Mémoire sur la détermination et la correction des élements de l'orbite d'un astre *

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One is occupied since a long time in the determination of the elements of the orbit of a star; and this problem, to which are brought back already the remarkable works of Newton and of Euler, has been in our day again a subject of deeper researches. To my knowledge, one of the most recent works on the correction of the elements of an orbit is the Memoir presented at the Institute by Mr. Yvon Villarceau, toward the end of the year 1845, and approved by the Academy. The Report made, in the name of the Commission, by Mr. Binet, offers a clear and precise résumé of the question and of the solutions that diverse authors have given to it. In the Memoir of Mr. Yvon Villarceau, the approximate linear equations whence are drawn the corrections of the elements are, following custom, deduced, by aid of the differential calculus, from the finite equations of movement, in which one makes the six elements vary by very small quantities. Of the equations thus formed are those that it is proper to use, when one wishes to obtain the values of the elements with a great exactitude. Besides, in order to draw the possible better part of these same equations, of which the number increases with the one of the given observations, it is proper to recur, for their solution, to one method which have the double advantage to be able to be easily practiced, and to offer a great certainty in the results of the calculation. As I have already remarked in the last session, these two conditions will be fulfilled, if one resolves the approximate equations of which there is question, by a process analogous to the one on which my new method of interpolation is applied. I will add that the same process furnishes again the means to calculate easily the probable errors introduced by the observations in the values of the geocentric longitude and latitude of a star. Finally, by aid of some artifices, which will be indicated here later, one is able not only to simplify the formulas relative to the determination or to the correction of the elements of an orbit, but also to make so that it becomes nearly impossible to commit the least error of calculation without being immediately warned of it by the formulas themselves.

We remark again that the method here above recalled would be able to be applied, if one wishes, no longer to the linear equations furnished by the diverse observations, but to those that one deduces from them by the method of *least squares*, whatever be besides the unknowns, represented, either by the six elements of the orbit of the

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. July 10, 2010 observed star, or by three geocentric longitudes [701] and latitudes each, as our young and illustrious colleague, Mr. Le Verrier, has proposed in a justly researched Memoir of the true friends of the science.

§ I. — *On the determination of the elements of an orbit of a star.*

Of the advantages inherent to the method of interpolation that I have applied to the development of the geocentric longitude and latitude of a star, one consists in this that one is unable to commit the least error without being warned almost immediately by the calculation. It is important to use for the determination of the elements of an orbit some formulas which present the same advantage, and which, being besides very simple, lend themselves easily to the use of logarithms. One satisfies these diverse conditions by operating as it follows.

We conserve the notations adopted in the preceding Memoir. After having calculated r, ρ and $D_t\rho = A\rho$, one will draw from the equations

$$\mathfrak{x} = \rho + R\cos\chi, \quad \mathfrak{y} = -R\sin\chi, \quad \mathfrak{z} = \rho\tan\theta, \tag{1}$$

the values of the coordinates $\mathfrak{x}, \mathfrak{y}, \mathfrak{z}$, and one will verify the exactitude of these values by aid of the formula

$$\mathfrak{x}^2 + \mathfrak{y}^2 + \mathfrak{z}^2 = r^2; \tag{2}$$

next one will draw from the equations

$$\left\{\begin{array}{lcl} \mathfrak{u} & = & D_t\rho + D_tR\cos\chi + RD_t\varpi\sin\chi, \\ \mathfrak{v} & = & \rho D_t\phi + D_tR\sin\chi + RD_t\varpi\cos\chi, \\ \mathfrak{w} & = & (D_t\rho + \rho D_t\phi)\tan\theta = (A + D_t\Theta)\mathfrak{z}, \end{array}\right. \tag{3}$$

the values of the speeds $\mathfrak{u}, \mathfrak{v}, \mathfrak{w}$; and one will verify the exactitude of these values by aid of the formula

$$G = \mathfrak{s}D_tR + \varsigma D_t\rho + \rho\mathfrak{y}D_t\chi + \mathfrak{z}^2 D_t\Theta, \tag{4}$$

