MÉMOIRE

SUR LE SYSTÈME DES VALEURS
QU'IL FAUT ATTRIBUER A DIVERS ÉLÉMENS DÉTERMINÉS PAR UN GRAND NOMBRE D'OBSERVATIONS, POUR QUE LA PLUS GRANDE DE TOUTES LES ERREURS, ABSTRACTION FAITE DU SIGNE, DEVIENNE UN MINIMUM.*

Augustin-Louis Cauchy

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Fr. Boscowich has shown how one was able to resolve the preceding question, in the case where one has only a single element to consider. Mr. Laplace has examined a similar question in the third Book of the *Méchanique céleste*, which treats of the figure of the terrestrial sphere, and has given an easy method in order to determine the elliptic figure, in which the greatest gap of the degrees of the meridian, setting aside the sign, becomes a minimum. One has in this case two elements to consider, instead of one alone. But the function of the elements which represents the errors is not the most general possible. There remains to extend the same theory to the case where this function becomes the most general of its kind, and where the number of the elements is superior to two. Mr. Laplace having well wished to show me this subject of researches, I myself have endeavored to respond to his expectation; and I am arrived to a general method which contains all the others, and which remains always the same, whatever be the number of elements that one considers. Such is the object of the Memoir which I have the honor to submit to the Class. Here is first in what consists the problem that the question is to resolve.

When, in order to determine an unknown element, for example a length, an angle, etc., one has made a great number of observations either on these elements themselves, or on some other quantities which depend on them, then each observation taken in part determines a particular value of the element. If one has already concluded, either from the observations that one considers, or from the other observations made previously, an approximate value to the element, in order to deduce from this value the truth, it will suffice to add a small correction to it that one can designate by the variable x. Each observation, taken separately and considered as exact, determines a particular value of the correction. But if, instead of considering this equation as exact, one supposes that it

*"Memoir on the system of values that it is necessary to attribute to diverse elements determined by a great number of observations, for which the greatest of all the errors, setting aside the sign, becomes a minimum." Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. July 6, 2010

is in error respecting the true result of a certain quantity, then the correction to make, or the variable x which represents it, will become a function of this error, and reciprocally. Hence, the error of each observation will be able in general to be expressed by a series ordered according to the powers of the variable, and in which, seeing the smallness of the correction to make, one will be able to be arrested at the first power of that. This error will be therefore represented by a binomial, of which the first term will be constant, and of which the second will contain only the first power of the variable x. If the given observations must serve to determine many elements instead of one, by designating the respective corrections of these by the variables $x, y, z, \ldots$ one will arrive in the same manner to represent each error by a polynomial of the first degree in $x, y, z, \ldots$ This put, *the question is to find for these variables a system of values*

$$x = \xi, \quad y = \eta, \quad z = \zeta, \quad \ldots,$$

such that the greatest of the polynomials that one considers, or that which returns to the same, the greatest of the errors that they represent, becomes, setting aside the sign, a minimum.

The problem is simplified considerably, when the polynomials which represent the errors are two by two equal and of contrary signs. Then, in fact, for the values determined of the variables $x, y, z, \ldots$ the greatest of the positive errors is equal to the greatest of the negative errors; and the question is reduced to determine the system of values of $x, y, z, \ldots$ for which the greatest of the positive errors becomes a minimum. If the errors are not two by two equals and of contrary signs, one will be able to restore this case in the preceding, by doubling by thought the number of errors and joining, to the polynomials which represent the given errors, other equal polynomials and of contrary signs, destined to represent the fictive errors that one has proposed to consider. If among the given errors there were found already many of them which were equals and of contrary sign, it would be useless to double the number of them. By means of the preceding artifice, one separates the difficulties which were able to be born from the distinction of the signs, and one is then authorized to consider a greater negative quantity as smaller than another lesser negative quantity. The proposed question is found thus, as one has already remarked, restored to the following:

> *$x, y, z, \ldots$ being the corrections of the elements that one considers, to determine for these variables a system of values such that the greatest of the positive errors becomes a minimum.*

I am going to expose in a few words the method which leads to the solution of this new problem.

Let $x = \xi$, $y = \eta$, $z = \zeta, \ldots$ be the values of the unknowns which resolve the question. Each of these values must be chosen among an infinity of others. It seems therefore first of all that, in order to arrive to the sought solution, it would be necessary to vary separately each of the corrections $x, y, z, \ldots$ and to examine what influence the variation of each of them is able to have concerning the increasing and the diminishing of the errors that one considers. One is able nonetheless to attain the end that one proposes, by being content to make a single correction vary, z for example, thus as we are going to show.

We suppose one moment the problem already resolved for a number of elements inferior by one unit to the one that one considers; we imagine moreover that one gives successively to z all the possible values from $z = -\infty$ to $z = +\infty$, and that for each value of z one determines the other variables $x, y, z, \dots$ by the condition that the greatest error becomes a minimum. One will obtain in this manner a sequence of systems of values of $x, y, z, \dots$, among which will be found necessarily the sought system; and, in order to obtain this last, it will suffice to choose, among the minima of the greatest errors corresponding to the diverse values of z, the one which is itself smaller than all the others. This last corresponds to the value ζ of z; and hence, if this value were known, there would be nothing more to do, and the question would be found thus restored to the case where one has one element less to consider. But, as one cannot hope to discover immediately the value ζ of z which satisfies the question, it will be necessary to begin by giving to z an arbitrary value, by supposing, for example, $z = 0$, and to determine the corresponding values of $x, y, \dots$, for the condition enunciated above, namely, that the greatest error becomes a minimum. After having thus obtained the minimum of the greatest errors for the value zero of z, no more will remain but to make z vary in a manner to make the minimum of which there is concern decrease, until this that it acquires the smallest value possible. The method that it is necessary to use in order to arrive to it is founded on the following theorem, demonstrated by Mr. Laplace:

> *Whatever be the number of elements contained in the errors that one considers, if one makes vary, either all these elements, or only some among them, and if one determines the values of the variable elements which render the greatest error a minimum, for the values of which there is concern, many errors will become at the same time equals among them and the greatest of all, and the number of these last will surpass always by at least one unit the number of variable elements.*

In order to show how one is able to make the application of this theorem to the proposed question, we suppose that one has three elements to consider. Let $x, y, z, \dots$ be the corrections of these three elements. Let n be the number of errors, and we designate those by $e_1, e_2, e_3, \dots e_n$. By virtue of that which precedes, one will begin by supposing in all the errors $z = 0$, and one will determine, under this hypothesis, the values of x and of y which render the greatest error a minimum. Let

$$x = \alpha, \quad y = \beta$$

be the values of which there is question. It follows from the preceding theorem that, for the values α, β and 0 of the variables $x, y, z, \dots$, three errors, for example

$$e_p, \quad e_q, \quad e_r,$$

will become equals among them and the greatest of all. Moreover, the double equation

$$e_p = e_q = e_r$$

will serve to determine the values of α and β of x and of y which correspond to $z = 0$. If, now, one makes z vary by a very small quantity more or less, the three errors e_p, e_q, e_r

will enjoy still the same property, that is to say that they will be always the greatest of all the values of x and of y which render the greatest error a minimum, and these same values will be yet determined by the double equation

$$e_p = e_q = e_r.$$

But, in order to make so that the common value of these three errors diminish, it will be able to arrive that one is obliged either to make increase or to make decrease z. We suppose, in order to fix the ideas, that this value diminishes when z increases, z continuing to increase, the errors e_p, e_q, e_r will diminish simultaneously, by remaining always the greatest of all, until a new error e_s arrives to equal them in order to surpass them next. Let γ_1 be the value of z for which the four errors e_p, e_q, e_r, e_s become equals among them; and we designate by α_1 and β_1 the corresponding values of x and of y. The system of values

$$x = \alpha_1, \quad y = \beta_1, \quad z = \gamma_1$$

will be determined by the triple equation

$$e = e_q = e_r = e_s$$

and, for the value of γ_1 of z, this system will be the one which renders the greatest error a minimum. Besides, it follows from the theorem enunciated above that, for the values of the unknowns which resolve the proposed question, four errors must be equals among them and the greatest of all. This last condition being satisfied, at the same time as the preceding, for the three values

$$x = \alpha_1, \quad y = \beta_1, \quad z = \gamma_1,$$

it is acceptable to research if these would not resolve the problem. One arrives to it in the following manner:

When one makes z increase to beyond γ_1, the three errors e_p, e_q, e_r cease to be conjointly the greatest of all for the values of x and of y which render the greatest error a minimum; and this property belongs then to two of among them taken conjointly with the new error e_s. One determines easily what are, among the three errors e_p, e_q, e_r the two which it is acceptable to choose for this object. Let, for example, e_q, e_r be these two errors; if one makes z increase to beyond γ_1, the common value of the three errors e_q, e_r, e_s will go by increasing or by diminishing. In the first case, the values $\alpha_1, \beta_1, \gamma_1$ of x, y and z will satisfy the proposed question. In the second case, the errors of which there is question will continue to decrease, by remaining always the greatest of all for the values of x and of y which render the greatest error a minimum, until a new error arrives to equal all three of them in order to surpass them next. Then one will obtain anew a triple equation among four errors. One will be able to judge, as previously, if the system of values of the unknowns determined by that triple equation satisfies the proposed equation. In the contrary case, by following always the same march, one will conclude by arriving to the solution of the problem.

