Mémoire sur deux formules générales, dont chacune permet de calculer rapidement des valeurs très-approchées des élments de l'orbite d'une planète ou d'une comète*

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After having shown, in a preceding session, how one is able, in certain cases, to restore the determination of the orbit of a star by the use of the formulas which contain only some derivatives of the first order, I have sought if it would not be possible to make a prompt and easy determination of these elements depend on the resolution of equations which no longer contain any derivative, and I have obtained, in fact, two very simple and very general equations, of which each fulfills the condition that I just indicated.

The two equations of which there is concern are able to be easily formed. If, in placing the origin of the coordinates at the center of the sun, one names x,y,z the coordinates of the observed star, the plane of the orbit of this star will be represented by a linear equation in x,y, z without constant term. Therefore, if one considered the star in three successive positions, the resultant formed with the corresponding values of x,y,z will be null. By equating this resultant to zero, one will obtain the first of the equations of which I have spoken. Besides, the coordinates x,y,z are able to be expressed as function of the givens of the observation and of the radius vector r, drawn from the sun to the star. [954] On the other hand, this radius vector depends only on the time and on three elements of the orbit, namely, on the eccentricity, on the great axis and on the instant of the passage to the perihelion. Therefore the found equation, joined to five observations of the star, will suffice in order to determine these three elements.

This is not all: if one names b the semi-parameter of the described curve, the difference $r-b$ will be yet linked to any two of the coordinates x,y,z by a linear equation which will not contain a constant term. There results from it that, in the found equation, one will be able to replace the three values of any one of the coordinates x,y,z by the three corresponding values of the difference $r-b$. One will obtain thus the second of the equations that I have announced. Besides, the radius r and the coordinates x,y,z are able to be considered as functions of the data from the observation and of the

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. July 10, 2010 two elements which determine the position of the plane of the orbit, that is to say as functions of the inclination and of the longitude of the ascendant node. Therefore the new equation, joined to five observations, will suffice in order to determine these two elements and the parameter of the described curve.

We see now what part one is able to draw from two general equations of which I am going to indicate the formation, and how one is able to be served by each of them, in order to determine with a great approximation the three elements among which it establishes a liaison.

I have shown, in the preceding sessions, that one is able, in general, to obtain with a great facility a first approximate value of the major axis, and, hence, of the other elements of the orbit, when the observed star is a planet, and I will add that of the known formulas one is able, in each case, to deduce from very simple equations which furnish by a rapid calculation some approximate values of the elements. This put, it is clear that, in order to determine with a grand approximation the three elements of the orbit, it will suffice to apply to one of the equations mentioned above the linear method, in correcting, by aid of five observations notably distant from each other, the first values obtained for the three elements of which there is concern.

ANALYSIS

§ I. — *On the means to obtain a first approximation in the calculation of the elements of the orbit of a star.*

We have previously indicated a means to obtain easily an approximate value of the distance from a star to the sun, when this star is a planet. In each case, one is able to determine approximately, with [955] a rather great facility, the distance from a planet or from a comet to the sun, or to the earth, or else still one or many elements of the orbit of the observed star, by starting from known formulas which serve to determine these distances or these elements as function of the geocentric longitude and latitude, and from their derivatives of the first and of the second order. In order to arrive there, it will suffice to substitute in each derivative a ratio in the differences, by having regard to the prescriptions that we are going to establish.

Let $t_{\prime}, t_{\prime\prime}$ be the epochs of two very near observations, and we put

$$t = \frac{t_{\prime} + t_{\prime\prime}}{2}, \quad \Delta t = \frac{t_{\prime\prime} - t_{\prime}}{2}. \tag{1}$$

Let, besides, $f(t)$ be any one of the variables of which the values are furnished by the observations. One will draw from formulas (1)

$$t_{\prime} = t - \Delta t, \quad t_{\prime\prime} = t + \Delta t;$$

and it will suffice to develop $f(t_{\prime})$, $f(t_{\prime\prime})$ following the ascending powers of Δt, in order to be assured that one has

$$f(t) = \frac{f(t_{\prime}) + f(t_{\prime\prime})}{2}, \tag{2}$$

and

$$D_t f(t) = \frac{f(t_{\prime\prime}) - f(t_{\prime})}{t_{\prime\prime} - t_{\prime}}, \tag{3}$$

or, that which reverts to the same,

$$D_t f(t) = \frac{\Delta f(t)}{\Delta t}, \tag{4}$$

by neglecting only, in the second members of equations (2), (3) and (4) some quantities proportional to the square and to the higher powers of Δt. It is good to remark: 1° that, in formula (4), Δt and $\Delta f(t)$ is able to be counted to represent, not only the halves of the two differences $t_{\prime\prime} - t_{\prime}$, $f(t_{\prime\prime}) - f(t_{\prime})$, but yet these differences themselves; 2° that, in each of the formulas (2), (4), the variable $f(t)$ is able to be replaced by its logarithm.

