Demonstration Élémentaire d'une Proposition Générale de la Théorie des Probabilités*

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Crelle, *Journal für die reine und angewandte Mathematik*, B. 33 (1846), pp. 259–267.

§1. The proposition, of which the demonstration will be the object of this note, is the following:

"One is able to assign always a number of trials such, that the probability of that which the ratio of the number of repetitions of the event E to the one of the trials will not deviate from the mean of the chances of E beyond some given limits, as tightened as are these limits, will be approached as much as one will wish to certitude."

This fundamental proposition of the theory of probabilities, containing as particular case the law of Jacques Bernoulli, is deduced by Mr. Poisson from a formula, which he obtains by calculating approximately the value of a rather complicated definite integral (See *Recherches sur les probabilités des jugements*, Chap. IV).

As ingenious as is the method employed by the celebrated Geometer, it does not furnish the limit of the error that his approximate analysis permits, and by this uncertainty over the value of the error the demonstration of the proposition lacks rigor.

I am going to show here how one is able to demonstrate rigorously this proposition by some totally elementary considerations.

§2. We suppose that $p_1, p_2, p_3, \dots p_\mu$ are the chances of the event E in μ consecutive trials, P_m the probability that E will arrive at least m times in these μ trials.

One will arrive, as one knows, to the expression of P_m by developing the product

$$(p_1 t + 1 - p_1)(p_2 t + 1 - p_2)(p_3 t + 1 - p_3) \dots (p_\mu t + 1 - p_\mu)$$

according to the powers of t by taking the sum of the coefficients of t^m, t^{m+1}, $\dots t^\mu$.

Thence result evidently these two properties of P_m:

1) This quantity contains $p_1, p_2, p_3, \dots p_\mu$ only to the degrees not superior to unity; 2) it is a symmetric function with respect to $p_1, p_2, p_3, \dots p_\mu$.

By virtue of the first property P_m will be able to be set under the form

$$U + Vp_1 + V_1 p_2 + W p_1 p_2,$$

where U, V, V_1, W are independent of p_1 and p_2; by virtue of the second, V and V_1 are equals. Therefore the form of the expression P_m is

$$U + V(p_1 + p_2) + W p_1 p_2,$$

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where U, V, W contain neither p_1 nor p_2. Following this it is easy to prove concerning the expression P_m the following theorem:

Theorem. "If p_1, p_2 are not equal, one is able, without changing the values of p_1+p_2, p_3,... p_μ, to increase that of P_m by taking $p_1=p_2$; or one is able to arrive to one of the following equations:

$$p_1=0, \qquad p_1=1,$$

without diminishing the value of P_m."

Demonstration. We have seen that the expression of P_m is able to be put under the form $U+V(p_1+p_2)+Wp_1p_2$, where U, V, W are independent of p_1 and p_2.

Now the formula $U+V(p_1+p_2)+Wp_1p_2$ presents always one of the three cases: $W>0$, $W=0$, $W<0$.

In the first case the sum p_1+p_2 remains the same, and the value of P_m increases from $\frac{1}{4}W(p_1-p_2)^2$, when one changes p_1, p_2 into $\frac{1}{2}(p_1+p_2)$, $\frac{1}{2}(p_1+p_2)$; because the difference

$$U+V\left[\frac{1}{2}(p_1+p_2)+\frac{1}{2}(p_1+p_2)\right]+W\frac{1}{2}(p_1+p_2)\frac{1}{2}(p_1+p_2) \\ -\{U+V(p_1+p_2)+Wp_1p_2\}$$

is reduced to $\frac{1}{4}W(p_1-p_2)^2$.