The errors e_p, e_q, e_r being supposed known, in order to discover the error e_s, it suffices evidently to seek that which, equaled to the first three, determine the smallest positive value of the variable z. But it is able to happen that, for this value of z, many errors, for example, $e_s, e_t, e_u, \dots$ become at the same time equals among them and to the first three. We designate always by γ_1 the value of which there is question. If one makes z increase beyond γ_1, three of the following errors

$$e_p, e_q, e_r, e_s, e_t, e_u, \quad \dots$$

will become conjointly the greatest of all for the values of x and of y which render the greatest error a minimum; and, concerning these three errors, two at most must be taken among the first three e_p, e_q, e_r. Excepting this restriction, the combination which contains the three new errors will be able to be any one of those that one forms by assembling three by three the errors

$$e_p, e_q, e_r, e_s, e_t, e_u, \quad \dots$$

In order to judge what is among these combinations that which merits the preference, one will suppose that the variable z increases beyond γ_1 by a positive, but indeterminate, quantity represented by k, and that the corresponding values α_1 and β_1 of the two other variables x and y receive at the same time the positive or negative increases g and h. The increases of the errors $e_p, e_q, e_r, e_s, e_t, e_u, \dots$, which, by hypothesis, were all equals among them, will be found then expressed by the homogeneous polynomials of the first degree in g, h and k; and it will suffice to determine the respective values of g and h for which the greatest of all becomes a minimum, k being an essentially positive quantity, one will be able, without any inconvenience, to divide by k each of these polynomials. The quotients will contain no longer variables but the ratios of the increases of x and of y to the one of z, and it will remain no more but to determine these two ratios in such manner that the greatest of the quotients that one considers becomes a minimum. Thus all the difficulties are found reduced to the solution of the general problem, in the case where one has only two elements to correct.

One will reduce likewise the difficulties that this last hypothesis presents to the difficulties which subsist, in the case where one has only a single element to consider. Finally one will reduce those to the determination of the greatest error for a given value of the variable, and then the proposed question will be found completely resolved.

One will be able likewise, in general, whatever be the number of the variables, to restore the proposed question to the case where one has one variable less to consider, and to lower next continuously the difficulty, until it disappears entirely. Thus, for example, if m represents the number of variables, one will begin by giving to one of them, that I will designate by z, an arbitrary value, by determining the others in a manner that the greatest error becomes a minimum. Then one will obtain a system of values for which m different errors will become equals among them and the greatest of all. One will make next z vary in a manner to make decrease the common value of the errors of which there is concern, until a new error arrives to equal them all, in order to surpass them next. Then one will obtain an equation among $m+1$ different errors; and one will judge easily if the values of the variables determined by this equation satisfy the proposed equation. In the contrary case, if one continues to make z vary always in the same sense, a new combination of m errors will replace the first; and, by following the same march, one will finish necessarily by arriving to the solution of the problem. The case where, for a like value of z, the number of errors equal among them and the greatest of all could come to surpass $m+1$, presents no difficulty that it is not always easy to resolve, by means of the artifice employed for this object under the hypothesis of three variables.

When one has a single element to correct, the preceding method is reduced to that which Fr. Boscowich has given, provided that one supposes the first value of z, which one is able to choose arbitrarily, equal to negative infinity. One is able nevertheless, in this case, to simplify the solution, by taking for first value of z that which renders equals among them the two errors where this variable has the greatest positive coefficient and the greatest negative coefficient.

If one has many elements to consider, the calculations become much simpler, in the case where some of these elements has the same coefficient, with sign excepted, in all the given errors. Thus, for example, if one considers two elements, and if the coefficient of the one among them is always equal to $+1$ or to -1, one arrives to a method similar to that which Mr. Laplace has given in order to determine, relative to the Earth, the elliptic figure in which the greatest gap of the degrees of the meridian become, setting aside the sign, a minimum. (A).

I join here the demonstration of the theorems that suppose the preceding method, and the formulas relative to the simplest cases.

I will finish by observing that, under the hypothesis of two and of three variables, the proposed question is able to receive a rather singular geometric interpretation. It is reduced then to one of the following two problems:

PROBLEM I. — *Being given the equations of the straight lines which form the sides of a polygon, to determine the lowest vertex of the polygon.*

PROBLEM II. — *Being given the equations of the planes which compose the faces of a polyhedron, to determine the lowest vertex of this same polyhedron.*

One can yet resolve, by the same analysis, the following problem:

Being given the equations of the planes which compose the faces of a pyramid, to determine the lowest of its edges. (B)

ADDITIONS

(A) We have remarked above that the values of the variables which resolve the proposed questions render always equals among them so many errors, plus one, as there are elements to consider. One could therefore, in rigor, discover the system of values demanded, by seeking, among those which satisfy the preceding condition, the one which renders the greatest error a minimum: but this method would be long and painful, and the number of the operations that it would require for a number m of elements would be equal to the number of combinations of errors taken $m+1$ by $m+1$. It is easy to see what advantage the preceding method exposed has on this last. Because, instead of using all the systems of values of the variables for which $m+1$ errors become equals among them, we have considered only a part of those where the equal errors become at the same time greatest of all. One can appreciate this advantage with some exactness by aid of a rather remarkable theorem, and of which the enunciation is here:

Let m be always the number of elements that one considers. We suppose that one combines successively the errors given one by one, two by two, three by three, etc. finally $m+1$ by $m+1$, and that one had only regard for the combinations formed from errors which are able to become simultaneously the greatest of all; the total number of combinations where the errors will enter in odd number will surpass by one unit the number of combinations where the errors will enter in even number.

Thus, for example, if one has a single element to consider, the number of errors which will be able to become successively the greatest of all will surpass by one unit the number of combinations two by two. If one has three elements to consider, the number of errors, plus the number of combinations three by three, will surpass by one unit the number of combinations two by two, and thus in sequence, ... Besides, it is easy to prove that the ratio of the number of combinations m by m to the number of combinations $m+1$ by $m+1$ surpasses always the half of the number of the elements increased by unity. This inequality, joined to the theorem enunciated above, suffices in order to show that the number of combinations $m+1$ by $m+1$ is not of an order higher than the number of combinations $m-1$ by $m-1$, when one has only regard to the combinations formed of errors which become simultaneously the greatest of all. One demonstrates, by this means, that, in the case of two variables, the number of operations that the proposed method requires increases only as the number of errors; while, by another method, it would increase as the cube of this last number. Likewise, in the case of three variables, the number of operations which the first method requires is not of an order higher than the square of the number of operations, while, by another method, it would be of the same order as the fourth power. In general, the order of which there is question is always lowered by the first method at least by two units. One is able likewise to show that, in many particular cases, the number of operations that it requires increases only as the number of observations. It is this which holds, for example, all the time that, in the given errors, the diverse variables, with the exception of one or two, have everywhere the same numeric coefficient.

In the case where one considers only two variables, the number of operations is never able to surpass the double of the number of errors. I am arrived to this theorem by three different paths; but one alone has led me to the determination of the number of operations that one is obliged to make when the number of variables is superior to two.

(B) Finally, the theorem of page 7 as particular cases, the following three:

1° *In a polygon open out of its two extremities, the number of sides surpasses by one unit the number of vertices.*

2° *In a polyhedron open out of its superior part, the number of faces, increased by the number of vertices, surpasses by one unit the number of edges.*

3° *If one reunites, around one another, many polyhedra, some closed, the others open, in such manner that each face is common to two different polyhedra, the number of polyhedra, increased by the number of edges, will surpass by one unit the number of faces increased by the number of vertices.*

There results from the first theorem that, in each polygon, the number of vertices is equal to the one of the sides. One deduces from the second the relation which Euler has discovered among the diverse elements of a convex polyhedron. The third theorem coincides with a theorem inserted in a Memoir that I have had the honor, three years ago, to present to the Class, and that it has deigned to welcome favorably.

Geometry could not go further, because it is limited to making the three dimensions of space vary. But Analysis, restoring the propositions that we just announced to the theory of combinations, furnishes the means to extend to any number of variables.

DEMONSTRATION OF THE THEOREMS WHICH THE METHOD EXPOSED IN THIS MEMOIR SUPPOSES.

THEOREM I. — *Whatever be the number of elements contained in the errors that one considers, if one makes vary, either all these elements, or only some among them, and if one determines the values of the variable elements which render the greatest error a minimum, for the values of which there is question, many errors would become at the same time equals among them and the greatest of all, and the number of these last will surpass always by at least one unit the number of the variable elements.*

Note. — One finds, in the calculation of probabilities, a demonstration of the preceding theorem, founded on this principle, that the values of $x, y, z, \dots$ which render the greatest error a minimum, render also a minimum the sum of the infinite powers of the errors. But one can also demonstrate directly this theorem by aid of the following considerations. Since I suppose that one does not set aside the sign of the errors, I will regard a greater negative quantity as smaller than another lesser negative quantity.

