Sur les Fractions Continues*

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In the month of October of the last year, I have had the honor to present to the Academy of Sciences one of the results of my researches on interpolation: it was a formula which represents approximately a sought function, according to many of its particular values, and of which the coefficients are determined by the conditions of the *Method of least squares*. That formula, as one sees by my writing inserted into the *Bulletin de l'Académie* (T. XIII, No. 13) under the title of *Note sur une formule d'Analyse*, is obtained by aid of the development of a certain function by continued fraction. Postponing that which touches on the consequences of this formula relative to interpolation to the end of my researches on this subject, I am going to consider it here in its relations with continued fractions, as expressing a particular property of these fractions.

I will commence with the deduction of the formula that I have presented without demonstration in the writing cited just now. Next I will show that which one is able to draw relative to the properties of the convergent fractions that one obtains by developing certain functions into continued fractions.

§1.

We commence our researches with the solution of the following question:

One knows some values of the function $F(x)$ for $n+1$ values of the variable, $x = x_0$, x_1, x_2,… x_n, and one supposes that the function is able to be represented by the formula

$$a + bx + cx^2 + \ldots + gx^{m-1} + hx^m,$$

the exponent m not surpass n. The concern is to find the coefficients of the formula by subjecting them to permit errors of the values $F(x_0)$, $F(x_1)$, $F(x_2)$,… $F(x_n)$, only the least possible influence on any value of $F(X)$.

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One obtains immediately this sequence of equations

$$\begin{aligned}
F(x_0) &= a + bx_0 + cx_0^2 + \ldots + gx_0^{m-1} + hx_0^m;\\
F(x_1) &= a + bx_1 + cx_1^2 + \ldots + gx_1^{m-1} + hx_1^m;\\
F(x_2) &= a + bx_2 + cx_2^2 + \ldots + gx_2^{m-1} + hx_2^m;\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
F(x_n) &= a + bx_n + cx_n^2 + \ldots + gx_n^{m-1} + hx_n^m;
\end{aligned}$$

In order to express the value of $F(X)$, by aid of these equations, we multiply them by some indeterminate factors λ_0, λ_1, λ_2,... λ_n, and we take the sum of them

$$\begin{aligned}
\lambda_0 F(x_0) \quad + &\quad \lambda_1 F(x_1) + \lambda_2 F(x_2) + \ldots + \lambda_n F(x_n)\\
= &\quad a(\lambda_0 + \lambda_1 + \lambda_2 + \ldots + \lambda_n)\\
+ &\quad b(\lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n)\\
+ &\quad c(\lambda_0 x_0^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2 + \ldots + \lambda_n x_n^2)\\
&\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
+ &\quad g(\lambda_0 x_0^{m-1} + \lambda_1 x_1^{m-1} + \lambda_2 x_2^{m-1} + \ldots + \lambda_n x_n^{m-1})\\
+ &\quad h(\lambda_0 x_0^m + \lambda_1 x_1^m + \lambda_2 x_2^m + \ldots + \lambda_n x_n^m).
\end{aligned}$$

If, now, we compare this sum to the expression of $F(X)$, which must be

$$F(X) = a + bX + cX^2 + \ldots + gX^{m-1} + hX^m,$$

we will find that in order to assure the relation

$$F(X) = \lambda_0 F(x_0) + \lambda_1 F(x_1) + \lambda_2 F(x_2) + \ldots + \lambda_n F(x_n),$$