In the two other cases one will not change the value of the sum p_1+p_2 and one will not diminish that of $U+V(p_1+p_2)+Wp_1p_2$, by changing p_1, p_2 into 0, p_1+p_2 or into 1, p_1+p_2-1; for

$$\begin{aligned} U+V[0+p_1+p_2] &+ W.0.(p_1+p_2)-\{U+V(p_1+p_2)+Wp_1p_2\} \\ &= -Wp_1p_2; \\ U+V[1+p_1+p_2-1] &+ W.1.(p_1+p_2-1)-\{U+V(p_1+p_2)+Wp_1p_2\} \\ &= -W(1-p_1)(1-p_2). \end{aligned}$$

But the values 0, p_1+p_2 will be able to be admitted for p_1, p_2 each time that p_1+p_2 not surpass 1; for they are then positive and do not surpass unity at all; in the contrary case where $p_1+p_2>1$, one will be able to change p_1 into 1, p_2 into p_1+p_2-1, that which proves the theorem announced. This theorem leads us next to the following:

Theorem. "The greatest value that P_m is able to have in the case where $p_1+p_2+p_3+\cdots+p_\mu=S$, corresponds to the values of $p_1, p_2, p_3, \ldots p_\mu$ given by the equations

$$p_1=0, \quad p_2=0, \quad p_\rho=0, \quad p_{\rho+1}=1, \quad p_{\rho+2}=1, \ldots p_{\rho+\sigma}=1,$$

$$p_{\rho+\sigma+1}=\frac{S-\sigma}{\mu-\rho-\sigma}, \quad p_{\rho+\sigma+2}=\frac{S-\sigma}{\mu-\rho-\sigma}, \ldots p_\mu=\frac{S-\sigma}{\mu-\rho-\sigma},$$

where ρ, σ designate certain numbers."

Demonstration. We suppose that π_1, π_2, π_3, $\ldots \pi_\mu$ are the system of values of p_1, $p_2, p_3, \ldots p_\mu$ which, verifying the equation

$$p_1+p_2+p_3+\cdots+p_\mu=S,$$

give the greatrial value of P_m and contain at the same time the greatest number possible of values equal to 1 and 0 under these conditions.

Let besides $\pi_1,\ \pi_2,\ \pi_3,\ \ldots \pi_\rho$ be those among the quantities $\pi_1,\ \pi_2,\ \pi_3,\ \ldots \pi_\mu$ which are equal to 0; $\pi_{\rho+1}$, $\pi_{\rho+2}$,$\ldots \pi_{\rho+\sigma}$ those which are equal to unity; all the others $\pi_{\rho+\sigma+1},\ \pi_{\rho+\sigma+2},\ \ldots \pi_\mu$ being according to the supposition different from 0 and 1, must be equal among themselves, as we are going to prove it just now.

In fact, if $\pi_{\rho+\sigma+1}$ is not equal to $\pi_{\rho+\sigma+2}$, it is possible, according to the preceding theorem, either to render P_m greater, without changing the sum

$$\pi_1 + \pi_2 + \cdots + \pi_{\rho+\sigma+1} + \pi_{\rho+\sigma+2} + \cdots + \pi_\mu,$$

by taking $\pi_{\rho+\sigma+1} = \pi_{\rho+\sigma+2}$ or by making $\pi_{\rho+\sigma+1}$ equal to 1 or 0, without diminishing the value of P_m.

But the one is contrary to the supposition that the system $\pi_1,\ \pi_2,\ \pi_3,\ \ldots \pi_\mu$ gives the greatest value to P_m under the condition

$$\pi_1 + \pi_2 + \pi_3 + \cdots + \pi_\mu = S;$$

the other is contrary to the supposition that of all the systems that have this property, $\pi_1,\ \pi_2,\ \pi_3,\ \ldots \pi_\mu$ is the one which contains the greatest number of values equal to 1 and 0. Therefore it is necessary that it be

$$\pi_{\rho+\sigma+1} = \pi_{\rho+\sigma+2} = \cdots = \pi_\mu.$$

But beyond these equations we have

$$\pi_1 = 0, \quad \pi_2 = 0, \ldots \pi_\rho = 0, \quad \pi_{\rho+1} = 1, \ldots \pi_{\rho+\sigma} = 1;$$

$$\pi_1 + \pi_2 + \pi_3 + \cdots + \pi_\mu = S;$$

whence results the equations of the proposed theorem.

