtracts equation [13] from equation [11] after having set that above under the form

$$\mathrm{B}^{(m+2p)} + 2\mathrm{B}^{(m+2p-2)} + 2\frac{3}{1}\mathrm{B}^{(m+2p-4)} + 2\frac{5}{1} \cdot \frac{4}{2}\mathrm{B}^{(m+2p-6)} + \cdots$$

. . . .

$$+ 2\frac{2p-2r-1}{1} \cdot \frac{2p-2r-2}{2} \cdot \frac{2p-2r-3}{3} \cdots \frac{p-r+1}{p-r-1}\mathrm{B}^{(m+2r)} + \text{ etc.} =$$
$$\frac{m+2p}{1} \cdot \frac{m+2p-1}{2} \cdot \frac{m+2p-2}{3} \cdots \frac{m+p+1}{p} - \mathrm{C}^{(2p-n)} -$$
$$\frac{k+2}{1}\mathrm{C}^{(2p-n-2)} - \frac{k+4}{1} \cdot \frac{k+3}{2}\mathrm{C}^{(2p-n-4)} - \frac{k+6}{1} \cdot \frac{k+5}{2} \cdot \frac{k+4}{3}\mathrm{C}^{(2p-n-6)}$$

. . . .

$$- \frac{k+2(p-n-s)}{1} \cdot \frac{k+2(p-n-s)-1}{2} \cdots \frac{k+p-n-s+1}{p-n-s}\mathrm{C}^{(n+2s)} - \text{ etc.} \quad [14]$$

all the terms of the remaining equation will be divisible by 2, and one will obtain after having executed this division

$$\mathrm{B}^{(m+2p-2)} + \frac{3}{1}\mathrm{B}^{(m+2p-4)} + \frac{5}{1}\cdot\frac{4}{2}\mathrm{B}^{(m+2p-6)} +$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\frac{2p-2r-1}{1}\cdot\frac{2p-2r-2}{2}\cdot\frac{2p-2r-3}{3}\cdots\frac{p-r+1}{p-r-1}\mathrm{B}^{(m+2r)} + \text{ etc.} =$$

$$\frac{m+2p-1}{1}\cdot\frac{m+2p-2}{2}\cdot\frac{m+2p-3}{3}\cdots\frac{m+p+1}{p-1} -$$

$$\mathrm{C}^{(2p-n-2)} - \frac{k+3}{1}\mathrm{C}^{(2p-n-4)} - \frac{k+5}{1}\cdot\frac{k+4}{2}\mathrm{C}^{(2p-n-6)} -$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{k+2(p-n-s)-1}{1}\cdot\frac{k+2(p-n-s)-2}{2}\cdot\frac{k-2(p-n-s)-3}{3}\cdots\frac{k+p-n-s+1}{p-n-s-1}\mathrm{C}^{(n+2s)} - \text{ etc.} \quad [15]$$

This equation must hold for all the values of p, one will be able to write $p+1$ instead of p, and one will have from it

$$\mathrm{B}^{(m+2p)} + \frac{3}{1}\mathrm{B}^{(m+2p-2)} + \frac{5}{1}\cdot\frac{4}{2}\mathrm{B}^{(m+2p-4)} +$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+2\frac{2p-2r+1}{1}\cdot\frac{2p-2r}{2}\cdot\frac{2p-2r-1}{3}\cdots\frac{p-r+2}{p-r}\mathrm{B}^{(m+2r)} + \text{ etc.} =$$

$$\frac{m+2p+1}{1}\cdot\frac{m+2p}{2}\cdot\frac{m+2p-1}{3}\cdots\frac{m+p+1}{p} -$$

$$\mathrm{C}^{(2p-n)} - \frac{k+3}{1}\mathrm{C}^{(2p-n-2)} - \frac{k+5}{1}\cdot\frac{k+4}{2}\mathrm{C}^{(2p-n-4)} -$$

$$\frac{k+7}{1}\cdot\frac{k+6}{2}\cdot\frac{k+5}{3}\mathrm{C}^{(2p-n-6)} - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$-\frac{k+2(p-n-s)+1}{1}\cdot\frac{k+2(p-n-s)}{2}\cdot\frac{k-2(p-n-s)-1}{3}\cdots\frac{k+p-n-s+2}{p-n-s}\mathrm{C}^{(n+2s)} - \text{ etc.} \quad [16]$$

54. By comparing this equation with equation [14] which is the same thing as

$$\mathrm{B}^{(m+2p)} + \frac{2}{1}\mathrm{B}^{(m+2p-2)} + \frac{4}{1}\cdot\frac{3}{2}\mathrm{B}^{(m+2p-4)} + \frac{6}{1}\cdot\frac{5}{2}\cdot\frac{4}{3}\mathrm{B}^{(m+2p-6)} +$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\frac{2p-2r}{1}\cdot\frac{2p-2r-1}{2}\cdot\frac{2p-2r-2}{3}\cdots\frac{p-r+1}{p-r}\mathrm{B}^{(m+2r)} + \text{ etc.} =$$

$$\frac{m+2p}{1}\cdot\frac{m+2p-1}{2}\cdot\frac{m+2p-2}{3}\cdots\frac{m+p+1}{p} -$$

$$\mathrm{C}^{(2p-n)} - \frac{k+2}{1}\mathrm{C}^{(2p-n-2)} - \frac{k+4}{1}\cdot\frac{k+3}{2}\mathrm{C}^{(2p-n-4)} - \frac{k+6}{1}\cdot\frac{k+5}{2}\cdot\frac{k+4}{3}\mathrm{C}^{(2p-n-6)} -$$

. . . .

$$-\frac{k+2(p-n-s)}{1}\cdot\frac{k+2(p-n-s)-1}{2}\cdot\frac{k-2(p-n-s)-2}{3}\cdots\frac{k+p-n-s+1}{p-n-s}\mathrm{C}^{(n+2s)}-\text{ etc.}$$

[17]

one will perceive easily that these two equations differ only by the numerators of the coefficients of which all the factors have increased by one unit through the operations which have led from equation [14] to equation [16]. The coefficients of this here prove the same change if one subtracts equation [17] from this equation, and if one substitutes next $p+1$ for p, because one has generally

$$\frac{2p-2r+1}{1}\cdot\frac{2p-2r}{2}\cdot\frac{2p-2r-1}{3}\cdots\frac{p-r+2}{p-r}-$$
$$\frac{2p-2r}{1}\cdot\frac{2p-2r-1}{2}\cdot\frac{2p-2r-2}{3}\cdots\frac{p-r+1}{p-r}=$$
$$\frac{p-r}{1}\cdot\frac{2p-2r}{2}\cdot\frac{2p-2r-1}{3}\cdot\frac{2p-2r-2}{4}\cdots\frac{p-r+2}{p-r}=$$
$$\frac{2p-2r}{1}\cdot\frac{2p-2r-1}{2}\cdot\frac{2p-2r-2}{3}\cdots\frac{p-r+2}{p-r-1},$$
$$\frac{m+2p+1}{1}\cdot\frac{m+2p}{2}\cdot\frac{m+2p-1}{3}\cdots\frac{m+p+2}{p}$$
$$-\frac{m+2p}{1}\cdot\frac{m+2p-1}{2}\cdot\frac{m+2p-2}{3}\cdots\frac{m+p+1}{p}=$$
$$\frac{m+2p}{1}\cdot\frac{m+2p-1}{2}\cdot\frac{m+2p-2}{3}\cdots\frac{m+p+2}{p-1},\text{ and}$$
$$\frac{k+2(p-n-s)+1}{1}\cdot\frac{k+2(p-n-s)}{2}\cdot\frac{k+2(p-n-s)-1}{3}\cdots\frac{k+p-n-s+2}{p-n-s}$$
$$-\frac{k+2(p-n-s)}{1}\cdot\frac{k+2(p-n-s)-1}{2}\cdot\frac{k+2(p-n-s)-2}{3}\cdots\frac{k+p-n-s+1}{p-n-s}=$$
$$\frac{k+2(p-n-s)}{1}\cdot\frac{k+2(p-n-s)-1}{2}\cdot\frac{k+2(p-n-s)-2}{3}\cdots\frac{k+p-n-s+2}{p-n-s-1}$$

this which reduces the remaining equation to

$$\mathrm{B}^{(m+2p-2)}+\frac{4}{1}\mathrm{B}^{(m+2p-4)}+\frac{6}{1}\cdot\frac{5}{2}\mathrm{B}^{(m+2p-6)}+$$

. . . .

$$+\frac{2p-2r}{1}\cdot\frac{2p-2r-1}{2}\cdot\frac{2p-2r-2}{3}\cdots\frac{p-r+2}{p-r-1}\mathrm{B}^{(m+2r)}+\text{ etc.}=$$
$$\frac{m+2p}{1}\cdot\frac{m+2p-1}{2}\cdot\frac{m+2p-2}{3}\cdots\frac{m+p+2}{p-1}-$$
$$\mathrm{C}^{(2r-n-2)}-\frac{k+4}{1}\mathrm{C}^{(2r-n-4)}-\frac{k+6}{1}\cdot\frac{k+5}{2}\mathrm{C}^{(2r-n-6)}-\cdots$$

. . . .

$$-\frac{k+2(p-n-s)}{1}\cdot\frac{k+2(p-n-s)-1}{3}\cdot\frac{k+2(p-n-s)-2}{4}\cdots\frac{k+p-n-s+2}{p-n-s-1}\mathrm{C}^{(n+2s)}-\text{ etc.}$$

or to

$$\mathrm{B}^{(m+2p)}+\frac{4}{1}\mathrm{B}^{(m+2p-2)}+\frac{6}{1}\cdot\frac{5}{2}\mathrm{B}^{(m+2p-4)}+\frac{8}{1}\cdot\frac{7}{2}\cdot\frac{6}{3}\mathrm{B}^{(m+2p-6)}+$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+\frac{2p-2r+2}{1}\cdot\frac{2p-2r+1}{2}\cdot\frac{2p-2r}{3}\cdots\frac{p-r+3}{p-r}\mathrm{B}^{(m+2r)}+\text{ etc.}=$$
$$\frac{m+2p+2}{1}\cdot\frac{m+2p+1}{2}\cdot\frac{m+2p}{3}\cdots\frac{m+p+3}{p}-$$
$$\mathrm{C}^{(2p-n)}-\frac{k+4}{1}\mathrm{C}^{(2p-n-2)}-\frac{k+6}{1}\cdot\frac{k+5}{2}\mathrm{C}^{(2p-n-4)}-\frac{k+8}{1}\cdot\frac{k+7}{2}\cdot\frac{k+6}{3}\mathrm{C}^{(2p-n-6)}-$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$-\frac{k+2(p-n-s)+2}{1}\cdot\frac{k-2(p-n-s)+1}{2}\cdot\frac{k+2(p-n-s)}{3}\cdots\frac{k+p-n-s+3}{p-n-s}\mathrm{C}^{(n+2s)}-\text{ etc.}$$

by writing $p+1$ instead of p.

55. If one pays attention that this increase by one unit in the factors of the numerators of these equations, is a necessary sequence of their form, one will be convinced easily that it takes place in each transformation that one is able to make successively, and that the different equations which result from it are consequently only some particular cases of a general formula that one will find by naming u the numerator of these transformations, departing from equation [17]. It will suffice to add u to each of the factors of the numerators of this equation, this which will give

$$\mathrm{B}^{(m+2p)}+\frac{u+2}{1}\mathrm{B}^{(m+2p-2)}+\frac{u+4}{1}\cdot\frac{u+3}{2}\mathrm{B}^{(m+2p-4)}+\frac{u+6}{1}\cdot\frac{u+5}{2}\cdot\frac{u+4}{3}\mathrm{B}^{(m+2p-6)}+$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+\frac{u+2p-2r}{1}\cdot\frac{u+2p-2r-1}{2}\cdot\frac{u+2p-2r-2}{3}\cdots\frac{u+p-r+1}{p-r}\mathrm{B}^{(m+2r)}+\text{ etc.}=$$
$$\frac{u+m+2p}{1}\cdot\frac{u+m+2p-1}{2}\cdot\frac{u+m+2p-2}{3}\cdots\frac{u+m+p+1}{p}-$$
$$\mathrm{C}^{(2p-n)}-\frac{u+k+2}{1}\mathrm{C}^{(2p-n-2)}-\frac{u+k+4}{1}\cdot\frac{u+k+3}{2}\mathrm{C}^{(2p-n-4)}-$$
$$\frac{k+u+6}{1}\cdot\frac{k+u+5}{2}\cdot\frac{k+u+4}{3}\mathrm{C}^{(2p-n-6)}-$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$-\frac{u+k+2(p-n-s)}{1}\cdot\frac{u+k+2(p-n-s)-1}{2}\cdot\frac{u+k+2(p-n-s)-2}{3}\cdots$$
$$\frac{u+k+p-n-s+1}{p-n-s}\mathrm{C}^{(n+2s)}-\text{ etc.}\quad[18]$$

56. Although the preceding demonstration is applied immediately only to the case where u is a positive whole number, it is easy to conclude from it by reasoning as we have done in regard to the analogous formula of the preceding problem, that that which we have just found also holds whatever be the value of u; one will be able therefore, finally to know immediately the simplest case, the sole one of which we have need, suppose that $u = -k$, this which will give

$$\mathrm{B}^{(m+2p)} + \frac{2-k}{1}\mathrm{B}^{(m+2p-2)} + \frac{4-k}{1}\cdot\frac{3-k}{2}\mathrm{B}^{(m+2p-4)} + \frac{6-k}{1}\cdot\frac{5-k}{2}\cdot\frac{4-k}{3}\mathrm{B}^{(m+2p-6)} +$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\frac{2p-2r-k}{1}\cdot\frac{2p-2r-k-1}{2}\cdot\frac{2p-2r-k-2}{3}\cdots\frac{p-r-k+1}{p-r}\mathrm{B}^{(m+2r)} + \text{etc.}$$

$$= \frac{m+2p-k}{1}\cdot\frac{m+2p-k-1}{2}\cdot\frac{m+2p-k-2}{3}\cdots\frac{m+p-k+1}{p}-$$

$$\mathrm{C}^{(2p-n)} - \frac{2}{1}\mathrm{C}^{(2p-n-2)} - \frac{4}{1}\cdot\frac{3}{2}\mathrm{C}^{(2p-n-4)} - \frac{6}{1}\cdot\frac{5}{2}\cdot\frac{4}{3}\mathrm{C}^{(2p-n-6)} -$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$-\frac{2(p-n-s)}{1}\cdot\frac{2(p-n-s)-1}{2}\cdot\frac{2(p-n-s)-2}{3}\cdots\frac{p-n-s+1}{p-n-s}\mathrm{C}^{(n+2s)} - \text{etc.}$$

$$= \frac{2p-n}{1}\cdot\frac{2p-n-1}{2}\cdot\frac{2p-n-2}{3}\cdots\frac{p-n+1}{p}-$$

$$\mathrm{C}^{(2p-n)} - 2\mathrm{C}^{(2p-n-2)} - 2\frac{3}{1}\mathrm{C}^{(2p-n-4)} - 2\frac{5}{1}\cdot\frac{4}{2}\mathrm{C}^{(2p-n-6)} -$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$-2\frac{2(p-n-s)-1}{1}\cdot\frac{2(p-n-s)-2}{2}\cdot\frac{2(p-n-s)-3}{3}\cdots\frac{p-n-s+1}{p-n-s-1}\mathrm{C}^{(n+2s)} - \text{etc.} \qquad [19]$$

by setting in the place of k its value $m+n$.

