SUR UNE APPLICATION CURIEUSE

DE

L'ANALYSE DES PROBABILITÉS

A LA DÉTERMINATION APPROXIMATIVE DES LIMITES DE LA PERTE RÉELLE EN HOMMES QU'ÉPROUVE UN CORPS D'ARMÉE PENDANT UN COMBAT*

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The object of this writing is to give some proper formulas to determine by approximation the number, or rather the rather tightened limits of the number of men, killed or wounded, for any period of a battle, and consequently before it is possible to have this number exactly. The means that we propose for this consists in taking at random in each regiment, battalion, squadron or company which must take part in the action, a certain number of individuals, who, during the combat, will each occupy their respective places. In order to equalize the chances as much as possible, it will be necessary, as it is fit, to take some precautions, as for example to arrange in such a way the number of men who one chooses in each branch is sensibly proportional to the total of that same kind of troups, that beyond these chosen men are distributed very nearly equally in each of the three files (lines of soldiers), seeing that they are not all three exposed in the same manner to the fire of the enemy. In a word, the more one will equalize the chances, the more the result obtained will be worthy of confidence. By taking the necessary measures, one will be likewise, during combat, to know how many, out of the chosen number of soldiers, there has been already dead and wounded of them. A simple rule of three will give then the probable number of the total of deaths and of wounded. Actually, if, departing from this probable number, one is given some limits more and less, one will be able to find the probability that the real number of men put disabled is comprehended within these limits. Now, if it happens that the probability that one attains for some rather narrow limits is sufficiently great, one will have an approximation which will be able perhaps to become useful. At the end of this Memoir we will consign some numerical results which set into evidence the degree of this approximation and the confidence that it merits.

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We ourselves will abstain from proposing some means in order to put into practice by our method; on this object, it is necessary to yield to the sagacity of the men of the art. It will be also to them to decide to what point the process in question is able to be put to profit. But we believe we must anticipate next that the grave inconvenience of the numerical calculation of the formulas, rather complicated by the nature itself of the problem, is able to be very easily isolated. For this there will be only to construct in advance a table which, at first inspection, will furnish the desired result. We will give, at the end of the Memoir, all the necessary details relative to the construction and to the form which it would be advantageous to give to a table of this kind.

The question that we ourselves propose to resolve analytically consists therefore in determining the probability that the loss in men not surpass certain limits, fixed in advance, in the same way the extent of these limits for a probability of which one will be acceptable to a *minimum*. Moreover, it will be proceeded by discussing with care, what must be the approximate number of men to choose out of the totality of combatants, in order to obtain rather precise results in practice.

Let N be the total of men who must take part in the action, and n the number of those who have been nominally chosen out of this total. At one determined epoch of combat one observes that, out of this number n, there are found i of them dead or wounded. In order to shorten the discourse we agree to designate these i individuals by the common denomination of men put *disabled*; there will remain therefore, out of this number n, $n-i$ combatants. This put, the different hypotheses that one will be able to make after the event observed on the total number of individuals put disabled and out of those and out of those who remain in the ranks, and who we will call *combatants*, will be the following

Hypotheses:	*Disabled:*	*Combatants:*
1^{st}	i	$N-i$
2^{nd}	$i+1$	$N-i-1$
3^{rd}	$i+2$	$N-i-2$
...	...	...
$(N-n+1)^{\text{st}}$	$i+N-n$	$n-i$

If one designates by x the probability of the soldier to have been put disabled for the time elapsed from the commencement of the action to the moment that one considers, $1-x$ will represent that of the contrary event. The values of x and of $1-x$, corresponding to the different hypotheses, will be:

Hypotheses:	*Disabled:*	*Combatants:*
1^{st}	$\frac{i}{N}$	$\frac{N-i}{N}$
2^{nd}	$\frac{i+1}{N}$	$\frac{N-i-1}{N}$
3^{rd}	$\frac{i+2}{N}$	$\frac{N-i-2}{N}$
...	...	...
$(N-n+1)^{\text{st}}$	$\frac{i+N-n}{N}$	$\frac{n-i}{N}$

We designate by P the probability *à priori* of the event observed; one will have

$$P = \frac{1.2.3\dots n}{1.2.3\dots i.1.2.3\dots(n-i)} \cdot x^i(1-x)^{n-i}.$$

By substituting successively into this formula the values of x relative to the different hypotheses, one will obtain the corresponding values of the probability of the observed event. Let P_μ be the value of P for the μ-th hypothesis. One will have

$$P_\mu = \frac{1.2.3\ldots n}{1.2.3\ldots i.1.2.3\ldots(n-i)} \cdot \frac{(i+\mu-1)^i(N-i-\mu+1)^{n-i}}{N^n}.$$

