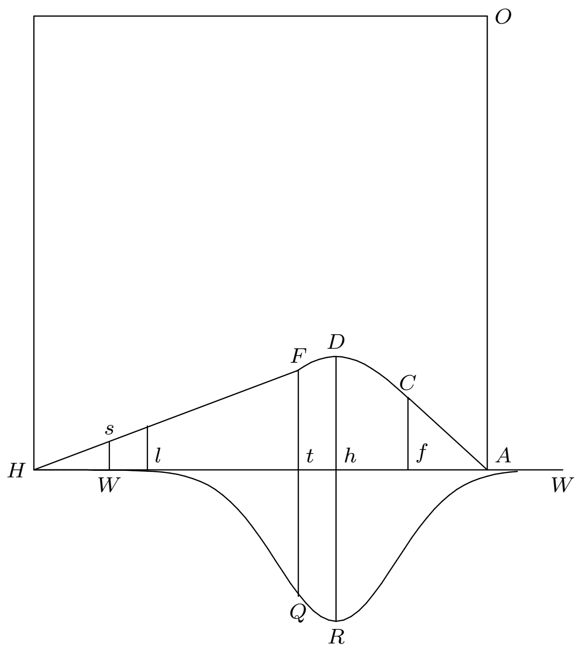
LII. *A Demonstration of the Second Rule in the Essay towards the Solution of a Problem in the Doctrine of Chances, published in the Philosophical Transactions,* Vol. LIII. *Communicated by the Rev. Mr.* Richard Price, *in a Letter to Mr.* John Canton, *M.A.F.R.S.*

Philosophical Transactions of the Royal Society of London **LIV** (1764) pp. 296–325

Dear Sir, Nov. 26, 1764

Read Dec. 6, 1764. I send you the following *Supplement to the Essay on a Problem in the Doctrine of Chances*, hoping that you may not think it improper to be communicatated to the Royal Society. I should not have troubled you again in this way had I not found that some additions to my former papers were necessary in order to explain some passages in them, and particularly what is hinted in the note at the end of the Appendix. "I have first given the deduction of Mr. Bayes' second rule chiefly in his own words; and then added, as briefly as possible, the demonstrations of several propositions, which seem to improve considerably the solution of the problem, and to throw light on the nature of the curve by the quadrature of which this solution is obtained." Perhaps, there is no reason for being very anxious about proceeding to further improvements. It would, however, be very agreeable to me to see a yet easier and nearer approximation to the value of the two series's in the first rule: but this I must leave abler persons to seek, chusing now entirely to drop this subject. The solution of the problem enquired after in the paper I have sent you has, I think, been hitherto a *desideratum* in philosophy of some consequence. To this we are now in a great measure helped by the abilities and skill of our late worthy friend; and thus are furnished with a necessary guide in determining the nature and proportions of unknown causes from their effects, and an effectual guard against one great danger to which philosophers are subject; I mean, the danger of founding conclusions on an insufficient induction, and of receiving just conclusions with more assurance than the number of experiments will warrant. I am, under a sense of the value of your friendship, heartily yours,

Richard Price.



Art. 1. If the curve ADH be divided into two parts by the ordinate Dh making Ah to Hh as p is to q; then taking $a=\frac{p}{n}$ and $b=\frac{q}{n}$ the ratio of the Area ADh to HO will be $\frac{a\times b^q}{p+1}\times\left(1+\frac{q}{p+2}\times\frac{p}{q}+\frac{q\times(q-1)\times p^2}{(p+2)\times(p+3)\times q^2}+\frac{q\times(q-1)\times(q-2)\times p^3}{(p+2)\times(p+3)\times(p+4)\times q^3}\right)$+&c. For the series $\frac{x^{p+1}r^q}{p+1}+\frac{q}{p+1}\times\frac{x^{p+2}\times r^{q-1}}{p+2}$+&c. in Prop. 10. Art. 2 of the Essay, which expresses the ratio of ACf to HO, becomes this series when $x=a=\frac{p}{n}$, $b=r=\frac{q}{n}$; that is when Cf has moved till it coincides with Dh and ACf becomes ADh. In like manner, from Art. 3. in the Essay, it appears that the ratio of HDh to HO is $\frac{a^p b^{q+1}}{q+1}\times\left(1+\frac{p}{q+2}\times\frac{q}{p}+\frac{p}{q+2}\times\frac{p-1}{q+3}\times\frac{q^2}{p^2}\right)$+&c.

From hence it follows that the ratio of the difference between ADh and HDh to HO is $\frac{a^p b^q}{n}$ multiplied by the difference between the series $\frac{p}{p+1}+\frac{q}{p+1}\times\frac{p^2}{pq+2q}+\frac{q\times(q-1)\times p^3}{(p+1)\times(p+2)\times(pq^2+3q^2)}$+&c. and the series $\frac{q}{q+1}+\frac{p}{q+1}\times\frac{q^2}{pq+2p}+\frac{p\times(p-1)\times q^3}{(q+1)\times(q+2)\times(p^2q+3p^2)}$+&c. the former series being to be subtracted from the latter, if HDh is greater than ADh, and *vice versa*.

2. The ratio of any term in the former of the two foregoing series to that which next but one follows the correspondent term in the latter is $\frac{pq+p}{p\times q}\times\frac{pq+2p}{p\times q}\times\frac{p\times q}{qp+q}\times\frac{pq+3p}{pq-q}\times\frac{pq}{pq+2q}\times\frac{pq+4p}{pq-2q}\times\frac{pq-p}{pq+3q}\times\frac{pq+5p}{pq-3q}\times\frac{pq-2p}{pq+4q}\times\frac{pq+6p}{pq-4q}$ &c. taking twice as many terms and four over as there are units in the number which expresses the distance of the term in the former series from its first term; which ratio if q be greater than p, it is evident must be greater than the ratio of equality. Wherefore, if from the second series you subtract the two first terms which together are less than two, the remainder is less than the former series; and

of consequence, the former series subtracted from the latter cannot leave a remainder so great as two. And therefore in this case, that is, when q is greater than p, by the preceding article, the ratio of $HDh - ADh$ to HO cannot be so great as $\frac{2a^p b^q}{n}$.