57. The form of the coefficients of the first member of this equation, shows that there exists a gap from the term for which $2p-2r-k=0$ of which the index $m+2r = m+2p-k = 2p-n$, to the one for which $p-r-k+1=0$, of which the index $m+2r = m+2(p-k+1) = 2p-m-2n+2 = 2p-n-k+2$, these terms, and all the intermediate terms are reduced to zero, because one of the factors of their coefficients vanishes, the first member is found thus divided into two parts, of which the first is able to be written thus:

$$\mathrm{B}^{(m+2p)} - \frac{k-2}{1}\mathrm{B}^{(m+2p-2)} + \frac{k-4}{1}\cdot\frac{k-3}{2}\mathrm{B}^{(m+2p-4)} - \frac{k-6}{1}\cdot\frac{k-5}{2}\cdot\frac{k-4}{3}\mathrm{B}^{(m+2p-6)} +$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\pm\frac{k-2(p-r)}{1}\cdot\frac{k-2(p-r)+1}{2}\cdot\frac{k-2(p-r)+2}{3}\cdots\frac{k-p+r-1}{p-r}\mathrm{B}^{(m+2r)} \mp \text{etc.}$$

until one arrives to a term of which the coefficient vanishes; the second part of the first member must commence at the term for which $p-r=k$, and $2(p-r)-k=k$, this term is

$$\frac{k}{1}\cdot\frac{k-1}{2}\cdot\frac{k-2}{3}\cdots\frac{1}{k}\mathrm{B}^{(2p-n-k)},$$

it will be consequently represented by the series

$$\frac{k}{1}\cdot\frac{k-1}{2}\cdot\frac{k-2}{3}\cdots\frac{1}{k}\mathrm{B}^{(2p-n-k)}+\frac{k+2}{1}\cdot\frac{k+1}{2}\cdot\frac{k}{3}\cdots\frac{2}{k+1}\mathrm{B}^{(2p-n-k-2)}+$$
$$\frac{k+4}{1}\cdot\frac{k+3}{2}\cdot\frac{k+2}{3}\cdots\frac{3}{k+2}\mathrm{B}^{(2p-n-k-4)}+\frac{k+6}{1}\cdot\frac{k+5}{2}\cdot\frac{k+4}{3}\cdots\frac{4}{k+3}\mathrm{B}^{(2p-n-k-6)}+$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\frac{2p-2r-k}{1}\cdot\frac{2p-2r-k-1}{2}\cdot\frac{2p-2r-k-2}{3}\cdots\frac{p-r-k+1}{p-r}\mathrm{B}^{(m+2r)}+\text{ etc.}$$

of which the first terms have been formed from the general term

$$\frac{2p-2r-k}{1}\cdot\frac{2p-2r-k-1}{2}\cdot\frac{2p-2r-k-2}{3}\cdots\frac{p-r-k+1}{p-r}\mathrm{B}^{(m+2r)}$$

by making successively $r=p-k$, $r=p-k-1$, $r=p-k-2$, etc.

58. It is easy to see that there is in all the terms of this series, k factors which are found at the same time in the numerator and in the denominator, and which are in the general term

$$p-r-k+1,\quad p-r-k+2,\quad p-r-k+3\ldots p-r,$$

this is why it is reduced to this simpler form

$$\mathrm{B}^{(2p-n-k)}+\frac{k+2}{1}\mathrm{B}^{(2p-n-k-2)}+\frac{k+4}{1}\cdot\frac{k+3}{2}\mathrm{B}^{(2p-n-k-4)}+$$
$$\frac{k+6}{1}\cdot\frac{k+5}{2}\cdot\frac{k+4}{3}\mathrm{B}^{(2p-n-k-6)}+\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+\frac{2p-2r-k}{1}\cdot\frac{2p-2r-k-1}{2}\cdot\frac{2p-2r-k-2}{3}\cdots\frac{p-r+1}{p-r-k}\mathrm{B}^{(m+2r)}+\text{ etc.}$$

and equation [19] becomes

$$\mathrm{B}^{(m+2p)}-\frac{k-2}{1}\mathrm{B}^{(m+2p-2)}+\frac{k-4}{1}\cdot\frac{k-3}{2}\mathrm{B}^{(m+2p-4)}-\frac{k-6}{1}\cdot\frac{k-5}{2}\cdot\frac{k-4}{3}\mathrm{B}^{(m+2p-6)}+$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\pm\frac{k-2(p-r)}{1}\cdot\frac{k-2(p-r)+1}{2}\cdot\frac{k-2(p-r)+2}{3}\cdots\frac{k-p+r-1}{p-r}\mathrm{B}^{(m+2r)}\mp\text{ etc.}$$
$$+\mathrm{B}^{(2p-n-k)}+\frac{k+2}{1}\mathrm{B}^{(2p-n-k-2)}+\frac{k+4}{1}\cdot\frac{k+3}{2}\mathrm{B}^{(2p-n-k-4)}+\frac{k+6}{1}\cdot\frac{k+5}{2}\cdot\frac{k+4}{3}\mathrm{B}^{(2p-n-k-6)}+$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\frac{2p-2r'-k}{1}\cdot\frac{2p-2r'-k-1}{2}\cdot\frac{2p-2r'-k-2}{3}\cdots\frac{p-r'+1}{p-r'-k}\mathrm{B}^{(m+2r')}+\text{ etc.}$$
$$=\frac{2p-n}{1}\cdot\frac{2p-n-1}{2}\cdot\frac{2p-n-2}{3}\cdots\frac{p-n+1}{p}-$$
$$\mathrm{C}^{(2p-n)}-2\mathrm{C}^{(2p-n-2)}-2\frac{3}{1}\mathrm{C}^{(2p-n-4)}-2\frac{5}{1}\cdot\frac{4}{2}\mathrm{C}^{(2p-n-6)}-$$
$$\cdots\cdots$$
$$-2\frac{2(p-n-s)-1}{1}\cdot\frac{2(p-n-s)-2}{2}\cdot\frac{2(p-n-s)-3}{3}\cdots\frac{p-n-s+1}{p-n-s-1}\mathrm{C}^{(n+2s)}-\text{ etc.,} \qquad [20]$$

where I have designated by r' the value of r smaller than $p-k$, while r continues to represent that which is greater than $\frac{2p-k}{2}$.

59. If one remembers now that $\mathrm{C}^{(2p-n)}$ is the number of arrangements of which $2p-n$ games are susceptible, under the assumption that the last achieves to ruin player C, without that neither he nor player B has been ruined in any of the preceding games; one will see that one is able to make in regard to $\mathrm{C}^{(2p-n)}$ that which we have done (47 and following) in regard to $\mathrm{B}^{(m+2p)}$. For that one will observe that the arrangements of which the number is represented by $\mathrm{C}^{(2p-n)}$ must be composed each of $p-n$ games won by player C, and of p games lost by the same player, since it is only under this last hypothesis that there remains a loss of them out of the $2p-n$ games, some n games which remove from him all his fortune. But one knows that $2p-n$ games are able to be partitioned in

$$\frac{2p-n}{1}\cdot\frac{2p-n-1}{2}\cdot\frac{2p-n-2}{3}\cdots\frac{p-n+1}{p}$$ [10]

different ways, into two groups, the one of p, and the other of $p-n$ games, the concern is no longer but to subtract from the number expressed by this formula, 1° the number of those of these arrangements which would suppose player C ruined at one of the preceding games, and that one finds by representing always the index of this game by $n+2s$, and by observing that player C has been able to be ruined only by some arrangements of s games won, and of $n+s$ games lost, of which the number is designated by $\mathrm{C}^{(n+2s)}$, and to which it is necessary to join $p-n-s$ games won, and as many games lost, this which is able to be executed from

$$\frac{2(p-n-s)}{1}\cdot\frac{2(p-n-s)-1}{2}\cdot\frac{2(p-n-s)-2}{3}\cdots\frac{p-n-s+1}{p-n-s},$$

or that which reverts to the same

$$2\frac{2(p-n-s)-1}{1}\cdot\frac{2(p-n-s)-2}{2}\cdot\frac{2(p-n-s)-3}{3}\cdots\frac{p-n-s+1}{p-n-s-1}$$

[10] One could take the equivalent and simpler expression

$$\frac{2p-n}{1}\cdot\frac{2p-n-1}{2}\cdot\frac{2p-n-2}{3}\cdots\frac{p+1}{p-n}$$

but it would lead less directly to the result that I myself propose to obtain.

different ways: one will have thus the formula

$$2\frac{2(p-n-s)-1}{1}\cdot\frac{2(p-n-s)-2}{2}\cdot\frac{2(p-n-s)-3}{3}\cdots\frac{p-n-s+1}{p-n-s-1}\text{C}^{(n+2s)},$$

in which it will be necessary to give successively to s all the values possible, in whole numbers, from $s=0$ to $s=p-n-1$. One has under this last supposition $n+2s=2p-n-2$, and it is evident that one would be able to assign to s a value no greater without rendering negative or null the number $p-n-s$ of games won and of games lost, between the game of which the rank is $n+2s$, and that of which the rank is $2p-n$.

60. We commence through the last of these substitutions, and we reunite all the results that they give successively, we will find for the pre-⟨vious...⟩[11]

$$2\text{C}^{(2p-n-2)}+2\frac{3}{1}\text{C}^{(2p-n-4)}+2\frac{5}{1}\cdot\frac{4}{2}\text{C}^{(2p-n-6)}+$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+2\frac{2(p-n-s)-1}{1}\cdot\frac{2(p-n-s)-2}{2}\cdot\frac{2(p-n-s)-3}{3}\cdots\frac{p-n-s+1}{p-n-s-1}\text{C}^{(n+2s)}+\text{ etc.}$$

2° the number of the arrangements which would have brought forth the ruin of player B before the game of which the rank is designated by $2p-n$. Those here are composed of a number $m+r'$ of games won by player C, and a number r' of games lost by the same player, who has ruined his adversary at the game of which the rank is $m+2r'$, and to which it is necessary to join $p-n-m-r'=p-k-r'$ games won by player C, and $p-r'$ games lost by the same player, in order to have the arrangements of $p-n$ games won, and of p lost, one will have therefore the formula

$$\frac{2p-2r'-k}{1}\cdot\frac{2p-2r'-k-1}{2}\cdot\frac{2p-2r'-k-2}{3}\cdots\frac{p-r'+1}{p-r'-k}\text{B}^{(m+2r')}$$

and by making successively

$$\begin{aligned}
r'&=p-k, &&\text{and}\quad m+2r'=m+2p-2k=2p-n-k,\\
r'&=p-k-1, &&\text{and}\quad m+2r'=2p-n-k-2,\\
r'&=p-k-2, &&\text{and}\quad m+2r'=2p-n-k-4,\ \text{etc.}
\end{aligned}$$

one will find that the second series to subtract is

$$\text{B}^{(2p-n-k)}+\frac{k+2}{1}\text{B}^{(2p-n-k-2)}+\frac{k+4}{1}\cdot\frac{k+3}{2}\text{B}^{(2p-n-k-4)}+$$
$$\frac{k+6}{1}\cdot\frac{k+5}{2}\cdot\frac{k+4}{3}\text{B}^{(2p-n-k-6)}+\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+\frac{2p-2r'-k}{1}\cdot\frac{2p-2r'-k-1}{2}\cdot\frac{2p-2r'-k-2}{3}\cdots\frac{p-r'+1}{p-r'-k}\text{B}^{(m+2r')}+\text{ etc.}$$

