Note sur les formules relatives à la détermination des orbites que décrivent les corps célestes *

M. Augustin Cauchy

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I myself propose, in a forthcoming Memoir, to demonstrate, by some numerical applications, the great advantages that my new method for the determination of the orbits of celestial bodies presents. I will limit myself, for the instant, to make, to the subject of the formulas that I have given or indicated in the preceding sessions, some remarks which will not be without utility.

We project, onto the plane of the ecliptic, the radius vector drawn from the earth to the observed star, and we name ρ the projection thus obtained. Let besides $2S$ be the area which the radius drawn from the sun to the star describes, in the unit of time. Let further $2U$, $2V$, $2W$ be the algebraic projections of the area $2S$ onto the coordinate planes, and ϖ that which W becomes, when one substitutes the earth to the star of which there is concern. In regard to the equation which makes known the value of $\frac{D_t\rho}{\rho}$, that is to say, in other terms, the value of the logarithmic derivative of ρ, the ratios of the constants

$$U, V, W - \varpi$$

[1003] to the distance ρ will be able to be immediately expressed as linear functions of ρ. Therefore, the value of ρ being one time determined by the resolution of the equation of the first degree to which it must satisfy, one will know the constants

$$U, V, W,$$

and, hence, the ratios of these constants, in the same way as the value of S. Besides U, V, W, S being known, one will know the position of the plane of the orbit; the north pole of this orbit being, on the celestial sphere, the point of which the longitude will have for tangent the ratio $\frac{V}{U}$, and of which the latitude will have for sine the ratio $\frac{W}{S}$. We add that, the semimajor axis a of the orbit being determined by aid of the equation of the lively forces, that is to say by aid of formula (8) of page 958,[1] one will be able, if one wishes, to deduce from the third law of Kepler, the time T of the revolution, and that the semiminor axis will have for value the ratio of the product ST to the semi-circumference described with the radius a. Moreover, ρ being known, one is able, in

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. July 8, 2010

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the same way as we said, to obtain immediately the value of the eccentricity ε, by aid of formula (9) on page 958, and then the semiminor axis will be found expressed by the product $a\sqrt{1-\varepsilon^2}$.

We speak now of the formula which we have given, in order to dispense the astronomers from calculating separately the corrections that the aberration of the light entails. This formula, substituted into a linear equation, of which the coefficients would be able to be deduced from four neighboring observations of the proposed star, seem, at very first, to require the use of a fifth observation, seeing that it contains the derivative of the value of ρ furnished by the equation of the first degree, or rather the part of this derivative which contains the derivatives of the first order of the longitude and of the latitude of the new star. But one is able to eliminate these derivatives of the fourth order, by means of the equation which the logarithmic derivative of ρ determines. Therefore four neighboring observations will suffice in order to determine the values, at least approximate, of the coefficients that the linear equation in ρ will contain, in the same case where one will have regard to the aberration of light.

I will remark finally that one is able with advantage to take for differential equations of the second order the complete equations of relative movement of the star that one considers around the sun, and to decompose each of the coordinates of this star into two parts, of which the first is the coordinate of the place where the observer is found placed. The distance which separates [1004] the star from the observer being projected onto the plane of the ecliptic, the projection ρ, thus obtained, and its two derivatives $D_t\rho$, $D_t^2\rho$ will be the only unknowns which will contain the three equations of movement. If besides, after having drawn from these equations the values of $D_t\rho$ and of $D_t^2\rho$, one equates $D_t^2\rho$ to the derivative of $D_t\rho$, one will arrive, by eliminating $D_t\rho$, to a new equation in ρ; and that will be able to be presented under a form such, that it is reduced, in the case where one neglects perturbations and parallax, to the equation found of the first degree. Now it is clear that this new equation in ρ will be able to be, in all cases, usefully employed and easily resolved, seeing that that of its roots which will resolve the problem will be confounded sensibly with the unique root of the equation of the first degree.

ANALYSIS

§ I. — *On the determination of the plane of the orbit.*

In conserving the notations adopted in the preceding Memoirs, we take always for the plane of x, y the plane of the ecliptic, for semiaxes of positive x and y, the straight lines drawn from the center of the sun to the first points of Aries and Cancer, and we suppose again the positive z measured on a perpendicular to the plane of the ecliptic on the side of the north pole. Let besides

x, y, z	be the coordinates of the star that one considers;
r	the distance of this star from the sun;
α, θ	the geocentric longitude and latitude of the star;
ι	the distance from this star to the earth;
ρ	the projection of this distance onto the plane of the ecliptic;
x, y	the coordinates of the earth;

R	the distance from the earth to the sun;
ϖ	the heliocentric longitude of the earth.