We observe now that the probable number of individuals put disabled being equal, by the theorem of Jakob Bernoulli, in the fourth term of the proportion $n : i = N : k$, on will have $k = \frac{Ni}{n}$. If $\frac{Ni}{n}$ is not whole, we take for k the integer contained in the fraction $\frac{Ni}{n}$. This put, we seek the probability that the real number of individuals put disabled, will be comprehended, inclusively, between the limits $k-\omega$ and $k+\omega$, ω designating an integer more or less great. In order to have this probability, which we will designate by p, we make use of the principle concerning the probability of the hypothesis. If one represents by Q_μ the probability of the μ-th hypothesis, one will have by this principle

$$Q_\mu = \frac{P_\mu}{P_1+P_2+P_3+\cdots+P_{N-n+1}}$$

We pass now to the determination of the probability p of the existence of any one of the hypotheses for which the total number of individuals put disabled is comprehended between the limits $k-\omega$ and $k+\omega$ inclusively. For this we will observe that to the numbers of individuals

$$k-\omega, \qquad k, \qquad k+\omega$$

correspond the hypotheses

$$(k-\omega-i+1)\text{-st}, \quad (k-i+1)\text{-st}, \quad (k+\omega-i+1)\text{-st},$$

so that if, in order to shorten the formulas, one supposes

$$k-\omega-i+1 = \omega_0 \text{ and } k+\omega-i+1 = \Omega,$$

one will have by the principles of the Calculus of Probabilities

$$p = Q_{\omega_0} + Q_{\omega_0+1} + Q_{\omega_0+2} + \cdots + Q_\Omega,$$

or else

$$p = \frac{P_{\omega_0} + P_{\omega_0+1} + P_{\omega_0+2} + \cdots + P_\Omega}{P_1+P_2+P_3+\cdots+P_{N-n+1}}$$

Let x' and x'' be the values of x corresponding to the hypotheses ω_0 and Ω; one will have

$$x' = \frac{k-\omega}{N} \text{ and } x'' = \frac{k+\omega}{N}.$$

Likewise, we represent by x_0 and X the values of x which correspond to the first and to the last hypotheses; consequently

$$x_0 = \frac{i}{N} \text{ and } X = \frac{i+N-n}{N}.$$

This put, by virtue of the formula which determines P_μ, one will be able to give to the preceding value of p the following form:

$$p = \frac{\overset{x=x''}{\underset{x=x'}{S}} x^i(1-x)^{n-i}}{\overset{x=X}{\underset{x=x_0}{S}} x^i(1-x)^{n-i}} \tag{1}$$

the numbers k, x', x'', x_0 and X being determined by the equations

$$k = \frac{Ni}{n}, \quad x' = \frac{k-\omega}{N}, \quad x'' = \frac{k+\omega}{N}, \quad x_0 = \frac{i}{N}, \quad X = \frac{i+N-n}{N}. \tag{2}$$

Thus, the ratio of these two sums, each taken inclusively between the limits which have just been designated, will represent the probability that, after the observed event, the number of individuals put disabled, our of a totality N, is comprehended between the limits $k-\omega$ and $k+\omega$, inclusively. The question is therefore reduced to calculating approximately formula (1), for its direct calculation, as it is evident besides, could not be carried out in general because of its excessive length.

In order to arrive to a degree of approximation that one is able to estimate, it is indispensable to agree in advance on the relative magnitude of the numbers N, n and ω, which, with i, are the givens of the question. The most natural hypothesis is to suppose that n and ω are of order $\sqrt{N}$. Thus, for example, if N were equal to 10000, one could take for n and ω some numbers which would not deviate too sensibly from 200, 300, 400 ..., by conforming besides to the requirements of practice. One will be able equally to suppose that the observed numbers i and $n-i$, always inferior to n, are of the same order $\sqrt{N}$, that is of the form $\lambda\sqrt{N}$, the coefficient of proportionality λ being a quantity of mean magnitude, which, often, is able to be inferior to unity. Moreover we will admit that the probability p must be determined with an approximation pushed to the quantities of order $\frac{1}{N}$, that is that we will neglect the quantities of this order, and hence those which will be proportional to $\frac{1}{n^2}$, $\frac{1}{i^2}$, $\frac{1}{(n-i)^2}$. This approximation, seeing the magnitude of the number N, will be, in general, very sufficient.

This put, it is easy to show that the characteristic S in formula (1) is able to be replaced by the characteristic of the definite integrals with a complementary term in the numerator. In fact, by making

$$y = x^i(1-x)^{n-i}, \quad y' = x'^i(1-x')^{n-i}, \quad y'' = x''^i(1-x'')^{n-i},$$

one will have first

$$\overset{x=x''}{\underset{x=x'}{S}} y = \sum_{x=x'}^{x=x''} y + y'';$$

moreover, by a known formula of Euler,

$$\sum_{x=x'}^{x=x''} y = \frac{1}{h}\int_{x'}^{x''} y\,dx + \frac{1}{2}(y''-y') + \frac{h}{12}\left[\left(\frac{dy''}{dx}\right) - \left(\frac{dy'}{dx}\right)\right] - \cdots$$

Therefore

$$\overset{x=x''}{\underset{x=x'}{S}} y = \frac{1}{h}\int_{x'}^{x''} y\,dx + \frac{1}{2}(y''-y') + \frac{h}{12}\left[\left(\frac{dy''}{dx}\right) - \left(\frac{dy'}{dx}\right)\right] - \cdots$$

If one now observes that h, which designates the finite increase of the probability x, is equal, in our question, to the fraction $\frac{1}{N}$, the preceding formula will take the form

$$\overset{x=x''}{\underset{x=x'}{S}} y = N\int_{x'}^{x''} y\,dx + \frac{1}{2}(y''-y') + \frac{1}{12N}\left[\left(\frac{dy''}{dx}\right) - \left(\frac{dy'}{dx}\right)\right] - \cdots \tag{3}$$

