Mémoire sur les résultats moyens d'un très-grand nombre d'observations

Mr. Augustin Cauchy*

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The regulation does not permit to insert this Memoir into the *Comptes rendus*, we limit ourselves to indicate summarily the principal results of it.

The author, adopting the notations of pages 264 and 265,[1] begins by reporting the formula

$$P = \frac{2}{\pi}\int_0^\infty \Phi(\theta)\frac{\sin\theta v}{\theta}d\theta, \tag{1}$$

in which one has

$$\Phi(\theta) = \phi(\lambda_1\theta)\phi(\lambda_2\theta)\dots\phi(\lambda_n\theta), \tag{2}$$

$$\phi(\theta) = 2\int_0^\kappa f(\varepsilon)\cos\theta\varepsilon\,d\varepsilon. \tag{3}$$

The function $f(\varepsilon)$, which represents the *index of probability* of the error ε, is subject to the condition

$$2\int_0^\kappa f(\varepsilon)\cos\theta\varepsilon\,d\varepsilon = 1, \tag{4}$$

to which one will satisfy, if one puts

$$f(\varepsilon) = K\varpi(\varepsilon), \tag{5}$$

$\varpi(\varepsilon)$ being an arbitrary function, but always positive, of ε, and K a positive constant determined by the formula

$$K = \frac{1}{2\int_0^\kappa \varpi(\varepsilon)d\varepsilon}. \tag{6}$$

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. July 7, 2010

[1] *Translator's note*: This refers to "Sur la probabilité des erreurs qui affectent des résultats moyens d'observations de même nature," item 527 of the *Oeuvres Complète*. The original text begins on page 264 of the *Comptes rendus*.

The *auxiliary function* $\phi(\theta)$, determined by formula (3), enjoys some remarkable properties. It is reduced to unity for a null value of θ; for each other value of θ, it is lowered numerically below unity; and if one puts

$$(7)\qquad [\phi(\theta)]^2 = \frac{1}{1+\rho\theta^2},$$

[382] the function ρ of θ will offer a positive value, whatever be θ. One will have, in particular, for $\theta = 0$,

$$(8)\qquad \rho = 2\mathrm{c},$$

the value of c being

$$(9)\qquad c = \int_0^\kappa \varepsilon^2 f(\varepsilon)\, d\varepsilon,$$

and for $\theta = \infty$

$$(10)\qquad \rho = \text{ or } > \left[\frac{1}{2f(\kappa)}\right]^2.$$

This put, let r be the smallest of the values of ρ; r will be always a positive quantity, and one will have constantly

$$(11)\qquad [\phi(\theta)]^2 = \text{ or } < \frac{1}{1+r\theta^2}.$$

When θ or $\theta\kappa$ are very small, one has

$$(12)\qquad \phi(\theta) = e^{-\varsigma\theta^2},$$

ς being the product of the constant c by a factor contained between the limits (18) of page 267, and for a stronger reason between the limits

$$(13)\qquad 1 - \left(\frac{\theta\kappa}{2}\right)^2, \quad \frac{1}{1-\mathrm{c}\theta^2}.$$

In order that the auxiliary function $\phi(\theta)$ is expressed in finite terms, it suffices that the function $\varpi(\varepsilon)$ is reduced to an entire function of ε, and of exponentials of which the exponents, real or imaginary, are proportionals to ε. The case where the function $\varpi(\varepsilon)$ is linear and of the form

$$(14)\qquad \varpi(\varepsilon) = a - b\varepsilon,$$

a, b being two positive constants, merits special attention. In this same case, one finds, by supposing $b = 0$,

$$(15)\qquad f(\varepsilon) = \frac{1}{2\kappa}, \quad \phi(\theta) = \frac{\sin\theta\kappa}{\theta\kappa}, \quad \mathrm{c} = \frac{1}{6}k^2,$$

and, by supposing $a = b\kappa$,

$$f(\varepsilon) = \frac{\kappa - \mathrm{c}}{\kappa^2}, \quad \phi(\theta) = \left(\frac{\sin\frac{\theta\kappa}{2}}{\frac{1}{2}\theta\kappa}\right)^2, \quad \mathrm{c} = \frac{1}{12}k^2, \tag{16}$$

We add that, in one and the other assumption, the value of r is given by [383] the formula

$$r = 2\mathrm{c}. \tag{17}$$

This last formula is deduced immediately from the following:

$$\frac{1}{\sin^2\theta} - \frac{1}{\theta^2} = \text{ or } > \frac{1}{3}, \tag{18}$$

to which one arrives by observing that, for some positive values of θ but inferior to $\frac{\pi}{2}$, the derivative of the function $\sin\theta\cos^{-\frac{1}{3}}\theta - \theta$, or, in other terms, the function $\frac{1}{3}\left(\cos^{-\frac{2}{3}}\theta - 1\right)^2\left(1 + 2\cos^{\frac{2}{3}}\theta\right)$, offers a value always positive.