the values of G, $\mathfrak{s}$, ς being

$$G = rD_tr = \mathfrak{u}\mathfrak{x} + \mathfrak{v}\mathfrak{y} + \mathfrak{w}\mathfrak{z}, \tag{5}$$

$$\left\{\begin{array}{lcl} \mathfrak{s} & = & R + \rho\cos\chi = \mathfrak{x}\cos\chi - \mathfrak{y}\sin\chi, \\ \varsigma & = & R\cos\chi + \rho\sec\theta = \mathfrak{x} + \mathfrak{z}\tan\theta; \end{array}\right. \tag{6}$$

[702] next finally one will draw from the equations

$$\omega = \sqrt{\mathfrak{u}^2 + \mathfrak{v}^2 + \mathfrak{w}^2}, \tag{7}$$

$$\mathfrak{U} = \mathfrak{w}\mathfrak{y} - \mathfrak{v}\mathfrak{z}, \quad \mathfrak{V} = \mathfrak{u}\mathfrak{z} - \mathfrak{w}\mathfrak{x}, \quad W = \mathfrak{v}\mathfrak{x} - \mathfrak{u}\mathfrak{y}, \tag{8}$$

$$H = \sqrt{\mathfrak{U}^2 + \mathfrak{V}^2 + W^2}, \quad I = \sqrt{\mathfrak{U}^2 + \mathfrak{V}^2}, \tag{9}$$

the values of the speed ω and of the areas $\mathfrak{U}, \mathfrak{V}, W, H, I$, and one will verify the exactitude of these values by aid of the formulas

$$\omega^2 r^2 = G^2 + H^2, \quad H^2 = I^2 + W^2. \tag{10}$$

This put, one will be able to determine the inclination ι and the longitude $\text{☋}$of the ascendant node by aid of the equations

$$\sin(\text{☋} - \phi) = \frac{\mathfrak{U}}{I}, \quad \cos(\text{☋} - \phi) = -\frac{\mathfrak{V}}{I}, \quad \cos\iota = \frac{W}{H}, \quad \sin\iota = \frac{I}{H}, \tag{11}$$

which are mutually verified, since they furnish at the same time the sine and the cosine of each of the angles ι, $\text{☋} - \phi$; next, after having drawn the value of $D_t r$ from the equation $D_t r = \frac{G}{r}$, one will determine the angle p by aid of the formulas

$$\sin p = \frac{\mathfrak{z}}{r\sin\iota}, \quad \cos p = \frac{\mathfrak{w}r - \mathfrak{z}D_t r}{H\sin\iota}, \tag{12}$$

which will verify again each other. We add that if one puts $\mathfrak{x} = \mathfrak{r}\cos\Phi$, $\mathfrak{y} = \mathfrak{r}\sin\Phi$, and, consequently,

$$\tan\Phi = \frac{\mathfrak{y}}{\mathfrak{x}}, \quad \mathfrak{r} = \frac{\mathfrak{x}}{\cos\Phi} = \frac{\mathfrak{y}}{\sin\Phi},$$

one will be able, in the first two of formulas (11), to substitute with advantage the following:

$$\cos(\phi + \Phi - \text{☋}) = \frac{r}{\mathfrak{r}}\cos p, \quad \sin(\phi + \Phi - \text{☋}) = \frac{r}{\mathfrak{r}}\sin p\cos\iota. \tag{13}$$

Finally, after having determined a, ε, by the equations

$$\frac{1}{a} = \frac{2}{r} - \frac{\omega^2}{K}, \quad 1 - \varepsilon^2 = \frac{H^2}{Ka}, \tag{14}$$

one will determine ψ and $p - \mathfrak{p}$ by aid of the formulas

$$\cos\psi = \frac{a-r}{a\varepsilon}, \quad \sin\psi = \frac{G}{\lambda a^2 \varepsilon}, \tag{15}$$