§3. We pass now to the research on the values of the expression of P_m which corresponds to

$$p_1 = 0, \quad p_2 = 0, \quad p_\rho = 0, \quad p_{\rho+1} = 1, \quad p_{\rho+2} = 1, \ldots p_{\rho+\sigma} = 1,$$

$$p_{\rho+\sigma+1} = \frac{S-\sigma}{\mu-\rho-\sigma}, \quad p_{\rho+\sigma+2} = \frac{S-\sigma}{\mu-\rho-\sigma}, \ldots p_\mu = \frac{S-\sigma}{\mu-\rho-\sigma}.$$

From the remark that we just made with respect to the expression P_m, it follows that the value of P_m which corresponds to

$$p_1 = 0, \quad p_2 = 0, \quad p_3 = 0, \quad p_{\rho+1} = 1, \quad p_{\rho+2} = 1, \ldots p_{\rho+\sigma} = 1,$$

$$p_{\rho+\sigma+1} = \frac{S-\sigma}{\mu-\rho-\sigma}, \quad p_{\rho+\sigma+2} = \frac{S-\sigma}{\mu-\rho-\sigma}, \ldots p_\mu = \frac{S-\sigma}{\mu-\rho-\sigma},$$

is the sum of the coefficients of t^m, $t^{m-1}, \ldots t^\mu$ in the development of the product

$$t^\sigma \left(\frac{S-\sigma}{\mu-\rho-\sigma} t + \frac{\mu-S-\rho}{\mu-\rho-\sigma}\right)^{\mu-\rho-\sigma},$$

and that, consequently, it is equal to

$$\frac{1.2\ldots(\mu-\rho-\sigma)}{1.2\ldots(m-\sigma).1.2.\ldots(\mu-m-\rho)}\left(\frac{S-\sigma}{\mu-\rho-\sigma}\right)^{m-\sigma}\left(\frac{\mu-S-\rho}{\mu-\rho-\sigma}\right)^{\mu-m-\rho}$$
$$\left\{1+\frac{\mu-m-\rho}{m-\sigma+1}\frac{S-\sigma}{\mu-S-\rho}+\frac{\mu-m-\rho}{m-\sigma+1}\frac{S-\sigma}{\mu-S-\rho}\frac{\mu-m-\rho-1}{m-\sigma+2}\frac{S-\sigma}{\mu-S-\rho}+\cdots\right.$$
$$\left.+\frac{\mu-m-\rho}{m-\sigma+1}\frac{S-\sigma}{\mu-S-\rho}\frac{\mu-m-\rho-1}{m-\sigma+2}\frac{S-\sigma}{\mu-S-\rho}\cdots\frac{1}{\mu-\rho-\sigma}\frac{S-\sigma}{\mu-S-\rho}\right\}.$$

Here is the expression which, in consequence of the preceding theorem, for certain positive whole numbers ρ and σ, will be the upper limit of all the values of P_m, in the case, where $p_1+p_2+p_3+\cdots+p_\mu=S$.

In noting that the value of the expression

$$1+\frac{\mu-m-\rho}{m-\sigma+1}\frac{S-\sigma}{\mu-S-\rho}+\frac{\mu-m-\rho}{m-\sigma+1}\frac{S-\sigma}{\mu-S-\rho}\frac{\mu-m-\rho-1}{m-\sigma+2}\frac{S-\sigma}{\mu-S-\rho}+\cdots$$
$$\cdots+\frac{\mu-m-\rho}{m-\sigma+1}\frac{S-\sigma}{\mu-S-\rho}\frac{\mu-m-\rho-1}{m-\sigma+2}\frac{S-\sigma}{\mu-S-\rho}+\cdots\frac{1}{\mu-\rho-\sigma}\frac{S-\sigma}{\mu-S-\rho}$$

is smaller than that of

$$1+\frac{\mu-m-\rho}{m-\sigma}\frac{S-\sigma}{\mu-S-\rho}+\left(\frac{\mu-m-\rho}{m-\sigma}\frac{S-\sigma}{m-\sigma-\rho}\right)^2+\cdots+\left(\frac{\mu-m-\rho}{m-\sigma}\frac{S-\sigma}{\mu-S-\rho}\right)^{\mu-m-\rho},$$

which is the development of

$$\frac{1-\left(\frac{\mu-m-\rho}{m-\sigma}\frac{S-\sigma}{\mu-S-\rho}\right)^{\mu-m-\rho+1}}{1-\frac{\mu-m-\rho}{m-\sigma}\frac{m-\sigma}{\mu-S-\rho}},$$

or of

$$\frac{(m-\sigma)}{(m-S)}\frac{(\mu-S-\rho)}{(\mu-\rho-\sigma)}\left[1-\left(\frac{\mu-m-\rho}{m-\sigma}\frac{S-\sigma}{m-\sigma-\rho}\right)^{\mu-m-\rho+1}\right],$$

we will arrive to this theorem.