By putting, for brevity, $\Theta = \tan\theta$, one will find

(1) $$x = \mathrm{x} + \rho\cos\alpha, \quad y = \mathrm{y} + R\sin\alpha, \quad z = \Theta\rho,$$

and

(2) $$\mathrm{x} = R\cos\varpi, \quad \mathrm{y} = R\sin\varpi.$$

Moreover, the equations of movement of the star that one considers will give

(3) $$D_t\rho = A\rho, \quad D_t^2\rho + \frac{\rho}{r^3} = B\rho, \quad \frac{1}{r^3} - \frac{1}{R^3} = C\rho,$$

[1005] A, B, C being three functions of x, y, $\alpha, D_t\alpha, D_t^2\alpha$, Θ, $D_t\Theta$, $D_t^2\Theta$, determined by formulas (6) and (7) of page 889. Finally, the first of the formulas (3) will entail the following

(4) $$D^2\rho = (A^2 + D_tA)\rho,$$

and from this last, joined to the formulas (3), one will draw

(5) $$C\rho = B - A - D_tA - \frac{1}{R^3}.$$

Let now $2S$ be the area which, in the unit of time, the radius vector r described, and $2U$, $2V$, $2W$ the algebraic projections of this area $2S$ on the coordinate planes. Let again $\mathfrak{w}$ be that which W becomes when one substitutes the earth to the star of which there is question. One will have

(6) $$U = yD_tz - zD_ty, \quad V = zD_tx - xD_tz, \quad W = xD_ty - yD_tx;$$

(7) $$\mathfrak{w} = \mathrm{x}D_t\mathrm{y} - \mathrm{y}D_t\mathrm{x};$$

and, as the quantities

$$U, V, W$$

will be respectively proportionals to the cosines of the angles formed by a perpendicular to the plane of the sought orbit with the semiaxes of the positive coordinates, it is clear that the knowledge of these quantities, or rather of their ratios, will give the position of this same plane. Besides, by virtue of formulas (1), joined to the equation

(8) $$D_t\rho = A\rho,$$

the coordinates x, y, z, and even their derivatives D_tx, D_ty, D_tz will be found immediately expressed as linear functions of ρ. Therefore, by virtue of formulas (6), joined to equations (1) and (8), the quantities U, V, W will be expressed by some functions of ρ, entire and of the second degree. But, in these functions, the independent parts of ρ will be reduced evidently to the values which the second members of formulas (6) acquire, when one puts $x = \mathrm{x}$, $y = \mathrm{y}$, $z = 0$, that is to say, to

$$0,\, 0,\, \mathfrak{w}.$$

Therefore, by virtue of formulas (6), joined to equations (1) and (8), the quantities

$$U,\, V,\, W - \mathfrak{w}$$

[1006] will be some functions of ρ, entire and of the second degree, which will vanish with ρ; so that the ratios

$$\frac{U}{\rho},\quad \frac{V}{\rho},\quad \frac{W-\mathfrak{w}}{\rho}$$

will be reduced to some linear functions of ρ. One will find effectively

$$(9)\quad \left\{\begin{aligned} \frac{U}{\rho} &= \mathrm{y}D_t\Theta - \Theta(D_t\mathrm{y} - A\mathrm{y}) + \rho(\sin\alpha\, D_t\Theta - \Theta\cos\alpha\, D_t\alpha),\\ \frac{V}{\rho} &= -\mathrm{x}D_t\Theta + \Theta(D_t\mathrm{x} - A\mathrm{x}) - \rho(\cos\alpha\, D_t\Theta - \Theta\sin\alpha\, D_t\alpha),\\ \frac{W-\mathfrak{w}}{\rho} &= (\mathrm{x}D_t\alpha + D_t\mathrm{y} - A\mathrm{y})\cos\alpha + (\mathrm{y}D_t\alpha - D_t\mathrm{x} + A\mathrm{x})\sin\alpha + \rho D_t\alpha. \end{aligned}\right.$$