We suppose now that $F(t)$ represents, no longer one of the variables of which the values are furnished by the observations, but one of their derivatives of the first order, or else yet a function of these derivatives and of the variables themselves. An approximate value of $F(t)$ will be able to be deduced easily from formulas (2) and (3). But formula (4) will no longer suffice to the determination, even approximate, of the function $D_t F(t)$, which will contain generally, [956] with one or many of the variables of which there is concern, their derivatives of the first and second order. In order to effect this determination, one must recur to two, or even to three pairs of observations, each group being composed of two observations very close to one another. Let t, t', t'' be the values of time corresponding to these three groups, each of these values being the arithmetic mean between the epochs of the two observations which compose one same group. If one has

$$t = \frac{t_{\prime} + t_{\prime\prime}}{2}$$

then, by neglecting only the quantities proportional to the square and to the higher powers of the differences $t - t'$, $t - t''$, one will find

$$D_t F(t) = \frac{F(t'') - F(t')}{t'' - t'}. \tag{5}$$

In the contrary case, one will have, under the same condition,

$$D_t F(t) = \frac{t - t'}{t'' - t'} \frac{F(t'') - F(t)}{t'' - t} + \frac{t'' - t}{t'' - t'} \frac{F(t) - F(t')}{t - t'}. \tag{6}$$

When one wishes to draw departing from these considerations in order to determine approximately the position of the plane of the orbit of a star, it is useful to introduce into the calculation certain auxiliary angles, as I am going to explicate in some words.

Let, at the end of time t,

ϕ and θ be the geocentric longitude and latitude of the observed star;
ι the inclination of its orbit, represented by an angle inferior to two rights;
d the longitude of the ascendant node;
ϖ the heliocentric longitude of the earth;
$\chi = \phi - \varpi$ the elongation;
R the distance from the earth to the sun;
H the area described by the radius vector r drawn from the sun to the observed star;
U, V, W the algebraic projections of this area onto the three coordinate planes of y, z, of z, x and of x, y, the last plane being the same plane as the ecliptic.

The linear equation which exists between the three constants U, V, W will furnish the means to calculate the ratio $\frac{U}{V}$ (*see* page 1008[1] of volume XXIII), and, [957] hence, the two elements $\mho$ and ι. In fact, let h be the quantity of which the numerical value is given by the formula

$$h = L(e)\sin 1'' = 0.00000210552,$$

e being the base of the hyperbolic logarithms, and the letter L indicating a logarithmic decimal. We suppose further that, the angles being expressed in sexagesimal seconds, one names τ an auxiliary angle determined by the formula

$$\tan\tau = \frac{D_t L \tan\theta}{h D_t \phi},$$

and we make

$$\Omega = \frac{\tan\theta}{\cos\tau} D_t\phi,$$

$$P = \Omega\sin(\tau+\chi), \quad Q = \Omega\cos(\tau+\chi),$$

$$\varsigma = \frac{1}{2} D_t L Q + D_t L R + \frac{h}{2}\frac{P}{Q} D_t\varpi,$$

$$\mathfrak{a} = D_t L\cos\phi - \varsigma, \quad \mathfrak{b} = D_t L\sin\phi - \varsigma, \quad \mathfrak{c} = D_t L\tan\theta - \varsigma,$$

$$\mathscr{P} = \frac{\mathfrak{a}}{\mathfrak{c}}\frac{\cos\phi}{\tan\theta}, \quad \mathscr{Q} = \frac{\mathfrak{b}}{\mathfrak{c}}\frac{\sin\phi}{\tan\theta}$$

one will find

(7) $$\tan\mho = \frac{\mathscr{Q}' - \mathscr{Q}}{\mathscr{P}' - \mathscr{P}} = \frac{\mathscr{Q}'' - \mathscr{Q}}{\mathscr{P}'' - \mathscr{P}},$$

$\mathscr{P}'$ and $\mathscr{Q}'$ or $\mathscr{P}''$ and $\mathscr{Q}''$ being that which $\mathscr{P}$ and $\mathscr{Q}$ become when t is changed into t' or into t''.

Moreover, by supposing the auxiliary angle Φ determined by the formula

$$\cot\Phi D_t\phi = \frac{\varsigma}{h} + \cot(\varpi+\mho) D_t\varpi - \frac{P}{Q} D_t\varpi,$$

one will find further

$$\tan\iota = \frac{\sin(\tau+\Phi)}{\sin(\tau+\mho+\Phi)}\tan\theta.$$

By aid of formulas (2), (4), (6), (7) and (8), and from the three groups of observations, composed each of two nearby observations, one will determine easily some approximate values of the elements ι and $\mho$. One will be able next to obtain for these same elements some corrections which will be already very little different from the true, by applying the linear method to the formulas given by Lagrange in the *Connaissance des Temps* for 1821, or, that which reverts to the same, formula (19) on page 703.[2]

[1] See "Notes sur les formules relatives à la détermination des orbites que décrivent les corps célestes." *Comptes Rendus Hebd. Séances Acad. Sci.* 23.

[2] See "Mémoire sur la détermination et la correction des éléments de l'orbite d'un astre." Comptes Rendus Hebd. Séances Acad. Sci. 25.