[11] A line of the text is apparently omitted here. The following expression makes use of equation [20]. RP

therefore

$$\mathrm{C}^{(2p-n)} = \frac{2p-n}{1}\cdot\frac{2p-n-1}{2}\cdot\frac{2p-n-2}{3}\cdots\frac{p-n+1}{p} -$$

$$2\mathrm{C}^{(2p-n-2)} - 2\frac{3}{1}\mathrm{C}^{(2p-n-4)} - 2\frac{5}{1}\cdot\frac{4}{2}\mathrm{C}^{(2p-n-6)} - \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$-2\frac{2(p-n-s)-1}{1}\cdot\frac{2(p-n-s)-2}{2}\cdot\frac{2(p-n-s)-3}{3}\cdots\frac{p-n-s+1}{p-n-s-1}\mathrm{C}^{(n+2s)} - \text{ etc.}$$

$$-\mathrm{B}^{(2p-n-k)} - \frac{k+2}{1}\mathrm{B}^{(2p-n-k-2)} - \frac{k+4}{1}\cdot\frac{k+3}{2}\mathrm{B}^{(2p-n-k-4)} - \frac{k+6}{1}\cdot\frac{k+5}{2}\cdot\frac{k+4}{3}\mathrm{B}^{(2p-n-k-6)} -$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$-\frac{2p-2r'-k}{1}\cdot\frac{2p-2r'-k-1}{2}\cdot\frac{2p-2r'-k-2}{3}\cdots\frac{p-r'+1}{p-r'-k}\mathrm{B}^{(m+2r')} - \text{ etc.} \qquad [21]$$

The equation that we just found is changed by transposition into

$$\mathrm{B}^{(2p-n-k)} + \frac{k+2}{1}\mathrm{B}^{(2p-n-k-2)} + \frac{k+4}{1}\cdot\frac{k+3}{2}\mathrm{B}^{(2p-n-k-4)} + \frac{k+6}{1}\cdot\frac{k+5}{2}\cdot\frac{k+4}{3}\mathrm{B}^{(2p-n-k-6)} +$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\frac{2p-2r'-k}{1}\cdot\frac{2p-2r'-k-1}{2}\cdot\frac{2p-2r'-k-2}{3}\cdots\frac{p-r'+1}{p-r'-k}\mathrm{B}^{(m+2r')} + \text{ etc.} =$$

$$\frac{2p-n}{1}\cdot\frac{2p-n-1}{2}\cdot\frac{2p-n-2}{3}\cdots\frac{p-n+1}{p} -$$

$$\mathrm{C}^{(2p-n)} - 2\mathrm{C}^{(2p-n-2)} - 2\frac{3}{1}\mathrm{C}^{(2p-n-4)} - 2\frac{5}{1}\cdot\frac{4}{2}\mathrm{C}^{(2p-n-6)} -$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$-2\frac{2(p-n-s)-1}{1}\cdot\frac{2(p-n-s)-2}{2}\cdot\frac{2(p-n-s)-3}{3}\cdots\frac{p-n-s+1}{p-n-s-1}\mathrm{C}^{(n+2s)} - \text{ etc.}$$

of which all the terms make part of equation [20]; it suffices therefore to subtract from this equation, by removing all these terms, this which gives

$$\mathrm{B}^{(m+2p)} - \frac{k-2}{1}\mathrm{B}^{(m+2p-2)} + \frac{k-4}{1}\cdot\frac{k-3}{2}\mathrm{B}^{(m+2p-4)} - \frac{k-6}{1}\cdot\frac{k-5}{2}\cdot\frac{k-4}{3}\mathrm{B}^{(m+2p-6)} +$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\pm\frac{k-2(p-r)}{1}\cdot\frac{k-2(p-r)+1}{2}\cdot\frac{k-2(p-r)+2}{3}\cdots\frac{k-p+r-1}{p-r}\mathrm{B}^{(m+2r)} \mp \text{ etc.}$$

$$= 0. \qquad [22]$$

61. One sees by the process that has led us to this equation, that one must prolong the first member in it only until one arrives to a term which vanishes by itself, this

which arrives as soon as r is smaller than $\frac{2p-k}{2}$, or when it is equal to it, whence it follows that when k is even, the last term is the one for which $r = p - \frac{k}{2} + 1$, this term is

$$\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{\frac{k}{2}}{\frac{k}{2} - 1} \mathrm{B}^{(m+2p-k+2)} = \frac{k}{2} \mathrm{B}^{(2p-n+2)}$$

equation [22] is composed, in this case, of $\frac{k}{2}$ terms, since r is susceptible of $\frac{k}{2}$ different values from $r = p - \frac{k}{2} + 1$, to $r = p$, but if k were odd, the last value of r would be $p - \frac{k-1}{2}$, and the corresponding term would be worth

$$\frac{1}{1} \cdot \frac{2}{2} \cdot \frac{3}{3} \cdots \frac{k - 1 - \frac{k-1}{2}}{\frac{k-1}{2}} \mathrm{B}^{(m+2p-k+2)} = \mathrm{B}^{(2p-n+1)},$$

in this case equation [22] would have $\frac{k-1}{2} + 1 = \frac{k+1}{2}$ terms, because it is there the number of values that one is able to give to r from $r = p - \frac{k-1}{2}$ to $r = p$, inclusively.

62. In both cases, the number of terms of the series of B which enter into equation [22], being constant, each of them is formed from the preceding, by virtue of an equation of the first degree of a determined number of terms, and the series of probabilities of player B

$$\mathrm{B}^{(m)} \frac{1}{(1+q)^m} + \mathrm{B}^{(m+2)} \frac{q}{(1+q)^{m+2}} + \mathrm{B}^{(m+4)} \frac{q^2}{(1+q)^{m+4}} + \cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+\mathrm{B}^{(m+2p)} \frac{q^p}{(1+q)^{m+2p}} + \text{ etc.}$$

is of the number of those that one calls recurrent. Every series of this kind being the development of a rational fraction, it suffices to determine the value of the fraction which corresponds to the series that we just found, in order to have the limit of the probabilities that the player B will finish by being ruined if he continues indefinitely to play.

63. The series being set under the form

$$\frac{1}{(1+q)^m} \left(\mathrm{B}^{(m)} + \mathrm{B}^{(m+2)} \frac{q}{(1+q)^2} + \mathrm{B}^{(m+4)} \frac{q^2}{(1+q)^4} + \mathrm{B}^{(m+6)} \frac{q^3}{(1+q)^6} + \right.$$
$$\left. \cdots\cdots\cdots\cdots \quad + \mathrm{B}^{(m+2p)} \frac{q^p}{(1+q)^{2p}} + \text{ etc.} \right)$$

it will be found ordered according to the successive powers of the quantity $\frac{q}{(1+q)^2}$, and according to the known theory of recurrent series, the equation

$$\mathrm{B}^{(m+2p)} - \frac{k-2}{1} \mathrm{B}^{(m+2p-2)} + \frac{k-4}{1} \cdot \frac{k-3}{2} \mathrm{B}^{(m+2p-4)} - \frac{k-6}{1} \cdot \frac{k-5}{2} \cdot \frac{k-4}{3} \mathrm{B}^{(m+2p-6)} +$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\pm \frac{k-2(p-r)}{1} \cdot \frac{k-2(p-r)+1}{2} \cdot \frac{k-2(p-r)+2}{3} \cdots \frac{k-p+r-1}{p-r} \mathrm{B}^{(m+2r)} \mp \text{ etc.} = 0.$$

of which the second member is able to be regarded as having been reduced to zero by transposition, will have for first member the denominator of the generating fraction of the series

$$\mathrm{B}^{(m)} + \mathrm{B}^{(m+2)} \frac{q}{(1+q)^2} + \mathrm{B}^{(m+4)} \frac{q^2}{(1+q)^4} + \mathrm{B}^{(m+6)} \frac{q^3}{(1+q)^6} + \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \mathrm{B}^{(m+2p)} \frac{q^p}{(1+q)^{2p}} + \text{ etc.}$$

in which one would have substituted the terms

$$\mathrm{B}^{(m+2p)}, \quad \mathrm{B}^{(m+2p-2)}, \quad \mathrm{B}^{(m+2p-4)}, \quad \mathrm{B}^{(m+2p-6)}, \ldots \mathrm{B}^{(m+2r)}, \text{etc.}$$

in the place of the successive powers

$$\frac{q^0}{(1+q)^0} = 1, \quad \frac{q^1}{(1+q)^2}, \quad \frac{q^2}{(1+q)^4}, \quad \frac{q^3}{(1+q)^6}, \cdots \frac{q^{p-r}}{(1+q)^{2p-2r}}, \text{etc.}$$

of the quantity $\frac{q}{(1+q)^2}$. One will obtain therefore the denominator of this fraction by substituting on the contrary

$$1, \quad \frac{q^1}{(1+q)^2}, \quad \frac{q^2}{(1+q)^4}, \quad \frac{q^3}{(1+q)^6}, \cdots \frac{q^{p-r}}{(1+q)^{2p-2r}}, \text{etc.}$$

in the place of

$$\mathrm{B}^{(m+2p)}, \quad \mathrm{B}^{(m+2p-2)}, \quad \mathrm{B}^{(m+2p-4)}, \quad \mathrm{B}^{(m+2p-6)}, \ldots \mathrm{B}^{(m+2r)}, \text{etc.}$$

in the first member of equation [22], this which will give

$$1 - \frac{k-2}{1} \cdot \frac{q}{(1+q)^2} + \frac{k-4}{1} \cdot \frac{k-3}{2} \cdot \frac{q^2}{(1+q)^4} - \frac{k-6}{1} \cdot \frac{k-5}{2} \cdot \frac{k-4}{3} \cdot \frac{q^3}{(1+q)^6} + \cdots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \pm \frac{k-2(p-r)}{1} \cdot \frac{k-2(p-r)+1}{2} \cdot \frac{k-2(p-r)+2}{3} \cdots \frac{k-p+r-1}{p-r} \cdot \frac{q^{p-r}}{(1+q)^{2p-2r}} \mp \text{ etc.}$$

in order to find the numerator of the same fraction, one will consider the series as the quotient of this numerator divided by the denominator that we just determined, whence one will conclude that it suffices in order to have the numerator to multiply the series by this denominator. One will execute therefore the multiplication thus as it follows:

$$\left\{\mathrm{B}^{(m)} + \mathrm{B}^{(m+2)}\frac{q}{(1+q)^2} + \mathrm{B}^{(m+4)}\frac{q^2}{(1+q)^4} + \mathrm{B}^{(m+6)}\frac{q^3}{(1+q)^6} + \cdots + \mathrm{B}^{(m+2r)}\frac{q^r}{(1+q)^{2r}} + \cdots \right.$$

$$\left. + \mathrm{B}^{(m+2p-6)}\frac{q^{p-3}}{(1+q)^{2p-6}} + \mathrm{B}^{(m+2p-4)}\frac{q^{p-2}}{(1+q)^{2p-4}} + \mathrm{B}^{(m+2p-2)}\frac{q^{p-1}}{(1+q)^{2p-2}} + \mathrm{B}^{(m+2p)}\frac{q^{p}}{(1+q)^{2p}} + \text{ etc.}\right\}$$

$$\times\left\{1 - \frac{k-2}{1}\cdot\frac{q}{(1+q)^2} + \frac{k-4}{1}\cdot\frac{k-3}{2}\cdot\frac{q^2}{(1+q)^4} - \frac{k-6}{1}\cdot\frac{k-5}{2}\cdot\frac{k-4}{3}\cdot\frac{q^3}{(1+q)^6} + \cdots \right.$$

$$\left. \pm\frac{k-2(p-r)}{1}\cdot\frac{k-2(p-r)+1}{2}\cdot\frac{k-2(p-r)+2}{3}\cdots\frac{k-p+r-1}{p-r}\cdot\frac{q^{p-r}}{(1+q)^{2p-2r}} \mp \text{ etc.}\right\}$$

$$\begin{array}{llllll}
\mathrm{B}^{(m)} & +\mathrm{B}^{(m+2)}\frac{q}{(1+q)^2} & +\mathrm{B}^{(m+4)}\frac{q^2}{(1+q)^4} & +\mathrm{B}^{(m+6)}\frac{q^3}{(1+q)^6} & +\cdots+\mathrm{B}^{(m+2p)}\frac{q^p}{(1+q)^{2p}} & + \text{ etc.}\\
 & -\frac{k-2}{1}\mathrm{B}^{(m)}\frac{q}{(1+q)^2} & -\frac{k-2}{1}\mathrm{B}^{(m+2)}\frac{q^2}{(1+q)^4} & -\frac{k-2}{1}\mathrm{B}^{(m+4)}\frac{q^3}{(1+q)^6} & -\cdots-\frac{k-2}{1}\mathrm{B}^{(m+2p-2)}\frac{q^p}{(1+q)^{2p}} & - \text{ etc.}\\
 & & +\frac{k-4}{1}\cdot\frac{k-3}{2}\mathrm{B}^{(m)}\frac{q^2}{(1+q)^4} & +\frac{k-4}{1}\cdot\frac{k-3}{2}\mathrm{B}^{(m+2)}\frac{q^3}{(1+q)^6} & +\cdots+\frac{k-4}{1}\cdot\frac{k-3}{2}\mathrm{B}^{(m+2p-4)}\frac{q^p}{(1+q)^{2p}} & + \text{ etc.}\\
 & & & -\frac{k-6}{1}\cdot\frac{k-5}{2}\cdot\frac{k-4}{3}\mathrm{B}^{(m)}\frac{q^3}{(1+q)^6} & -\cdots-\frac{k-6}{1}\cdot\frac{k-5}{2}\cdot\frac{k-4}{3}\mathrm{B}^{(m+2p-6)}\frac{q^p}{(1+q)^{2p}} & - \text{ etc.}\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
 & & & \multicolumn{2}{r}{\pm\frac{k-2(p-r)}{1}\cdot\frac{k-2(p-r)+1}{2}\cdot\frac{k-2(p-r)+2}{3}\cdots\frac{k-p+r-1}{p-r}\mathrm{B}^{(m+2r)}\frac{q^p}{(1+q)^{2p}}} & \mp \text{ etc.}\\
 & & \mp \text{ etc.} & & & \mp \text{ etc.}
\end{array}$$