3. The curve ADH being as before divided into two parts ADh and HDh, let the ordinates Cf and Ft be placed on each side of Dh and at the same distance from it, and let z be the ratio of hf or ht to AH. Then if y, x and r be respectively the ratios of Cf, Af and Hf to AH, by the nature of the curve $y = x^p r^q$. But because the ratio of Ah to AH is a, and that of hf to AH is z, the ratio of $Ah - hf$ $(= Af)$ to AH is $a - z$. Wherefore $a - z = x$. And in like manner $b + z = r$. But $y = x^p r^q$, and y is the ratio of Cf to AH. Wherefore the ratio of Cf to AH is $(a-z)^p \times (b+z)^q$. And in like manner the ratio of Ft to AH is $(a+z)^p \times (b-z)^q$ And consequently Cf is to Ft as $(a-z)^p \times (b+z)^q$ is to $(a+z)^p \times (b-z)^q$.

4. If q is greater than p, $(a+z)^p \times (b-z)^q$ is greater than $(a-z)^p \times (b+z)^q$, and the ratio between them increases as z increases. For the hyperbolic logarithm of that ratio taken as usual, and then instead of p and q putting na and nb because $a = \frac{p}{n}$ and $b = \frac{q}{n}$ (See Art. 1) you will find to be $2n$ multiplied by the series $\frac{b^2-a^2}{3b^2a^2} \times z^3 + \frac{b^4-a^4}{5b^4a^4} \times z^5 + \frac{b^6-a^6}{7b^6a^6} \times z^7$ &c. which logarithm when q is greater than p, and therefore b greater than a has all its terms positive, and so much the greater as z is greater; and therefore it is the logarithm of a ratio greater than that of equality, and which increases as z increases.

5. By Art. 3. Ft is to Cf as $(a+z)^p \times (b-z)^q$ is to $(a-z)^p \times (b+z)^q$. And by Art. 4. $(a+z)^p \times (b-z)^q$ is greater than $(a-z)^p \times (b+z)^q$, and the ratio between them increases as z increases, if q is greater than p. Wherefore, upon this supposition, also Ft is greater than Cf, and the ratio between them increases as z or ht and hf increases, and consequently this will be true also concerning the areas described by them as their equal abscissas ht and hf increase. Wherefore, when q is greater than p, $DhtF$ is greater than $DhfC$, and the ratio between them increases as $hf = ht$ increases.

6. Because Ah is to Hh as p is to q, when q is greater than p, Hh is greater than Ah. In Hh therefore taking hl equal to Ah, by the preceding Art. the part of the figure HDh which insists upon hl will be greater than ADh, and the ratio of that part of HDh to ADh will be greater than the ratio of $DhtF$ to $DhfC$. Consequently, much more (q being greater than p) the whole figure HDh is greater than ADh, and the ratio of HDh to ADh is greater than that of $DhtF$ to $DhtC$.

7. When q is greater than p, $(1 - \frac{n^2z^2}{pq})^{\frac{n}{2}}$ is greater than $(1 - \frac{nz}{p})^p \times (1 + \frac{nz}{q})^q$ and less than $(1 - \frac{nz}{q})^q \times (1 + \frac{nz}{p})^p$. For the fluxion of $(1 - \frac{n^2z^2}{pq})^{\frac{n}{2}}$ is $-\frac{n^3z\dot{z}}{pq} \times (1 - \frac{n^2z^2}{pq})^{\frac{n}{2}-1}$ and the fluxion of $(1 - \frac{nz}{p})^p \times (1 + \frac{nz}{q})^q$ (because $p + q = n$) is $-\frac{n^3z\dot{z}}{pq} \times (1 - \frac{nz}{p})^{p-1} \times (1 + \frac{nz}{q})^{q-1}$. Wherefore $(1 - \frac{n^2z^2}{pq})^{\frac{n}{2}}$ is to $(1 - \frac{nz}{p})^p \times (1 + \frac{nz}{q})^q$ as the fluxion of the former multiplied by $(1 - \frac{n^2z^2}{pq})$ to the fluxion of the latter multiplied by $((1 - \frac{nz}{p}) \times (1 + \frac{nz}{q})$ or) $1 - \frac{nz}{p} + \frac{nz}{q} - \frac{n^2z^2}{pq}$. From which analogy, because q is greater than p, it is plain that $(1 - \frac{nz}{p})^p \times (1 + \frac{nz}{q})^q$ varies at

a greater rate in respect of its own magnitude than $(1-\frac{n^2z^2}{pq})^{\frac{n}{2}}$ does. And, because their fluxions as found out before have negative sign before them, they both decrease as z increases; consequently, if they are ever equal, as z increases the latter must be the largest. But when $z=0$ they are each equal to 1. In like manner the other part of this article appears. And hence, since $a=\frac{p}{n}$ and $b=\frac{q}{n}$, it is manifest that $a^pb^q\times(1-\frac{n^2z^2}{pq})^{\frac{n}{2}}$ is greater than $(a-z)^p\times(b+z)^q$ and less than $(a+z)^p\times(b-z)^q$, when q is greater than p.

8. Suppose now further that the curve RQW be described meeting the lines Dh, Ft, ht produced in R, Q, W, in such manner that Ft, which is to Cf as $(a+z)^p\times(b-z)^q$ to $(a-z)^p\times(b+z)^q$ (Vid. Art. 3.) shall be to Qt as $(a+z)^p\times(b-z)^q$ to $a^pb^q\times(1-\frac{n^2z^2}{pq})^{\frac{n}{2}}$ wherever the points t and f fall at equal distances from h. And it is manifest by the foregoing Art. that Qt must always be greater than Cf, and less than Ft. And of consequence the same must be true concerning the areas described by their motion while their abscissas increase. Wherefore $RhtQ$ is greater than $DhfC$, and less than $DhtF$.