From these last formulas joined to equation (5), one will deduce immediately the values of U, V, W, and one will be able to obtain next the value of S by aid of the formula

$$(10)\qquad S = \sqrt{U^2 + V^2 + W^2}.$$

On the other hand, if one names

$$\chi \text{ and } \iota$$

the heliocentric longitude and latitude of the north pole of the orbit described by the star that one considers, one will have

$$(11)\qquad \frac{U}{S} = \cos\chi\cos\iota,\quad \frac{V}{S} = \sin\chi\cos\iota,\quad \frac{W}{S} = \sin\iota,$$

consequently

$$(12)\qquad \tan\chi = \frac{V}{U};$$

and it is clear that, the values of U, V, W, S being known, one will draw immediately the value of χ from formula (12), next the value of ι of any one of formulas (11). We add that formulas (9) and (12) will give

$$(13)\qquad \tan\chi = \frac{-\mathrm{x}D_t\Theta + \Theta(D_t\mathrm{x} - A\mathrm{x}) - \rho(\cos\alpha\, D_t\Theta + \Theta\sin\alpha\, D_t\alpha)}{\mathrm{y}D_t\Theta - \Theta(D_t\mathrm{y} - A\mathrm{y}) + \rho(\sin\alpha\, D_t\Theta - \Theta\cos\alpha\, D_t\alpha)}$$

One would be able, moreover, to arrive again to the value of $\tan\chi$, which will furnish equation (13), by aid of another method that we are going to indicate.

It suffices to add among them formulas (9), respectively multiplied by the factors

$$\cos\alpha, \quad \sin\alpha, \quad \Theta,$$

[1007] in order to eliminate at the same time from these formulas the quantities $D_t\mathrm{x}$, $D_t\mathrm{y}$ and A. One finds thus:

$$U\cos\alpha + V\sin\alpha + (W - \mathfrak{w})\Theta = \Lambda\rho, \tag{14}$$

the value of Λ being

$$\Lambda = (\mathrm{x}\cos\alpha + \mathrm{y}\sin\alpha)\Theta D_t\alpha - (\mathrm{x}\sin\alpha - \mathrm{y}\cos\alpha)D_t\Theta, \tag{15}$$

or, that which reverts to the same,

$$\Lambda = R[\Theta\cos(\alpha - \varpi)D_t\alpha - \sin(\alpha - \mathfrak{w})D_t\Theta]. \tag{16}$$

Besides, by differentiating twice in succession equation (14), and by having regard to formulas (4), (8), one will find

$$\begin{cases} UD_t\cos\alpha + VD_t\sin\alpha + (W - \mathfrak{w})D_t\Theta = (A\Lambda + D_t\Lambda)\rho, \\ UD_t^2\cos\alpha + VD_t^2\sin\alpha + (W - \mathfrak{w})D_t^2\Theta = (A^2\Lambda + \Lambda D_tA + 2AD_t\Lambda + D_t^2\Lambda)\rho, \end{cases} \tag{17}$$

Now it is clear that formulas (14), (17) will suffice in order to determine the mutual ratios of the four quantities

$$U, V, W - \mathfrak{w}, \rho.$$

There is more: by putting, for brevity,

$$\frac{\Lambda}{\Theta} = \lambda, \quad \frac{\cos\alpha}{\Theta} = \mu, \quad \frac{\sin\alpha}{\Theta} = \nu, \tag{18}$$

one draws from formula (14)

$$U\mu + V\nu + W - \mathfrak{w} = \lambda\rho. \tag{19}$$

On the other hand, by differentiating equation (19), one will find

$$UD_t\mu + VD_t\nu = (A\lambda + D_t\lambda)\rho, \tag{20}$$

or, that which reverts to the same,

$$\rho = \frac{UD_t\mu + VD_t\nu}{A\lambda + D_t\lambda}. \tag{21}$$

This put, the equation

$$A = \frac{D_t\rho}{\rho} = D_t\ln(\rho)$$

will give evidently

$$\frac{UD_t^2\mu + VD_t^2\nu}{UD_t\mu + VD_t\nu} = A + \frac{D_t(A\lambda + D_t\lambda)}{A\lambda + D_t\lambda}, \tag{22}$$

[1008] or, that which reverts to the same,

$$\frac{D_t^2\mu + \tan\chi\, D_t^2\nu}{D_t\mu + \tan\chi\, D_t\nu} = A + \frac{D_t(A\lambda + D_t\lambda)}{A\lambda + D_t\lambda}, \tag{23}$$

next one will conclude from it

$$\tan\chi = -\frac{(A\lambda + D_t\lambda)D_t^2\mu - [(A^2 + D_tA)\lambda + 2AD_t\lambda + D_t^2\lambda]D_t\mu}{(A\lambda + D_t\lambda)D_t^2\nu - [(A^2 + D_tA)\lambda + 2AD_t\lambda + D_t^2\lambda]D_t\nu}. \tag{24}$$