Theorem "For certain whole and positive numbers ρ and σ the value of the expression

$$\frac{1.2\ldots(\mu-\rho-\sigma)}{1.2\ldots(m-\sigma).1.2.\ldots(\mu-m-\rho)}\left(\frac{S-\sigma}{\mu-\rho-\sigma}\right)^{m-\sigma}\left(\frac{\mu-S-\rho}{\mu-\rho-\sigma}\right)^{\mu-m-\rho+1}$$
$$\frac{(m-\sigma)}{(m-S)}\left[1-\left(\frac{\mu-m-\rho}{m-\sigma}\frac{S-\sigma}{\mu-S-\rho}\right)^{\mu-m-\rho+1}\right]$$

surpasses the value P_m of the probability that in the μ trials the event E having the chances $p_1, p_2, p_3, \ldots p_\mu$, will arrive at least m times, where S is the sum $p_1+p_2+p_3+\cdots+p_\mu$."

§4. We stop ourselves at the case, where m surpasses $S+1$. According to the last theorem we have

$$P_m < \frac{1.2\ldots(\mu-\rho-\sigma)}{1.2\ldots(m-\sigma).1.2.\ldots(\mu-m-\rho)}\left(\frac{S-\sigma}{\mu-\rho-\sigma}\right)^{m-\sigma}\left(\frac{\mu-S-\rho}{\mu-\rho-\sigma}\right)^{\mu-m-\rho+1}$$
$$\frac{(m-\sigma)}{(m-S)}\left[1-\left(\frac{\mu-m-\rho}{m-\sigma}\frac{S-\sigma}{\mu-S-\rho}\right)^{\mu-m-\rho+1}\right]$$

and more so

$$(1)\qquad P_m < \frac{1.2\ldots(\mu-\rho-\sigma)}{1.2\ldots(m-\sigma).1.2.\ldots(\mu-m-\rho)}\left(\frac{S-\sigma}{\mu-\rho-\sigma}\right)^{m-\sigma}\left(\frac{\mu-S-\rho}{\mu-\rho-\sigma}\right)^{\mu-m-\rho+1}\frac{(m-\sigma)}{(m-S)}$$

But m being greater than $S+1$, the value of the expression

$$\frac{1.2\ldots(\mu-\rho-\sigma)}{1.2\ldots(m-\sigma).1.2.\ldots(\mu-m-\rho)}\left(\frac{S-\sigma}{\mu-\rho-\sigma}\right)^{m-\sigma}\left(\frac{\mu-S-\rho}{\mu-\rho-\sigma}\right)^{\mu-m-\rho+1}\frac{m-\sigma}{m-S}$$

will increase with the diminution of the positive whole numbers ρ and σ.

In fact, if we divide by this expression the value that it takes after the changing of σ into $\sigma-1$, we will find for their ratio

$$\frac{\mu-\rho-\sigma+1}{m-\sigma}\frac{(S-\sigma+1)^{m-\sigma+1}}{(S-\sigma)^{m-\sigma}}\frac{(\mu-\rho-\sigma)^{\mu-\rho-\sigma+1}}{(\mu-\rho-\sigma+1)^{\mu-\rho-\sigma+2}},$$

or else

$$\frac{S-\sigma+1}{m-\sigma}\left(\frac{S-\sigma+1}{S-\sigma}\right)^{m-\sigma}\left(\frac{\mu-\rho-\sigma}{\mu-\rho-\sigma+1}\right)^{\mu-\rho-\sigma+1},$$