Demonstration. — Let $x, y, z, \dots$ be the corrections of the elements that one supposes variables, and we designate by $\xi, \eta, \zeta, \dots$ the values of the elements which render the greatest positive error a minimum. Finally let $e_1, e_2, \dots, e_n$ be the given errors in number equal to n, and we suppose generally

$$e_r = a_r + b_r x + c_r y + d_r z + \dots,$$

whatever be the value of r. If one makes $x, y, z, \dots$ increase by arbitrary quantities $g, h, k, \dots$, the increase of the error e_r will be

$$b_r g + c_r h + d_r k + \dots$$

We suppose now that this error becomes the greatest of all for the values of $\xi, \eta, \zeta, \dots$ of the variables $x, y, z, \dots$ It is easy to prove that, for these same values, many other errors $e_s, e_t, , \dots$ will be equal to it: because, if this equality did not hold, the error e_r would remain the greatest of all for the values of $x, y, z, \dots$ very near to $\xi, \eta, \zeta, \dots$ Besides, if one makes $\xi, \eta, \zeta, \dots$ increase by very small but arbitrary quantities $g, h, k, \dots$, one will be able to fix always the signs of these quantities in a manner that the increase of e_r or

$$b_r g + c_r h + d_r k + \dots$$

becomes negative, and is changed into a diminution. Hence, the error e_r would be able yet to diminish by remaining greatest of all, and $\xi, \eta, \zeta, \dots$ would not be the values of $x, y, z, \dots$ which render the greatest error a minimum, this which is contrary to the hypothesis.

It remains to show that the number of errors which will become equals for the values $\xi, \eta, \zeta, \dots$ of the variables $x, y, z, \dots$ will be always superior at least by one unit to that of these same variables.

In fact, we designate by m the number of variables which one considers, and let

$$e_r, e_s, e_t, \ldots$$

be the errors which become at the same time equals among them and the greatest of all, when one supposes

$$x = \xi, \quad y = \eta, \quad z = \zeta, \quad \ldots$$

If the number of these errors surpasses m by one unit, the multiple equation

$$e_r = e_s = e_t = \cdots$$

will suffice in general in order to determine completely the values $\xi, \eta, \zeta, \ldots$ of the variables $x, y, z, \ldots$ But, in the contrary case, one will be able to give to these same variables some very near values of $\xi, \eta, \zeta, \ldots$ which satisfy always the equation of which there is concern, and for which the errors $e_r, e_s, e_t, \ldots$ are always the greatest of all. In order to obtain these new values, one will make $\xi, \eta, \zeta, \ldots$ increase by very small but indeterminate quantities $g, h, k, \ldots$ The increases corresponding to the errors $e_r, e_s, e_t, \ldots$ will be

$$\begin{array}{l} b_r g + e_r h + d_r k + \ldots, \\ b_s g + e_s h + d_s k + \ldots, \\ b_t g + e_t h + d_t k + \ldots, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{array}$$

and by hypothesis they must all be equal among them. This equality will determine some of the quantities $g, h, k, \ldots$ as function of the others; and, if one eliminates the first of the one of the increases of which there is concern, those which will remain after the elimination will remain entirely arbitrary. For the rest, the result will be the same, whatever be the one of the increases that one considers; and it is easy to see that this result will not contain a constant term. Hence, by giving appropriate signs to those of the quantities $g, h, k, \ldots$ which are found included, one will be able always to make so that it is negative, that is to say that it represents a diminution. Thus, in this case, the errors $e_r, e_s, e_t, \ldots$ could yet diminish by remaining the greatest of all; and $\xi, \eta, \zeta, \ldots$ would not be the values of $x, y, z, \ldots$ which render the greatest error a minimum; this which is contrary to the hypothesis.

The preceding demonstration could hold parallelly, if, the errors $e_r, e_s, e_t, \ldots$ being in number equal to $m+1$, the multiple equation

$$e_r = e_s = e_t = \ldots$$

did not suffice to determine the values of the variables represented by

$$\xi, \quad \eta, \quad \zeta, \quad \ldots$$

THEOREM II. — *The problem which is the object of the preceding Memoir can never admit but one solution, unless to admit an infinite number.*

Demonstration. — We imagine that one gives successively to z all the possible values from $-\infty$ to $+\infty$, and that for each value of z one determines the other variables $x, y, z, \ldots$ for the condition that the greatest error becomes a minimum. One will have in this manner the minima of the greatest errors corresponding to the diverse values of z, and one will be able always, by the preceding method, to obtain a minimum smaller than those which precede it and those which follow it. This put, it will be easy to prove that no other minimum can enjoy the same property. In fact, let ζ be the value of z corresponding to the one that one considers; and we suppose that one gives successively to z all the values possible from $z = \zeta$ to $z = \infty$, I say that the minimum of the greatest of all for the value of x, y, ... which render the greatest errors will go always by increasing. Because, if it was otherwise, this minimum would cease to increase for a certain value of z that I will designate by γ. Let now e_p, e_q, e_r, ... be the errors which are equals among them, and the greatest of all for the values of x, y, ... which render the greatest error a minimum, at the moment where z is at the point to attain the value γ. These errors will be in number equal to the one of the variables $x, y, z, \ldots$; and, hence, whatever be the value of z, the equation

$$e_p = e_q = e_r = \ldots$$

will determine always the values of x and of y which render a minimum the greatest of the errors $e_p, e_q, e_r, \ldots$ By virtue of this same equation, the values of $x, y, z, \ldots$become proportionals to z, the common value of the errors $e_p, e_q, e_r, \ldots$ will become also proportional to z; and, since this value increases when z is at the point to attain the value γ, it will increase still when one will make z increase beyond γ. Thus, when one is limited to consider the errors $e_p, e_q, e_r, \ldots$, if for the value γ of z the minimum of the greatest errors is designated by M, for a value of z superior to γ, this minimum will become superior to M. We suppose now that instead of considering only the errors $e_p, e_q, e_r, \ldots$, one has at the same time regard to all the given errors. The minimum of the greatest errors corresponding to a given value of z will be able only to increase when one will pass from the first hypothesis to the second. Hence, in this last case, for the values of z superior to γ, the minimum of the greatest errors will be always superior to M. This minimum will not be able to cease to increase for a certain value γ of z. But on the contrary it will go always by increasing from $z = \zeta$ to $z = \infty$. One could prove likewise that it will increase always from $z = \zeta$ to $z = -\infty$. Thus, among the minima corresponding to the diverse values of z, one alone is smaller than those which precede it and those which follow it, and that one alone resolves the proposed question.

The preceding demonstration supposes that z comes to vary on all sides from ζ, the minimum of the greatest errors begin to increase as soon as one gives to z a value greater or lesser than ζ. But it could happen that before increasing the minimum of which there is concern remain some times stationary. Then one would obtain an infinity of minima all equal among them, and corresponding to an infinity of values of z. In all the cases, as soon as one time the minimum of the greatest errors have begun to increase, it is no longer able to stop. Thus, when the question becomes indeterminate, all the values of z which resolve it are found contained between two given limits, and the minimum of the greatest errors conserve always the same value between these two limits.

THEOREM III. — *We suppose that the error e_p becomes the greatest of all: 1° for the values $\alpha_1, \beta_1, \gamma_1, \ldots$ of $x, y, z, \ldots$; 2° for the values $\alpha_2, \beta_2, \gamma_2, \ldots$ of the same variables; if one designates by α a value of x contained between α_1 and α_2, and by β, γ, … the corresponding values that one obtains for $y, z, \ldots$ by making $x = \alpha$ in the equations*

$$\frac{y-\beta_1}{\beta_2-\beta_1} = \frac{x-\alpha_1}{\alpha_2-\alpha_1},$$
$$\frac{z-\gamma_1}{\gamma_2-\gamma_1} = \frac{x-\alpha_1}{\alpha_2-\alpha_1},$$
$$\cdots\cdots\cdots\cdots$$

the error e_p will be again the greatest of all for the values α, β,γ, … of the variables $x, y, z, \ldots$

Demonstration. — In fact, we suppose that one gives successively to x all the possible values from $x = -\infty$ to $x = +\infty$, and that for each value of x one determines the values of $y, z, \ldots$ by the equations

$$(1) \qquad \begin{cases} \dfrac{y-\beta_1}{\beta_2-\beta_1} = \dfrac{x-\alpha_1}{\alpha_2-\alpha_1}, \\ \dfrac{y-\gamma_1}{\gamma_2-\gamma_1} = \dfrac{x-\alpha_1}{\alpha_2-\alpha_1}, \\ \cdots\cdots\cdots\cdots \end{cases}$$

one will obtain an infinity of systems of values of $x, y, z, \ldots$ among which the three systems will be found contained

$$\begin{array}{llll} \alpha, & \beta, & \gamma, & \ldots, \\ \alpha_1, & \beta_1, & \gamma_1, & \ldots, \\ \alpha_2, & \beta_2, & \gamma_2, & \ldots, \end{array}$$