Now, it is easy to show that the second member of this equation will be reduced simply to its first two terms

$$N\int_{x'}^{x''} y\,dx + \frac{1}{2}(y''-y')$$

if one agrees, as we just said, to reject the quantities of an order equal and superior to $\frac{1}{N}$ with respect to the one that one conserves. In fact, by virtue of a known theorem of the Integral Calculus, one has first, by observing that $x' = \frac{k-\omega}{N}$ and $x'' = \frac{k+\omega}{N}$,

$$N\int_{x'}^{x''} y\,dx = N(x''-x')\overset{x''}{\underset{x'}{M}} y = 2\omega \overset{x''}{\underset{x'}{M}} y,$$

the notation $\overset{x''}{\underset{x'}{M}} y$ designating the arithmetic mean of the function

$$y = x^i 1-x)^{n-i}$$

for all the values of x comprehended between $\frac{k-\omega}{N}$ and $\frac{k+\omega}{N}$, by supposing that this variable increases in a continuous manner between these limits. The ratio of the second term $\frac{1}{2}(y''-y')$ of formula (3) to this first $2\omega \overset{x''}{\underset{x'}{M}} y$ will be therefore of the order $\frac{1}{\sqrt{N}}$, since ω, by hypothesis, is of order $\sqrt{N}$. Thus this second term must be conserved.

We calculate now the third term

$$\frac{1}{12N}\left[\left(\frac{dy''}{dx}\right) - \left(\frac{dy'}{dx}\right)\right].$$

Since one has

$$\frac{dy}{dx} = (i-nx)x^{i-1}(1-x)^{n-i-1},$$

one will obtain by setting in the place of x' and x'' their values (2)

$$\frac{1}{12N}\left[\left(\frac{dy''}{dx}\right) - \left(\frac{dy'}{dx}\right)\right] =$$

$$\frac{1}{12N}\left[\left(i - n\frac{k+\omega}{N}\right)x''^{i-1}(1-x'')^{n-i-1} - \left(i - n\frac{k-\omega}{N}\right)x'^{i-1}(1-x')^{n-i-1}\right];$$

this formula, by virtue of the equality $k = \frac{Ni}{n}$, will be reduced to

$$\frac{1}{12N}\left[\left(\frac{dy''}{dx}\right) - \left(\frac{dy'}{dx}\right)\right] = -\frac{n\omega}{12N^2}\left[{x''}^{i-1}(1-x'')^{n-i-1} + {x'}^{i-1}(1-x')^{n-i-1}\right].$$

Now, this term being of order $\frac{1}{N\sqrt{N}}$ with respect to the first $2\omega\overset{x''}{\underset{x'}{M}}y$, must be rejected, in consequence of which one will have simply with the desired approximation

$$\overset{x''}{\underset{x'}{S}}y = N\int_{x'}^{x''} y\,dx + \frac{1}{2}(y'' + y'),$$

for the terms which follow the third in formula (3) will be yet smaller, as it is easy to be assured directly.

We consider now the denominator of formula (1). One will have as above

$$\overset{X}{\underset{x_0}{S}}y = N\int_{x_0}^{X} y\,dx + \frac{1}{2}(Y + y_0) + \frac{1}{12N}\left[\left(\frac{dY}{dx}\right) - \left(\frac{dy_0}{dx}\right)\right] - \cdots \qquad (4)$$

y designating the same function $x^i(1-x)^{n-i}$, and Y and y_0 being determined by the equations

$$Y = X^i(1-X)^{n-i}, \qquad y_0 = x_0^i(1-x_0)^{n-i}.$$

Now, as

$$x_0 = \frac{i}{N} \text{ and } X = \frac{i+N-n}{N},$$

one will have

$$N\int_{x_0}^{X} y\,dx = (N-n)\overset{X}{\underset{x_0}{M}}y$$

and the term following $\frac{1}{2}(Y + y_0)$ being with respect to $(N-n)\overset{X}{\underset{x_0}{M}}y$ of the order $\frac{1}{N-n}$, or, that which reverts to the same, of the order $\frac{1}{N}$ since n is only of order $\sqrt{N}$, must be rejected. For greater reason one will be correct to omit the terms of formula (4) which follow the second. Therefore, finally,

$$p = \frac{N\int_{x'}^{x''} x^i(1-x)^{n-i}dx + \frac{1}{2}\left[{x''}^i(1-x)^{n-i} + {x'}^i(1-x')^{n-i}\right]}{N\int_{x_0}^{X} x^i(1-x)^{n-i}dx} \qquad (5)$$

the numbers x', x'', k, x_0 and X being determined by formulas (2).

We occupy ourselves now with the approximate calculation of the two integrals which enter in formula (5). We begin by that which is found in the numerator.

If one supposes

$$x = \frac{i}{n} + z, \text{ whence } 1 - x = \frac{n-i}{n} - z,$$

z will be a rather small quantity, since its limits will be

$$z' = x' - \frac{i}{n} = \frac{k-\omega}{N} - \frac{i}{N} = -\frac{\omega}{N}$$
$$z'' = x'' - \frac{i}{n} = \frac{k+\omega}{N} - \frac{i}{N} = +\frac{\omega}{N}$$

because $k = \frac{Ni}{n}$. Therefore

$$\int_{x'}^{x''} x^i (1-x)^{n-i} dx = \int_{-\frac{\omega}{N}}^{+\frac{\omega}{N}} \left(\frac{i}{n} + z\right)^i \left(\frac{n-i}{n} - z\right)^{n-i} dz.$$