[703]

$$\cos\frac{p-\mathfrak{p}}{2} = \left(\frac{a}{r}\right)^{\frac{1}{2}}(1-\varepsilon)^{\frac{1}{2}}\cos\frac{\psi}{2}, \quad \sin\frac{p-\mathfrak{p}}{2} = \left(\frac{a}{r}\right)^{\frac{1}{2}}(1+\varepsilon)^{\frac{1}{2}}\sin\frac{\psi}{2}. \tag{16}$$

Then there will remain no more but to determine the element c by aid of the formula

$$c = \psi - \varepsilon\sin\psi - \lambda t,$$

which is reduced simply to

$$c = \psi - \varepsilon\sin\psi,$$

when one supposes the time t counted starting from the epoch of mean observation.

I will make, in ending, a last remark. If one names $\mathfrak{O}, \mathfrak{Q}$ the algebraic projections of the area H onto the planes perpendicular to the radius vectors r, R drawn from the earth to the observed star, and from the sun to the earth, one will have

$$\mathfrak{O} = \mathfrak{U}\cos\theta + W\sin\theta, \quad \mathfrak{Q} = U\cos\chi - \mathfrak{V}\sin\chi; \tag{17}$$

and by putting, for brevity,

$$\Lambda = D_t\phi\cos\chi - D_t\Theta\sin\chi, \quad P = R^2 D_t\varpi\sin\theta, \quad Q = \frac{1}{2}\Lambda R^2\sin 2\theta,$$

one will draw from formulas (1), (3), (8),

$$\rho\mathfrak{O} = -R\mathfrak{Q}\cos\theta, \quad \mathfrak{O} - P = \Lambda R\rho\sin\theta; \tag{18}$$

consequently,

$$\mathfrak{O}(\mathfrak{O} - P) + Q\mathfrak{Q} = 0. \tag{19}$$

Let now $\rho, \mathfrak{O}_{\prime}, \mathfrak{Q}_{\prime}$ and $\rho, \mathfrak{O}_{\prime\prime}, \mathfrak{Q}_{\prime\prime}$ that which $\rho, \mathfrak{O}, \mathfrak{Q}$ becomes when one attributes to time t two new values $t_{\prime}$, $t_{\prime\prime}$. If one takes for unknowns $\rho, \rho_{\prime}$, $\rho_{\prime\prime}$ or $\mathfrak{O}, \mathfrak{O}_{\prime}$, $\mathfrak{O}_{\prime\prime}$, the three quantities $\mathfrak{Q}, \mathfrak{Q}_{\prime}$, $\mathfrak{Q}_{\prime\prime}$ will be able to be considered as functions of these unknowns; and equation (19), joined to those that one deduces from it, when at time t one substitutes $t_{\prime}$ or $t_{\prime\prime}$, will furnish a means to determine with $\rho, \rho_{\prime}$, $\rho_{\prime\prime}$ the elements of the orbit of the observed star.

Moreover, I will return, in another article, to this subject, to which is brought back more or less directly to the work of some geometers, and particularly a Memoir of Lagrange, inserted into the *Connaissance des Temps* for 1821.[1]

§ II. — *On the correction of the elements of the orbit of a star.*

We suppose the six elements $a, \varepsilon, c, \mathfrak{p}$, ♉, ι determined approximately [704] by aid of the formulas indicated in § I. In order to obtain the conditions to which the corrections $\delta a, \delta\varepsilon$, δc, $\delta\mathfrak{p}$, δ♉, $\delta\iota$ of these same elements, supposed very small, must satisfy, it will suffice to transform into approximate linear equations the finite equations of movement of the observed star. We enter to this subject in some details.

Let, as in the preceding Memoir, ϕ, θ be the geocentric longitude and latitude of the observed star, r the distance from this star to the sun, and p its heliocentric longitude measured in the plane of the orbit, starting from the ascendant node. The values of r and of p will be determined, at the end of time t, by the formulas (2) on page 654,[2] as function of $t, a, \varepsilon, c, \mathfrak{p}$, and by the formulas (8) on page 655, as function of ϕ, θ, ♉, ι. We name $\partial r, \partial p$ the increases that the calculated values of r and of p take as we just said, when one passes from formulas (2) to formulas (8). Let, besides $\delta r, \delta p$ be the variations of r and of p which correspond, by virtue of formulas (2), to some very small

[1] *Translator's note*: This must refer to "Nouvelle méthode pour déterminer l'orbite des comètes d'après les observations," *Connaissance des temps..., pour l'an* 1821 (1818), pp.469–483.