After having established, as we have just said it, the principal properties of the auxiliary function $\phi(\theta)$, the author researches that which becomes the *probability* P in the special case where the number n of the observations become very great, and where, the factors $\lambda_1, \lambda_2, \ldots, \lambda_n$ being each very small of order $\frac{1}{n}$, or of an order inferior, the sum of their squares

$$\Lambda = \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2, \tag{19}$$

is a very small quantity of order $\frac{1}{n}$. In this case, to formula (1) one is able, without obvious error, to substitute other formulas which furnish some values very near to the probability P. Thus, for example, if one names Θ a very great number of order superior to the one of $\sqrt{n}$, but inferior to the order of n, one will have obviously, for very great values of n,

$$P = \frac{2}{\pi}\int_0^{\Theta} \Phi(\theta)\frac{\sin\theta v}{\theta}d\theta; \tag{20}$$

and, if one names λ the greatest of the factors $\lambda_1, \lambda_2, \ldots, \lambda_n$, the difference between the values of P furnished by equations (1) and (20) will be inferior, setting aside the sign, to the product,

$$\frac{1}{\pi\mathfrak{N}}e^{-\mathfrak{N}}, \tag{21}$$

$\mathfrak{N}$ being a number determined by the formula

$$\mathfrak{N} = \frac{1}{2}\frac{r\Lambda\Theta^2}{1 + r\lambda^2\Theta^2}, \tag{22}$$

[384] and, consequently, a very great number, since of the two products $\Lambda\Theta^2$, $\lambda\Theta$, the first will be very great and the second very small.

Moreover, if Θ is a very great number of which the order is inferior not only to the one of n, but also to the order of $n^{\frac{3}{4}}$, one will have again, obviously, for some very great values of n,

$$P = \frac{2}{\pi}\int_0^{\Theta} e^{-s\theta^2}\frac{\sin\theta v}{\theta}d\theta; \tag{23}$$

the value of s being

$$s = \mathrm{c}\Lambda; \tag{24}$$

and the difference between the values of P furnished by equations (20) and (23) will be inferior, setting aside the sign, to the product

$$\frac{2h\sqrt{3}}{\pi}\ln\left(\frac{\theta v}{\sqrt{3}} + \sqrt{1+\frac{\theta^2 v^2}{3}}\right), \tag{25}$$

h being the greatest of the two differences

$$e^{\frac{1}{4}s\lambda^2\Theta^4\kappa^2} - 1, \qquad 1 - e^{-\frac{\mathrm{c}s\lambda^2\Theta^4}{1+\mathrm{c}\lambda^2\Theta^2}}. \tag{26}$$

This put, if one attributes to the limit κ, a finite value, the product (25) will be, for some very great values of n, of the same order as the quantity $\Lambda\lambda^2\Theta^4\ \ln(\Theta)$, consequently, of the same order as the two quantities $\frac{\Theta^4\ln(\Theta)}{n^2}$, $\frac{\Theta^4}{n^3}$, which will become very small when the order of Θ will be inferior to the one of $n^{\frac{3}{4}}$.

Finally one will have obviously, for some very great values of n,

$$P = \frac{2}{\pi}\int_0^{\infty} e^{-s\theta^2}\frac{\sin\theta v}{\theta}d\theta = \frac{2}{\sqrt{\pi}}\int_0^{\frac{v}{2\sqrt{s}}} e^{-\theta^2}d\theta, \tag{27}$$

and the difference between the values of P furnished by formulas (23), (27) will be inferior, setting aside the sign, to the ratio

$$\frac{1-e^{-s\theta^2}}{\pi s\Theta^2}, \tag{28}$$

which will be of the order $\frac{n}{\Theta^2}$, and consequently very small when the order of Θ will be inferior to the one of $n^{\frac{1}{2}}$.

[385] Therefore, finally, if, the value of the limit κ being finite, one attributes to the factors $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the numerical values of the order of $\frac{1}{n}$, or of an inferior order, but such, that the sum Λ of their squares is of the order of $\frac{1}{n}$, then, for some very great values of n, the probability P will be generally determined with a great approximation by formula (27). If besides one assigns to the function $f(\varepsilon)$, which represents the index of probability of the error ε, a determined form, one will be able to find a limit superior to the error that one will commit, when to formula (1) one will substitute formula (27). One will be able, for example, to take for this limit the sum of the expressions (21),

(25), (28), Θ being a very great number, of which the order superior to the one of $n^{\frac{1}{2}}$ is inferior to the order of $n^{\frac{3}{4}}$.

We observe now that one draws from formulas (3), (9) and (24)

$$c = \frac{1}{2}\eta\kappa^2, \tag{29}$$

$$\frac{v}{2\sqrt{s}} = \frac{1}{\sqrt{2\eta\Lambda}}\frac{v}{\kappa}, \tag{30}$$

η being a number inferior to unity. Now it follows from formula (30) that, if one attributes to the limit v a value comparable to the limit κ, by putting, for example, $v = \kappa$, or $v = \frac{1}{2}\kappa$, or $v = \frac{1}{3}\kappa$, ..., the superior limit $\frac{v}{2\sqrt{s}}$ of the integral contained in the last member of formula (27) will be, for some very great values of n, a very great number of an order at least equal to the order of $\sqrt{n}$. Therefore then the probability P will be very near to certitude 1. This consequence subsists besides whatever be the form attributed to the function $f(\varepsilon)$.

The diverse formulas which we just transcribed permit again to value, by reducing them to their correct value, the advantages that one is able to derive from the use of such or such system of factors, by consequence of such or such method. But, forced to stop us here, we will postpone that which we would have to say on this point to another article.