Let u be this integral; one will have

$$u = \frac{i^i (n-i)^{n-i}}{n^n} \int_{-\frac{\omega}{N}}^{+\frac{\omega}{N}} \left(1 + \frac{nz}{i}\right)^i \left(1 - \frac{nz}{n-i}\right)^{n-i} dz.$$

We represent the binomials under the exponential form

$$\left(1 + \frac{nz}{i}\right)^i = e^{i \log\left(1 + \frac{nz}{i}\right)}$$
$$\left(1 - \frac{nz}{n-i}\right)^{n-i} = e^{(n-i) \log\left(1 + \frac{nz}{n-i}\right)}$$

and we develop next the logarithms into series; we will have

$$\left(1 + \frac{nz}{i}\right)^i \left(1 - \frac{nz}{n-i}\right)^{n-i} =$$
$$e^{-\frac{n^2 z^2}{2}\left(\frac{1}{i} + \frac{1}{n-i}\right) + \frac{n^3 z^3}{3}\left(\frac{1}{i^2} - \frac{1}{(n-i)^2}\right) - \frac{n^4 z^4}{4}\left(\frac{1}{i^3} + \frac{1}{(n-i)^3}\right) + \cdots}$$
$$= e^{-\frac{n^3 z^2}{2i(n-i)}} \cdot e^{\alpha z^3 - \beta z^4 + \cdots}$$

by making for brevity

$$\alpha = \frac{n^3}{3}\left(\frac{1}{i^2} - \frac{1}{(n-i)^2}\right), \quad \beta = \frac{n^4}{4}\left(\frac{1}{i^3} - \frac{1}{(n-i)^3}\right), \ldots$$

By developing the second exponential into series, one will obtain

$$\left(1 + \frac{nz}{i}\right)^i \left(1 - \frac{nz}{n-i}\right)^{n-i} = e^{-\frac{n^3 z^2}{2i(n-i)}} \cdot \left\{1 + \alpha z^3 - \beta z^4 + \cdots\right\},$$

and consequently

$$u = \frac{i^i (n-i)^{n-i}}{n^n} \left\{ \int_{-\frac{\omega}{N}}^{+\frac{\omega}{N}} e^{-\frac{n^3 z^2}{2i(n-i)}} dz + \alpha \int_{-\frac{\omega}{N}}^{+\frac{\omega}{N}} e^{-\frac{n^3 z^2}{2i(n-i)}} z^3 \, dz \right.$$
$$\left. - \beta \int_{-\frac{\omega}{N}}^{+\frac{\omega}{N}} e^{-\frac{n^3 z^2}{2i(n-i)}} z^4 \, dz + \cdots \right\}.$$

As the first and the third of these integrals each contain an even function of the variable z, one has only to double them by taking zero for the inferior limit. As for the second integral, as the function under the $\int$ sign is odd, the integral taken between the limits equal and of contrary signs $-\frac{\omega}{N}$ and $+\frac{\omega}{N}$, will be reduced to zero. Let be

$$\frac{n^3z^2}{2i(n-i)} = t^2, \text{ or else } t = \frac{n\sqrt{n}}{\sqrt{2i(n-i)}}.z;$$

by making

$$\frac{n\sqrt{n}}{\sqrt{2i(n-i)}} \cdot \frac{\omega}{N} = T,$$

the preceding value of u will be reduced to

$$u = \frac{2i^i(n-i)^{n-i}}{n^n} \cdot \frac{\sqrt{2i(n-i)}}{n\sqrt{n}} \left\{ \int_0^T e^{-t^2}dt - 4\beta\frac{i^2(n-i)^2}{n^6}\int_0^T e^{-t^2}t^4dt + \cdots \right\}.$$

Now, it is easy to show that the second term

$$4\beta\frac{i^2(n-i)^2}{n^6}\int_0^T e^{-t^2}t^4dt = \frac{n^2-3ni+3i^2}{ni(n-i)}\int_0^T e^{-t^2}.t^4\,dt$$

will be of the order $\frac{1}{N\sqrt{N}}$ with respect to the first

$$\int_0^T e^{-t^2}dt.$$

In fact, since e^{-t^2} does not change sign, one will have

$$\int_0^T e^{-t^2}.t^4\,dt = T_0^4\int_0^T e^{-t^2}dt,$$

T_0 designating a certain mean of t, comprehended between the limits 0 and T. Moreover, as T, generally, by virtue of the equation $T = \frac{n\sqrt{n}}{\sqrt{2i(n-i)}} \cdot \frac{\omega}{N}$, is of the order $\frac{1}{\sqrt[4]{N}}$, its fourth power will be of order $\frac{1}{N}$; thus T_0^4, inferior to T^4, will be all the more of this same order $\frac{1}{N}$. As for the factor $\frac{n^2-3ni+3i^2}{ni(n-i)}$, it is of the order $\frac{1}{\sqrt{N}}$; therefore the second term that we will consider being of the order $\frac{1}{N\sqrt{N}}$ with respect to the first, must be rejected, and we will have simply

$$\int_{x'}^{x''} x^i(1-x)^{n-i}dx = \frac{2i^i(n-i)^{n-i}}{n^n} \cdot \frac{\sqrt{2i(n-i)}}{n\sqrt{n}}\int_0^T e^{-t^2}dt, \qquad (6)$$

T being determined by the equality

$$T = \frac{n\sqrt{n}}{\sqrt{2i(n-i)}} \cdot \frac{\omega}{N} \qquad (7)$$

Moreover, it could happen that the value of T would be sensibly equal to unity, and even superior to this number. In this case the integral $\int_0^T e^{-t^2} t^4 dt$, and sometimes also $\int_0^T e^{-t^2} t^6 dt$, must be conserved in the calculation. We will return again onto this circumstance at the end of our Memoir.