64. The last of the columns that we have written in the product represent them all, this is why we will have been able ourselves to dispense with writing even the first columns of this product that it will have given us, when we would have need of it, by making successively $p=0$, $p=1$, $p=2$, $p=3$, etc. Now the coefficient of $\frac{q^p}{(1+q)^{2p}}$ in this column is precisely the same thing as the part of the first member of equation [20] which precedes the gap, by transposing the rest of this member, one finds that this coefficient is equal to

$$\frac{2p-n}{1}\cdot\frac{2p-n-1}{2}\cdot\frac{2p-n-2}{3}\cdots\frac{p-n+1}{p}-$$

$$\mathrm{C}^{(2p-n)}-2\mathrm{C}^{(2p-n-2)}-2\frac{3}{1}\mathrm{C}^{(2p-n-4)}-2\frac{3}{1}\cdot\frac{4}{2}\mathrm{C}^{(2p-n-6)}-$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$-2\frac{2(p-n-s)-1}{1}\cdot\frac{2(p-n-s)-2}{2}\cdot\frac{2(p-n-s)-3}{3}\cdots\frac{p-n-s+1}{p-n-s-1}\mathrm{C}^{(n+2s)}-\text{ etc.}$$

$$-\mathrm{B}^{(2p-n-k)}-\frac{k+2}{1}\mathrm{B}^{(2p-n-k-2)}-\frac{k+4}{1}\cdot\frac{k+3}{2}\mathrm{B}^{(2p-n-k-4)}-\frac{k+6}{1}\cdot\frac{k+5}{2}\cdot\frac{k+4}{3}\mathrm{B}^{(2p-n-k-6)}-$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$-\frac{2p-2r'-k}{1}\cdot\frac{2p-2r'-k-1}{2}\cdot\frac{2p-2r'-k-2}{3}\cdots\frac{p-r'+1}{p-r'-k}\mathrm{B}^{(m+2r')}-\text{ etc.}$$

a value which is reduced to zero, according to that which one has seen (60) by virtue of equation [21], as soon as this last commences to take place, that is, as soon as $\mathrm{C}^{(2p-n)}$, and the other terms of like nature are not null; all the columns of the preceding product vanish therefore by themselves, immediately as one is arrived to some terms for which $\mathrm{C}^{(2p-n)}$, $\mathrm{C}^{(2p-n-2)}$, $\mathrm{C}^{(2p-n-4)}$, etc., and $\mathrm{B}^{(2p-n-k)}$, $\mathrm{B}^{(2p-n-k-2)}$, $\mathrm{B}^{(2p-n-k-4)}$, etc. ceasing to be reduced to zero.

65. $\mathrm{C}^{(2p-n)}$ is the first of these quantities which satisfies this condition, that arrives when $p=n$, since one has then $\mathrm{C}^{(2p-n)}=\mathrm{C}^{(n)}=1$, it is necessary to take account only of the columns for which p is smaller than n, erasing in the general value of the coefficient $\frac{q^p}{(1+q)^{2p}}$, the terms that this assumption makes vanish, it is reduced to

$$\mathrm{B}^{(m+2p)}-\frac{k-2}{1}\mathrm{B}^{(m+2p-2)}-\frac{k-4}{1}\cdot\frac{k-3}{2}\mathrm{B}^{(m+2p-4)}-\frac{k-6}{1}\cdot\frac{k-5}{2}\cdot\frac{k-4}{3}\mathrm{B}^{(m+2p-6)}+$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\pm\frac{k-2(p-r)}{1}\cdot\frac{k-2(p-r)+1}{2}\cdot\frac{k-2(p-r)+2}{3}\cdots\frac{k-p+r-1}{p-r}\mathrm{B}^{(m+2r)}\mp\text{etc.}=$$
$$\frac{2p-n}{1}\cdot\frac{2p-n-1}{2}\cdot\frac{2p-n-2}{3}\cdots\frac{p-n+1}{p}.\quad[23]$$

66. This new value must be yet null, by the vanishing of its factors from $p=n-1$ to $\frac{n}{2}$, or to $p=\frac{n+1}{2}$, inclusively, according as n is even or odd; there will remain therefore in the product that we just found only the columns for which p has a value

smaller than $\frac{n}{2}$ and these columns will be reduced each to a single term, by means of equation [23], which gives it by writing successively 0, 1, 2, 3, etc., in the place of p,

$$\mathrm{B}^{(m)} = 1,$$
$$\mathrm{B}^{(m+2)} - \frac{k-2}{1}\mathrm{B}^{(m)} = \frac{2-n}{1} = -\frac{n-2}{1},$$
$$\mathrm{B}^{(m+4)} - \frac{k-2}{1}\mathrm{B}^{(m+2)} + \frac{k-4}{1}\cdot\frac{k-3}{2}\mathrm{B}^{(m)} = \frac{4-n}{1}\cdot\frac{3-n}{2} = \frac{n-4}{1}\cdot\frac{n-3}{2},$$
$$\mathrm{B}^{(m+6)} - \frac{k-2}{1}\mathrm{B}^{(m+4)} + \frac{k-4}{1}\cdot\frac{k-3}{2}\mathrm{B}^{(m+2)} - \frac{k-6}{1}\cdot\frac{k-5}{2}\cdot\frac{k-4}{3}\mathrm{B}^{(m)} = \frac{6-n}{1}\cdot\frac{5-n}{2}\cdot\frac{4-n}{3} = -\frac{n-6}{1}\cdot\frac{n-5}{2}\cdot\frac{n-4}{3}$$

and in general

$$\mathrm{B}^{(m+2p)} - \frac{k-2}{1}\mathrm{B}^{(m+2p-1)} - \frac{k-4}{1}\cdot\frac{k-3}{2}\mathrm{B}^{(m+2p-4)} - \frac{k-6}{1}\cdot\frac{k-5}{2}\cdot\frac{k-4}{3}\mathrm{B}^{(m+2p-4)} + \ldots\ldots\ldots$$
$$\pm\frac{k-2(p-r)}{1}\cdot\frac{k-2(p-r)+1}{2}\cdot\frac{k-2(p-r)+2}{3}\cdots\frac{k-p+r-1}{p-r}\mathrm{B}^{(m+2r)} \mp \text{ etc.} = \pm\frac{n-2p}{1}\cdot\frac{n-2p+1}{2}\cdot\frac{n-2p+2}{3}\cdots\frac{n-p-1}{p},$$

the numerator of the generating fraction of the series

$$\mathrm{B}^{(m)} + \mathrm{B}^{(m+2)}\frac{q}{(1+q)^{(m+2)}} + \mathrm{B}^{(m+4)}\frac{q^2}{(1+q)^{(m+4)}} + \mathrm{B}^{(m+6)}\frac{q^3}{(1+q)^{(m+6)}} + \ldots\ldots\ldots + \mathrm{B}^{(m+2p)}\frac{q^p}{(1+q)^{(m+2p)}} + \text{ etc.}$$

is therefore equal to

$$1 - \frac{n-2}{1}\frac{q}{(1+q)^2} + \frac{n-4}{1}\cdot\frac{n-3}{2}\frac{q^2}{(1+q)^4} - \frac{n-6}{1}\cdot\frac{n-5}{2}\cdot\frac{n-4}{3}\frac{q^3}{(1+q)^6} + \pm\frac{n-2p}{1}\cdot\frac{n-2p+1}{2}\cdot\frac{n-2p+2}{3}\cdots\frac{n-p-1}{p}\cdot\frac{q^p}{(1+q)^{(2p)}} \mp \text{ etc.}$$

and as the denominator, of which we have just found the value, is able, because p, r, and consequently $p-r$, are absolutely indeterminate, to be written thus

$$1 - \frac{k-2}{1}\frac{q}{(1+q)^2} + \frac{k-4}{1}\cdot\frac{k-3}{2}\frac{q^2}{(1+q)^4} - \frac{k-6}{1}\cdot\frac{k-5}{2}\cdot\frac{k-4}{3}\frac{q^3}{(1+q)^6} + \ldots\ldots\ldots \pm\frac{k-2p}{1}\cdot\frac{k-2p+1}{2}\cdot\frac{k-2p+2}{3}\cdots\frac{k-p-1}{p}\cdot\frac{q^p}{(1+q)^{(2p)}} \mp \text{ etc.}$$

one will have the sum of the probabilities that player B will be ruined $\frac{1}{(1+q)^m}\times$

$$\frac{1-\frac{n-2}{1}\frac{q}{(1+q)^2}+\frac{n-4}{1}\cdot\frac{n-3}{2}\frac{q^2}{(1+q)^4}-\cdots\pm\frac{n-2p}{1}\cdot\frac{n-2p+1}{2}\cdot\frac{n-2p+2}{3}\cdots\frac{n-p-1}{p}\cdot\frac{q^p}{(1+q)^{2p}}\mp\text{etc.}}{1-\frac{k-2}{1}\frac{q}{(1+q)^2}+\frac{k-4}{1}\cdot\frac{k-3}{2}\frac{q^2}{(1+q)^4}-\cdots\pm\frac{k-2p}{1}\cdot\frac{k-2p+1}{2}\cdot\frac{k-2p+2}{3}\cdots\frac{k-p-1}{p}\cdot\frac{q^p}{(1+q)^{(2p)}}\mp\text{ etc.}}$$

67. By reasoning as we just did for player B, in regard of player C, one will find that the sum of the probabilities that his last will be ruined, represented to the present by

$$\frac{q^n}{(1+q)^n}\Bigg(\mathrm{C}^{(n)}+\mathrm{C}^{(n+2)}\frac{q}{(1+q)^2}+\mathrm{C}^{(n+4)}\frac{q^2}{(1+q)^4}+\mathrm{C}^{(n+6)}\frac{q^3}{(1+q)^6}+\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\quad+\mathrm{C}^{(n+2p)}\frac{q^p}{(1+q)^{2p}}+\text{ etc.}\Bigg)$$

is equal to $\frac{q^n}{(1+q)^n}\times$

$$\frac{1-\frac{m-2}{1}\cdot\frac{q}{(1+q)^2}+\frac{m-4}{1}\cdot\frac{m-3}{2}\cdot\frac{q^2}{(1+q)^4}-\cdots\pm\frac{m-2p}{1}\cdot\frac{m-2p+1}{2}\cdot\frac{m-2p+2}{3}\cdots\frac{m-p-1}{p}\cdot\frac{q^p}{(1+q)^{2p}}\mp\text{etc.}}{1-\frac{k-2}{1}\cdot\frac{q}{(1+q)^2}+\frac{k-4}{1}\cdot\frac{k-3}{2}\cdot\frac{q^2}{(1+q)^4}-\cdots\pm\frac{k-2p}{1}\cdot\frac{k-2p+1}{2}\cdot\frac{k-2p+2}{3}\cdots\frac{k-p-1}{p}\cdot\frac{q^p}{(1+q)^{2p}}\mp\text{ etc.}}$$

68. We multiply now above and below by $(1+q)^{m+n-1}=(1+q)^{k-1}$,[12] the two values that we just found for these two sums of probabilities, the first will become

$$\frac{\begin{array}{l}(1+q)^{n-1}-\frac{n-2}{1}q(1+q)^{n-3}+\frac{n-4}{1}\cdot\frac{n-3}{2}q^2(1+q)^{n-5}-\cdots\\ \qquad\pm\frac{n-2p}{1}\cdot\frac{n-2p+1}{2}\cdot\frac{n-2p+2}{3}\cdots\frac{n-p-1}{p}q^p(1+q)^{n-2p-1}\mp\text{etc.}\end{array}}{\begin{array}{l}(1+q)^{k-1}-\frac{k-2}{1}q(1+q)^{k-3}+\frac{k-4}{1}\cdot\frac{k-3}{2}q^2(1+q)^{k-5}-\cdots\\ \qquad\pm\frac{k-2p}{1}\cdot\frac{k-2p+1}{2}\cdot\frac{k-2p+2}{3}\cdots\frac{k-p-1}{p}q^p(1+q)^{k-2p-1}\mp\text{ etc.}\end{array}}$$

[12]One will be assured that this multiplication suffices in order to make vanish the fractions contained in the numerators and in the common denominator of these two quantities, if one pays attention that $2p$ which represents the exponent of $1+q$, in their general terms

$$\frac{n-2p}{1}\cdot\frac{n-2p+1}{2}\cdot\frac{n-2p+2}{3}\cdots\frac{n-p-1}{p}\cdot\frac{q^p}{(1+q)^{2p}},$$
$$\frac{m-2p}{1}\cdot\frac{m-2p+1}{2}\cdot\frac{m-2p+2}{3}\cdots\frac{m-p-1}{p}\cdot\frac{q^p}{(1+q)^{2p}},\text{ and}$$
$$\frac{k-2p}{1}\cdot\frac{k-2p+1}{2}\cdot\frac{k-2p+2}{3}\cdots\frac{k-p-1}{p}\cdot\frac{q^p}{(1+q)^{2p}},$$

must be necessarily smaller than n in the first, than m in the second, and than k in the third, in order that the coefficients of these terms not vanish.

and the second

$$q^n \times \frac{\begin{array}{r}(1+q)^{m-1} - \dfrac{m-2}{1} q(1+q)^{m-3} + \dfrac{m-4}{1} \cdot \dfrac{m-3}{2} q^2 (1+q)^{m-5} - \cdots \\ \pm \dfrac{m-2p}{1} \cdot \dfrac{m-2p+1}{2} \cdot \dfrac{m-2p+2}{3} \cdots \dfrac{m-p-1}{p} q^p (1+q)^{m-2p-1} \mp \text{etc.}\end{array}}{\begin{array}{r}(1+q)^{k-1} - \dfrac{k-2}{1} q(1+q)^{k-3} + \dfrac{k-4}{1} \cdot \dfrac{k-3}{2} q^2 (1+q)^{k-5} - \cdots \\ \pm \dfrac{k-2p}{1} \cdot \dfrac{k-2p+1}{2} \cdot \dfrac{k-2p+2}{3} \cdots \dfrac{k-p-1}{p} q^p (1+q)^{k-2p-1} \mp \text{ etc.}\end{array}}$$

The numerators and the common denominator of these new values, being some particular cases of the formula

$$(1+q)^{x-1} - \frac{x-2}{1} q(1+q)^{x-3} + \frac{x-4}{1} \cdot \frac{x-3}{2} q^2 (1+q)^{x-5} - \frac{x-6}{1} \cdot \frac{x-5}{2} \cdot \frac{x-4}{3} q^3 (1+q)^{x-7} +$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots \pm \frac{x-2p}{1} \cdot \frac{x-2p+1}{2} \cdot \frac{x-2p+2}{3} \cdots \frac{x-p-1}{p} q^p (1+q)^{x-2p-1} \mp \text{ etc.}$$

we see first if this last would not be able to be reduced to a simpler form.