Now formulas (13) and (14) is confounded with one another, when one substitutes, in the first, the value of ρ drawn from formula (5), and in the second, the values of λ, μ, ν, drawn from formulas (15) and (18), by having regard besides to the two equations

$$D_t^2\mathrm{x} + \frac{\mathrm{x}}{R^3} = 0, \quad D_t^2\mathrm{y} + \frac{\mathrm{y}}{R^3} = 0. \tag{25}$$

It is good to observe that, if one eliminates ρ between formula (14) and the first of equations (17), one will find

$$\begin{cases} U[D_t\cos\alpha - (A + D_t\ln\Lambda)\cos\alpha] + V[D_t\sin\alpha - (A + D_t\ln\Lambda)\sin\alpha] \\ \qquad + [D_t\Theta - (A + D_t\ln\Lambda)\Theta](W - \mathfrak{w}) = 0. \end{cases} \tag{26}$$

This linear equation, among the constants U, V, $W - \mathfrak{w}$, is one of those which Mr. Michal has obtained [*see* page 973]. Besides, in this same equation, the coefficients of U, V, W contain the quantities

$$\alpha, \quad D_t\alpha, \quad D_t^2\alpha, \quad \Theta, \quad D_t\Theta, \quad D_t^2\Theta$$

of which the values are able to be determined, at least approximately, by aid of three neighboring observations. Finally, it is clear that two equations of the form (26), constructed by aid of two series of observations, will suffice in order to determine the mutual ratios of the three constants

$$U, V, W - \mathfrak{w}.$$

A third equation of the same form, constructed by aid of a third series of observations, and joined to the first two equations, would be able to serve only to test those, and not to determine the values of the three constants, as Mr. Michal has appeared to believe [page 973]. We add that, if the second series of observations is approached indefinitely from the first, the ratios of the three constants

$$U, V, W - \mathfrak{w}$$

[1009] will be found determined by equation (26) joined to its derivative, or, that which reverts to the same, by the two formulas what one draws from equations (17), by substituting the value of ρ furnished by equation (14). Therefore then one will obtain, for value of the ratio,

$$\frac{V}{U} = \tan\chi,$$

that which formulas (13) and (24) give simultaneously.

§II — *On the correction which the aberration of light requires.*

Let always r be the distance from the earth to the observed star, and ρ the projection of this distance onto the plane of the ecliptic. Let, moreover,

$$\rho = K \tag{1}$$

be the value of ρ furnished by equation (5) from § I. K will be able to be considered as a function of the sole variable quantities

$$R,\, \varpi,\, D_t\varpi;$$
$$\alpha,\, D_t\alpha,\, D_t^2\alpha,\, D_t^3\alpha,\, \theta,\, D_t\theta,\, D_t^2\theta,\, D_t^3\theta,$$

of which the first three are related to the movement of the earth, and the others to the movement of the observed star. Moreover, as the value K of ρ to be to verify the equation

$$D_t\rho = A\rho,$$

one will have identically

$$D_tK = AK. \tag{2}$$

Finally, it is clear that, in the derivative D_tK, one will be able to distinguish two parts, of which the one G, relative to the movement of the earth, and produced by the variation of the quantities

$$R,\, \varpi,\, D_t\varpi,$$

will contain two new derivatives

$$D_tR,\, D_t^2\varpi,$$

while the other part H, relative to the movement of the observed star, will be produced by the variation of the quantities

$$\alpha,\, D_t\alpha,\, D_t^2\alpha,\, D_t^3\alpha,\, \theta,\, D_t\theta,\, D_t^2\theta,\, D_t^3\theta,$$