The integral which is found in the denominator of formula (5) is not able to be calculated my means of equation (6) because its limits x_0 and X deviate too much from the probable value $\frac{i}{n}$ which corresponds to the maximum of the function $y = x^i(1-x)^{n-i}$. In order to have this integral, we decompose it first in the following manner:

$$\int_{x_0}^{X} y\,dx = \int_0^1 y\,dx - \int_0^{x_0} y\,dx - \int_X^1 y\,dx. \tag{8}$$

This put, we will show that $\int_{x_0}^{X} y\,dx$ is reduced sensibly to $\int_0^1 y\,dx$, the two other integrals, seeing their smallness, are able to be neglected. But we calculate previously, with the approximation of which we are agreed, the integral

$$\int_0^1 x^i(1-x)^{n-i} dx.$$

By setting it under the known form

$$\int_0^1 x^i(1-x)^{n-i} dx = \frac{1.2.3\ldots i.1.2.3\ldots(n-i)}{1.2.3\ldots(n+1)},$$

and by developing the products by means of the formula of *Stirling*, one will have

$$\int_0^1 x^i(1-x)^{n-i} dx = e.\frac{i^i(n-i)^{n-i}\sqrt{2\pi i(n-i)}}{(n+1)^{n+\frac{3}{2}}} \cdot \frac{\left(1+\frac{1}{12i}\right)\left(1+\frac{1}{12(n-i)}\right)}{1+\frac{1}{12(n+1)}}.$$

We have conserved in this development the terms of the order $\frac{1}{i}$, $\frac{1}{n-i}$, $\frac{1}{n}$, and rejected those of order $\frac{1}{i^2}$, $\frac{1}{(n-i)^2}$, $\frac{1}{n^2}$, ... which correspond to the quantities of order $\frac{1}{N}$, according to the convention established above.

We transform now the term $(n+1)^{n+\frac{3}{2}}$ which is found in the denominator of the preceding formula. One will have successively

$$\begin{aligned}
(n+1)^{n+\frac{3}{2}} &= n^{n+\frac{3}{2}}\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)^{\frac{3}{2}}, \\
\left(1+\frac{1}{n}\right)^n &= 1+1+\left(1-\frac{1}{n}\right)\frac{1}{1.3}+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\frac{1}{1.2.3} \\
&\quad +\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\frac{1}{1.2.3.4}+\cdots \\
&= 1+\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\cdots \\
&\quad -\frac{1}{n}\left(\frac{1}{1.2}+\frac{1+2}{1.2.3}+\frac{1+2+3}{1.2.3.4}+\cdots\right)+\cdots
\end{aligned}$$

Now, it is easy to note that the series

$$\frac{1}{1.2}+\frac{1+2}{1.2.3}+\frac{1+2+3}{1.2.3.4}+\frac{1+2+3+4}{1.2.3.4.5}+\cdots$$

represent the development of $\frac{1}{2}e$; in fact, its general term being

$$\frac{1+2+3+\cdots+(\mu-1)}{1.2.3\ldots\mu}=\frac{\frac{\mu(\mu-1)}{2}}{1.2.3\ldots\mu}=\frac{1}{2}\frac{1}{1.2.3\ldots(\mu-2)},$$

one will have evidently

$$\frac{1}{1.2}+\frac{1+2}{1.2.3}+\frac{1+2+3}{1.2.3.4}+\frac{1+2+3+4}{1.2.3.4.5}+\cdots=\frac{1}{2}\left(1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\cdots\right)=\frac{1}{2}e.$$

We observe in passing that this rather curious transformation leads also the the following expression:

$$e=2\left[\frac{1}{2}+\frac{1}{2.3}+\frac{2}{1.3}+\frac{1}{2.3.4}+\frac{2}{1.3.4}+\frac{3}{1.2.4}+\frac{1}{2.3.4.5}+\frac{2}{1.3.4.5}+\frac{3}{1.2.4.5}+\frac{4}{1.2.3.5}+\cdots\right].$$

Thus, by neglecting the terms of order $\frac{1}{n^2}$, or, that which reverts to the same, of the order $\frac{1}{N}$, one will have

$$\left(1+\frac{1}{N}\right)^n=e-\frac{1}{2}\cdot\frac{e}{n},$$

and hence

$$(n+1)^{n+\frac{3}{2}}=e.n^{n+\frac{3}{2}}\left(1-\frac{1}{2n}\right)\left(1+\frac{1}{n}\right)^{\frac{3}{2}}=e.n^{n+\frac{3}{2}}\left(1+\frac{1}{n}\right).$$

If one substitutes this value into the formula which expresses the integral sought, and if moreover one replaces $\frac{1}{12(n+1)}$ by $\frac{1}{12n}$, that which is permitted, one will obtain

$$\int_0^1 x^i(1-x)^{n-i}dx=\frac{i^i(n-i)^{n-1}}{n^{n+1}}\cdot\frac{\sqrt{2\pi i(n-i)}}{\sqrt{n}}\cdot\frac{\left(1+\frac{1}{12i}\right)\left(1+\frac{1}{12(n-i)}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{12n}\right)},$$

and as

$$\frac{\left(1+\frac{1}{12i}\right)\left(1+\frac{1}{12(n-i)}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{12n}\right)}=1-\frac{13i(n-i)-n^2}{12i(n-i)n}+\cdots$$

will have finally

$$\int_0^1 x^i(1-x)^{n-i}dx=\frac{i^i(n-i)^{n-1}}{n^{n+1}}\cdot\frac{\sqrt{2\pi i(n-i)}}{\sqrt{n}}\left(1-\frac{13i(n-i)-n^2}{12i(n-i)n}\right),\quad(9)$$

whence one concludes that this integral is of the order of magnitude $\frac{1}{\sqrt{n}}$ or $\frac{1}{\sqrt[4]{N}}$.