9. Since Ft is to Qt as $(a+z)^p\times(b-z)^q$ to $a^pb^q\times(1-\frac{n^2z^2}{pq})^{\frac{n}{2}}$; and $(a+z)^p\times(b-z)^q$ (by Art. 3.) expresses the ratio of Ft to AH; the ratio of Qt to AH must be $a^pb^q\times 1-\frac{n^2z^2}{pq})^{\frac{n}{2}}$, and as has been all along supposed z is the ratio of ht to AH. Wherefore, by squaring the curve $RhtQ$, it will appear that the ratio of $RhtQ$ to HO is $a^pb^q\times\big(z-\frac{n^3z^3}{2.3pq}+\frac{n-2}{4}\times\frac{n^5z^5}{2.5p^2q^2}-\frac{n-2}{4}\times\frac{n-4}{6}\times\frac{n^7z^7}{2.7p^3q^3}+\&c.\big)$ which (if $m=\frac{n^3}{2pq}$) is $a^pb^q\times\frac{\sqrt{2pq}}{n\sqrt{n}}\times\big(mz-\frac{m^3z^3}{3}+\frac{n-2}{2n}\times\frac{m^5z^5}{5}-\frac{n-2}{2n}\times\frac{n-4}{3n}\times\frac{m^7z^7}{7}+\frac{n-2}{2n}\times\frac{n-4}{3n}\times\frac{n-6}{4n}\times\frac{m^9z^9}{9}-\&c.\big)$ which last series when $\frac{n^2z^2}{pq}=1$, and consequently the ordinate Qt vanishes, becomes $B-\frac{B^3}{3}+\frac{B^2-1}{2B^2}\times\frac{B^5}{5}-\frac{B^2-1}{2B^2}\times\frac{B^2-2}{3B^2}\times\frac{B^7}{7}+\&c.$ taking $B^2=\frac{n}{2}$.

10. If $B^2=\frac{n}{2}$ the ratio of the whole figure $RQWh$ to HO is $\frac{\sqrt{2pq}}{n\sqrt{n}}\times a^pb^q\times\big(B-\frac{B^2-1}{2B^2}\times 5-\&c.$ Now, (by Prop. 10. Art. 4 of the Essay) the ratio of $ACFH$ to HO is $\frac{1}{n+1}\times\frac{1}{E}$, E being the coefficient of that term of the binomial $(a+b)^n$ expanded in which occurs a^pb^q. Wherefore, the ratio of $RQWh$ to $ACFH$ is $\frac{n+1}{n}\times\frac{\sqrt{2pq}}{\sqrt{n}}\times Ea^pb^q\times\big(B-\frac{B^3}{3}+\frac{B^2-1}{2B^2}\times\frac{B^5}{5}\&c.\big)$ Put G now for the coefficient of the middle term of the same binomial, and if $p=q=\frac{n}{2}$, $E=G$, $a=\frac{1}{2}=b$ the area $RRQWh$ is equal to half $ACFH$; for then Qt, Ft, Cf are all equal, and consequently the areas $RQWh$, HDh and ADh. Wherefore, the series $B-\frac{B^3}{3}+$ &c. is equal to $\frac{\sqrt{2n}}{n+1}\times\frac{2^{n-1}}{G}$. But the series $B-\frac{B^3}{3}+$ &c. (because $B^2=\frac{n}{2}$) does not alter whatever p and q are, whilst their sum n remains the same. Wherefore, in all cases, the ratio of $RQWh$ to $ACFH$ is $\frac{\sqrt{pq}}{n}\times\frac{Ea^pb^q}{G}\times 2^n$.

11. By Prop. 10. Art. 4. of the Essay, the ratio of $ACFH$ to HO* is $\frac{1}{n+1}\times\frac{1}{E}$; and by Art. 9. the ratio of $RhtQ$ to HO is $a^pb^q\times\frac{\sqrt{2pq}}{n\sqrt{n}}\times\big(mz-\frac{m^3z^3}{3}+\frac{n-2}{2n}\times\frac{m^5z^5}{5}\&c.\big)$. Wherefore, the ratio of $RhtQ$ to $ACFH$ is $\frac{n+1}{n}\times$

*It is hoped that the imperfection of the figure all along referred to will be excused. The lines Rh and Dh should appear equal; and it will be found presently, that the curve line $ACDFH$ should have been drawn from F and C convex towards AH.

$\frac{\sqrt{2pq}}{\sqrt{n}} \times Ea^pb^q \times \left(mz - \frac{m^3z^3}{3} + \frac{n-2}{2n} \times \frac{m^5z^5}{5} - \frac{n-2}{2n} \times \frac{n-4}{3n} \times \frac{m^7x^7}{7} + \&c.\right)$ Likewise, by Art. 10. the ratio of $RQWh$ to $ACFH$ is $\frac{\sqrt{2pq}}{n} \times \frac{Ea^pb^q}{G} \times 2^n$. Wherefore the ratio of $RhtQ$ to $RQWh$ is $\frac{n+1}{\sqrt{n}} \times \frac{\sqrt{2}}{2n} \times G \times \left(mz - \frac{m^3z^3}{3} + \&c.\right)$.