Moreover, whatever be the system that one considers, the difference between the error e_p and any other error e_q will be a polynomial of the first degree in $x, y, z, \ldots$; and, if, one substitutes into it for $y, z, \ldots$ their values in x deduced from equations (1), this difference will become simply a polynomial in x of the first degree or of the form

$$Ax + B.$$

Now, if this polynomial remains positive for the values of α_1 and α_2 of x, it is clear that it will be yet positive for each value of x contained between α_1 and α_2. If therefore the error e_p is superior to each other e_q for the two systems

$$\begin{array}{llll} \alpha_1, & \beta_1, & \gamma_1, & \ldots, \\ \alpha_2, & \beta_2, & \gamma_2, & \ldots, \end{array}$$

it will yet be superior to all the others for the system

$$\alpha, \quad \beta, \quad \gamma, \quad \ldots$$

Corollary I. — If two, three, ..., or a greater number of errors $e_p, e_q, e_r, \ldots$ are equals among them and the greatest of all: 1° for the values $\alpha_1, \beta_1, \gamma_1, \ldots$ of the variables $x, y, z, \ldots$; 2° for the values $\alpha_2, \beta_2, \gamma_2, \ldots$ of the same variables, they enjoy still the same property for the values $\alpha, \beta, \gamma, \ldots$ of $x, y, z, \ldots$ provided however that these values satisfy equations (1), and that α is contained between α_1 and α_2.

In fact, that which one has said above concerning the error e_p is able to be applied equally to the errors $e_q, e_r, \ldots$ Moreover, each of the differences

$$\begin{gathered} e_p - e_q, \\ e_p - e_r, \\ \cdots\cdots\cdots \end{gathered}$$

becoming, in virtue of equations (1), a polynomial in x of the first degree, is not able to be null for the values α_1 and α_2 of x without being equally null for each other value α of the same variable. Hence, the errors $e_p, e_q, e_r, \ldots$ remain constantly equals among them for all the systems of values of $x, y, z,$ which satisfy the equations (1).

Corollary II. — If, for the value α_1 of the variable x, one is able to determine the other variables $y, z, \ldots$ in a manner that the error e_p becomes the greatest of all, and that one arrives to fulfill the same condition by giving to x the value α_2; one will be able again to arrive there by giving to x any one of the values contained between α_1 and α_2.

Corollary III. — If, for the values α_1 and α_2 of the variable x, one is able to determine the other variables $y, z, \ldots$ in a manner that the errors $e_p, e_q, e_r, \ldots$ become simultaneously superior to all the others, one will be able again to fulfill the same condition by giving to x any one of the values contained between α_1 and α_2.

Corollary IV. — If one considers a combination formed from l errors $e_p, e_q, e_r, \ldots$ and if, for two systems of different values of the variables $x, y, z, \ldots$ all the errors which form this combination become equals among them and the greatest of all, one will be able, by passing from one to the other system by insensible degrees, to obtain an infinity of different systems each contained between the first two, and for which the errors $e_p, e_q, e_r, \ldots$ will remain the greatest of all. Moreover, for each of these systems the values of $x, y, z, \ldots$ will satisfy always the multiple equation

$$e_p = e_q = e_r = \ldots$$

This multiple equation will determine many of the variables $x, y, z, \ldots$ as functions of the others, and the number of those which will be thus determined will be, in general, inferior by one unit to the number of errors $e_p, e_q, e_r, \ldots$, that is to say equal to

$$l - 1.$$

But it will be able to become less. If one designates this number by $k - 1$, k will be that which we will call henceforth the *order of the combination formed with the errors* $e_p, e_q, e_r, \ldots$ This order will indicate therefore, in general, the number of the errors contained in the combination that one considers: but it is able to become inferior to it, without being nevertheless ever null. It would be reduced to unity, if one had $l = 1$, that is to say if one is limited to consider an isolated error.

In that which will follow, we will occupy ourselves no longer but with errors which become the greatest of all, or with the combinations formed of errors which enjoy simultaneously this property. As for any system of values of x, y, z, … it is necessary that one, or two, or three, … or a greater number of errors become superior to all the others, to each system of values will correspond always a combination of a certain order. This put, it follows from that which precedes that the different systems which correspond to one same combination are always reunited into one same group and, consequently, contained between certain limits. The determination of these limits is the object of the following proposition:

THEOREM IV. — *The systems of values of* $x, y, z, \ldots$ *which correspond to the combinations of order* k *have for respective limits the systems which correspond to the combinations of order* $k+1$.

Demonstration. — In fact, we consider first simple errors, by having regard only to those which are able to become, each separately, superior to all the others. As it is necessary that each system of values correspond at least to one of these errors, all the systems of possible values will be found apportioned into groups, if I am able thus to express myself, among the diverse errors of which there is concern. In some of these groups, the values of the variables will remain finite always. In others, they will be able to be extended to infinity. Moreover, as one will not be able to exit from a group without passing into another, each group will be necessarily encircled by many others, which will be neighboring or contiguous with it. This put, the systems which are common to two neighboring groups and which correspond to the combinations of the second order will be evidently the limits of the errors which correspond to the simple errors or to the combinations of the first order. If one designates under the name of *contiguous errors* those which correspond to some neighboring groups, one will be able to say again that two contiguous errors have always for common limit a combination of the second order.

The different systems which correspond to the combinations of the second order, as those which correspond to the simple errors, can, in certain cases, admit only finite values of the variables, and, in other cases, many of these systems would be extended to infinity. If one designates under the name of *errors* and *definite combinations* those which can correspond only to some systems of finite values of the variables, and those which are in the contrary case under the name of *errors* and *indefinite combinations*, one will recognize without difficulty that one indefinite combination of the second order is able to serve as limit only to two indefinite errors.

We consider now the diverse combinations of the second order which serve as limits to one same simple error, and we suppose that one traverses the different systems which correspond to these combinations in a continuous manner, that is to say by making the variables increase or decrease by insensible degrees. As, in this case, one will not be able to quit the systems which correspond to a combination of the second order without encountering those which correspond to another combination of the second order, one will find in the passage from one to the others of the intermediate systems which will serve as limits to them. These intermediate systems will be those which correspond to the combinations of the third order. If one calls *contiguous combinations of the second* *order* those which correspond to some neighboring systems, one will be able to say that two contiguous combinations of the second order have always for common limit a combination of the third order.

By continuing likewise, one will show that the systems corresponding to a combination of order k have always for limits some other systems corresponding to some combinations of order $k+1$; this which one is able also to express by saying that a combination of order k has always for limits other combinations of order $k+1$. Moreover, these limits belong at the same time to the given combination and to other neighboring or contiguous combinations. Finally, an indefinite combination of order $k+1$ is able to serve as limit only to some indefinite combinations of order k.

If one designates by m the number of given variables, $m+1-k$ will be the number of variables which remain arbitrary in the systems which correspond to a combination of order k. Hence, there will remain only a single arbitrary variable in the systems corresponding to the combinations of order m. The diverse values that this variable will be able to receive will be contained between two fixed limits, of which one will be able to extend to infinity; and each of these limits, when it will be finite, will determine, for the variables $x, y, z, \ldots$, a system of values corresponding to a combination of order $m+1$. Thus each combination of order m has for limits two combinations of order $m+1$, unless to one of these limits the values of the variables do not become infinite; and, in this case, the other limit is always a combination of order $m+1$.

If one considers now the combinations of this last order, one will find that, in the corresponding systems, there no longer remains arbitrary variables, but that the variables are entirely determined. These combinations are therefore of the highest order that one is able to admit. Moreover, one has shown that it was among the systems corresponding to the combinations of this order that one must have sought the one which resolves the proposed question; and the method that we have indicated for the solution of the problem is reduced in fact to test successively many of the combinations of which there is concern. The number of these tests has therefore for limit the number of combinations of order $m+1$, and it could not increase more rapidly than this last number. Thus, in order to have a limit of the number of tests that the method requires, it matters to know how the number of the combinations of order $m+1$ increases with the number of simple errors. We will give, in this regard, the following theorems:

THEOREM V. — *Whatever be the number of elements that one considers, the number of combinations of odd order will surpass always by one unit the number of combinations of even order.*

(One supposes always that one had regard only to the combinations formed of errors which are able to become simultaneously the greatest of all.)