this which, being set under the form

$$\frac{1}{1+\frac{m-S-1}{S-\sigma+1}}e^{-(m-\sigma)\log\left(1-\frac{1}{S-\sigma+1}\right)+(\mu-\rho-\sigma+1)\log\left(1-\frac{1}{\mu-\rho-\sigma+1}\right)},$$

is reduced to

$$\frac{1}{1+\frac{m-S-1}{S-\sigma+1}}e^{\frac{m-S-1}{S-\sigma+1}+\frac{1}{2}\left\{\frac{m-\sigma}{(S-\sigma+1)^2}-\frac{1}{\mu-\rho-\sigma+1}\right\}+\frac{1}{3}\left\{\frac{m-\sigma}{(S-\sigma+1)^3}-\frac{1}{(\mu-\rho-\sigma+1)^2}\right\}+\cdots}$$

Now, this value is evidently greater than 1; for

$$\frac{1}{1+\frac{m-S-1}{S-\sigma+1}}e^{\frac{m-S-1}{S-\sigma+1}}$$

is equal to

$$\frac{1+\frac{m-S-1}{S-\sigma+1}+\frac{1}{2}\left(\frac{m-S-1}{S-\sigma+1}\right)^2+\frac{1}{2.3}\left(\frac{m-S-1}{S-\sigma+1}\right)^3+\cdots}{1+\frac{m-S-1}{S-\sigma+1}},$$

and this here surpasses unity, because, m being, by assumption, greater than $S+1$, $m-S+1$ will have a positive value.

As for the values of

$$\frac{m-\sigma}{(S-\sigma+1)^2}-\frac{1}{\mu-\rho-\sigma+1},\quad \frac{m-\sigma}{(S-\sigma+1)^3}-\frac{1}{(\mu-\rho-\sigma+1)^2},\ldots,$$

they are positive, seeing that, by assumption, $m-\sigma$ surpasses $S-\sigma+1$, and $S-\sigma+1$ is not able to surpass $\mu-\rho-\sigma+1$; for otherwise $\frac{S-\sigma}{\mu-\rho-\sigma}$, which is the value of a certain probability (see §2), would be greater than unity.

We ourselves are therefore convinced that with the diminution of σ the value of the expression

$$\frac{1.2\ldots(\mu-\rho-\sigma)}{1.2\ldots(m-\sigma).1.2.\ldots(\mu-m-\rho)}\left(\frac{S-\sigma}{\mu-\rho-\sigma}\right)^{m-\sigma}\left(\frac{\mu-S-\rho}{\mu-\rho-\sigma}\right)^{\mu-m-\rho+1}\frac{m-\sigma}{m-S}$$

increases. The same holds with respect to ρ.

We conclude thence that for $m>S+1$ the value of the expression

$$\frac{1.2\ldots(\mu-\rho-\sigma)}{1.2\ldots(m-\sigma).1.2.\ldots(\mu-m-\rho)}\left(\frac{S-\sigma}{\mu-\rho-\sigma}\right)^{m-\sigma}\left(\frac{\mu-S-\rho}{\mu-\rho-\sigma}\right)^{\mu-m-\rho+1}\frac{m-\sigma}{m-S}$$

in inequality (1) is not able to surpass that which corresponds to $\rho=0$, $\sigma=0$ and which is equal to

$$\frac{1.2\ldots\mu}{1.2\ldots m.1.2.\ldots(\mu-m)}\left(\frac{S}{\mu}\right)^{m}\left(\frac{\mu-S}{\mu}\right)^{\mu-m+1}\frac{m}{m-S}.$$

We are able therefore to deduce from inequality (1) this here:

$$(2)\qquad P_m<\frac{1.2\ldots\mu}{1.2\ldots m.1.2.\ldots(\mu-m)}\left(\frac{S}{\mu}\right)^{m}\left(\frac{\mu-S}{\mu}\right)^{\mu-m+1}\frac{m}{m-S},$$

where m is supposed greater than $S+1$.

§5. But one knows, that the value of the product $1.2\ldots(x-1).x$ is smaller than $2.53x^{x+\frac{1}{2}}e^{-x+\frac{1}{12x}}$ and greater than $2.50x^{x+\frac{1}{2}}e^{-x}$.[1]

[1]Here is how one arrives very simply to this result.

