PROBLÈME DE COMBINAISONS

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Having taken at random, in space, n points a, b, c, $\ldots$; one demands what will be the number N of new points A, B, C, $\ldots$ which result from the intersections three by three of the planes passing each through three of the points a, b, c, $\ldots$?

We designate generally by $C_{m,n}$ the number of combinations n *by* n of m letters.

The number p of the planes will be $C_{n,3}$.

If these planes were taken arbitrarily, they would be cut three by three, in a number of points equal to $C_{p,2}$.

But through each of the n given points a, b, c, $\ldots$ there passes evidently $C_{n-1,2}$ planes; we designate this number by q.

These planes would give place, if they were any, at $C_{q,2}$ points of encounter, which are reduced here to a single point.

Therefore, the number N will be given by the formula

$$N = C_{p,3} - nC_{q,3}.$$

By effecting the calculations, one finds

$$N = \frac{5}{9}(n^3 - 19n + 6)C_{n+1,6}$$

or

$$N = 20[14C_{n+3,9} + C_{n+1,6}].$$

If one considered n points a, b, c, $\ldots$ all situated in a given plane, one would be able to demand likewise what is the number N of new points A, B, C, $\ldots$ which result from the intersections two by two of the straight lines passing each through two of the points a, b, c, $\ldots$; and one will find, without difficulty, $N = 3C_{n,4}$.