69. By reversing the order of the factors of which the numerators of the coefficients of its different terms are composed, and by developing the powers of $1+q$, one will set first this quantity under the following form:

$$\begin{array}{r}
1 + \frac{x-1}{1} q + \frac{x-1}{1} \cdot \frac{x-2}{2} q^2 + \frac{x-1}{1} \cdot \frac{x-2}{2} \cdot \frac{x-3}{3} q^3 + \cdots + \frac{x-1}{1} \cdot \frac{x-2}{2} \cdot \frac{x-3}{3} \cdot \frac{x-4}{4} \cdots \frac{x-p}{p} q^p + \text{ etc.} \\
-\frac{x-2}{1} q - \frac{x-2}{1} \cdot \frac{x-3}{1} q^2 - \frac{x-2}{1} \cdot \frac{x-3}{1} \cdot \frac{x-4}{2} q^3 - \cdots - \frac{x-2}{1} \cdot \frac{x-3}{1} \cdot \frac{x-4}{2} \cdot \frac{x-5}{3} \cdots \frac{x-p-1}{p-1} q^p + \text{ etc.} \\
+\frac{x-3}{1} \cdot \frac{x-4}{2} q^2 + \frac{x-3}{1} \cdot \frac{x-4}{2} \cdot \frac{x-5}{1} q^3 + \cdots + \frac{x-3}{1} \cdot \frac{x-4}{2} \cdot \frac{x-5}{1} \cdot \frac{x-6}{2} \cdots \frac{x-p-2}{p-2} q^p + \text{ etc.} \\
-\frac{x-4}{1} \cdot \frac{x-5}{2} \cdot \frac{x-6}{3} q^3 + \cdots + \frac{x-4}{1} \cdot \frac{x-5}{2} \cdot \frac{x-6}{3} \cdot \frac{x-7}{1} \cdots \frac{x-p-3}{p-3} q^p + \text{ etc.} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\pm \frac{x-p-1}{1} \cdot \frac{x-p-2}{2} \cdot \frac{x-p-3}{3} \cdots \frac{x-2p}{p} q^p \pm \text{ etc.} \\
\mp \text{etc.}
\end{array}$$

one will observe next that a and t representing any two numbers, one has

$$(1-a)^{-t-1} = 1 + \frac{t+1}{1} a + \frac{t+2}{1} \cdot \frac{t+1}{2} a^2 + \frac{t+3}{1} \cdot \frac{t+2}{2} \cdot \frac{t+1}{3} a^3 + \cdots + \frac{t+p}{1} \cdot \frac{t+p-1}{2} \cdot \frac{t+p-2}{3} \cdots \frac{t+1}{p} a^p + \text{ etc.}$$

and $(1-a)^t = 1 - \frac{t}{1} a + \frac{t}{1} \cdot \frac{t-1}{2} a^2 - \frac{t}{1} \cdot \frac{t-1}{2} \frac{t-2}{3} a^3 + \cdots \pm \frac{t}{1} \cdot \frac{t-1}{2} \cdot \frac{t-2}{3} \cdots \frac{t-p+1}{p} a^p \mp \text{ etc.}$

these two equations multiplied by one another give $(1-a)^{-1}$ or $\frac{1}{1-a}=$

$$\begin{aligned}
1+\tfrac{t+1}{1}a+\tfrac{t+2}{1}\cdot\tfrac{t+1}{2}a^2+\tfrac{t+3}{1}\cdot\tfrac{t+2}{2}\cdot\tfrac{t+1}{3}a^3+\cdots+\tfrac{t+p}{1}\cdot\tfrac{t+p-1}{2}\cdot\tfrac{t+p-2}{3}\tfrac{t+p-3}{4}\cdots\tfrac{t+1}{p}a^p+\text{ etc.}\\
-\tfrac{t}{1}a-\tfrac{t+1}{1}\cdot\tfrac{t}{1}a^2-\tfrac{t+2}{1}\cdot\tfrac{t+1}{1}\cdot\tfrac{t}{2}a^3-\cdots-\tfrac{t+p-1}{1}\cdot\tfrac{t+p-2}{1}\cdot\tfrac{t+p-3}{2}\cdots\tfrac{t+1}{p-1}\cdot\tfrac{t}{1}a^p-\text{ etc.}\\
+\tfrac{t}{1}\cdot\tfrac{t-1}{2}a^2+\tfrac{t+1}{1}\cdot\tfrac{t}{2}\cdot\tfrac{t-1}{2}a^3+\cdots+\tfrac{t+p-2}{1}\cdot\tfrac{t+p-3}{2}\cdots\tfrac{t+1}{p-2}\cdot\tfrac{t}{1}\cdot\tfrac{t-1}{2}a^p+\text{ etc.}\\
-\tfrac{t}{1}\cdot\tfrac{t-1}{2}\cdot\tfrac{t-2}{3}a^3-\cdots-\tfrac{t+p-3}{1}\cdots\tfrac{t+1}{p-3}\cdot\tfrac{t}{1}\cdot\tfrac{t-1}{2}\cdot\tfrac{t-2}{3}a^p-\text{ etc.}\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\pm\tfrac{t}{1}\cdot\tfrac{t-1}{2}\cdot\tfrac{t-2}{3}\cdots\tfrac{t-p+1}{p}a^p\pm\text{ etc.}\\
\mp\text{ etc.}
\end{aligned}$$

but one knows that

$$\frac{1}{1-a}=1+a+a^2+a^3+\cdots+a^p+\text{ etc.}$$

these two developments of one same quantity must be identical whatever be the value of a, one is able to deduce from it this sequence of equations

$$\begin{gathered}
\frac{t+1}{1}-\frac{t}{1}=1,\\
\frac{t+2}{1}\cdot\frac{t+1}{2}-\frac{t+1}{1}\cdot\frac{t}{1}+\frac{t}{1}\cdot\frac{t-1}{2}=1,\\
\frac{t+3}{1}\cdot\frac{t+2}{2}\cdot\frac{t+1}{3}-\frac{t+2}{1}\cdot\frac{t+1}{2}\cdot\frac{t}{1}+\frac{t+1}{1}\cdot\frac{t}{1}\cdot\frac{t-1}{2}-\frac{t}{1}\cdot\frac{t-1}{2}\cdot\frac{t-2}{3}=1,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\frac{t+p}{1}\cdot\frac{t+p-1}{2}\cdot\frac{t+p-2}{3}\cdot\frac{t+p-3}{4}\cdots\frac{t+1}{p}-\frac{t+p-1}{1}\cdot\frac{t+p-2}{2}\cdot\frac{t+p-3}{3}\cdots\frac{t+1}{p-1}\cdot\frac{t}{1}+\\
\frac{t+p-2}{1}\cdot\frac{t+p-3}{2}\cdots\frac{t+1}{p-2}\cdot\frac{t}{1}\cdot\frac{t-1}{2}-\frac{t+p-3}{1}\cdots\frac{t+1}{1}\cdot\frac{t}{1}\cdot\frac{t-1}{2}\cdot\frac{t-2}{3}+\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\cdots\cdots\cdots\pm\frac{t}{1}\cdot\frac{t-1}{2}\cdot\frac{t-2}{3}\cdots\frac{t-p+1}{p}=1
\end{gathered}$$

and so forth.

70. These equations holding independently from one another, and for each value of t, one is able to suppose
in the first $t=x-2$,
in the second $t=x-3$,
in the third $t=x-4$,
and in general in the last $t=x-p-1$, this which gives by substituting

$$\frac{x-1}{1}-\frac{x-2}{1}=1,$$

$$\frac{x-1}{1}\cdot\frac{x-2}{2}-\frac{x-2}{1}\cdot\frac{x-3}{1}+\frac{x-3}{1}\cdot\frac{x-4}{2}=1,$$

$$\frac{x-1}{1}\cdot\frac{x-2}{2}\cdot\frac{x-3}{3}-\frac{x-2}{1}\cdot\frac{x-3}{1}\cdot\frac{x-4}{2}+\frac{x-3}{1}\cdot\frac{x-4}{2}\cdot\frac{x-5}{1}-\frac{x-4}{1}\cdot\frac{x-5}{2}\cdot\frac{x-6}{3}=1,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{x-1}{1}\cdot\frac{x-2}{2}\cdot\frac{x-3}{3}\cdot\frac{x-4}{4}\cdots\frac{x-p}{p}-\frac{x-2}{1}\cdot\frac{x-3}{1}\cdot\frac{x-4}{2}\cdot\frac{x-5}{3}\cdots\frac{x-p-1}{p-1}+$$

$$\frac{x-3}{1}\cdot\frac{x-4}{2}\cdot\frac{x-5}{1}\cdot\frac{x-6}{2}\cdots\frac{x-p-2}{p-2}-\frac{x-4}{1}\cdot\frac{x-5}{2}\cdot\frac{x-6}{3}\cdot\frac{x-7}{1}\cdots\frac{x-p-3}{p-3}+$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\pm\frac{x-p-1}{1}\cdot\frac{x-p-2}{2}\cdot\frac{x-p-3}{3}\cdots\frac{x-2p}{p}=1,\ \text{etc.}$$

71. by comparing the first members of these equations with the different columns of the value that we have found just now for

$$(1+q)^{x-1}-\frac{x-2}{1}q(1+q)^{x-3}+\frac{x-4}{1}\cdot\frac{x-3}{2}q^2(1+q)^{x-5}-\frac{x-6}{1}\cdot\frac{x-5}{2}\cdot\frac{x-4}{3}q^3(1+q)^{x-7}+$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\pm\frac{x-2p}{1}\cdot\frac{x-2p+1}{2}\cdot\frac{x-2p+2}{3}\cdots\frac{x-p-1}{p}q^p(1+q)^{x-2p-1}\mp\ \text{etc.}$$

one sees that

$$(1+q)^{x-1}-\frac{x-2}{1}q(1+q)^{x-3}+\frac{x-4}{1}\cdot\frac{x-3}{2}q^2(1+q)^{x-5}-\frac{x-6}{1}\cdot\frac{x-5}{2}\cdot\frac{x-4}{3}q^3(1+q)^{x-7}+$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\pm\frac{x-2p}{1}\cdot\frac{x-2p+1}{2}\cdot\frac{x-2p+2}{3}\cdots\frac{x-p-1}{p}q^p(1+q)^{x-2p-1}\mp\ \text{etc.}=$$

$$1+q+q^2+q^3+\cdots\cdots\cdots\cdots+q^{x-1},$$

by making successively $x=n$, $x=m$, and $x=k$, one will reduce to a very simple form the numerators and the common denominators of the probabilities found above (68), so that the limit of the probabilities contrary to player B, will be expressed by

$$\frac{1+q+q^2+q^3+\cdots q^{n-1}}{1+q+q^2+q^3+\cdots q^{k-1}},$$

and that of the probabilities contrary to player C, by

$$q^n\times\frac{1+q+q^2+q^3+\cdots q^{m-1}}{1+q+q^2+q^3+\cdots q^{k-1}}=\frac{q^n+q^{n+1}+q^{n+2}+q^{n+3}+\cdots q^{k-1}}{1+q+q^2+q^3+\cdots q^{k-1}},$$

because $m+n=k$.

72. The sum of the two probabilities that we just calculated, is evidently equal to unity, that is to certitude, so that one is not able to doubt that one of the players is finished by being ruined. In regard to the advantage that the inequality of their fortunes gives to the richest, it is necessary in order to determine it to suppose all the rest equal between the two players, and consequently $q = 1$. The numerator of the first fraction is reduced then to n units, because it contains n terms; the numerator of the second and the common denominator are reduced respectively to m and to k units, and by recalling that $k = m + n$, one sees that the two fractions become

$$\frac{n}{m+n} \qquad \text{and} \qquad \frac{m}{m+n},$$

now $m : n$ is the ratio of the fortune of player B to that of player C, the probability that each player, in an equal game, will ruin his adversary, is therefore in direct ratio of his fortune.

73. When q is not equal to one, one is able to reduce to two terms the numerator and the denominator of each fraction by multiplying them by $q - 1$, one has thus $\frac{q^n-1}{q^k-1}$, for the probability that C will ruin B, and $\frac{q^k-q^n}{q^k-1}$ in order that B will ruin C.

74. If one would wish to know the ratio which must exist, in each game, between the chances favorable to each player, in order that there result from it in favor of the less rich, an advantage which tends constantly to compensate the inequality that the difference of their two fortunes sets, without ever giving to him more hope than there would remain to his adversary, it would be necessary to determine q in a manner that there was equality between the two fractions

$$\frac{1 + q + q^2 + q^3 + \cdots q^{n-1}}{1 + q + q^2 + q^3 + \cdots q^{k-1}},$$

and

$$\frac{q^n + q^{n+1} + q^{n+2} + q^{n+3} + \cdots q^{k-1}}{1 + q + q^2 + q^3 + \cdots q^{k-1}},$$

this which would be done by resolving the equation of degree $k - 1$

$$q^{k-1} + q^{k-2} + q^{k-3} + \cdots + q^n - q^{n-1} - q^{n-2} - q^{n-3} - \cdots - 1 = 0.$$

By comparing the two fractions

$$\frac{q^n - 1}{q^k - 1}, \qquad \text{and} \qquad \frac{q^k - q^n}{q^k - 1},$$

one would have found

$$q^k - 2q^n + 1 = 0,$$

an equation of a simpler form, but of a degree higher than the preceding, and which contains the factor $q - 1$, extraneous to the question.