[2] See "Application des formules que fournit la nouvelle méthode d'interpolation à la résolution d'un système d'équations linéares approximatives, et, en particulier, à la correction des élements de l'orbite d'un astre." *Comptes Rendus Hebd. Séances Acad. Sci.* 25.

variations δa, $\delta\varepsilon$, δc, $\delta\mathfrak{p}$ of the four elements $a, \varepsilon, c, \mathfrak{p}$. Let, on the contrary, dr, dp be the variations of r and of p which correspond, by virtue of formulas (8), to some very small variations $\delta\forall$, $\delta\iota$ of the elements $\forall$ and ι. If the values of ϕ, θ corresponding to one observation, consequently to a given value of t, are not affected by any error, one will have, for this value of t,

$$\partial r + dr = \delta r, \quad \partial p + dp = \delta p,$$

or that which reverts to the same,

(1) $$\delta r - dr = \partial r, \quad \delta p - dp = \partial p.$$

Such is the general form of the two linear equations that will form each observation among the six corrections δa , $\delta\varepsilon$, δc, $\delta\mathfrak{p}$, $\delta\forall$, $\delta\iota$. By applying to the equations thus formed the method of resolution indicated in the preceding Memoir, not only one will obtain, for the six elements, some corrections which should inspire a great confidence, but, moreover, the known terms $\partial r, \partial p$ will give birth to some finite differences of diverse orders, of which the last will represent the probable errors introduced in r and p by the errors of which ϕ, θ are found affected by virtue of the given observations. Consequently, it will become easy to form two new linear equations proper to furnish the probable errors of ϕ and of θ corresponding to each observation.

It is good to observe that by virtue of formulas (2) on page 654, r is a sole function of a, ε, c, t. Therefore the first of the formulas (1) will not [705] contain the correction $\delta\mathfrak{p}$, and will be able to determine separately the corrections of the five elements a, ε, c, $\mathfrak{p}$, $\forall$, ι, if the concern is to fix the orbit of a star which has been observed more than four times. In order to be in a state to apply to the determination of this orbit the first of the formulas (1), it suffices to understand the values of δr and dr expressed as linear functions of the corrections δa, $\delta\varepsilon$, δc, $\delta\forall$, $\delta\iota$. Now if, one puts, for brevity,

(2) $$\delta r = \mathscr{A}\delta a + \mathscr{E}\delta\varepsilon + \mathscr{C}\delta c, \quad dr = -\Omega\delta\mathrm{d} - \mathscr{I}\delta\iota,$$

in order to reduce the first of the equations (1) to the form

(3) $$\mathscr{A}\delta a + \mathscr{E}\delta\varepsilon + \mathscr{C}\delta c + \Omega\delta\mathrm{d} + \mathscr{I}\delta\iota = \partial r,$$

and if, besides, one conserves the notations adopted in the preceding paragraph, one will draw from formulas (2) on page 654

(4) $$\mathscr{A} = \frac{r}{a} - \frac{3}{2}\frac{a}{r}\lambda t\varepsilon\sin\psi, \quad \mathscr{E} = -a\cos(p-\mathfrak{p}), \quad \mathscr{C} = \frac{a\varepsilon}{(1-\varepsilon^2)}\sin(p-\mathfrak{p}),$$

and from formulas (8) on page 655

(5) $$\mathscr{I} = \frac{\rho\varsigma}{R\sin(\varpi-\mathrm{d})}\frac{\sin p}{\sin\iota}, \quad \Omega = -\frac{\rho\varsigma}{R\sin(\varpi-\mathrm{d})}\cos p.$$