Demonstration. — It follows from the preceding theorem: 1° that the simple errors, compared among them two by two, have for respective limits combinations of the second order; 2° that the combinations of the second order which serve as limits to one same error, being compared two by two, have for respective limits combinations of the third order, etc. One will find likewise that the combinations of the third order which serve as limits to one same combination of the second order, being compared among them two by two, have for respective limits combinations of the fourth order, etc.; and, if one designates always by m the number of variable elements, one will see again that the combinations of the order m which serve as limits to a like combination of order $m-1$ have for respective limits combinations of the order $m+1$. Finally each combination of order m will have for limits two combinations of order $m+1$, unless one of these limits is not extended toward infinity. If therefore one increases by one unit the total number of combinations of order $m+1$, in order limits to take place which diverge toward infinity, one will find placed into some circumstances completely similar to those which would take place, if one had to consider only errors and definite combinations.

This put, we designate respectively by
M_1 the number of simple errors or combinations of the first order,
M_2 the number of combinations of the second order,
....,
M_m the number of combinations of the m^{th} order,
M_{m+1} the number of combinations of the $(m+1)^{\text{st}}$ order.

$M_{m+1}+1$ will be this last number increased by unity; and, in order to demonstrate the theorem enunciated above, it will suffice to show that one has

$$M_1+M_3+\cdots+M_m = M_2+M_4+\cdots+(M_{m+1}+1), \tag{1}$$

if m is an odd number, and

$$M_1+M_3+\cdots+(M_{m+1}+1) = M_2+M_4+\cdots+(M_m+2), \tag{2}$$

if m is an even number.
These two equations are contained in the following

$$M_1-M_2+M_3-\cdots\pm M_m\mp M_{m+1}=1, \tag{3}$$

of which it is necessary to prove exactness.

I observe first that, if the theorem contained in this equation is true, whatever be m, relatively to the total number of errors that one considers and of their respective combinations, there will subsist yet among the combinations of the second order or of a higher order, which belong to a like indefinite error. Thus, for example, if one designates by

$$P_2, \qquad P_3, \qquad P_4, \qquad \ldots, \qquad P_m, \qquad P_{m+1}$$

the combinations of the diverse orders in which the indefinite error e_p enters, one will have

$$P_2-P_3+P_4-\cdots\mp P_m\pm P_{m+1}=1. \tag{4}$$

One sees in fact, by that which has been said above, that the conditions to which the quantities $P_2,P_3,\ldots,P_m,P_{m+1}$ must satisfy are entirely similar to those in which the quantities $M_1,M_2,\ldots,M_{m+1}$ are subjected. If therefore these conditions are sufficient in order to establish, relatively to the last kind of quantities, the theorem proposed, they will suffice also in order to establish it relatively to the first.

If the error e_p instead of being an indefinite error, was a definite error, it would be necessary, by virtue of the remark made above, to diminish by one unit the number of combinations of the highest order and, hence, equation (4) would become

$$P_2 - P_3 + P_4 - \cdots \mp P_m \pm (P_{m+1} - 1) = 1. \tag{5}$$

an equation in which one must admit the superior sign, if m is an odd number, and the inferior sign in the contrary case.

It is again good to remark that the theorem contained in equation (3) is only a particular case of another more general theorem, of which here is the enunciation:

We suppose that among the given errors one may choose many of them e_p, e_q, $e_r, \ldots$, all contiguous to one another, and we designate respectively by

$$N_1, \qquad N_2, \qquad N_3, \qquad \ldots, \qquad N_m, \qquad N_{m+1}$$

the numbers of these same errors and of the combinations which contain them, by having care however to increase the number of combinations of order $m+1$ by one unit; if, among the errors e_p, e_q, $e_r, \ldots$ there is found some indefinite of them, one will have always

$$N_1 - N_2 + N_3 - \cdots \pm N_m \mp N_{m+1} = \mp 1; \tag{6}$$

the superior sign must be admitted when m will be an odd number, and the inferior sign when m will be an even number.

In order to deduce equation (3) from equation (6), it will suffice to suppose that the sequence of errors $e_p, e_q, e_r, \ldots$ contains all the definite and indefinite errors, with the exception of one alone. One has in fact, under this hypothesis,

$$N_1 = M_1 - 1, \quad N_2 = M_2, \quad N_3 = M_3, \quad \ldots, \quad N_m = M_m, \quad N_{m+1} = M_{m+1} + 1;$$

and these values, substituted into equation (6), reproduce equation (3).

If equation (6) was one time demonstrated, applying to it the reasonings which have served to deduce equation (4) from equation (3), one could obtain the following theorem:

Let e_p be any error. Let $e_q, e_r, e_s, \ldots$ be many other definite or indefinite errors, contiguous among them and to the error e_s, and we designate by

$$p_2, \quad p_3, \quad \ldots, \quad p_m, \quad p_{m+1}$$

the numbers of combinations of the diverse orders where enter, with the error e_p, one or many of the errors e_q, e_r, $e_s, \ldots$, by having need however to increase the last number by one unit, if some of the combinations that one considers are indefinite; one will have

$$p_2 - p_3 + p_4 - \cdots \mp p_m \pm p_{m+1} = \pm 1, \tag{7}$$

the superior sign must prevail if m is an odd number, and the inferior sign in the contrary case.

It is easy to understand that this last equation contains equations (4) and (5), just as equation (6) contains equation (3).

The theorem contained in equation (4), and all the other theorems reported above, rest uniquely, as one comes to see, on the one which contains equation (6). It will suffice therefore to demonstrate this last in order to establish all the others.

This put, we imagine first that the theorem contained in equation (6) had been demonstrated for a number of elements inferior by one unit to the one that one considers, or equal to $m-1$. Equations (4), (5) and (7) will be found, that way likewise, sufficiently established; and, hence, if one designates by e_p any definite error, and by

$$P_2, \quad P_3, \quad \dots, \quad P_m, \quad P_{m+1},$$

the numbers of combinations of the diverse orders which contain this same error, one will have

$$P_2 - P_3 + P_4 - \cdots \mp P_m \pm (P_{m+1} - 1) = 1. \tag{5}$$

Let now e_q be another definite error, contiguous to the error e_p; we designate by

$$Q_2, \quad Q_3, \quad Q_4, \quad \dots, \quad Q_m, \quad Q_{m+1}$$

the numbers of combinations of the diverse orders which contain the error e_q, and by

$$Q_2 - Q_2', \quad Q_3 - Q_3', \quad \dots, \quad Q_m - Q_m', \quad Q_{m+1} - Q_{m+1}'$$

the numbers of those which contain the error e_q with the error e_p;

$$Q_2', \quad Q_3', \quad \dots, \quad Q_m', \quad Q_{m+1}'$$

will be the numbers of those which contain the error e_q without the error e_p. One will have besides, by virtue of equation (5),

$$Q_2 - Q_3 + \cdots \mp Q_m \pm (Q_{m+1} - 1) = 1,$$

and, by virtue of equation (7),

$$(Q_2 - Q_2') - (Q_3 - Q_3') + \cdots \pm (Q_{m+1} - Q_{m+1}') = \pm 1.$$

If one subtracts these two equations the one from the other, one will find

$$Q_2' - Q_3' + Q_4' - \cdots \mp Q_m' \pm Q_{m+1}' = 1. \tag{8}$$

We consider again a third definite error e_r contiguous to one of the first two, or to both together. We designate by

$$R_2, \quad R_3, \quad \dots, \quad R_m, \quad R_{m+1}$$

the numbers of combinations of the diverse orders where the error e_r enters, and by

$$R_2'', \quad R_3'', \quad \dots, \quad R_m'', \quad R_{m+1}''$$

the numbers of combinations where it enters without any of the errors e_p, e_q; one will have, by virtue of equations (5) and (7),

$$R_2 - R_3 + \cdots \mp R_m \pm (R_{m+1} - 1) = 1,$$
$$(R_2 - R_2'') - (R_3 - R_3'') + \cdots \mp (R_m - R_m'') \pm (R_{m+1} - R_{m+1}'') = \pm 1,$$

and, hence,

$$R_2'' - R_3'' + R_4'' - \cdots \mp R_m'' \pm R_{m+1}'' = 1. \tag{9}$$

By continuing likewise, and considering successively many definite errors e_p, e_q, e_r, e_s, ..., one will obtain a sequence of equations similar to equations (8) and (9). Besides, if one designates by N the number of errors e_p, e_q, e_r, e_s, ..., and by

$$N_2, \quad N_3, \quad \ldots, \quad N_m, \quad N_{m+1}$$

the numbers of combinations of diverse orders which contain these same errors, one will have evidently

$$\begin{array}{lllll}
N_1 & = 1 & +1 & +1 & + \cdots, \\
N_2 & = P_2 & +Q_2' & +R_2'' & + \cdots, \\
N_3 & = P_3 & +Q_3' & +R_3'' & + \cdots, \\
\multicolumn{5}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots,} \\
N_m & = P_m & +Q_m' & +R_m'' & + \cdots, \\
N_{m+1} & = P_{m+1} & +Q_{m+1}' & +R_{m+1}'' & + \cdots.
\end{array}$$

This put, if one adds among them equations (5), (8), (9), ... or

$$P_2 - P_3 + P_4 - \cdots \mp P_m \pm (P_{m+1} - 1) = 1,$$
$$Q_2' - Q_3' + Q_4' - \cdots \mp Q_m' \pm (Q_{m+1}' - 1) = 1,$$
$$R_2'' - R_3'' + R_4'' - \cdots \mp R_m'' \pm (R_{m+1}'' - 1) = 1,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

one will have equation (6), namely

$$N_2 - N_3 + N_4 - \cdots \mp N_m \pm (N_{m+1} - 1) = N_1,$$

or, that which reverts to the same,

$$N_1 - N_2 + N_3 - \cdots \pm N_m \mp N_{m+1} = \mp 1.$$

If, among the errors e_p, e_q, e_r, ... which we have supposed each definite, some became indefinite, it would suffice, in order to have regard to this circumstance, to increase, in the preceding equation, the value of N_{m+1} by one unit.