75. In the case where one would suppose infinite the fortune of one of the two players, that for example of player C, one would have $n = \frac{1}{0}$. Then the number m remaining finite, the fraction $\frac{m}{m+n}$, which expresses the probability that this player will be ruined, would vanish, and the fraction $\frac{n}{m+n}$, which expresses the probability that he will ruin his adversary would become equal to 1, so that this last probability would be equivalent to certitude; player B would be found then precisely in the same case as in the first problem that we have resolved, where one would suppose that he played indifferently against all the players with whom he would be found in the case of being measured. It is evident, in fact, as we have just said [6], that these players are able then to be considered as a single adversary of whom the fortune would be infinite, and here is why the player of this first problem should necessarily be ruined. The preceding calculations accord perfectly with these results, because we have seen that the first n terms of the series of B, are the same as those of the series of A, whence it follows that these two series are identical when $n = \frac{1}{0}$.

76. By supposing always the game equal, and consequently $q = 1$, and making $m = n$, as that holds in the case where the two players are equally rich, the two fractions $\frac{m}{m+n}$ and $\frac{n}{m+n}$ become equals and both are reduced to $\frac{1}{2}$. The probability of being ruined is therefore the same for the two players; and as nothing diminishes the totality of their fortunes, the danger to which they expose themselves, must be regarded as compensated by the hope that each of them has to double his fortune. It is in this sense that I have said (6) that the game presented in this case no absolute disadvantage, although it is always imprudent to risk so all that which one possessed in the view of growing rich. The same compensation would hold, when the two players are unequally rich, if one could regard the loss of his fortune as a misfortune proportional to the absolute value of this fortune; for by multiplying the fortune of the player B by the probability of his ruin, such as it has been determined (72), and by making the same operation in regard to player C, one finds two products expressed by the same fraction $\frac{mn}{m+n}$, and consequently equal between them. But if the misfortune of losing his fortune is in general more sensible, when this fortune is more considerable, it is not at all in the ratio of its absolute value, it is solely because of the new needs that the men themselves make in measure as they acquire riches, of the rank that they accustom themselves to occupy in society, etc.: considerations of which it is impossible to make any numerical evaluation, and which seem to me must be absolutely rejected from the purely mathematical theory of the game, as I have already observed (3). The misfortune which menaces the players, being the same for both, nothing is able to compensate the advantage of the probability which exists in favor of the richest, according to the preceding calculations and the constant experience of the ordinary results of the game.[13]

[13] All the world knows the trivial proverb, to which this experience has given place $\langle \ldots \rangle$. [It would seem that one or more lines of text are missing here since the footnote breaks off. The proverb may be this one: "Le jeu est le fils de l'avarice, et le père du desespoir." That is, "The game is the son of greed, and the father of despair." RP]

APPENDIX

77. I myself have proposed to join to the preceding Memoir some applications of the formulas which are demonstrated to diverse questions foreign to the theory of probabilities, finally to leave no doubt on the utility that one is able to withdraw from these formulas, in some researches very different from those which have led me; but this utility must only be indicated in a work such as the one here, I have thought that it sufficed to give of them a single example. A formula known for a long time, but of which I have found no part of compete demonstration,[14] has presented to me one that I have preferred to every other, because it has furnished the occasion to insist on the advantages that one would withdraw from this formula, of one brought back, in the manner that I will explicate soon, many theories until the present scattered and independent of one another, in all the works which treat it.

78. One knows that in the case of the whole and positive exponent, the formula of the binomial of Newton is able to be set under this form

$$\begin{aligned}(a+b)^n = a^n + bn + \frac{n}{1}ab(a^{n-2}+b^{n-2}) + \frac{n}{1}\cdot\frac{n-1}{2}a^2b^2(a^{n-4}+b^{n-4}) + \\ \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}a^3b^3(a^{n-6}+b^{n-6}) + \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ +\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdots\frac{n-p+1}{p}a^pb^p(a^{n-2p}+b^{n-2p}) + \text{ etc.} \quad [24]\end{aligned}$$

it gives then the value of any power of the sum $a+b$ as function of the product ab and of the sums of the powers

$$\begin{aligned}a^n+b^n,\quad a^{n-2}+b^{n-2},\quad a^{n-4}+b^{n-4},\quad a^{n-6}+b^{n-6},\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots a^{n-2p}+b^{n-2p},\text{ etc.}\end{aligned}$$

the formula that I myself propose to demonstrate, gives on the contrary the value of a^n+b^n, as function of the product ab and of the quantities

$$\begin{aligned}(a+b)^n,\quad (a+b)^{n-2},\quad (a+b)^{n-4},\quad (a+b)^{n-6},\ldots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (a+b)^{n-2p}\text{ etc.}\end{aligned}$$

under this point of view it is so to speak the inverse of the formula of the binomial. One finds easily by induction that

$$a^n+b^n = (a+b)^n - \frac{n}{1}ab(a+b)^{n-2} + \frac{n}{1}\cdot\frac{n-3}{2}a^2b^2(a+b)^{n-4} -$$

[14]Castillon, in the Mémoires de Berlin, is himself occupied with this formula, but the demonstration that he gives of it, although quite superior to that which is found on the same subject in some elementary books, rests entirely on a calculation by induction, of which it is impossible to follow the march, and one encounters in it at each step some reductions and some transformations of which one sees not at all the turn.

$$\frac{n}{1}\cdot\frac{n-5}{2}\cdot\frac{n-4}{3}a^3b^3(a+b)^{n-6}+\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\pm\frac{n}{1}\cdot\frac{n-2p+1}{2}\cdot\frac{n-2p+2}{3}\cdots\frac{n-p-1}{p}a^pb^p(a+b)^{n-2p}\mp\ \text{etc.}\quad [25]$$

in order to demonstrate in a complete and general manner we will consider the second member of this equation as a function of a and of b that the concern is to bring back to a more simple form, and the end that we propose will be fulfilled if we find that it is reduced in fact to $a^r + b^r$.

79. By writing successively $n-2, n-4, n-6, \dots, n-2p$, etc. in the place of n in equation [24], we will have the values of

$$(a+b)^{n-2}, \quad (a+b)^{n-4}, \quad (a+b)^{n-6}, \cdots (a+b)^{n-2p}, \text{ etc.}$$

by substituting them in it, so that that of $(a+b)^n$, in the function that we wish to reduce to its most simple expression, we will change it into

$$\begin{array}{lll}
a^n+b^n+\frac{n}{1}ab(a^{n-2}+b^{n-2}) & +\frac{n}{1}\cdot\frac{n-1}{2}a^2b^2(a^{n-4}+b^{n-4}) & +\cdots+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdots\frac{n-p+1}{p}a^pb^p(a^{n-2p}+b^{n-2p})+\text{ etc.}\\
\qquad -\frac{n}{1}ab(a^{n-2}+b^{n-2}) & -\frac{n}{1}\cdot\frac{n-2}{1}a^2b^2(a^{n-4}+b^{n-4}) & -\cdots-\frac{n}{1}\cdot\frac{n-2}{1}\cdot\frac{n-3}{2}\cdot\frac{n-4}{3}\cdots\frac{n-p}{p-1}a^pb^p(a^{n-2p}+b^{n-2p})-\text{ etc.}\\
 & +\frac{n}{1}\cdot\frac{n-3}{1}a^2b^2(a^{n-4}+b^{n-4}) & +\cdots+\frac{n}{1}\cdot\frac{n-3}{1}\cdot\frac{n-4}{1}\cdot\frac{n-5}{2}\cdot\frac{n-6}{3}\cdots\frac{n-p-1}{p-2}(a^{n-2p}+b^{n-2p})+\text{ etc.}\\
 & & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
 & & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
 & & \pm\frac{n}{1}\cdots\frac{n-2p+1}{2}\cdot\frac{n-2p+2}{3}\cdots\frac{n-p-1}{p}a^pb^p(a^{n-2p}+b^{n-2p})\pm\text{ etc.}\\
 & & \mp\text{ etc.}
\end{array}$$

that we will represent for brevity by X.

80. If we take again now equation [5], if we calculated some last terms in it, by making successively $r = 0$, $r = 1$, $r = 2$, $r = 3$, etc. until $r = p$, in the general term

$$\frac{u+2p-2r}{1}\cdot\frac{u+2p-2r-1}{2}\cdot\frac{u+2p-2r-2}{3}\cdots$$
$$\frac{u+p-r+1}{p-r}\cdot\frac{m}{1}\cdot\frac{m+2r-1}{2}\cdot\frac{m+2r-2}{3}\cdots\frac{m+r+1}{r}$$

and if we wrote the terms thus found in an order inverse to the one which had been followed in equation [5], we will have

$$\frac{u+2p}{1}\cdot\frac{u+2p-1}{2}\cdot\frac{u+2p-2}{3}\cdots\frac{u+p+1}{p}+\frac{u+2p-2}{1}\cdot\frac{u+2p-3}{1}\cdot$$
$$\frac{u+2p-4}{3}\cdots\frac{u+p}{p-1}\cdot\frac{m}{1}+\frac{u+2p-4}{1}\cdot\frac{u+2p-5}{2}\cdot\frac{u+2p-6}{3}\cdots\frac{u+p-1}{p-2}\cdot\frac{m}{1}\cdot\frac{m+3}{2}+$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+\frac{m}{1}\cdot\frac{m+2p-1}{2}\cdot\frac{m+2p-2}{3}\cdots\frac{m+p+1}{p}=$$
$$\frac{u+m+2p}{1}\cdot\frac{u+m+2p-1}{2}\cdot\frac{u+m+2p-2}{3}\cdots\frac{u+m+p+1}{p}.$$

We suppose $m = -n$, and we will write first the factors where between this letter, there will come

$$\frac{u+2p}{1}\cdot\frac{u-2p-1}{2}\cdot\frac{u-2p-2}{3}\cdots\cdot\frac{u+p+1}{p}-\frac{n}{1}\cdot\frac{u+2p-2}{2}\cdot\frac{u+2p-3}{2}\cdot$$
$$\frac{u+2p-4}{3}\cdots\frac{n+p}{p-1}+\frac{n}{1}\cdot\frac{n-3}{2}\cdot\frac{n+2p-4}{1}\cdot\frac{n+2p-5}{2}\cdot\frac{n+2p-6}{3}\cdots\frac{n+p-1}{p-2}-$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\pm\frac{n}{1}\cdot\frac{n-2p+1}{2}\cdot\frac{n-2p+2}{3}\cdots\frac{n-p-1}{p}=$$
$$\frac{u+2p-n}{1}\cdot\frac{u+2p-n-1}{2}\cdot\frac{u+2p-n-2}{3}\cdots\frac{u+p-n+1}{p}.$$

The value of u being arbitrary, one is able to take $u = n - 2p$, or $u + 2p = n$, the second member disappears under this supposition by the vanishing of its first factor, and one has

$$\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdots\frac{n-p+1}{p}\cdot\frac{n}{1}\cdot\frac{n-2}{1}\cdot\frac{n-3}{2}\cdot\frac{n-4}{3}\cdots\frac{n-p}{p-1}\cdots$$
$$\frac{n}{1}\cdot\frac{n-3}{2}\cdot\frac{n-4}{1}\cdot\frac{n-5}{2}\cdot\frac{n-6}{3}\cdots\frac{n-p-1}{p-2}-\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\pm\frac{n}{1}\cdot\frac{n-2p+1}{2}\cdot\frac{n-2p+2}{3}\cdots\frac{n-p-1}{p}=0; \quad [26]$$

the first member of this equation being precisely the same thing as the sum of the coefficients of $a^p b^p(a^{n-2p} + b^{n-2p})$, in the value that we just found (79) for X, it is evident that the last of the columns that we have written in this value is equal to zero, and as this column represents all the others, that it gives it immediately supposing successively $p = 1$, $p = 2$, $p = 3$, etc., to $p = \frac{n}{2}$ or $\frac{n-1}{2}$ according as n is even or odd, it follows that the value of X is reduced, thus as we ourselves had proposed to demonstrate it to $a^n + b^n$.