12. By Art. 2.6. When q is greater than p, the ratio of $HDh - ADh$ to HO is less than $\frac{2a^pb^q}{n}$. And by Prop. 10. Art. 4. of the Essay, the ratio of $ACFH$ or $HDh + ADh$ to HO is $\frac{1}{n+1} \times \frac{1}{E}$. Wherefore, the sum of these two ratios, or the ratio of $2HDh$ to HO, is less than $\frac{1}{n+1} \times \frac{1}{E} + \frac{2a^pb^q}{n}$; and the difference between them, or the ratio of $2ADh$ to HO is greater than $\frac{1}{n+1} \times \frac{1}{E} - \frac{2a^pb^q}{n}$. Wherefore, the ratio of $2HDh$ to $2ADh$, or that of HDh to ADh, is less than that of $\frac{1}{n+1} \times \frac{1}{E} + \frac{2a^pb^q}{n}$ to $\frac{1}{n+1} \times \frac{1}{E} - \frac{2a^pb^q}{n}$, which is equal to the ratio of $1 \times 2Ea^pb^q + \frac{2Ea^pb^q}{n}$ to $1 - 2Ea^pb^q - \frac{2Ea^pb^q}{n}$. But the ratio of HDh to ADh, by Art. 6. is greater than the ratio of $DhtF$ to $DhfC$, when q is greater than p. Wherefore, much more when q is greater than p, the ratio of $DhtF$ to $DhfC$ will be less than that of $1 + 2Ea^pb^q + \frac{2Ea^pb^q}{n}$ to $1 - 2Ea^pb^q - \frac{2Ea^pb^q}{n}$. And because, by Art. 8. $RhtF$ is a mean between $DhtF$ and $DhfC$, the ratio of $DhtF$ to $RhtQ$ will be less than that of $1 + 2Ea^pb^q + \frac{2Ea^pb^q}{n}$ to $1 - 2Ea^pb^q - \frac{2Ea^pb^q}{n}$. And the ratio of $DhfC$ to $RhtQ$ will be greater than that of $1 - 2Ea^pb^q - \frac{2Ea^pb^q}{n}$ to $1 + 2Ea^pb^q + \frac{2Ea^pb^q}{n}$.

RULE II.

If nothing is known of an event but that it has happened p times and failed q in $p+q$ or n trials, and q be greater than p; and from hence I judge that the probability of its happening in a single trial lies between $\frac{p}{n}$ and $\frac{p}{n} + z$, (if $m^2 = \frac{n^3}{2pq}$, $a = \frac{p}{n}$, $b = \frac{q}{n}$, E the coefficient of the term in which occurs a^pb^q when $(a+b)^n$ is expanded, and $\Sigma = \frac{n+1}{n} \times \frac{\sqrt{2pq}}{\sqrt{n}} \times Ea^pb^q \times \left(mz - \frac{m^3z^3}{3} + \frac{n-2}{2n} \times \frac{m^5z^5}{5} - \frac{n-2}{2n} \times \frac{n-4}{3n} \times m^5z^57 + \&c.\right)$ my chance to be in the right is greater than Σ, and less than $\Sigma \times \frac{1+2Ea^pb^q+\frac{2Ea^pb^q}{n}}{1-2Ea^pb^q-\frac{2Ea^pb^q}{n}}$. For by Art. 11. compared with the value of Σ here set down, the ratio of $RhtQ$ to $ACFH$ is Σ. But by Art. 8. $DhtF$ is greater than $RhtQ$, and by Art. 12. the ratio of $DhtF$ to $RhtQ$ is less than that of $1 + 2Ea^pb^q + \frac{2Ea^pb^q}{n}$ to $1 - 2Ea^pb^q - \frac{2Ea^pb^q}{n}$. From whence it is plain that the ratio of $DhtF$ to $ACFH$ is greater than Σ, and less than $\Sigma \times \frac{1+2Ea^pb^q+\frac{2Ea^pb^q}{n}}{1-2Ea^pb^q-\frac{2Ea^pb^q}{n}}$. But, as appears from the 10th Proposition in the Essay, the chance that the probability of the event lies between $\frac{p}{n}$ and $\frac{p}{n} + z$ (that is, between the ratio of Ah to AH, and that of At to AH) is the ratio of $DhtF$ to $ACFH$. Wherefore, the chance I am right in my guess is greater than Σ and less than $\Sigma \times \frac{1+2Ea^pb^q+\frac{2Ea^pb^q}{n}}{1-2Ea^pb^q-\frac{2Ea^pb^q}{n}}$.

In like manner, 2dly, the same things proposed, if I judge that the probability of the event lies between $\frac{p}{n}$ and $\frac{p}{n} - z$, my chance to be right is less than Σ, and greater than $\Sigma \times \frac{1-2Ea^pb^q-\frac{2Ea^pb^q}{n}}{1+2Ea^pb^q+\frac{2Ea^pb^q}{n}}$. This is manifest as the other case, because $DhfC$ is less than $RhtQ$, but the ratio between them is greater than that of $1 - 2Ea^pb^q - \frac{2Ea^pb^q}{n}$ to $1 + 2Ea^pb^q + \frac{2Ea^pb^q}{n}$.

3dly, If, other things supposed as before, p is greater than q, and I judge the probability of the event lies between $\frac{p}{n}$ and $\frac{p}{n}+z$, my chance to be right is less than Σ, and greater than $\Sigma\times\frac{1-2Ea^pb^q-\frac{2Ea^pb^q}{n}}{1+2Ea^pb^q+\frac{2Ea^pb^q}{n}}$. But if I judge it lies between $\frac{p}{n}$ and $\frac{p}{n}-z$, my chance is greater than Σ, and less than $\Sigma\times\frac{1+2Ea^pb^q+\frac{2Ea^pb^q}{n}}{1-2Ea^pb^q-\frac{2Ea^pb^q}{n}}$. And if $p=q$, which ever of these ways I guess, my chance, my chance is Σ exactly. This may be proved in the same manner with the foregoing cases, where q is greater than p, or may be proved from them by considering the happening and failing of an event, as the same with the failing and happening of its contrary.

4thly, Other things supposed the same, whether q be greater or less than p, and I judge the probability of the event lies between $\frac{p}{n}+z$ and $\frac{p}{n}-z$, my chance is greater than $\frac{2\Sigma}{1+2Ea^pb^q+\frac{2Ea^pb^q}{n}}$, and less than $\frac{2\Sigma}{1-2Ea^pb^q-\frac{2Ea^pb^q}{n}}$. This is an evident corollary from the cases already determined. And here, if $p=q$, my chance is 2Σ exactly.

Thus far I have transcribed Mr. Bayes.