There results from the preceding calculations that if equation (6) is true for a number of variables equal to $m-1$, it will be yet true for a number of variables equal to m. Besides, if the number of variables is reduced to unity, each error will have for limits two combinations of the second order; and each combination of the second order will be a limit common to two contiguous errors. Then, if one considers many defined and contiguous errors in number equal to N_1, and if N_2 is the number of combinations of the second order where they are found contained, one will have evidently

$$N_2 = N_1 + 1.$$

Because, all the errors of which there is question being found then contained between two determined limits, if one makes the unique variable increase from the first limit to the second, the diverse values of this variable will correspond successively: 1° to the combination of the second order which forms the first limit; 2° to one of the errors that one considers; 3° to a new combination of the second order; 4° to another error, etc.; finally, to the combination of the second order which forms the last limit. But as, having to arrive to this last combination, one will have encountered alternately some combinations and some errors, there results from it that the number of combinations will surpass by one unit the one of the errors that one considers. One will have therefore

$$N_2 = N_1 + 1 \quad \text{or} \quad N_1 - N_2 = -1.$$

Equation (6), being thus verified for the case of one variable, will be true, by virtue of that which precedes, for the case of two variables and, hence, for the case of three, of four, etc., and, in general, of any number of variables.

Equation (6) being true, equation (3) will be equally, since, in order to obtain it, it will suffice to suppose in equation (6) the number of errors $e_p, e_q, e_r, \dots$ equal to the total number of definite and indefinite errors diminished by one unit.

Scholium. — We have already remarked the geometric interpretation which was able to receive theorem (3) in the case where one considers one, two, or three variables. One is able also to present this theorem under a simple and analytic form at the same time, whatever be the number of variables, by enunciating it as it follows.

We suppose that having combined among them, in diverse ways, one by one, two by two, three by three, …, m by m the indices

$$1, \quad 2, \quad ,3, \quad \dots, \quad M_1,$$

one forms from these observations many sequences in number equal to m. Let there be respectively

[1]	$1,$	$2,$	$3,$	$\dots,$	M_1
[2]	$a_1,$	$a_2,$	$a_3,$	$\dots,$	$a_{M_1},$
[3]	$b_1,$	$b_2,$	$b_3,$	$\dots,$	$b_{M_1}.$
	$\dots,$	$\dots,$	$\dots,$	$\dots,$	$\dots$

these same sequences, which we indicate respectively by the numerals [1], [2], [3], …, $[m-1]$, $[m]$, and of which each term represents one of the combinations of which there is question. We suppose, moreover, that the first sequence being uniquely composed of the indices themselves, each term of the second is formed by a reunion of two indices, and that the terms of any one of the other sequences comprehend each of the indices contained in two or many terms of the preceding sequence, in a manner that one ends always by exhausting the terms of one sequence, by writing successively near the ones from the others those to which one or many terms of the preceding series belong in common. We suppose next that one supposes, in the sequences [2], [3], ..., [m]: 1° all the terms which do not contain the index α; 2° the index α and all those which are not found with the index α in one of the terms of the sequence [2]; and that after the deletions of which there is concern, the sequences [2], [3], [4], ..., [m] fulfill the same conditions to which the preceding sequences [1], [2], [3], ..., [$m-1$] satisfied. We suppose further that one suppresses anew, in the sequences [3], [4], ..., [m]: 1° all the terms which do not contain the index β; 2° the index β and all those which are not found with the index β in one of the terms of the sequence [3]; and that after these new deletions, the sequences [3], [4], ..., [m] fulfill the conditions to which in first place the sequences [1], [2], ..., [$m-2$] satisfied; finally that one has operated, with the same success, many consecutive deletions similar to the preceding, in a manner to conserve only the sequences [$m-1$] and [m], reduced, first, to a sequence of isolated indices, and second, to some combinations of these same indices considered two by two; and we imagine that, under this hypothesis, each index of the sequence [$m-1$] may reappear in two different terms of the sequence [m]. If the deletions indicated above succeed equally whatever be the indices $\alpha, \beta, \ldots$ and whatever be the order established between these same indices, then, in designating by

$$M_1, \quad M_2, \quad M_3, \quad \ldots, \quad M_{m-1}, \quad M_m$$

the numbers of the terms of the sequences

$$[1], \quad [2], \quad [3], \quad \ldots, \quad [m-1], \quad m],$$

one will have

$$M_1 - M_2 + M_3 - \cdots \pm M_{m-1} \mp (M_m - 1) = 1, \tag{10}$$

the superior sign must be admitted, if m is even, and the inferior sign, if m is an odd number. We have here diminished M_m by one unit, because the case that we consider corresponds to the one where all the errors would be definite.

Example. — We consider the three sequences of combinations

$$\begin{array}{c|l} [1] & 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \\ [2] & (1,2), \quad (1,3), \quad (2,3), \quad (4,5), \quad (4,6), \quad (5,6), \quad (1,4), \quad (2,5), \quad (3,6), \\ [3] & (1,2,3), \quad (4,5,6), \quad (1,4,2,5), \quad (2,5,3,6), \quad (1,4,3,6). \end{array}$$

It is easy to see that these three sequences satisfy the required conditions. Because: 1° The terms of the second result from the combinations two by two of the terms of the first, and each term of the third contains the indices contained within two terms of the second. 2° If, in the sequences [2] and [3], one suppresses all the terms which do not contain the index 1, and if, in the other terms, one conserves only the indices 2, 3 and 4 which are contained, with index 1, in the first, the second and the third term of the sequence [2]; the sequences [2] and [3] will become

$$\begin{array}{c|l} [2] & 2, \quad 3, \quad 4, \\ [3] & (2,3), \quad (2,4), \quad (3,4), \end{array}$$

Hence, the sequence [2] will no longer be formed but with isolated indices; the sequence [3], but of the combinations two by two of these same indices; and, moreover, each term of the sequence [2] will be contained in two different terms of the sequence [3]. 3° It is easy to be assured that one will obtain similar results if, instead of deleting the terms which do not contain the index 1, one deletes those which do not contain any of the other indices 2, 3, 4, ... Finally, before or after the deletions, one is able to exhaust all the terms of the sequence [3], by writing after one another those to which appear in common one or many terms of the sequence [2]; and the terms of the second sequence enjoy again the same property relatively to those of the first. This put, as the numbers of terms of the series

$$[1], \quad [2], \quad [3]$$

are respectively

$$6, \quad 9, \quad 5,$$

one must have, by virtue of equation (10),

$$6-9+(5-1)=1,$$

this which is exact.

THEOREM VI. — *If one designates by m the number of variable elements, each definite error will be found comprised at least in* $m+1$ *combinations of order* $m+1$.

Demonstration. — 1° If one considers first only one element alone, each definite error will have for limits two combinations of the second order, this which verifies the enunciated theorem.

2° We suppose that one considers two elements; and let e_p be any one definite error. Let (e_p, e_q) be one of the combinations of the second order which serve as limits to it. This combination of the second order will have itself for limits two combinations of the third order, which we designate by

$$(e_p, e_q, e_r), \quad (e_p, e_q, e_s).$$

Let $\alpha_1, \beta_1; \alpha_2, \beta_2$ be the values of the two given variables x, y, which satisfy the two double equations

$$e_p = e_q = e_r, \quad e_p = e_q = e_s;$$

the equation $e_p = e_q$ will be equivalent to this one

$$\frac{x-\alpha_1}{\alpha_2-\alpha_1} = \frac{y-\beta_1}{\beta_2-\beta_1};$$

and, if one gives to the variables x and y the values which satisfy this equation, x being contained between α_1 and α_2, the two errors e_p, e_q will become simultaneously the greatest of all. Now, if, instead supposing $e_p = e_q$, one supposes

$$e_p = e_q + \delta,$$

δ being a very small positive quantity, and if one imagines always the values of x and of y contained between those which the double equations

$$e_p = e_q + \delta = e_r, \quad e_p = e_q + \delta = e_s;$$

determine it is clear that the error e_p will remain superior to all the others, and that it will surpass even the error e_q conjointly with the error e_r, if one supposes

$$e_p = e_q + \delta = e_r,$$

and, conjointly with the error e_s, if one supposes

$$e_p = e_q + \delta = e_s.$$

If, now, one makes δ increase, by supposing always $e_p = e_q + \delta = e_r$, the errors e_p, e_r will continue to be conjointly the greatest of all, until this that a new error e_t or else the error e_s itself finishes by equaling them both for a like system of values of x and of y; and it is this which will arrive always necessarily, since, the error e_p being supposed definite, δ is not able to increase indefinitely without that the error e_p ceases to be the greatest. One will obtain therefore, by this way, a new combination of the third order, namely (e_p, e_q, e_t), (e_t being able to be equal to e_s), which will contain the error e_p; and, hence, the three combinations of the third order

$$(e_p, e_q, e_r), \quad (e_p, e_q, e_s), \quad (e_p, e_q, e_t)$$

will contain the definite error e_p, this which verifies the enunciated theorem.

