Note sur un Problème de Combinaisons

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Mr. Brianchon has just inserted, in the XXV$^{\text{th}}$ notebook of the *Journal de l'École Polytechnique*, a rather extended Memoire on the *determination of the number of terms of the power m of a polynomial of name n*.

The formula to which he arrives, and which is not, I believe, new, is able to be demonstrated in the following manner.

The general term of the development of $(a+b+c+\cdots+t)^m$ is, as one knows,

$$\frac{1.2.3\ldots m}{1.2.3\ldots\alpha.1.2\ldots\beta\ldots 1.2\ldots\theta} a^\alpha b^\beta c^\gamma \cdots t^\theta, \tag{1}$$

by putting

$$\alpha+\beta+\gamma+\cdots+\theta = m. \tag{2}$$

The number of terms of this development is the one of the solutions, in whole non-negative numbers, of equation (2), which contains n unknowns, n designating the number of terms of the proposed polynomial; or the number of ways in which it is possible to form a sum m, with n positive or null whole numbers; or finally, the number of combinations that one is able to effect with n different letters, by taking them m by m, and by supposing that each letter is able to enter 0, 1, 2, ... times into each term. It is under this last point of view that I consider the question; and I designate by N the number sought.

In order to find this number, I observe that, in order to form all the combinations of which there is concern, one would be able to use the following means:

a, b, c, being in order to fix the ideas, three letters that there is concern to arrange 7 by 7:

1°. Let us take the quantity $a'b'c'd'e'f'g'$, which contain seven accented letters, written in alphabetical order;

2°. In any term equal to the one there, we erase 1, 2 or 3 letters (and in general, n letters at most, if n is $< m$, m letters at most, if n is $> m$); then we replace each erased letter by one of the letters a, b, c (and in general, by one of the n letters $a, b, c, \ldots t$), by taking care that, in each term thus formed, the letters without accent do not offer alphabetical inversion; that none is found repeated; and that one sequence of accented letters is always preceded by a letter without accent (this which requires that one erase always the letter a').

We will obtain thus a sequence of terms such as

$$ab'c'be'f'g', \quad abc'd'e'cg', \quad bb'c'd'cf'g', \quad \text{etc.} \tag{A}$$

3 $^\circ$. Finally, in each of the terms of the sequence (A), we replace each accented letter by the letter without accent which precedes it. We will have the new sequence:

$$aaabbbb, \quad abbbbcc, \quad bbbbccc, \quad \text{etc.} \tag{B}$$

If one has effected on the quantity $a'b'c'd'e'f'g'$ the indicated operations, in all the possible ways, the sequence (B) will contain all the combinations demanded, without that there are omitted of them there, nor repeated: we will suppress the demonstration, which is quite simple.

Now, the sequence (A), which contains as many terms as the sequence (B), contains all the combinations of the 6 letters b', c', d', f', g', and of the 3 letters a, b, c, taken 7 by 7. Therefore in general, $N = C_{n+m-1,m}$; namely

$$N = \frac{n+m-1}{1} \cdot \frac{n+m-2}{2} \cdots \frac{n+1}{m-1} \cdot \frac{n}{m}, \tag{3}$$

or

$$N = \frac{m+1}{1} \cdot \frac{m+2}{2} \cdots \frac{n+m-1}{n-1}. \tag{4}$$

Formulas (3) and (4), give the number of terms of the development of which (1) is the general term.