It appears, from the Appendix to the Essay, that the rule here demonstrated, though of great use, does not give the required chance within limits sufficiently narrow. It is therefore necessary to look out for a contradiction of these limits; and this, I think, we shall discover by the help of the following deductions; which, for the sake of greater distinctness, I shall give as a continuation of the foregoing Articles.

13. The ratio of the fluxion of $(1-\frac{n^2z^2}{pq})^{\frac{n}{2}}$ to the fluxion of $(1+\frac{nz}{p})^p\times(1-\frac{nz}{q})^q$ is

$$\frac{(1-\frac{n^2z^2}{pq})^{\frac{n}{2}-1}}{(1+\frac{nz}{p})^{p-1}\times(1-\frac{nz}{q})^{q-1}}$$

and the ratio of the fluxion of $(1-\frac{nz}{p})^p\times(1+\frac{nz}{q})^q$ to the fluxion of $(1-\frac{n^2z^2}{pq})^{\frac{n}{2}}$ is

$$\frac{(1-\frac{nz}{p})^{p-1}\times(1+\frac{nz}{q})^{q-1}}{(1-\frac{n^2z^2}{pq})^{\frac{n}{2}-1}}$$

This will immediately appear from Art. 7.

14. While z is increasing from nothing till $\frac{n^2z^2}{pq}$ becomes equal to unity, these two ratios are at first greater than the ratio of equality, and increase as z increases, till they come to a *maximum*. Afterwards they decrease untill they become first equal to the ratio of equality, and then less. This is proved by finding the hyperbolic logarithms of these ratios, the hyperbolic logarithm of the first is the series $\left(\frac{q-1}{q}-\frac{p-1}{p}\right)\times nz+\left(\frac{q-1}{q^2}+\frac{p-1}{p^2}-\frac{n-2}{pq}\right)\times\frac{n^2z^2}{2}+\left(\frac{q-1}{q^3}-\frac{p-1}{p^3}\right)\times\frac{n^3z^3}{3}+\left(\frac{q-1}{q^4}+\frac{p-1}{p^4}-\frac{n-2}{p^2q^2}\right)\times\frac{n^4z^4}{4}+\left(\frac{q-1}{q^5}-\frac{p-1}{p^5}\right)\times\frac{n^5z^5}{5}+\left(\frac{q-1}{q^6}+\frac{p-1}{p^6}-\frac{n-2}{p^3q^3}\right)\times\frac{n^6z^6}{6}+$&c.. The hyperbolic logarithm of the second ratio is the series $\left(\frac{q-1}{q}-\frac{p-1}{p}\right)\times nz-\left(\frac{q-1}{q^2}+\frac{p-1}{p^2}-\frac{n-2}{pq}\right)\times\frac{n^2z^2}{2}+\left(\frac{q-1}{q^3}-\frac{p-1}{p^3}\right)\times\frac{n^3z^3}{3}-\left(\frac{q-1}{q^4}+\frac{p-1}{p^4}-\frac{n-2}{p^2q^2}\right)\times\frac{n^4z^4}{4}+$&c. It will appear from examining these two serieses (q all along supposed greater than p) that while z is small the value of each of them is positive, and increases

as z increases till it becomes a *maximum*, after which it decreases till it becomes nothing, and after that negative; which demonstrates this article.

15. The former of the two ratios in Art. 13. (q being greater than p) is at first, while z is increasing from nothing, less than the second ratio; and does not become equal to it, till some time after both ratios have been the greatest possible.

Upon considering the two serieses in the last Art. it will appear that the first term of the first series is always positive, the second negative, the third also negative, after which the terms become alternately positive and negative. On the other hand, it will appear that in the second series the two first terms are always positive, and all that follow negative. But as the serieses converge very fast when z is small, the second term being negative in the first series and positive in the second, has a greater effect in making the first series less than the second, than can be compensated for by the terms being afterwards alternately negative and positive in the one, and all negative in the other. It will further appear from considering the two serieses, that the first must continue less than the second 'till z becomes so large as to make the fourth term equal to the second, in which circumstances the two serieses are nearly equal. Afterwards, as z goes on to increase, the value of both lessens continually; but the second now decreasing fastest, as before it increased fastest, becomes first nothing. After which, the other series becomes nothing; and after that both remain negative. From hence it is easy to infer this Article.

16. What has now been shewn of the ratio of the fluxion of $(1-\frac{n^2z^2}{pq})^{\frac{n}{2}}$ to the fluxion of $(1+\frac{nz}{p})^p \times (1-\frac{nz}{q})^q$ compared with the ratio of the fluxion of $(1-\frac{nz}{p})^p \times (1+\frac{nz}{q})^q$ to the fluxion of $(1-\frac{n^2z^2}{pq})^{\frac{n}{2}}$ is also true of the ratio of the fluxion of $a^pb^q \times (1-\frac{n^2z^2}{pq})^{\frac{n}{2}}$ (or Qt in the figure) to the fluxion of $(a+z)^p \times (b-z)^q$ (or Ft) compared to the ratio of the fluxion of $(a-z)^p \times (b+z)^q$ (or Cf) to the fluxion of $a^pb^q \times (1-\frac{n^2z^2}{pq})^{\frac{n}{2}}$ or Qt; the latter quantities being only the former multiplied by the common and permanent quantity a^pb^q. It appears, therefore, that if we conceive Ft, Qt, Cf (Vid. Fig.) to move with equal and uniform velocities, from Dh to Rh along AH, in order to generate the areas HDh, RWh, ADh; Cf will at first not only decrease faster than Qt, and Qt than Ft; but the ratio of the rate at which Cf decreases to the rate at which Qt decreases, will be greater than the ratio of the rate at which Qt decreases to the rate at which Ft decreases. It appears also that after some time, first Cf and Qt, and then Qt and Ft will come to decrease equally; after which, Qt will decrease faster than Cf, and Ft faster than Qt.