We show now that the last two integrals of the second member of equation (8) must be rejected. We begin with the integral

$$\int_0^{x_0} x^i(1-x)^{n-i}dx.$$

As the function $x^i(1-x)^{n-i}$ has only a single *maximum*, corresponding to $x = \frac{i}{n}$, it follows from it that this function is constantly increasing from $x = 0$ to $x = \frac{i}{n}$, and decreasing next to $x = 1$. Thus, since the value of $x_0 = \frac{i}{N}$ is inferior to $\frac{i}{n}$, the function $x^i(1-x)^{n-i}$, between the limits 0 and x_0, will attain its maximum for $x = x_0$, and will be consequently $x_0(1-x_0)^{n-i}$. On the other side one has

$$\int_0^{x_0} x^i(1-x)^{n-i}dx = x_0 \overset{x_0}{\underset{0}{M}} x^i(1-x)^{n-i},$$

and as besides, according to that which was just said,

$$\overset{x_0}{\underset{0}{M}} x^i(1-x)^{n-i} < x_0^i(1-x_0)^{n-i},$$

one will have also

$$\int_0^{x_0} x^i(1-x)^{n-i}dx < x_0.x_0^i(1-x_0)^{n-i}.$$

Setting for x_0 its value $\frac{i}{N}$, this inequality will be reduced to

$$\int_0^{x_0} x^i(1-x)^{n-i}dx < \left(\frac{i}{N}\right)^{i+1}\left(1-\frac{i}{N}\right)^{n-i}.$$

The integral (9) is, as we have already remarked, a quantity of the order $\frac{1}{\sqrt[4]{N}}$. As for the integral $\int_0^{x_0} x^i(1-x)^{n-i}dx$, its order of magnitude, by virtue of the preceding inequality, will be, generally, inferior to $\left(\frac{i}{N}\right)^{i+1}$, or else to $\frac{1}{N^{\frac{i+1}{2}}}$. Now, as i is a whole number, composed at least of some simple units, the integral in question will be totally insensible with respect to $\int_0^1 x^i(1-x)^{n-i}dx$.

By operating as we just did, one will show equally that the integral $\int_X^1 x^i(1-x)^{n-i}dx$ must be rejected. In fact, by setting it under the form

$$\int_X^1 x^i(1-x)^{n-i}dx = (1-X)\overset{1}{\underset{X}{M}} x^i(1-x)^{n-i},$$

and by observing that

$$\overset{1}{\underset{X}{M}} x^i(1-x)^{n-i} < X^i(1-X)^{n-i} = \left(\frac{n-i}{N}\right)^{n-i}\cdot\left(1-\frac{n-i}{N}\right)^i,$$

one will have

$$\int_X^1 x^i(1-x)^{n-i}dx < \left(\frac{n-i+1}{N}\right)^{n-i}\cdot\left(1-\frac{n-i}{N}\right)^i.$$

Therefore this integral will be all the more a quantity of the order

$$\left(\frac{1}{\sqrt{N}}\right)^{n-i+1} = \frac{1}{N^{\frac{n-i+1}{2}}},$$

which, seeing its extreme smallness, must be rejected. Thus, of the three integrals which compose the second member of the equation (8), it is necessary to conserve only the first, that is the integral (9).

Therefore, definitely, by substituting the values (6) and (9) into formula (5), and by dividing high and low by

$$N.\frac{i^i(n-i)^{n-i}}{n^n} \cdot \frac{\sqrt{2i(n-i)}}{n\sqrt{n}} \cdot \sqrt{\pi},$$

one will obtain

$$p = \frac{\frac{2}{\sqrt{\pi}}\int_0^T e^{-t^2}dt + \frac{n^{n+1}\sqrt{n}}{2Ni^i(n-i)^{n-i}\sqrt{2\pi i(n-i)}}\left[x''^i(1-x'')^{n-i} + x'^i(1-x')^{n-i}\right]}{1 - \frac{13i(n-i)-n^2}{12i(n-i)n}}, \tag{10}$$

with the following determinations:

$$k = \frac{Ni}{n}, \quad x' = \frac{k-\omega}{N}, \quad x'' = \frac{k+\omega}{N}, \quad T = \frac{n\sqrt{n}}{\sqrt{2i(n-i)}} \cdot \frac{\omega}{N}. \tag{11}$$

Here is therefore the probability p determined with the degree of precision desired. If the sought number was ω, for a probability of which the *minimum* would be fixed in advance, then, by observing that $\frac{2}{\sqrt{\pi}}\int_0^T e^{-t^2}dt$ is the dominating term in formula (10), one would find the approximate value of T, corresponding to this *minimum*, and this would suffice to the end that one has proposed; thence, by virtue of the equality $T = \frac{n\sqrt{n}}{\sqrt{2i(n-i)}} \cdot \frac{\omega}{N}$, one would determine next the number ω as function of the other data of the question.