§ II. — *On the means to obtain, with a great approximation, the elements of an orbit, by aid of five observations.*

We suppose that, the center of the sun being taken for origin of the coordinates, and the plane of the ecliptic for the plane of x, y, one counts z positive on the side of the north pole; and let, at the end of time t,

x, y, z be the coordinates of the observed star;

r the distance from this star to the sun;

b the semi-parameter of the orbit described;

H the double of the area described during the unit of time by the radius r;

U, V, W the algebraic projections of this area onto the coordinate planes.

The equation of the plane of the orbit will be

$$Ux+Vy+Wz=0. \tag{1}$$

We imagine now that by aid of one or two accents placed at the base of each variable, one designates the value that this variable takes when one replaces t by $t_{\prime}$ or by $t_{\prime\prime}$. Formula (1) will continue to subsist when one will replace x, y, z by $x_{\prime}$, $y_{\prime}$, $z_{\prime}$ or by $x_{\prime\prime}$, $y_{\prime\prime}$, $z_{\prime\prime}$, and one will have, consequently,

$$xy_{\prime}z_{\prime\prime}-xy_{\prime\prime}z_{\prime}+x_{\prime}y_{\prime\prime}z-x_{\prime}yz_{\prime\prime}+x_{\prime\prime}yz_{\prime}-x_{\prime\prime}y_{\prime}z=0. \tag{2}$$

One is able besides, as one knows, to arrive directly to this equation, by observing that the volume of the tetrahedron which has for edges the radius vectors r, $r_{\prime}$, $r_{\prime\prime}$ vanished, since the orbit is a plane. We add that, in formula (2), x, y, z are able to be expressed as a linear function of the distance τ from the star to the earth, the coefficients being from the data of the observation; and as r, τ are besides linked between them by a known equation of the second degree, there results from it that the coordinates x, y, z are able to be considered as functions of r. Therefore also these coordinates are able to be considered as functions of time, and of the three elements on which the radius r depends, namely, on the major axis, of the epoch of the passage of the star at the perihelion and of the eccentricity. Therefore, by supposing already known the approximate values of these three elements, one will be able to correct separately by aid of the linear method applied to formula (2), and from five observations.

We remark now that, if to the three variables x, y, z one joins the variable $r-b$, one will be able to establish, between this fourth variable and any two of the other three, a linear equation which, as formula (1), [959] will not contain a constant term. Therefore formula (2) will continue to subsist if one replaces the three values of one of the coordinates by the three values corresponding to $r-b$. One will have, for example,

$$(x_{\prime}y_{\prime\prime}-x_{\prime\prime}y_{\prime})(r-b)+(x_{\prime\prime}y-xy_{\prime\prime})(r_{\prime}-b)+(xy_{\prime}-x_{\prime}y)(r_{\prime\prime}-b)=0. \tag{3}$$

or, that which reverts to the same,

$$b=\frac{(x_{\prime}y_{\prime\prime}-x_{\prime\prime}y_{\prime})r+(x_{\prime\prime}y-xy_{\prime\prime})r_{\prime}+(xy_{\prime}-x_{\prime}y)r_{\prime\prime}}{x_{\prime}y_{\prime\prime}-x_{\prime\prime}y_{\prime}+x_{\prime\prime}y-xy_{\prime\prime}+xy_{\prime}-x_{\prime}y} \tag{4}$$

Besides, as we have said, x, y, z are able to be expressed as a linear function of the distance τ from the star to the earth, the coefficients being from the data from observation.

Effectively, if one conserves the notations adopted in § 1, one will have

$$(5) \qquad x = R\cos\varpi + \tau\cos\phi\cos\theta, \quad y = R\sin\varpi + \tau\cos\phi\cos\theta, \quad z = \tau\sin\theta.$$

On the other hand, from equation (1), joined to formulas (4), one will draw

$$(6) \qquad \tau = -R\frac{(U\cos\phi + V\sin\phi)\cos\theta + W\sin\theta}{U\cos\varpi + V\sin\varpi}$$

Finally, the areas U, V, W are linked to the elements ♉, ι, b by the formulas

$$U = W\sin\text{♉}\tan\iota, \quad V = -W\cos\text{♉}\tan\iota,$$
$$U^2 + V^2 + W^2 = H^2 = Kb,$$

K being the attractive force of the sun; and, consequently, formula (5) gives

$$(7) \qquad \tau = R\frac{\sin\theta\cot\iota - \sin(\phi - \text{♉})}{\sin(\varpi - \text{♉})}$$

Therefore τ, hence, x, y, z are able to be considered as some functions of the elements ♉, ι and of the data of the observation. Finally, one is able to say as much of the radius r determined by the formula

$$r = \sqrt{x^2 + y^2 + z^2}.$$

This put, it is clear that by supposing already known approximate values of the three elements ι, ♉, b, or even only two elements ι and ♉, one will be able to correct them immediately by aid of the linear method applied to formula (2) or (4), and from five observations.