17. The curves DFH, RQW, DCA, have each of them a point of contrary flexure; and the value of z, or of the equal abscissas at that point, is in all three $\frac{\sqrt{pq}}{\sqrt{n^3-n^2}}$. This may be found in the common manner, by putting the second fluxions of the ordinates equal to nothing. In the single case, when either p or q is equal to unity, one of these points vanishes, or coincides with A or H.

18. At the points of contrary flexure (q being greater than p) the ratio of the fluxion of Qt to the fluxion of Ft is a *maximum*; and the same is true of

the ratio of the fluxion of Cf to the fluxion of Qt. This is found by making the fluxions of the logaritms of these ratios, or of

$$\frac{(1-\frac{n^2z^2}{pq})^{\frac{n}{2}-1}}{(1+\frac{nz}{p})^{p-1}\times(1-\frac{nz}{q})^{q-1}}, \quad \text{and} \quad \frac{(1-\frac{nz}{p})^{p-1}\times(1+\frac{nz}{q})^{q-1}}{(1-\frac{n^2z^2}{pq})^{\frac{n}{2}-1}}$$

equal to nothing: which will give the value of z equal to $\frac{\sqrt{pq}}{\sqrt{n^3-n^2}}$, or the same with the value of z at the points of contrary flexure.

19. At the points of contrary flexure, the ratio of the fluxion of Cf to the fluxion of Qt, is greatest in comparison of the ratio of the fluxion of Qt to the fluxion of Ft. This is proved by finding the value of z when the fluxion of the former ratio divided by the latter, or of

$$\frac{(1-\frac{n^2z^2}{p})^{p-1}\times(1+\frac{n^2z^2}{q})^{q-1}}{(1-\frac{n^2z^2}{pq})^{n-2}}$$

is nothing, which will still give $z=\frac{\sqrt{pq}}{\sqrt{n^3-n^2}}$. The reason, therefore, in the nature of the curve, which, as the ordinates flow, keeps at first the excess of Ft above Qt less than the excess of Qt above Cf, operates with the greatest force at the points of contrary flexure.

20. The greatest part of the area $RQWh$ lies between Rh, and the ordinate at the point of contrary flexure. By Art. 11 the ratio of $RhtQ$ to $RQWh$ is $\frac{n+1}{\sqrt{n}}\times\frac{\sqrt{2}}{2^n}\times G\times\big(mz-\frac{m^3z^3}{3}+\frac{n-2}{2n}\times\frac{m^5z^5}{5}-$ &c. Substitute here $\sqrt{\frac{pq}{n^3-n^2}}$ for z, and $\frac{2^n}{\sqrt{nK}\times H}$[†] for G (K being the ratio of the quadrantal arc to radius, and H the ratio whose hyperbolic logarithm is $\frac{3}{12n}-\frac{15}{360n^3}+\frac{63}{1260n^5}$ &c.[‡]) and the ratio of $RhtQ$ to $RQWh$ at the point of contrary flexure, will be $\frac{n+1}{\sqrt{n}\times\sqrt{n-1}}\times\frac{.797884}{H}\times\big(1-\frac{n}{2.3.(n-1)}+\frac{n\times(n-2)}{2.5.4(n-1)^2}-\frac{n(n-2)(n-4)}{2.3.7.8(n-1)^3}+\frac{n(n-2)(n-4)(n-6)}{2.3.4.9.16(n-1)^4}-$ &c. Now when n is little, the value of this expression will be considerably greater than .6822. It approaches to this continually as n increases; and when n is large, it may be taken for this exactly. Thus when $n=6$, this expression is equal to .804. When $n=110$, it is equal to .6903. If we would know the ratio of $RhtQ$ to $RQWh$, when Cf comes to decrease no faster in respect of Qt, than Qt decreases in respect of Ft; that is, when the excess of Qt above Cf, is greatest in comparison of the excess of Ft above Qt, it may be found (by putting the fourth term of the series in the 14$^{\text{th}}$ Art. equal to the second term, and then finding the value of z) to be about .8426, when n, p, and q are considerable; and in other cases greater.

Coroll. 'Tis easy to gather from hence that in like manner the greatest part of the area ADH lies between the two ordinates at the points of contrary

[†] This is always the true value of G; but it would be too tedious to give the demonstration of this here.

[‡] Vid. the Second Rule in the Essay, Phil. Trans. Vol. LIII.

flexure.[§]

21. $RhtQ$ is greater than the arithmetical mean between $DhtF$ and $DhfC$. This appears from the latter part of Art. 19. for what is there proved of the ordinates must hold true of the contemporary areas generated by them. And though beyond the points at which the ratio of the decrease of Qt to the decrease of Ft comes to an equality with the ratio of the decrease of Qt to the decrease of Cf, the excess of Ft above Qt begins to grow larger than before in respect of the excess of Qt above Cf; yet as it appears from the last article, that above five sixths of the areas $RQWh$ and $ACFH$ are generated before the ordinates come to these points, and as also beyond these points the said ratios, 'till they become negative and for some time afterwards, are but small; the effect produced before towards rendering the excess of $DhtF$ above $RbtQ$ always less than the excess of $RhtQ$ above $DhfC$, will be such as cannot be compensated for afterwards.

A further proof of this will appear from considering that even when $RhtQ$ is increased to $RQWh$, it is but little short of the arithmetical mean between ADh and HDh. For from Art. 11. and 20. it may be inferred that the ratio of the whole area $RQWh$ to this mean, or to $\frac{ACFH}{2}$, is $h \times H$, which is never far from the ratio of equality, but when both p and q are of any considerable magnitude, it is very nearly the ratio of equality. For example; when $n = 110$, $q = 100$, $p = 10$, it is .9938.