[1010] and will contain two new derivatives, namely

$$D_t^4\alpha,\, D_t^4\theta.$$

We add that, in formula (2), combined with the identical equation

$$D_tK = G + H, \tag{3}$$

one will draw immediately

$$G + H = AK. \tag{4}$$

These principles being admitted, we examine attentively the nature of the correction which the aberration of light requires. According to that which has been said in the preceding session, one will be able to introduce immediately into the calculation of the values of $\alpha, D_t\alpha, \ldots, \theta, D_t\theta \ldots$, drawn from the observations, if in equation (1) one substitutes the following:

$$\rho = \frac{K}{1 - \frac{H}{\delta \cos\theta}}, \tag{5}$$

δ being the speed of light. Now, at the very first, the approximate determination of the quantity H which contains $D_t^4\alpha$ and $D_t^4\theta$, would seem to require five observations of the star, made in some epochs near one another, that is to say one observation more than the approximate determination of K. But, it is important to remark, one is able to substitute into formula (5) the value of H drawn from equation (4), and one finds then

$$\rho = \frac{K}{1 - \frac{AK-G}{\delta \cos\theta}}, \tag{6}$$

or very nearly

$$\rho = K\left(1 + \frac{AK-G}{\delta\cos\theta}\right). \tag{7}$$

Now, in the second member of formula (6) or (7), the only variable quantities which are related to the movement of the observed star are those which were already contained in the value of K, namely,

$$\alpha, D_t\alpha, D_t^2\alpha, D_t^3\alpha, \theta, D_t\theta, D_t^2\theta, D_t^3\theta,$$

that is to say from the quantities of which the approximate values are able to be deduced from four observations made in some instants near one another. [1011]

§ III. — *On the determination of the orbit which a star describes around the sun, in the case where one takes account of the perturbing actions, and of the position that the observer occupies on the surface of the earth.*

The center of the sun being take for origin of the coordinates, and the plane of the ecliptic for the plane of x, y we name always x, y, z the coordinates of the observed star. Let, moreover, X, Y, Z be the algebraic projections of the accelerative force which solicits this star in its relative movement around the sun. The equations of this movement will be

$$D_t^2x + X = 0, \quad D_t^2y + Y = 0, \quad D_t^2z + Z = 0. \tag{1}$$

Let be besides, at the end of time t,

τ the distance from the observer to the star that one considers;

ρ the projection of this distance onto the plane of the ecliptic;

α, θ the longitude and the latitude of the star, measured with respect to the place that the observer occupies; finally,

x, y, z the coordinates of this same place. One will have

$$(2) \qquad x = \mathrm{x} + \tau\cos\alpha\cos\theta, \quad y = \mathrm{y} + \tau\sin\alpha\cos\theta, \quad z = \mathrm{z} + \tau\sin\theta,$$

$$(3) \qquad \rho = \rho\cos\theta,$$

and, hence,

$$(4) \qquad x = \mathrm{x} + \rho\cos\alpha, \quad y = \mathrm{y} + \rho\sin\alpha, \quad z = \mathrm{z} + \Theta\rho,$$

the value of Θ being

$$(5) \qquad \Theta = \tan\theta.$$

On the other hand, if one takes for unity the mass of the sun, and if one names r the distance from the sun, to the observed star, one will have, not only

$$(6) \qquad r^2 = x^2 + y^2 + z^2,$$

but also

$$(7) \qquad X = \frac{x}{r^3} + \mathfrak{X}, \quad Y = \frac{y}{r^3} + \mathfrak{Y}, \quad Z = \frac{z}{r^3} + \mathfrak{Z},$$

$\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$ being some functions of t and of ρ, which will be of the order of the perturbing forces. This put, by naming R the distance from the earth to the sun, one will draw from equations (1), joined to formulas (4) and (7),

$$(8) \qquad D_t\rho = A\rho, \quad D_t^2\rho + \frac{\rho}{r^2} = B\rho, \quad \frac{1}{r^3} + \frac{1}{R^3} = C\rho,$$

[1012] the values of the coefficients A, B, C being determined by the system of formulas

$$(9) \qquad \begin{cases} C\mathrm{x} + [B - (D_t\alpha)^2]\cos\alpha - (D_t^2\alpha + 2AD_t\alpha)\sin\alpha + \mathfrak{L} = 0, \\ C\mathrm{y} + [B - (D_t\alpha)^2]\sin\alpha - (D_t^2\alpha + 2AD_t\alpha)\cos\alpha + \mathfrak{M} = 0, \end{cases}$$

$$(10) \qquad B\Theta + 2AD_t\Theta + D_t^2\Theta + \mathfrak{N} = 0,$$

and the values of $\mathfrak{L}, \mathfrak{M}, \mathfrak{N}$ being

$$(11) \qquad \mathfrak{L} = \frac{\mathfrak{X} + D_t^2\mathrm{x} + \frac{\mathrm{x}}{R^3}}{\rho}, \quad \mathfrak{M} = \frac{\mathfrak{Y} + D_t^2\mathrm{y} + \frac{\mathrm{y}}{R^3}}{\rho}, \quad \mathfrak{N} = \frac{\mathfrak{L} + D_t^2\mathrm{z} + \frac{\mathrm{z}}{R^3}}{\rho}.$$