81. The preceding demonstration would offer only little interest, if the whole did not announce that the diverse applications that the formula that is the object of it presents, are alone able to give to algebra, and particularly to the algebraic resolution of the equations, all the perfection of which this part of mathematics is susceptible. One finds in all the works where it is treated with some extent, the solution of the reciprocal equations; some methods in order to resolve the equations of the third degree, and those of the higher degrees of which the roots are able to be determined by the same processes; the examination of the cases where these methods become useless; of the formulas for the extraction of the roots of the quantities into rational parts and into irrational or imaginary parts, etc. But one sets no relation between these different objects, one presents them not at all as simple applications of one same formula, this which would contribute at the same time to simplify the study of them, and to engrave them more easily in memory. Nothing would be however easier if one is attached to deduce them from equation [25], of which there are so many immediate corollaries. This manner to consider them has appeared to present to me some results too advantageous in order to not enter here into some details which will be able to give a just idea of it; but I must say before a word on the application of the same formula to the determination of the symmetric functions of the two roots of any equation of the second degree, $x^2 - gx + h = 0$. By naming a and b these two roots, one will have $a + b = g$, $ab = h$, and each symmetric function of a and of b will be able to be represented by $a^s b^r + a^r b^s$, in order to find from it the value it will be necessary first to suppose in equation [25], $n = s - r$, this which will give

$$a^{s-r} + b^{s-r} = g^{s-r} - \frac{s-r}{1} g^{s-r-2} h + \frac{s-r}{1} \cdot \frac{s-r-3}{2} g^{s-r-4} h^2 -$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\pm \frac{s-r}{1} \cdot \frac{s-r-2p+1}{2} \cdot \frac{s-r-2p+2}{3} \cdots \frac{s-r-p-1}{p} g^{s-2p} b^p \mp \text{ etc.}$$

one will multiply next this equation by $a^r b^r = h^r$, and one will have

$$a^s b^r + a^r b^s = g^{s-r} h^r - \frac{s-r}{1} g^{s-r-2} h^{r+1} + \frac{s-r}{1} \cdot \frac{s-r-3}{2} g^{s-r-4} h^{r+2} -$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\pm \frac{s-r}{1} \cdot \frac{s-r-2p+1}{2} \cdot \frac{s-r-2p+2}{3} \cdots \frac{s-r-p-1}{p} g^{s-r-2p} h^{r+p} \mp \text{ etc.}$$

The last term of this formula is found, when $s - r$ is even, by making $2p = s - r$, or

$p = \frac{s-r}{2}$, this last term is

$$\pm\frac{s-r}{1}\cdot\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdots\frac{\frac{s-r}{2}-1}{\frac{s-r}{2}}h^{r+\frac{s-r}{2}} = \pm 2h^{\frac{s+r}{2}}.$$

When $s-r$ is odd it is necessary in order to have the last term to suppose $p = \frac{s-r-1}{2}$, this which gives for the value of this term

$$\pm\frac{s-r}{1}\cdot\frac{2}{2}\cdot\frac{3}{3}\cdot\frac{4}{4}\cdots\frac{\frac{s-r-1}{2}}{\frac{s-r-1}{2}}gh^{r+\frac{s-r-1}{2}} = \pm\frac{s-t}{1}gh^{\frac{s+r-1}{2}}.$$

In both cases the upper sign corresponds to the even values of p, that is to $s-r = 4n$, and to $s-r = 4n+1$, while the lower holds when p is odd, that is when $s-r = 4n+2$, or when $s-r = 4n+3$.

82. The reciprocal equations, considered under the most general point of view, are those of which the first member is a symmetric and homogeneous function, of the unknowns and of a quantity that one supposes ordinarily equal to unity, but that we will represent by c, in order to give more regularity and more generality to the calculation, each reciprocal equation will be found thus comprised in the formula

$$x^m + pcx^{m-1} + qx^2x^{m-2} + \cdots + qc^{m-2}x^2 + pc^{m-1}x + c^m = 0,$$

or that which reverts to the same

$$x^m + c^m + pcx(x^{m-2} + c^{m-2}) + qc^2x^2(x^{m-4} + c^{m-4}) + \text{ etc.} = 0.$$

The form of this equation shows that it is divisible by $x + c$ all the time that m is odd, and as the quotient is a reciprocal equation of which the degree is even, it follows from it that the general solution of the equations of this kind is restored to that of the reciprocal equations of even degree, which are all represented by the formula

$$x^{2r} + c^{2r} + pcx(x^{2r-2} + c^{2r-2}) + qc^2x^2(x^{2r-4} + c^{2r-4}) + \text{ etc.} = 0,$$

it reduces the solution of it from this here to that of equations of degree r, by dividing it by c^rx^r, this which gives

$$\frac{x^r}{c^r} + \frac{c^r}{x^r} + p\left(\frac{x^{r-1}}{c^{r-1}} + \frac{c^{r-1}}{x^{r-1}}\right) + q\left(\frac{x^{r-2}}{c^{r-2}} + \frac{c^{r-2}}{x^{r-2}}\right) + \text{ etc.} = 0,$$

and by substituting in the place of the quantities

$$\frac{x^r}{c^r} + \frac{c^r}{x^r}, \qquad \frac{x^{r-1}}{c^{r-1}} + \frac{c^{r-1}}{x^{r-1}}, \qquad \frac{x^{r-2}}{c^{r-2}} + \frac{c^{r-2}}{x^{r-2}}, \text{ etc.}$$

the values that one finds by supposing successively $n = r$, $n = r-1$, $n = r-2$, etc. in the equation

$$\frac{x^n}{c^n} + \frac{c^n}{x^n} = z^n - \frac{n}{1} z^{n-2} + \frac{n}{1} \cdot \frac{n-3}{2} z^{n-4} - \frac{n}{1} \cdot \frac{n-5}{2} \cdot \frac{n-4}{3} z^{n-6} +$$

$$\cdots \pm \frac{n}{1} \cdot \frac{n-2p+1}{2} \cdot \frac{n-2p+2}{3} \cdots \frac{n-p-1}{p} z^{n-2p} \mp \text{ etc.}$$

which is nothing other than equation [25], in which one has made $a = \frac{x}{c}$, $b = \frac{c}{x}$ and consequently $ab = 1$, and $a + b = \frac{x}{c} + \frac{c}{x}$, a quantity which we have represented for brevity by z. The equation in z which will result from these substitutions will be only of degree r, less than half the degree of the equation in x; it is thus that the resolution of the reciprocal equations of any degree m, is reduced to that of the equations of degree $\frac{m}{2}$ or $\frac{m-1}{2}$ according as m is even or odd, because as soon as one has the r values of z, one finds $2r$ values of x by virtue of the equation

$$\frac{x}{c} + \frac{c}{x} = z, \quad \text{or} \quad x^2 - zcx + c^2 = 0,$$

and one has moreover $x = -c$ in the case where m is odd.

83. By making in equation [25] $a^n + b^n = k$, $ab = h$, and $a + b = z$, it will become

$$z^n - \frac{n}{1} h z^{n-2} + \frac{n}{1} \cdot \frac{n-3}{2} h^2 z^{n-4} - \frac{n}{1} \cdot \frac{n-5}{2} \cdot \frac{n-4}{3} h^3 z^{n-6} + \cdots$$

$$\pm \frac{n}{1} \cdot \frac{n-2p+1}{2} \cdot \frac{n-2p+2}{3} \cdots \frac{n-p-1}{p} h^p z^{n-2p} \mp \text{ etc.} = k; \quad [27]$$

an equation among z, h and k, where one is able to regard z as unknown. The solution of this equation is linked with that of the equation of which a and b represent all the roots, and that one finds immediately by considering a^n and b^n as the two roots of one same equation of the second degree, and by combining the two equations

$$a^n + b^n = k, \quad a^n b^n = h^n$$

this which gives

$$a^{2n} - ka^n + h^n = 0, \quad \text{or} \quad b^{2n} - kb^n + h^n = 0. \quad [28]$$

one sees in fact that each value of a and of b give to it one of z, by virtue of the equation $z = a + b = a + \frac{h}{a}$, and that reciprocally if one had all the values of z, one would find those of a and b by drawing two of these last from each value of z, by the resolution of the equation of the second degree

$$a^2 - za + h = 0, \quad \text{or} \quad b^2 - zb + h = 0, \quad [29]$$

One restores ordinarily the solution of equation [27] to that of equation [28], because this last is reduced with simple extractions, according as one has completed the square of which the first two terms are $a^{2n} - ka^n$, this is why one regards as entirely resolved the equations of these two forms

$$a^{2n} - ka^n + h^n = 0,$$

$$\text{and } z^n - \frac{n}{1}hz^{n-2} + \frac{n}{1}\cdot\frac{n-3}{2}h^2z^{n-4} - \frac{n}{1}\cdot\frac{n-5}{2}\cdot\frac{n-4}{3}h^3z^{n-6} +$$

$$\pm\frac{n}{1}\cdot\frac{n-2p+1}{2}\cdot\frac{n-2p+2}{3}\cdots\frac{n-p-1}{p}h^pz^{n-2p} \mp \text{ etc.} = k,$$

of which the second is especially remarkable in that which it becomes when $n = 3$,

$$z^3 - 3hz - k = 0,$$

an equation which contains all those of the third degree, after one has made the second term vanish.

84. It is thus that the preceding formulas lead to the general solution of the equations of this degree, they give equally the expression of the roots of the equations of odd degrees of which one is able to make all the even terms vanish,[15] and that this operation restores to the equations that one finds by supposing successively $n = 5$, $n = 7$, etc., namely

$$z^5 - 5hz^3 + 5h^2z - k = 0,$$
$$z^7 - 7hz^5 + 14h^2z^3 - 7h^3z - k = 0,$$
$$\text{etc.} \qquad\qquad \text{etc.}$$

all this is well known, in the same way as the uselessness of this process in the case to which one has given the name of the irreducible case; the extractions to which one is led becoming then inexecutables, one must regard as absolutely illusory, not only the solution of equation [27], but also of equation [28]. In fact, the end that one must propose in the algebraic solution of the equations, is to find a formula which presents the table of one sequence of operations by the aid of which one is able to calculate all the roots, each under the form which is proper to it; that is the exact values of the rational roots and of the imaginary roots with rational real part, and the approximate values of those which are irrational reals, or imaginary with irrational real part. Each expression of the roots of an equation which does not fulfill this end is able to be of no use in practice, and must be rejected as to indicating only some inexecutable operations. This is that which arrives in regard to the equations that we examine, when one is led to extract odd roots from quantities with real part and with imaginary part; the algebra which gives the means to extract by approximation all sorts of roots of a real quantity, and only the square root of an imaginary quantity, has aid of two formulas

$$\sqrt{a+b\sqrt{-1}} = \pm\left(\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}} + \sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}\sqrt{-1}\right) \qquad [30]$$

[15]The method of Tschirnaus furnishes a very simple means of succeeding in the equations of the fifth degree, the equation which one has resolved on account of making the second and the fourth term vanish climbs only to the third degree.

and

$$\sqrt{a-b\sqrt{-1}} = \mp\left(\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}} - \sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}\sqrt{-1}\right), \qquad [31]$$

present none in order to determine the other roots of these quantities, independently of the same equations of which they should give the solution; so that after having found the algebraic expressions of the roots demanded, one is able to test the calculations without being restored by a vicious circle to the same equations that one was first proposed to resolve.[16] After having exhausted all the combinations that this subject could present, the mathematicians are agreed to recognize that one ought have in the irreducible case, no regard to the formulas which express the roots of equations [27], and to resolve directly this equation by the method of the commensurable divisors or by the methods of approximation; it seems to me that they would have to reject equally the algebraic expressions of the roots of the equations of the form of equation [28], since these expressions contain the indication of an inexecutable operation, and that if they have not the inconvenience of giving a real quantity under an imaginary form, they have the one of giving an imaginary quantity that one knows to be susceptible to being restored to the form $a + b\sqrt{-1}$, under a form all different, this which is also harmful in practice, whence one seeks the imaginary roots only in order to know separately the two parts. One must therefore regard the solution of the equations of the form $a^{2n} - ka^n + h^n = 0$, as incomplete in this that it extends not at all to the case of which we speak, and it appears that if one has paid only a little attention to this imperfection of one method that one sees completely announced as if it were complete and general, that comes from this that all the roots are then imaginaries, and that one is in general much less occupied with the means to find these roots under the form which is proper to them, than of those which lead to the determination of the real roots; we will see if the preceding theory offered something more satisfying in regard to the equations that we examine.

85. The solution of equation [28] and that of equation [27], are so much dependent on one another, that as soon as the first is no longer able to serve to determine the roots of equation [27], it is necessary on the contrary to have recourse to that here in order to find the roots of the first. It suffices for the rest to know one alone of the roots of equation [27], in order to find all those of equation [28]; one seeks it first by the method of commensurable divisors, and when the equation has no rational roots at all, one has recourse to the methods of approximation; by means of this value of z, and by resolving the equation $a^2 - za + h = 0$, one obtains two of them a or b, which give next all the others by multiplying each by the $n-1$ roots of the unit of the degree n, which are not equal to one. But in order to apply this method to any equation of the number of those which one resolves ordinarily by the manner of equations of the second degree, it is necessary first to restore it to the form $a^{2n} - ka^n + h^n = 0$, that is, that it is necessary to prepare it in a manner that its last term is an exact power of degree

[16]The tables of sines offer in truth the same facilities for these extractions, that the tables of logarithms for the extractions of the roots of the real quantities. But I speak here only of the means drawn from the ordinary calculus, which took the place in the present case of the usage of these last tables, and which could not compensate for the one of the tables of the sines.

n without this precaution the coefficients of the equation in z would be irrationals, this which would complicate much the solution of this equation, and would give only an approximate value in some cases where one is able to have an exact radical expression and indicating in it only some exactable operations in order to set it in number. Let therefore $x^{2n} - fx^n + g = 0$ be an equation in which the value of x^n is imaginary, and of which the solution by the ordinary method becomes useless, it will be necessary first to see if n is even or odd. In the first case n being of the form $2^r i$, where i designates an odd number, one will make $x^{2^r} = y$, and consequently $x^{2^r i}$ or $x^n = y^i$, this which will restore the solution of the equation proposed to that of the equation $y^{2i} - fy^i + g = 0$, that one will obtain by the method that we are going to apply to the equation $x^{2n} - fx^n + g = 0$, by supposing n odd. By making $x = \frac{a}{g^{\frac{n-1}{2n}}}$, this equation becomes $\frac{a^{2n}}{g^{a-1}} - f\frac{a^n}{g^{\frac{n-1}{2}}} + g = 0$, one $a^{2n} - fg^{\frac{n-1}{2}}a^n + g^n = 0$, of which the last term is an exact power of degree n, and which contains only rational coefficients, because n being odd, $\frac{n-1}{2}$ is a whole number; one will form therefore the equation in z, which will be

$$z^n - \frac{n}{1}gz^{n-2} + \frac{n}{1}\cdot\frac{n-3}{2}g^2z^{n-4} - \frac{n}{1}\cdot\frac{n-5}{2}\cdot\frac{n-4}{3}g^3z^{n-6} +$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\pm\frac{n}{1}\cdot\frac{n-2p+1}{1}\cdot\frac{n-2p+2}{2}\cdots\frac{n-p-1}{p}g^pz^{n-2p} \mp \text{ etc.} = fg^{\frac{n-1}{2}};$$

after one will have found one of the values of z, and after one will have concluded from it all those of a, as we just explicated, one will determine those of x, by aid of the formula $x = \frac{a}{g^{\frac{n-1}{2n}}}$, or $x = \frac{a}{\sqrt[n]{g^{\frac{n-1}{2}}}}$; the denominator of this expression is in truth irrational, but it is always easy to calculate the real value, the only one of which one has need, this real value is unique because the quantity $g^{\frac{n-1}{2n}}$ is rational, and because the index n of the radical is odd. In the case where n would be even, the preceding method would not give the values of x, but only those of x^{2^r}, this is why one would calculate only two of these values, corresponding to the two values of u deduced from the same value of z, and would extract from each of them r times the square root, by formulas [30] and [31], one would have thus two values of x which would give all the others by multiplying them by the roots of unity of degree n, different from one.