Before passing to the numerical applications of formula (10), we will observe that its calculation will be facilitated by means of the tables of the integral $\int_T^\infty e^{-t^2}dt$ which are found in the *Analyse des réfractions astronomiques et terrestres* by Kramp. If one wishes more precision than these tables include, one will make use either of the known method of interpolation, or of the different series which serve from development in the integrals $\int_0^T e^{-t^2}dt$ and $\int_T^\infty e^{-t^2}dt$. As to the other terms of formula (10), one will be able to determine them by aid of logarithmic calculation.

We suppose first that the body of the army which must take part in combat, is 10000 men, out of which one chooses 100 of them, and that in a determined epoch, out of these 100 men, 20 have been put disabled. One will have the following data: $N = 10000$, $n = 100$, $i = 20$, $n - i = 80$, $k = 2000$.

We admit moreover, that we wish to determine the probability that the total number of men, put disabled, will not deviate beyond 100 of the found number 2000, or, in other terms, that this number will be comprehended between 1900 and 2100. One will have

$$\omega = 100, \text{ and consecutively } T = \frac{1}{\sqrt{32}} = 0.1767\ldots$$

By making use of the known methods, one finds

$$\frac{2}{\sqrt{\pi}} \int_0^T e^{-t^2} dt = 1 - \frac{2}{\sqrt{\pi}} \int_T^\infty e^{-t^2} dt = 0.1973\ldots$$

One will have also

$$x'' = \frac{21}{100}, \qquad x' = \frac{19}{100},$$

and consequently the second term of the numerator of formula (10), a term that we will designate by A, will be able to be put under the form

$$A = \frac{1}{800\sqrt{2\pi}} \left\{ \left(\frac{21}{20}\right)^{20} \cdot \left(\frac{79}{80}\right)^{80} + \left(\frac{19}{20}\right)^{20} \cdot \left(\frac{81}{80}\right)^{80} \right\}.$$

Now, one will have

$$\sqrt{2\pi} = 2.5066\ldots, \qquad \frac{1}{800\sqrt{2\pi}} = \frac{1}{2008.28} = 0.00049\ldots$$

$$\left(\frac{21}{20}\right)^{20} \left(\frac{79}{80}\right)^{80} = 0.9699\ldots, \qquad \left(\frac{19}{20}\right)^{20} \cdot \left(\frac{81}{80}\right)^{80} = 0.9684\ldots$$

$$\left(\frac{21}{20}\right)^{20} \cdot \left(\frac{79}{80}\right)^{80} + \left(\frac{19}{20}\right)^{20} \cdot \left(\frac{81}{80}\right)^{80} = 1.9634\ldots$$

Therefore

$$A = 0.00095\ldots$$

We determine now the term which must be subtracted from 1 in the denominator of formula (10). One will find

$$\frac{13i(n-i) - n^2}{13i(n-i)n} = \frac{108}{19200} = 0.0056\ldots$$

Consequently

$$p = \frac{0.1973 + 0.0008}{1 - 0.0086} = 0.199\ldots$$

Thus, according to the conditions of our problem, the sought probability is only around $\frac{1}{5}$; it will be evidently too feeble in order that one is able reasonably to found oneself on it. We see therefore that the hypothesis that we just made on the relative magnitude of the numbers N, n and ω are not at all able to lead us to the end that we ourselves have proposed. In order to obtain a stronger probability, it would be necessary to increase the number n of men that one chooses, or else render a greater interval 2ω of the limits; but it will be worth better yet to increase at the same time the two numbers n and ω.

We suppose, for example, that out of the same total $N = 10000$ men, one chooses 400 of them, that is 40 out of 1000 or 4 percent, and that moreover one takes $\omega = 200$. We admit that the observed number of individuals put disabled is 80. One will have

$$N = 10000, \quad n = 400, \quad i = 80, \quad n - i = 320, \quad k = 2000, \quad \omega = 200,$$

and hence

$$T = \frac{400\sqrt{400}}{\sqrt{2.80.320}} \cdot \frac{200}{10000} = \frac{1}{\sqrt{2}} = 0.7071\ldots$$

$$\int_0^T e^{-t^2} dt = 0.6050, \qquad \frac{2}{\sqrt{\pi}} \int_0^T e^{-t^2} dt = 0.6926\ldots$$

Moreover, as

$$x'' = \frac{22}{100}, \qquad x' = \frac{18}{100},$$

the term to add to the integral $\frac{2}{\sqrt{\pi}} \int_0^T e^{-t^2} dt$ will be, all reduction done,

$$A = \frac{1}{400\sqrt{2\pi}} \left\{ \left(\frac{11}{10}\right)^{80} \cdot \left(\frac{39}{40}\right)^{320} + \left(\frac{9}{10}\right)^{80} \cdot \left(\frac{41}{40}\right)^{320} \right\}.$$