We add that one will draw from formulas (8)

$$(12) \qquad C\rho = B - A^2 - D_tA - \frac{1}{R^3}.$$

If, by reducing to zero the perturbing forces, one made the point coincide, of which the coordinates are designated by x, y, z, with the center of the earth, one would have

$$(13)\qquad \begin{cases} \mathfrak{X} = 0, & \mathfrak{Y} = 0, & \mathfrak{Z} = 0; \\ D_t^2 \mathrm{x} + \dfrac{\mathrm{x}}{R^3} = 0, & D_t^2 \mathrm{y} + \dfrac{\mathrm{y}}{R^3} = 0, & D_t^2 \mathrm{z} + \dfrac{\mathrm{z}}{R^3} = 0; \end{cases}$$

and, hence,

$$\mathfrak{L} = 0, \quad \mathfrak{M} = 0, \quad \mathfrak{N} = 0.$$

Therefore then, the values of the coefficients A, B, C, furnished by equations (9), would become independent of ρ, and the value of ρ would be found immediately given by equation (12), that is to say by an equation of the first degree.

In the general case where the conditions (13) cease to be verified, the three sums

$$(14)\qquad D_t^2 \mathrm{x} + \frac{\mathrm{x}}{R^3} \quad D_t^2 \mathrm{y} + \frac{\mathrm{y}}{R^3}, \quad D_t^2 \mathrm{z} + \frac{\mathrm{z}}{R^3},$$

are some known functions of t; but

$$\mathfrak{X},\ \mathfrak{Y},\ \mathfrak{Z},\ \mathfrak{L},\ \mathfrak{M},\ \mathfrak{N}$$

are reduced to some known functions of t and of ρ. Therefore, in passing from the [1013] particular case, where $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$ vanish, to the general case, one will see the coefficients

$$A, B, C$$

which were first independent of ρ, to acquire very small increases which will be represented by some determined functions of t and of ρ. We add that in regard to the first of the formulas (8), the very small increase of $D_t A$ will be able itself to be expressed as function of t and of the unknown ρ. This put, it is clear that, in the general case, one will be able to determine again the unknown ρ by aid of formula (12), which, without being then of the first degree with respect to ρ, will offer, at least, a root very little different from a value of ρ furnished by a linear equation. Besides, this root being common to formula (12) and to the last of formulas (8), one will be able to deduce from these formulas by making the radicals disappear, and recurring next to the method of greatest common divisor.

It is good to remark that, in the case where the observed star is a comet, and where one takes account of a single perturbing force, namely, of the action exercised on that comet by the earth, the values of $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ are sensibly proportional to $\frac{1}{\rho^2}$. Therefore then the three ratios $\frac{\mathfrak{X}}{\rho}$, $\frac{\mathfrak{Y}}{\rho}$, $\frac{\mathfrak{Z}}{\rho}$ will be sensibly proportional to $\frac{1}{\rho^3}$, and, by neglecting the quantities comparable to the square of the perturbing force, one will see formula (12) is reduced to an equation in ρ of the fourth degree.

We have, in that which precedes, set aside corrections which the aberration of light requires; and formula (12) supposes that the values of α and θ have each undergone the correction due to this cause. But it will suffice to apply to equation (12) the principles established in the preceding Memoir, in order to transform it into a new equation in which one will be able to substitute immediately the values of α, θ, $D_t\alpha, D_t\theta, \ldots$ drawn from observations.

We will make here a last remark, relative to formulas (18) and (23) from pages 891 and 892. The first of these two formulas contain only, with the angles α and θ, their derivatives of the first and second order, that is to say of the quantities of which the approximate values are able to be determined by aid of three observations. If one wishes that formula (23) on page 892 enjoys in the same property, it will be necessary to eliminate from this formula, by aid of equation (12), the derivative D_tA contained in the value of $D_t\Omega$. But then one will obtain an equation ρ which will be identical. Therefore formula (23) [1014] on page 892 is nothing but an equation of third degree, of which the first member is exactly divisible by

$$\rho - K,$$

K being the value of ρ that equation (12) on page 890 furnishes.