By dividing respectively the values of the expressions $\frac{1.2\ldots(x-1).x}{x^{x+\frac{1}{2}}e^{-x+\frac{1}{12x}}}$, $\frac{1.2\ldots(x-1).x}{x^{x+\frac{1}{2}}e^{-x}}$ corresponding to $x=n+1$, by their values, which correspond to $x=n$, one finds for their ratios

$$\left(\frac{n}{n+1}\right)^{n+\frac{1}{2}}\cdot e^{1+\frac{1}{12}\left(\frac{1}{n}-\frac{1}{n+1}\right)},\quad \left(\frac{n}{n+1}\right)^{n+\frac{1}{2}}\cdot e,$$

this which is reduced to

$$e^{\left(\frac{n}{n+1}\right)\log\frac{n}{n+1}+1+\frac{1}{12}\left(\frac{1}{n}-\frac{1}{n+1}\right)},\quad e^{\left(n+\frac{1}{2}\right)\log\frac{n}{n+1}+1}$$

or finally to

$$e^{\left(\frac{1}{12}-\frac{3}{2.4.5}\right)\frac{1}{(n+1)^4}+\left(\frac{1}{12}-\frac{4}{2.5.6}\right)\frac{1}{(n+1)^5}+\cdots},\quad e^{-\frac{1}{12(n+1)^2}-\frac{1}{12(n+1)^3}-\cdots}.$$

According to this the value of the expression

$$\frac{1.2\ldots\mu}{1.2\ldots m.1.2.\ldots(\mu-m)}\left(\frac{S}{\mu}\right)^m\left(\frac{\mu-S}{\mu}\right)^{\mu-m+1}\frac{m}{m-S}$$

is smaller than

$$\frac{2.53e^{\frac{1}{12\mu}}}{(2.50)^2}\frac{\mu^{\mu+\frac{1}{2}}}{m^{m+\frac{1}{2}}(\mu-m)^{\mu-m+\frac{1}{2}}}\left(\frac{S}{\mu}\right)^m\left(\frac{\mu-S}{\mu}\right)^{\mu-m+1}\frac{m}{m-S},$$

and the more so smaller than

$$\frac{\frac{1}{2}\mu^{\mu+\frac{1}{2}}}{m^{m+\frac{1}{2}}(\mu-m)^{\mu-m+\frac{1}{2}}}\left(\frac{S}{\mu}\right)^m\left(\frac{\mu-S}{\mu}\right)^{\mu-m+1}\frac{m}{m-S};$$

because the greatest value of $e^{\frac{1}{12\mu}}$, which is $e^{\frac{1}{12}}$, the product

$$\frac{2.53}{(2.50)^2}e^{\frac{1}{12\mu}}$$

is yet smaller than $\frac{1}{2}$.

One has therefore according to (2):

$$P_m<\frac{\frac{1}{2}\mu^{\mu+\frac{1}{2}}}{m^{m+\frac{1}{2}}(\mu-m)^{\mu-m+\frac{1}{2}}}\left(\frac{S}{\mu}\right)^m\left(\frac{\mu-S}{\mu}\right)^{\mu-m+1}\frac{m}{m-S},$$

or, that which is the same:

$$P_m<\frac{1}{2(m-S)}\sqrt{\frac{m(\mu-m)}{\mu}}\left(\frac{S}{m}\right)^m\left(\frac{\mu-S}{\mu-m}\right)^{\mu-m+1}.$$

The first quantity being greater than unity, the second smaller, it is clear that when x increases, the value of $\frac{1.2\ldots(x-1).x}{x^{x+\frac{1}{2}}e^{-x+\frac{1}{12x}}}$ increases also and that of $\frac{1.2\ldots(x-1).x}{x^{x+\frac{1}{2}}e^{-x}}$ diminishes.