22. The ratio of $DhtF$ to $RhtQ$ is less than that of $1 + 2Ea^pb^q + \frac{2Ea^pb^q}{n}$ to one. For by Art. 12. the ratio of $DhtF$ to $DhfC$ is less than that of $1 + 2Ea^pb^q + \frac{2Ea^pb^q}{n}$ to $1 - 2Ea^pb^q - \frac{2Ea^pb^q}{n}$. But by the last Art. $RhtQ$ is greater than the arithmetical mean between $DhtF$ and $DhfC$, and 1 is exactly the arithmetical mean between $1 + 2Ea^pb^q + \frac{2Ea^pb^q}{n}$ and $1 - 2Ea^pb^q - \frac{2Ea^pb^q}{n}$. From whence this Article is plain.

23. The ratio of $DhtF$ to $ACFH$ is greater than Σ, and less than $\Sigma \times \left(1 + 2Ea^pb^q + \frac{2Ea^pb^q}{n}\right)$. For $DhtF$ being greater than $RhtQ$, the ratio of it to $ACFH$ must be greater than the ratio of $RhtQ$ to $ACFH$, or greater than Σ. Also; since the ratio of $RhtQ$ to $ACFH$ is equal to Σ; the ratio of $DhtF$ to $RhtQ$ is less than the ratio of $1 + 2Ea^pb^q + \frac{2Ea^pb^q}{n}$ to 1; it follows that the ratio compounded of the ratio of $RhtQ$ to $ACFH$, and of $DhtF$ to $RhtQ$, that is, the ratio of $DhtF$ to $ACFH$ must be less than $\Sigma \times \left(1 + 2Ea^pb^q + \frac{2Ea^pb^q}{n}\right)$.

24. The ratio of $DhtF + DhfC$ to $ACFH$ (that is, the chance for being right in judging that the probability of an event perfectly unknown, which has

[§]From this Article may be inferred a method of finding at once, without any labour, whereabouts it is reasonable to judge the probability of an unknown event lies, about which a given number of experiments have been made. For when neither p nor q are very small, or even not less than 10, it will be nearly an equal chance, that the probability of the event lies between $\frac{p}{n} + \frac{\sqrt{pq}}{\sqrt{2n^3-2n^2}}$ and $\frac{p}{n} - \frac{\sqrt{pq}}{\sqrt{2n^3-2n^2}}$. It will be the odds of two to one that it lies between $\frac{p}{n} + \frac{\sqrt{pq}}{\sqrt{n^3-n^2}}$ and $\frac{p}{n} - \frac{\sqrt{pq}}{\sqrt{n^3-n^2}}$; and the odds of five to one that it lies between $\frac{p}{n} + \frac{\sqrt{2pq}}{\sqrt{n^3-n^2}}$ and $\frac{p}{n} - \frac{\sqrt{2pq}}{\sqrt{n^3-n^2}}$. For instance; when $p = 1000$, $q = 100$, there will be nearly an equal chance, that the probability of the event lies between $\frac{10}{11} + \frac{1}{163}$ and $\frac{10}{11} - \frac{1}{163}$; two to one that it lies between $\frac{10}{11} + \frac{1}{115}$ and $\frac{10}{11} - \frac{1}{115}$; and five to one that it lies between $\frac{10}{11} + \frac{1}{81}$ and $\frac{10}{11} - \frac{1}{81}$.

happened p and failed q times in $p+q$ or n trials, lies somewhere between $\frac{p}{n}+z$ and $\frac{p}{n}-z$) is greater than $\frac{2\Sigma}{1+2Ea^pb^q+\frac{2Ea^pb^q}{n}}$, and less than 2Σ. The former part of this Art. has already been proved, Art. 12. The latter part is evident from Art. 21. For $RhtQ$ being greater than the arithmetical mean between $DhtF$ and $DhfC$, $2RhtQ$ must be greater than $DhtF+DhfC$; and consequently the ratio of $2RhtQ$ to $ACFH$, greater than the ratio of $DhtF+DhfC$ to $ACFH$.¶

It will be easily seen that this Article improves considerably the rule given in Art. 12. But we may determine within still narrower limits whereabouts the required chance must lie, as will appear from the following Articles.

25. If c and d stand for any two fractions, whenever the fluxion of $c\times FT$ is greater than the fluxion of $d\times Cf$ (Vid. fig.) $c\times Ft+d\times Cf$ will be greater than Qt. For in the same manner with Art. 6. it will appear that $c\times Ft+d\times Cf$ is to Qt, as the fluxion of $c\times Ft\times(1+\frac{nz}{p})\times(1-\frac{nz}{q})$ together with the fluxion of $d\times Cf\times(1-\frac{nz}{p})\times(1+\frac{nz}{q})$ to the fluxion of $Qt\times(1-\frac{n^2z^2}{pq})$. Now since $1-\frac{n^2z^2}{pq}$ is the arithmetical mean between $(1+\frac{nz}{p})\times(1-\frac{nz}{q})$ and $(1-\frac{nz}{p})\times(1+\frac{nz}{q})$, it is plain, that were the fluxion of $c\times Ft$ equal to the fluxion of $d\times Cf$, $c\times Ft+d\times Cf$ would decrease in respect of its own magnitude at the same rate with Qt; and, therefore, since at first equal, they would always continue equal. But the fluxion of $c\times Ft$ being greater than the fluxion of $d\times Cf$ by supposition, and (since q greater than) $p(1+\frac{nz}{p})\times(1-\frac{nz}{q})$, also greater than $(1-\frac{nz}{p})\times(1+\frac{nz}{q})$, it follows that the fluxion of $c\times Ft\times(1+\frac{nz}{p})\times(1-\frac{nz}{q})$ added to the fluxion of $d\times Cf\times(1-\frac{nz}{p})\times(1+\frac{nz}{q})$ is greater than these two fluxions multiplied by $(1-\frac{n^2z^2}{pq})$; and, therefore, greater, than the fluxion of $Qt\times(1-\frac{n^2z^2}{pq})$; and, therefore, $c\times Ft+d\times Cf$ greater than Qt.