Now

$$\left(\frac{11}{10}\right)^{80} \cdot \left(\frac{39}{40}\right)^{320} = 0.6207\ldots, \quad \left(\frac{9}{10}\right)^{80} \cdot \left(\frac{41}{40}\right)^{320} = 0.5902\ldots$$

$$\left(\frac{11}{10}\right)^{80} \cdot \left(\frac{39}{40}\right)^{320} + \left(\frac{9}{10}\right)^{80} \cdot \left(\frac{41}{40}\right)^{320} = 1.2109\ldots$$

$$\frac{1}{400\sqrt{2\pi}} = 0.0009\ldots$$

Therefore one will have

$$A = (1.2109\ldots) \times (0.0009\ldots) = 0.0012.$$

The term to subtract from 1 in the denominator of formula (10) will be

$$\frac{13i(n-i) - n^2}{12i(n-i)n} = 0.0014.$$

Consequently

$$p = \frac{0.6926 + 0.0012}{1 - 0.0014} = 0.684.$$

Thus, the probability that by choosing 400 men out of 10000 combatants, one will not be deceived more than 200 men, either by more or by less, out of the total lost, will be more than $\frac{2}{3}$. But although this probability is superior to $\frac{1}{2}$, it is yet too feeble in order to be able to account with confidence on the extent of the limits admitted. It will be necessary therefore, anew, to increase one of the numbers n or ω, or, as we have already noted, both at once. We suppose that one is arrested at 5 percent for the number n, and at $2\frac{1}{2}$ percent for the number ω. One will have $N = 10000$, $n = 500$, $\omega = 250$. Let moreover the number observed $i = 100$, and hence $n - i = 400$. One will find

$$T = \frac{5\sqrt{5}}{8\sqrt{2}} = 0.9882\ldots$$

$$\frac{2}{\sqrt{\pi}}\int_0^T e^{-t^2}dt = 1 - \frac{2}{\sqrt{\pi}}\int_T^\infty e^{-t^2}dt = 1 - 0.16235\ldots = 0.83765\ldots$$

The complementary term of the numerator of formula (10) will be found equal to 0.00081..., and the term subtracted from the unity in the denominator to 0.00112... Therefore

$$p = \frac{0.83765\ldots + 0.00061\ldots}{1 - 0.00112\ldots} = 0.839\ldots$$

Here is therefore arrived to us a probability very nearly equal to $\frac{9}{10}$. By making again to incur a slight increase to the ratios $\frac{n}{N}$ and $\frac{\omega}{N}$, one will attain a probability superior to $\frac{9}{10}$, which, in ordinary practice, will suffice certainly to the end that we ourselves propose.

We say now some words on the construction of a table which would serve to judge, at first inspection, on the extent of the limits of the probable loss in dead and wounded. We will propose for that to consider n as a quantity of order $\sqrt{N}$, and such, that the ratio $\frac{n}{N}$ is constant, equal, for example, to $\frac{5}{100}$ as in the last case, or, that which would be worth yet more, to a number superior to $\frac{5}{100}$, for the precision of the result. Under this last hypothesis, the limit $T = \frac{n\sqrt{n}}{\sqrt{2i(n-i)}} \cdot \frac{\omega}{N}$ will surpass generally *unity*; for that same, in the approximate calculation of the integral $\int_{x'}^{i} x'' x^i (1-x)^{n-i} dx$ transformed, it will be necessary to conserve the term $\int_0^T e^{-t^2} t^4\, dt$, that we have omitted in considering T as a quantity of the order $\frac{1}{\sqrt[4]{t}}$. It would be able likewise to be done as one was obliged to have regard to the term containing the integral $\int_0^T e^{-t^2} t^6\, dt$. This would bring a modification to our formula (10), a modification which would not present the least difficulty, because, one knows that in general $\int_0^T e^{-t^2} t^{2m}\, dt$ is expressed very easily by means of $\int_0^T e^{-t^2}\, dt$, already calculated in advance.

We return to that which we had to say with respect to the table of probable losses. Having already admitted that the ratio of the men chosen to the one of the total of the combatants is invariable, one will suppose moreover the probability p very nearly constant, superior, for example, to $\frac{9}{10}$ or to every other fraction that one will be convenient to choose. This put, the table will contain two arguments: the number N, total of the combatants, and the number i which represents the number observed of the individuals put disabled. The sought number will be then ω, that is the deviation more or less from the probable number $k = \frac{Ni}{n}$ of the total real loss. From this manner, the table could be set under the form of the ordinary abacus of Pythagoras. The first horizontal rank would be destined, for example, to the numbers representing the total of the combatants,and the first vertical rank at left would designate the the different observed numbers of disabled individuals. The case of the encounter of the two corresponding ranks, horizontal and vertical, would contain the number ω, or, better yet, the two limits $k - \omega$ and $k + \omega$ of the probable real loss.

The argument i being the observed number, one could take successively for this number the one of the men dead or wounded, by setting even among them such difference as one will wish, so many that there were of officers or of soldiers, that they would bring to the infantry or to the cavalry, etc. The same table will be able to serve also to determine the number of men killed either by cold steel, or by fire arms, thus to many other usages, independently even of the military art, and which will present themselves naturally by themselves.

The judges competent in this matter will find perhaps that our process is difficult to set into execution during combat; we ourselves will refer to their advice on this point. However, in each case, we believe that the table that we propose will not be without some practical utility, in this that it will furnish a very simple means in order to judge in a rather approximative manner, immediately after the end of the action, from the different losses endured by the army so many with men as with horses, etc. This knowledge, before having the precise number, will be able, perhaps, to have already some value.

In order to define better the calculation and the construction of *the table of probable losses*, it would be necessary to be put in relation with some persons, to whom the practical details of the question are familiar. Lacking necessary enlightenment on this subject, we have considered the problem only under its purely theoretic point of view.