86. The process that we just indicated, and which is able alone to convey the true solution of the equations of the form $x^{2n} - fx^n + g = 0$, when the value that they give for x^n is imaginary, is able also to be employed when this value is real; but it is only in the case where the equation in z has a commensurable divisor, that it presents more advantages than the solution by the ordinary method, it gives then the values of x under a simpler form, and of which the calculation is less complicated than the one of the expressions deduced from this method. One sees by reuniting all that which we just said that in order to resolve conveniently an equation of the form $x^{2n} - fx^n + g = 0$, it is necessary first to draw the value of x^n if it is imaginary, one will be able to employ only the method of article 85; if it is real, it will be necessary next to serve oneself of the same method, to calculate the equation in z, and to seek if it would have a

commensurable divisor; it is only in the case where one would not find it at all, that it would be necessary to have recourse to the march indicated in all the elementary books in order to resolve the equations of this form. One will obtain from it thus all the roots under the simplest form, and one will never be obliged to resort to the method of extractions of the roots of the quantities into rational parts and into irrational or imaginary parts, which have been invented only in order to compensate as much as it was possible for the defects of the ordinary solution. The determination of these sorts of roots, although become useless to the resolution of the equations of which we speak, is besides too interesting in itself in order to not say a word; the method that I am going to give for arriving to it will be a quite simple application of the preceding theory, it will have on the ordinary method the advantage of being truly analytic, in this that it will suppose not at all that one knows in advance the form of the sought root, as one has been until the present obliged to do it, without having to demonstrate that this form was the only one which was appropriate, and that it was impossible to obtain a result more satisfying by assigning to it another form.

87. Let the radical quantity of the second degree be $a + \sqrt{b}$, which is real or imaginary according as the sign of b in order to extract the root of degree n, one will represent this root by x and one will have $a + \sqrt{b} = x^n$, or $b = x^{2n} - 2ax^n + a^2$, that is $x^{2n} - 2ax^n + a^2 - b = 0$, an equation of the form of those which we just resolved, but which would lead only to a vicious circle if one sought from it the solution by the ordinary method, it will be necessary therefore to employ only that of n° 85, which will give in all the cases a value of x of the form $p + \sqrt{q}$, where p and q will be rational only when the equation in z will have a commensurable divisor. If it does not have it and if b is positive, the calculation of the expression $p + \sqrt{q}$ would be more difficult than the one of the approximate value of $\sqrt[n]{a + \sqrt{b}}$ by immediate extraction, it will be therefore useless to follow this march; but it will not be likewise in the case where b being negative, $a + \sqrt{b}$ would be imaginary, for it presents then the only way to find the diverse roots of this quantity, under the form to which one must restore all the imaginaries, this which fills the vacuum that they leave in the set of arithmetic operations, that indicate the different formulas used in algebra, the impossibility where one is to find directly the approximate values of the two parts of the odd roots of the imaginary quantities, it seems that one has not yet enough sense, despite the continual usage that one is obliged to make of these quantities, that they are an essential part of the theory of the calculus, and that this theory will never be complete, as much as one will not have some easy and uniform means to submit them to the same operations that one executes on the other numbers.

88. The last application that we will make of the preceding formulas, will have for object the equation [27]. We will deduce from it some very simple relations between the different roots of this equation, by aid of which we will be able to calculate all of them, as soon as we will know one of them alone. Let t be one of the values of z, we will have for two of the values of a or of b the two roots of the equation $a^2 - ta + h = 0$, and we will be able to take indifferently $a = \frac{t \pm \sqrt{t^2 - 4h}}{2}$, b being then equal to $\frac{t \mp \sqrt{t^2 - 4h}}{2}$; we will conclude from it $a + b = t$ and $a - b = \pm\sqrt{t^2 - 4h}$; in order

to have the other values of a and of b, it will be necessary to multiply those that we just found by the roots of unity of degree n, and as one has $z = a+b = a+\frac{h}{a}$, one will find all the values of z by substituting into this last equation in the place of a, $\frac{t\pm\sqrt{t^2-4h}}{2}$ multiplied by the n^{th} roots of unity, let $p+q\sqrt{-1}$ be one of these roots, $p-q\sqrt{-1}$ will be the other of them, and the product of these two roots will be a third of them; but this product is real and positive; it is therefore necessarily equal to unity, this which reduces the general value of z, $z = a(p+q\sqrt{-1}) + \frac{h}{a(p+q\sqrt{-1})} = a(p+q\sqrt{-1}) + \frac{b}{(p+q\sqrt{-1})}$, to $z = a(p+q\sqrt{-1}) + b(p-q\sqrt{-1}) = p(a+b) + q(a-b)\sqrt{-1} = pt \pm q\sqrt{4h-t^2}$, this very simple value will give all those of z, by putting in the place of p and of q the different values represented by these letters: p and q being some real quantities, the values of z will be all reals, whatever be the number n, when $4h$ will be greater than t^2, that is, all the time that the value of a, $a = \frac{t\pm\sqrt{t^2-4h}}{2}$, will be imaginary; if, on the contrary, a is real, t^2 will be greater than $4h$, and z will be imaginary, unless $q = 0$, that which is able to take place only for two values, when n is even, and for one alone, when this number is odd; in this last case, a will be real or imaginary under the same circumstances as a^n, whence it follows that then the equation in z will have all its roots real, or will have only a single one of them according as the value of a^n will be imaginary or real. This well-known theorem, but for which one has ordinarily recourse to the consideration of the trigonometric lines, is found thus demonstrated in a purely algebraic and very simple manner.

89. Such is the manner by which it seems to me that this part of algebra must be presented in the treatises where one wishes to give a just idea of all the branches of this science, and to apply oneself rather to develop some fertile principles in consequences, than to offer separately some theories of which one indicates not at all the liaison, and which, although very interesting, each in particular, are fixed only with much difficulty in the memory of those who study them. It is true that in order to restore to equation [25] all that which we just deduced from it, it would be necessary to be able to demonstrate this equation as generally as we just did, without having recourse to some formulas dependent of the theory of probabilities, of which the beginners have ordinarily no idea. Here is a very simple demonstration which is able to leave nothing to desire in this regard, and to which I have been led by the process to which I have been obliged to have recourse in order to reduce to its form the simplest formula of n° 69.

90. After having developed the quantity X, as we have done (79), all the demonstration consists in showing that it is reduced to $a^n + b^n$, that is, to show that any column, of which the coefficient is represented in general by

$$\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}\cdots\frac{n-p+1}{p} - \frac{n}{1}\cdot\frac{n-2}{1}\cdot\frac{n-3}{2}\cdot\frac{n-4}{3}\cdots\frac{n-p}{p-1} +$$
$$\frac{n}{1}\cdot\frac{n-3}{1}\cdot\frac{n-4}{1}\cdot\frac{n-5}{2}\cdots\frac{n-p-1}{p-2} - \frac{n}{1}\cdot\frac{n-5}{2}\cdot\frac{n-4}{3}\cdot\frac{n-6}{1}\cdots\frac{n-p-2}{p-3} +$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots \pm \frac{n}{1}\cdot\frac{n-2p+1}{2}\cdot\frac{n-2p+2}{3}\cdots\frac{n-p-1}{p},$$

vanishes, whatever be the value of p. We suppress the common factor $\frac{n}{1}$, and we arrange all the factors of the numerators in a manner that the greatest in each term are always the first, the concern will be to demonstrate that

$$\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}\cdots\frac{n-p+1}{p}-\frac{n-2}{1}\cdot\frac{n-3}{2}\cdot\frac{n-4}{3}\cdots\frac{n-p}{p-1}+$$
$$\frac{n-3}{2}\cdot\frac{n-4}{1}\cdot\frac{n-5}{2}\cdots\frac{n-p-1}{p-2}-\frac{n-4}{2}\cdot\frac{n-5}{3}\cdot\frac{n-6}{1}\cdots\frac{n-p-2}{p-3}+$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\pm\frac{n-p-1}{2}\cdot\frac{n-p-2}{3}\cdot\frac{n-p-3}{4}\cdots\frac{n-2p+1}{p}=0.$$

this which is done thus: by multiplying the one by the other the two equations

$$\begin{aligned}\frac{1}{(1-a)^t}=&(1-a)^{-t}=1+\frac{t}{1}a+\frac{t+1}{1}\cdot\frac{t}{2}a^2+\frac{t+2}{1}\cdot\frac{t+1}{2}\cdot\frac{t}{3}a^3+\cdots\\&+\frac{t+p-1}{1}\cdot\frac{t+p-2}{2}\cdot\frac{t+p-3}{3}\cdot\frac{t+p-4}{4}\cdots\frac{t}{p}a^p+\text{ etc.}\end{aligned}$$

and

$$\begin{aligned}(1-a)^t=&1-\frac{t}{1}a+\frac{t}{1}\cdot\frac{t-1}{2}a^2-\frac{t}{1}\cdot\frac{t-1}{2}\cdot\frac{t-2}{3}a^3+\cdots\\&\pm\frac{t}{1}\cdot\frac{t-1}{2}\cdot\frac{t-2}{3}\cdot\frac{t-3}{4}\cdots\frac{t-p+1}{p}a^p\mp\text{ etc.}\end{aligned}$$

one has

$$\begin{aligned}1=1&+\frac{t}{1}a+\frac{t+1}{1}\cdot\frac{t}{2}a^2+\frac{t+2}{1}\cdot\frac{t+1}{2}\cdot\frac{t}{3}a^3+\cdots\\&+\frac{t+p-1}{1}\cdot\frac{t+p-2}{2}\cdot\frac{t+p-3}{3}\cdot\frac{t+p-4}{4}\cdots\frac{t}{p}a^p+\text{ etc.}\\&-\frac{t}{1}a-\frac{t}{1}\cdot\frac{t}{1}a^2-\frac{t+1}{1}\cdot\frac{t}{2}\cdot\frac{t}{1}a^3-\cdots\\&-\frac{t+p-2}{1}\cdot\frac{t+p-3}{2}\cdot\frac{t+p-4}{3}\cdots\frac{t}{p-1}\cdot\frac{t}{1}a^p-\text{ etc.}\\&+\frac{t}{1}\cdot\frac{t-1}{2}a^2+\frac{t}{1}\cdot\frac{t}{1}\cdot\frac{t-1}{2}a^3+\cdots\\&+\frac{t+p-3}{1}\cdot\frac{t+p-4}{2}\cdots\frac{t}{p-2}\cdot\frac{t}{1}\cdot\frac{t-1}{2}a^p+\text{ etc.}\\&-\frac{t}{1}\cdot\frac{t-1}{2}\cdot\frac{t-2}{3}a^3-\cdots-\frac{t-p-4}{1}\cdots\frac{t}{p-3}\cdot\frac{t}{1}\cdot\frac{t-1}{2}\cdot\frac{t-2}{3}a^p-\text{ etc.}\\&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\&\pm\frac{t}{1}\cdot\frac{t-1}{2}\cdot\frac{t-2}{3}\cdot\frac{t-3}{4}\cdots\frac{t-p+1}{p}a^p\pm\text{ etc.}\\&+\text{ etc.}\end{aligned}$$

this equation must be identical, whatever be the values of a and of t, it is necessary that all the columns which are found after 1 in the second member, vanish of themselves, this which gives by equating only to zero that which represent them all, and by supposing the common factor $\frac{t}{1}$.

$$\frac{t+p-1}{2}\cdot\frac{t+p-2}{3}\cdot\frac{t+p-3}{4}\cdot\frac{t+p-4}{5}\cdots\frac{t+p}{p}-\frac{t+p-2}{1}\cdot\frac{t+p-3}{2}\cdot\frac{t+p-4}{3}\cdots\frac{1}{p-1}+$$
$$\frac{t+p-3}{1}\cdot\frac{t+p-4}{2}\cdots\frac{1}{p-2}\cdot\frac{t-1}{2}-\frac{t+p-4}{1}\cdots\frac{t}{p-3}\cdot\frac{t-1}{2}\cdot\frac{t-2}{3}+$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\pm\frac{t-1}{2}\cdot\frac{t-2}{3}\cdot\frac{t-3}{4}\cdots\frac{t-p+1}{p}=0.$$

By making $t=n-p$, and by changing the order of the factors of the denominator, one will see easily that this equation reverts to

$$\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}\cdots\frac{n-p+1}{p}-\frac{n-2}{1}\cdot\frac{n-3}{2}\cdot\frac{n-4}{3}\cdots\frac{n-p}{p-1}+$$
$$\frac{n-3}{2}\cdot\frac{n-4}{1}\cdot\frac{n-5}{1}\cdots\frac{n-p-1}{p-2}-\frac{n-4}{2}\cdot\frac{n-5}{3}\cdot\frac{n-6}{1}\cdots\frac{n-p-2}{p-3}+$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\pm\frac{n-p-1}{2}\cdot\frac{n-p-2}{3}\cdot\frac{n-p-3}{4}\cdots\frac{n-2p+1}{p}=0$$

which is precisely that which the concern was to demonstrate.

END