26. If we suppose three continued arithmetical means between Cf and Ft $\left(\frac{3Cf+Ft}{4},\frac{Cf+Ft}{2},\frac{3Ft+Cf}{4}\right)$ Qt will be greater than the second, and less than the third, if p is greater than 1. That Qt will be greater than the second has been already proved; and that it will be less than the third, will be an immediate consequence from the last Article, if it can be shewn that the fluxion of $\frac{3Ft}{4}$ is greater than the fluxion of $\frac{Cf}{4}$. This will appear in the following manner. The ratio of the fluxion of Cf to the fluxion of Ft is by Art. 7. and 14.

$$\frac{(1-\frac{nz}{p})^{p-1}\times(1+\frac{nz}{q})^{q-1}}{1+\frac{nz}{p}^{p-1}\times(1-\frac{nz}{q})^{q-1}}.$$

The hyperbolic logarithm of this ratio is $(\frac{1}{p}-\frac{1}{q})\times 2nz-(\frac{1}{p^2}-\frac{1}{p^3}-\frac{1}{q^2}+\frac{1}{q^3})\times\frac{2n^3z^3}{3}-(\frac{1}{p^4}-\frac{1}{p^5}-\frac{1}{q^4}+\frac{1}{q^5})\times\frac{2n^5z^5}{5}$, &c. This ratio by Art. 18. is greatest at the point of contrary flexure, or when $z=\frac{\sqrt{pq}}{\sqrt{n^3-n^2}}$. Substitute this for z in the series, and it will become $(\frac{1}{p}-\frac{1}{q})\times\frac{2\sqrt{pq}}{\sqrt{n-1}}-(\frac{1}{p^2}-\frac{1}{p^3}-\frac{1}{q^2}+\frac{1}{q^3})\times\frac{2p^{\frac{3}{2}}\times q^{\frac{3}{2}}}{3\times(n-1)^{\frac{3}{2}}}-$&c. which, therefore, expresses the logarithm of the ratio when greatest, and will easily discover it in every case. 'Tis apparent that the value of this series is

¶This Art. is true, whether p be greater or less than q.

greatest when p is least in respect of q. Suppose then $p = 2$, and q infinite. In this case, the value of the series will be 1.072, and the number answering to this logarithm is not greater than 2.92. The fluxion, therefore, of Cf, when greatest, cannot be three times the contemporary fluxion of Ft; from whence it follows that the fluxion of $\frac{3Ft}{4}$ must be greater than the fluxion of $\frac{Cf}{4}$.

It is easy to see how these demonstrations are to be varied when q is less than p, and how in this case similar conclusions can be drawn. Or, the same conclusions will in this case immediately appear, by changing p into q and q into p, which will not make any difference in the demonstrations.

In the manner specified in this Article we may always find within certain limits how near the value of Qt comes to the arithmetical mean between Ft and Cf, which limits grow narrower and narrower, as p and q are taken larger, or their ratio comes nearer to that of equality, 'till at last, when p and q are either very great or equal, Qt coincides with this mean. Thus, if either p or q is not less than 10; that is, in all cases, where it is not practicable without great difficulty to find the required chance exactly by the first rule, Qt will be greater than the fourth, and less than the fifth of seven arithmetical means between Cf and Ft.

27. The arithmetical means mentioned in the last Article may be conceived as ordinates describing areas at the same time with Qt; and what has been proved concerning them is true also of the areas described by them compared with $RhtQ$.

28. If either p or q is greater than 1, the true chance that the probability of an unknown event which has happened p times and failed q in $(p+q)$ or n trials, should lie somewhere between $\frac{p}{n} + z$ and $\frac{p}{n} - z$ is less than 2Σ, and greaterer than

$$\Sigma + \frac{\Sigma \times (1 - 2Ea^p b^q - \frac{2Ea^p b^q}{n})}{1 + Ea^p b^q + \frac{Ea^p b^q}{n}}.$$

If either p or q is greater than 10, this chance is less than 2Σ, and greater than

$$\Sigma + \frac{\Sigma \times (1 - 2Ea^p b^q - \frac{2Ea^p b^q}{n})}{1 + \frac{1}{2}Ea^p b^q + \frac{2Ea^p b^q}{2n}}.$$

This is easily proved in the same manner with Art. 12, 23, 24.

That it may appear how far what has been now demonstrated improves the solution of the present problem, let us take the fifth case mentioned in the Appendix to the Essay, and enquire what reason there is for judging that the probability of an event concerning which nothing is known, but that it has happened 100 times and failed 1000 times in 1100 trials, lies between $\frac{10}{11} + \frac{1}{110}$ and $\frac{10}{11} - \frac{1}{110}$. The second rule as given in Art. 12. informs us, that the chance‖ for this must lie between .6512, (or the odds of 186 to 100) and .7700 (or the odds of 334 to 100). but from this last Art. it will appear that the required

‖In the Appendix, this chance, as discovered by Mr. Bayes's second rule, is given wrong, in consequence of making m^2 equal to $\frac{n^3}{pq}$, whereas it should have been taken equal to $\frac{n^3}{2pq}$ as appears from Article 8.

chance in this case must lie between 2Σ, and

$$\Sigma + \Sigma \times \frac{1 - Ea^p b^q - \frac{2Ea^p b^q}{n}}{1 + \frac{1}{10} Ea^p b^q + \frac{Ea^p b^q}{10n}};$$

or, between .6748 and .7057; that is, between the odds of 239 to 100, and 207 to 100.

In all cases when z is small, and also whenever the disparity between p and q is not great 2Σ is almost exactly the true chance required. And I have reason to think, that even in all other cases, 2Σ gives the true chance nearer than within the limits now determined. But not to pursue this subject any further; I shall only add that the value of 2Σ may be always calculated very nearly, and without great difficulty; for the approximations to the value of $Ea^p b^q$, and of the series $m - z\frac{m^3 z^3}{3} + \frac{n-2}{2n} \times \frac{m^5 z^5}{5}$, &c.** given in the Essay, are sufficiently accurate in all cases where it is necessary to use them.

**In the expression for this last approximation there is an error of the press which should be corrected; for the sign before the fourth term should be − and not +.