3 ° We suppose that one considers three elements. Let e_p be any definite error and (e_p, e_q) one of the combinations of the second order which contains this error. As the equation $e_p = e_q$ leaves only two arbitrary variables, one will prove, as in the preceding case, that the combination of the second order of which there is concern belongs to three combinations of the fourth order. Let

$$(e_p, e_q, e_r, e_s), \quad (e_p, e_q, e_s, e_t), \quad (e_p, e_q, e_r, e_t)$$

be respectively these three combinations. If one gives to the three variables x, y. z some values which satisfy the equation

$$e_p = e_q,$$

and which are comprehended between the limits determined by the three multiple equations

$$e_p = e_q = e_r = e_s, \quad e_p = e_q = e_s = e_t, \quad e_p = e_q = e_r = e_t,$$

is to say the values for which one has

$$e_p = e_q > e_r, e_s, e_t;$$

the errors e_p, e_q will become simultaneously the greatest of all.

Now, if, instead of supposing $e_p = e_q$, one supposes $e_p = e_q + \delta$, δ being a very small positive quantity, and that one gives to x, y, z some values comprehended between those which determine the three multiple equations

$$e_p = e_q + \delta = e_r = e_s, \quad e_p = e_q + \delta = e_s = e_t, \quad e_p = e_q + \delta = e_r = e_t,$$

it is clear that the error e_p will remain superior to the others, and that it will surpass the error e_q conjointly with the errors e_r, e_s, if one supposes

$$e_p = e_q + \delta = e_r = e_s.$$

If now one makes δ increase, by supposing always

$$e_p = e_q + \delta = e_r = e_s,$$

the errors e_p, e_r, e_s will continue to be conjointly the greatest of all, until this that the error e_t or a new error e_u arrive to equal them; this which will finish necessarily by arriving, since the error e_p is supposed definite. Then, one will obtain a fourth combination of the fourth order, which will contain the error e_p, this which will verify the enunciated theorem.

By reasoning in the same manner, one will finish by demonstrating the theorem, whatever be the number of elements that one considers.

We have supposed, in that which precedes, that the combinations of the second order contained only two errors, those of the third order, three errors, etc. But it is easy to see that the same conclusions would subsist, if the number of errors of one or many combinations became superior to their order.

THEOREM VII. — *Let $e_p, e_q, e_r, \ldots$ be many errors, definite or indefinite, comprehended in one same combination of order $m+1$, each of them being able to become separately the greatest of all. Let, moreover, e_u be a fictive error which is equal to the errors $e_p, e_q, e_r, \ldots$ when these become equals among them, that is to say for the system of values of $x, y, z, \ldots$ which correspond to the combination that one considers. If the fictive error e_u becomes superior to all the others for some systems of values which rendered previously the error e_p the greatest of all, the difference $e_u - e_p$ will be necessarily positive for some of the systems of values which correspond to those of the combinations of the order m where the errors $e_q, e_r, \ldots$ enter conjointly with the error e_p.*

Demonstration. — In fact, the systems which correspond to the error e_p, that is to say to those for which the error e_p became superior to all the others, will be found now separated into two groups more. For one of these groups, one will have

$$e_p > e_u$$

and, for the other,

$$e_p < e_u.$$

Each of these groups will have for limits systems corresponding to some combinations of the second order, these of the systems corresponding to some combinations of the third order, and thus in sequence ..., until this that finally one arrives to some combinations of the order m, which will have themselves for limits the combination of the order $m+1$ that one considers. The difference $e_u - e_p$ will be therefore positive for some of the systems which correspond to those of the combinations of the order m where the errors $e_q, e_r, \ldots$ are found comprehended with the error e_p.

If the two groups of which we have spoken are reunited into one, one will have, for this last group, $e_u > e_p$, and the preceding conclusions will hold *a fortiori*.

THEOREM VIII. — *Let e_p be an error which becomes, for certain systems of values, superior to all the others. Let moreover*

$$(e_p, e_q, e_r, e_s, \ldots)$$

be one of the combinations of order $m+1$ which contain the error e_p. One will be able to imagine always a fictive error e_u which becomes equal to each of the errors $e_p, e_q, e_r, e_s, \ldots$ for the system of values which correspond to the preceding combination, and which is inferior to e_p for each other system corresponding to this last error.

Demonstration. — This theorem appears true and general. But it will suffice, for our object, to demonstrate it in the case where the number of combinations of order m, which contains the error e_p, and which has for limit the combination given of order $m+1$, does not surpass m.

This put, we designate by $\alpha, \beta, \gamma, \ldots$ the system of values of $x, y, z, \ldots$ which correspond to the combination of order $m+1$

$$(e_p, e_q, e_r, e_s, \ldots)$$

Let moreover

$$\begin{aligned} e_p &= a_p + b_p x + c_p y + d_p z + \cdots, \\ e_q &= a_q + b_q x + c_q y + d_q z + \cdots, \\ e_r &= a_r + b_r x + c_r y + d_r z + \cdots, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

We make likewise

$$e_u = a_u + b_u x + c_u y + d_u z + \cdots;$$

$a_u, b_u, c_u, d_u, \ldots$ being indeterminate coefficients. Finally we designate by A the common value of the errors $e_p, e_q, e_r, \ldots$ which correspond to the system of values $\alpha, \beta, \gamma, \ldots$ Since one supposes, in this case, the error e_u equal to the others, one will have

$$a_u + b_u \alpha + c_u \beta + d_u \gamma + \cdots = A.$$

This equation will serve to determine a_u, when one will know $b_u, c_u, d_u, \ldots$ There remains to determine these last coefficients in a manner that, for each system corresponding to error e_p and different from $\alpha, \beta, \gamma, \ldots$ the difference $e_p - e_u$ is positive.

We make, for more convenience,

$$x = \alpha + x', \quad y = \beta + y', \quad z = \gamma + z', \quad \ldots$$

One will have, in this case,

$$\begin{aligned} e_p &= A + b_p x' + c_p y' + d_p z' + \cdots, \\ e_q &= A + b_q x' + c_q y' + d_q z' + \cdots, \\ e_r &= A + b_r x' + c_r y' + d_r z' + \cdots, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ e_u &= A + b_u x' + c_u y' + d_u z' + \cdots, \end{aligned}$$

If one equals among them the preceding values of those of the errors $e_q,\ e_r, e_s, \ldots$ which enter with e_p into one same combination of order m, one will have a multiple equation, and this multiple equation will determine the ratios

$$\frac{y'}{x'}, \frac{z'}{x'}, \ldots$$

which agrees in all the systems corresponding to this combination. Let $k, l, \ldots$ be these same ratios. If one supposes that the errors comprehended in the combinations of which there is concern become superior to all the others for some positive values of $x' = x - \alpha$, the common value of these diverse errors, corresponding to any one value of x'e, will be of the form

$$A + Bx',$$

provided that one supposes

$$(1) \qquad B = b_p + c_p k + d_p l + \cdots = b_q + c_q k + d_q l + \cdots = b_r + c_r k + d_r l + \cdots ;$$

and, as in the contrary case one must have

$$e_u < e_p = A + Bx',$$

it will be necessary to suppose

$$b_u + c_u k + d_u l + \cdots < B.$$

If therefore one designates by δ a very small quantity, that one will be able besides to choose at will, and if one makes

$$B(1 - \delta) = B',$$

one will be able to suppose

$$(2) \qquad b_u + c_u k + d_u l + \cdots = B'.$$

This first equation will establish among the unknowns $b_u, c_u, d_u, \ldots$ a relation by virtue of which the difference $e_p - e_u$ will remain positive for the system of values corresponding to one of the combinations of order m which contain the error e_p, and which have for limit the combination given of order $m + 1$.

We suppose now that the number of combinations of this kind does not surpass m; one will be able to form as many equations parallel to equation (2) as there will be of similar combinations, and to determine the values of the unknowns $b_u, c_u, d_u, \ldots$ in a manner that all these equations are satisfied. Then one will be assured that the difference $e_p - e_u$ remains positive for all the systems of values corresponding to the combinations of which there is concern. Hence, this difference will be positive for all the systems of values which corresponded to the error e_p. Because, if, for some among them, it became negative, it was it again, by virtue of Theorem VII, for some of the systems corresponding to the combinations of order m that one considers.