Therefore for all the values of x, less than s, one will have

$$\frac{1.2\ldots(x-1).x}{x^{x+\frac{1}{2}}e^{-x+\frac{1}{12x}}}<\frac{1.2\ldots(s-1).s}{s^{s+\frac{1}{2}}e^{-s+\frac{1}{12s}}},\quad\frac{1.2\ldots(x-1).x}{x^{x+\frac{1}{2}}e^{-x}}>\frac{1.2\ldots(s-1).s}{s^{s+\frac{1}{2}}e^{-s}}$$

and, consequently,

$$\text{(A)}\qquad 1.2.\ldots(x-1).x<Te^{-\frac{1}{12s}}x^{x+\frac{1}{2}}e^{-x+\frac{1}{12x}},\quad 1.2\ldots(x-1).x>Tx^{x+\frac{1}{2}}e^{-x},$$

where T designates the value of the expression $\frac{1.2\ldots(s-1).s}{s^{s+\frac{1}{2}}e^{-s}}$.

We put $s=\infty$ and we name T_0 the value of $\frac{1.2\ldots(s-1).s}{s^{s+\frac{1}{2}}e^{-s}}$ for $s=\infty$; it follows from (A) that for all the finite values of x one will have

$$1.2.\ldots(x-1).x<T_0x^{x+\frac{1}{2}}e^{-x+\frac{1}{12x}},\quad 1.2\ldots(x-1).x>T_0x^{x+\frac{1}{2}}e^{-x},$$

where T_0 is a constant.

By making in these inequalities $x=10$, one will find that T_0 is greater than 2.50 and less than 2.53; consequently the preceding inequality gives

$$1.2.\ldots(x-1).x<2.53x^{x+\frac{1}{2}}e^{-x+\frac{1}{12x}},\quad 1.2\ldots(x-1).x>2.50x^{x+\frac{1}{2}}e^{-x}.$$

This inequality gives the following theorem:

Theorem. "If the chances of the event E in μ consecutive trials are $p_1, p_2, p_3, \dots p_\mu$, and if their sum is S, the value of the expression

$$\frac{1}{2(m-S)}\sqrt{\frac{m(\mu-m)}{\mu}}\left(\frac{S}{m}\right)^m\left(\frac{\mu-S}{\mu-m}\right)^{\mu-m+1},$$

for m greater than $S+1$, surpasses always the probability that E will arrive at least m times in these μ trials."

By changing m, p_1, p_2, p_3,... p_μ, S into $\mu-n$, $1-p_1$, $1-p_2$, $1-p_3$,... $1-p_\mu$, $\mu-S$, it follows from this theorem that, if the sum $1-p_1+1-p_2+1-p_3+\dots+1-p_\mu$ is equal to $\mu-S$, the value of the expression

$$\frac{1}{2(S-n)}\sqrt{\frac{n(\mu-n)}{\mu}}\left(\frac{\mu-S}{\mu-n}\right)^{\mu-n}\left(\frac{S}{n}\right)^{n+1}$$

for $\mu-n>\mu-S+1$ surpass that of the probability that the event contrary to E will arrive at least $\mu-n$ times in μ trials, where $p_1, p_2, p_3, \dots p_\mu$ are the chances of E.

By observing that the conditions

$$1-p_1+1-p_2+1-p_3+\dots+1-p_\mu=\mu-S; \quad \mu-n>\mu_S+1$$

is reduced to

$$p_1+p_2+p_3+\dots+p_\mu=S; \quad n<S-1,$$

and that the event contrary to E does not arrive at least $\mu-n$ times in μ trials, if E presents itself in these trials no more than n times, we will arrive to the following theorem:

Theorem. "If the chances of the event E in μ consecutive trials are $p_1, p_2, p_3, \dots p_\mu$, and if their sum is S, the value of the expression

$$\frac{1}{2(S-n)}\sqrt{\frac{n(\mu-n)}{\mu}}\left(\frac{\mu-S}{\mu-n}\right)^{\mu-n}\left(\frac{S}{n}\right)^{n+1}$$

for n smaller than $S-1$, will surpass always that of the probability that E will arrive in these trials no more than n times."

§6. But the repetition of the event E is able to take place only in one of these three cases: either the event will return at least m times, or it will not return more than n times, or finally it will return more than n times and less than m times.

Therefore the probability of the last case will be determined by the difference between unity and the sum of the probabilities of the first two cases.

Therefore, as consequence of the last two theorems, results the following:

Theorem. "If the chances of the event E in μ consecutive trials are $p_1, p_2, p_3, \dots p_\mu$, and if their sum is S, the probability that the number of repetitions of the event E in these μ trials will be less than m and greater than n, will surpass, for m greater than $S+1$ and for n smaller than $S-1$, the value of the expression

$$1-\frac{1}{2(m-S)}\sqrt{\frac{m(\mu-m)}{\mu}}\left(\frac{S}{m}\right)^m\left(\frac{\mu-S}{\mu-m}\right)^{\mu-m+1}-\frac{1}{2(S-n)}\sqrt{\frac{n(\mu-n)}{\mu}}\left(\frac{\mu-S}{\mu-n}\right)^{\mu-n}\left(\frac{S}{n}\right)^{n+1}.$$

In order to deduce from this theorem the proposition enunciated at the beginning of the note, we note that the ratio of the number of repetitions of the event E in μ trials to the number μ does not attain the limits

$$\frac{S}{\mu} + z \text{ and } \frac{S}{\mu} - z,$$

if E in these trials arrives less than $S + \mu z$ and more than $S - \mu z$ times.

But the probability that this here takes place, will surpass (according to the last theorem), for $z > \frac{1}{\mu}$, the value of the expression

$$\begin{aligned} 1 \quad &- \quad \frac{1}{2\mu z}\sqrt{\frac{(S+\mu z)(\mu - S - \mu z)}{\mu}}\left(\frac{S}{S+\mu z}\right)^{S+\mu z}\left(\frac{\mu - S}{\mu - S - \mu z)}\right)^{\mu - S - \mu z + 1} \\ &- \quad \frac{1}{2\mu z}\sqrt{\frac{(S-\mu z)(\mu - S + \mu z)}{\mu}}\left(\frac{\mu - S}{\mu - S + \mu z}\right)^{\mu - S + \mu z}\left(\frac{S}{\mu - \mu z)}\right)^{S - \mu z + 1} \end{aligned}$$

which is able to be set under the form

$$1 - \frac{1-p}{2z\sqrt{\mu}}\sqrt{\frac{p+z}{1-p-z}}H^{\mu} - \frac{p}{2z\sqrt{\mu}}\sqrt{\frac{1-p+z}{p-z}}H_1^{\mu}, \tag{3}$$

where one has made for brevity $\frac{S}{\mu} = p$ and

$$\begin{aligned} \left(\frac{p}{p+z}\right)^{p+z}\left(\frac{1-p}{1-p-z}\right)^{1-p-z} &= H; \\ \left(\frac{1-p}{1-p+z}\right)^{1-p+z}\left(\frac{p}{p+z}\right)^{p-z} &= H_1. \end{aligned} \tag{4}$$

The equations (4) will give us for the natural logarithms of H, H_1 the following series:

$$-\frac{z^2}{2p}\left(1 - \frac{1}{3}\frac{z}{p}\right) - \frac{z^4}{12p^2}\left(1 - \frac{3}{5}\frac{z}{p}\right) - \cdots - \frac{z^2}{2(1-p)} - \frac{z^3}{6(1-p)^2} - \cdots$$

and

$$-\frac{z^2}{2p} - \frac{z3}{6p^2} - \cdots - \frac{z^2}{2(1-p)}\left(1 - \frac{1}{3}\frac{z}{1-p}\right) - \frac{z^4}{12(1-p)^2}\left(1 - \frac{3}{5}\frac{z}{1-p}\right) - \cdots$$

whence it is clear that H, H_1 have some values less than 1.

It follows thence that the expression (3) approaches indefinitely toward 1 by the increase of μ, in a manner that one will render its difference from 1 much smaller than Q, by taking for μ any number greater than

$$\frac{\log\left[Q.\frac{z}{1-p}\sqrt{\frac{1-p-z}{p+z}}\right]}{\log H} \text{ and } \frac{\log\left[Q.\frac{z}{p}\sqrt{\frac{p-z}{1-p+z}}\right]}{\log H_1}.$$

We are therefore arrived to the rigorous demonstration of the proposition which is the object of this note.