Corollary I. — One will be able always to determine the coefficients of the fictive error e_u in a manner that this error is inferior to e_p for all the systems of values which render the error e_p superior to the others, excepting however the one which renders the errors e_p, e_q, e_r, … equals among them, and for which one will have again $e_u = e_p$. Moreover, since, for the given values of the variables $x, y, x, \ldots$ the difference

$$e_p - e_u$$

will depend on the difference $B - B' = B\delta$ and on all the similar differences each of which are able to become so small that one will judge it suitable, and that the coefficients of $e_p - e_u$, namely

$$b_p - b_u, \quad c_p - c_u, \quad d_p - d_u, \quad \ldots$$

are, thus as one is able to conclude from equations (1) and (2), of the same order as these differences; one sees that the difference

$$e_p - e_u$$

will be able itself to become less than each given quantity.

Corollary II. — We consider a system of values for which one has

$$e_q > e_p;$$

one will be able always to determine the coefficients of e_u in a manner that the difference

$$e_p - e_u$$

is inferior (setting aside the sign) to

$$e_q - e_p$$

and, hence, in a manner that e_u is inferior to e_q. Thus one will be able always to make so that the error e_u never becomes greatest of all, if it is for the system of values which corresponds to the combination of order $m+1$ that one considers, and for which one will have at the same time

$$e_p = e_q = e_r = \cdots = e_u.$$

Corollary III. — The error e_u being determined as we just said, the differences

$$e_p - e_u, \quad e_q - e_u, \quad e_r - e_u, \quad \ldots$$

will be all equal to zero for the system of values

$$\alpha, \quad \beta, \quad \gamma, \quad \ldots$$

which corresponds to the combination which one considers. But, for each other system, one or many of these differences will become positive, and if one increases indefinitely the value of $x - \alpha$, by leaving all the same values in the ratios

$$\frac{y-\beta}{x-\alpha}, \quad \frac{z-\gamma}{x-\alpha}, \quad \ldots,$$

that of the differences

$$e_p - e_u, \quad e_q - e_u, \quad \ldots$$

which will be positive, will finish by becoming greater than each given quantity.

Corollary IV. — The error e_u being always determined in the same manner, let e_v be a second fictive error, and we make

$$e_v = e_u + \varepsilon;$$

ε being a very small quantity, but arbitrary. Then, for the system of values $\alpha, \beta, \gamma, \ldots$ and for the neighboring systems, the error e_v will become superior to all the others. Moreover, as for the infinity of values of $x-\alpha$, $y-\beta$, $z-\gamma$,... some of the differences

$$e_p - e_u, \quad e_q - e_u, \quad \ldots$$

become positive and infinite, and that on the contrary the difference $e_p - e_u$ is always constant, one sees that for greater values $x-\alpha$, $y-\beta$, $z-\gamma$, ... some of the differences

$$e_p - e_v, \quad e_q - e_v, \quad \ldots$$

become positive. Hence, the systems of values which render the error e_p superior to the others is not able to be extended to infinity. This error will be therefore a definite error. Finally it is easy to see that the combinations of order k which would contain some of the errors $e_p, e_q, e_r, \ldots$ comprehended in the combination of order $m+1$ that one considers, will be found, by the addition of the error e_v transformed into some combinations of order $k+1$.

THEOREM IX. — *If one designates by* m *the number of variable elements, each combination of order* $m+1$ *will serve as limit, at least, to* $m+1$ *combinations of order* m.

Demonstration. — Let

$$(e_p, e_q, e_r, e_s, \ldots)$$

be the combination of order $m+1$ that one considers, and let e_p be one of the errors contained in this combination, an error which will be able to become superior to all the others. If the number of combinations of order m which contain the error e_p is superior to m, the theorem will be verified immediately; but, if this number is not superior to m, one will be able, by virtue of the preceding proposition (corollary IV), to imagine a fictive error e_v which is definite and which surpasses all the others for the system of values

$$\alpha, \quad \beta, \quad \gamma, \quad \ldots$$

corresponding to the combination of order $m+1$ that one considers. Moreover, if the fictive error e_v is determined by the method that we have indicated, then each of the combinations of order k which belonged to the given combination of order $m+1$ will become, by the addition of the error e_v, a combination of order $k+1$. Hence, the number of combinations of order $m+1$ which will contain the definite error e_v will be equal to the number of combinations of order m which had for common limit the given combination of order $m+1$. Besides, by virtue of theorem VI, the number of combinations of order $m+1$ which contain a like definite error is at least equal to $m+1$. It will be likewise of the number of combinations of order m which have a same limit; this which verifies the enunciated theorem.

When the number of errors contained in the combination of order $m+1$ that one considers is only equal to $m+1$, one is able again to demonstrate easily theorem IX in the following manner:

Let $e_p, e_q, e_r, \ldots$ be the errors contained in one same combination of order $m+1$, in number equal to $m+1$. These errors will become simultaneously the greatest of all, if one determines the variables $x, y, z, \ldots$ by the equation

$$e_p = e_q = e_r = \cdots$$

But, if one designates by δ a very small and positive quantity, and if one determines $x, y, z, \ldots$

$$e_p + \delta = e_q = e_r = \cdots,$$

the error e_p will become inferior to the others, and the errors e_q, e_r, $e_s, \ldots$ will be simultaneously the greatest of all. In the same case, the combination

$$(e_q, e_r, e_s, \ldots)$$

will be of order m. One will obtain therefore a combination of order m formed of errors which become simultaneously the greatest of all, if in the given combination

$$(e_p, e_q, e_r, e_s, \ldots)$$

one suppresses the first error e_p. One would arrive again to the same conclusions if, instead of suppressing the error e_p, one would suppress the error e_q, or the error $e_r, \ldots$, or some one of the other given errors. These last being, by hypothesis, in number equal to $m+1$, one will obtain, by these diverse suppressions, $m+1$ combinations of order m, which all will be found contained in the given combination.

Corollary. — Let M_{m+1} be the total number of combinations of order $m+1$, and M_m the total number of combinations of order m, as many definite as indefinite. Since each combination of order $m+1$ contains at least $m+1$ combinations of order m, and since each combination of order m has for limits two combinations of order $m+1$, if it is definite, and one alone, if it is indefinite; one will have

$$(m+1)M_{m+1} < 2M_m, \qquad \text{or} \qquad M_m > \frac{m+1}{2}M_{m+1}.$$

This inequality joined to equation (3) of theorem V serves to determine a limit to the number of operations that the method exposed in this Memoir requires.

THEOREM X. — *The number of operations that the method exposed in this Memoir requires is not of an order higher than the number of combinations* $m-1$ *by* $m-1$ *of the given errors, m being the number of variable elements.*

Demonstration. — In fact, we suppose that in having regard only to the combinations formed of errors which are able to become simultaneously the greatest of all, one designates by M_1 the number of the simple errors or combinations of the first order, and by

$$M_2, \quad M_3, \quad \ldots, M_m, \quad M_{m+1},$$

the numbers of combinations of the second, of the third, ..., finally of the m^{th} and of the $(m+1)^{\text{st}}$ order. One will have, by virtue of theorem V,

$$M_{m+1} - M_m + M_{m-1} - \cdots \pm M_2 \mp M_1 \pm 1 = 0,$$

and by virtue of theorem IX,

$$M_m > \frac{m+1}{2} M_{m+1}.$$

If one adds member to member the equation and the inequality preceding, one will have

$$M_{m-1} - M_{m-2} + \cdots \pm M_2 \mp M_1 \pm 1 > \frac{m-1}{2} M_{m+1},$$

whence one concludes

$$(1) \qquad M_{m+1} < \frac{2}{m-1}(M_{m-1} - M_{m-2} + \cdots \pm M_2 \mp M_1 \pm 1);$$

the superior sign must be admitted if m is an odd number, and the inferior sign in the contrary case. Besides M_{m+1} represents, as we have already remarked, the limit of the number of operations to make, and M_{m-1} indicates the number of combinations of order $m-1$, which is either equal or inferior to the number of combinations $m-1$ by $m-1$ of the given errors or of a part of these errors. The preceding inequality verifies therefore the enunciated theorem.

Corollary I. — If one has two variable elements, it will be necessary to suppose $m = 2$, and the preceding inequality will become

$$M_3 < 2M_1 - 2;$$

Hence, the number of operations to make will not be able to surpass $2M_1$ or the double to the number of errors.

Corollary II. — If one supposes $m = 3$, one will have

$$M_4 < M_2 - M_1 + 1;$$

Hence, the number of operations to make will not be able to be of an order superior to the number of combinations two by two, that is to say to the square of the number of errors.

Corollary III. — If one supposes $m = 4$, one will have

$$M_5 < \frac{2}{3}(M_3 - M_2 + M_1 - 1);$$

Hence, the number of operations to make will not be able to be of an order superior to M_3 or to the cube of the number of errors, etc.