it suffices that the factors λ_0, λ_1, λ_2,... λ_n satisfy the equations

$$(1) \qquad \left\{\begin{array}{lllll}
\lambda_0 & +\lambda_1 & +\lambda_2 & +\ldots+\lambda_n & = 1,\\
\lambda_0 x_0 & +\lambda_1 x_1 & +\lambda_2 x_2 & +\ldots+\lambda_n x_n & = X,\\
\lambda_0 x_0^2 & +\lambda_1 x_1^2 & +\lambda_2 x_2^2 & +\ldots+\lambda_n x_n^2 & = X^2,\\
\ldots & & & & \\
\ldots & & & & \\
\lambda_0 x_0^{m-1} & +\lambda_1 x_1^{m-1} & +\lambda_2 x_2^{m-1} & +\ldots+\lambda_n x_n^{m-1} & = X^{m-1}\\
\lambda_0 x_0^m & +\lambda_1 x_1^m & +\lambda_2 x_2^m & +\ldots+\lambda_n x_n^m & = X^m,
\end{array}\right.$$

When $m = n$, these equations determine completely the factors λ_0, λ_1, λ_2,... λ_n, since the number of the ones and of the others is the same. In this case the system of factors thus calculated is the only one which is able to form the coefficients of the general expression

$$F(X) = \lambda_0 F(x_0) + \lambda_1 F(x_1) + \lambda_2 F(x_2) + \ldots + \lambda_n F(x_n).$$

If, on the contrary, $m < n$, these equations will be able to be satisfied in an infinity of ways, and each system of values assigned to the factors λ_0, λ_1, λ_2,... λ_n, in the formula

$$F(X) = \lambda_0 F(x_0) + \lambda_1 F(x_1) + \lambda_2 F(x_2) + \ldots + \lambda_n F(x_n).$$

will furnish a particular expression of $F(X)$. But, according to the last condition of the problem, it is necessary to choose, among all the expressions of $F(X)$, that in which the errors of the values $F(x_0)$, $F(x_1)$, $F(x_2)$,... $F(x_n)$ have the minimum influence on the sought magnitude $F(X)$. Now one knows, by the theory of probabilities, that one will arrive to this end, by subjecting the factors λ_0, λ_1, λ_2,... λ_n of $F(X)$ to reduce to the minimum the sum

$$k_0^2\lambda_0^2 + k_1^2\lambda_1^2 + k_2^2\lambda_2^2 + \ldots + k_n^2\lambda_n^2,$$

in which k_0^2, k_1^2, k_2^2,... k_n^2 designate some quantities proportional to the means of the squares of the errors of the values $F(x_0)$, $F(x_1)$, $F(x_2)$,... $F(x_n)$. One sees that, for more generality, we suppose different from one another the laws of errors of these $n+1$ quantities. If the law of probability is the same for all, one has in this case

$$k_0 = k_1 = k_2 = \ldots = k_n,$$

and one is able to reduce these multipliers to unity.

The solution of the question is found restored by that which precedes to express $F(X)$ by the formula

$$F(X) = \lambda_0 F(x_0) + \lambda_1 F(x_1) + \lambda_2 F(x_2) + \ldots + \lambda_n F(x_n),$$

by determining the factors λ by the equations (1) and by the condition of the minimum of the sum

$$k_0^2\lambda_0^2 + k_1^2\lambda_1^2 + k_2^2\lambda_2^2 + \ldots + k_n^2\lambda_n^2.$$

We note, in passing, that this condition is able to be extended to the case itself, in which $m = n$. For the factors λ are then completely determined by the equations (1), and the condition of the minimum of the sum of squares requires nothing more; that which accords with that which has already been said on this particular case.

Arriving to the calculation of the factors λ_0, λ_1, λ_2,... λ_n, we suppose that $\theta(x)$ is an entire function of x, which for the values x_0, x_1, x_2,... x_n of x, takes respectively the values $\frac{1}{k_0}$, $\frac{1}{k_1}$, $\frac{1}{k_2}$,... $\frac{1}{k_n}$. The sum

$$k_0^2\lambda_0^2 + k_1^2\lambda_1^2 + k_2^2\lambda_2^2 + \ldots + k_n^2\lambda_n^2.$$

will be written under the form

$$\frac{\lambda_0^2}{\theta^2(x_0)} + \frac{\lambda_1^2}{\theta^2(x_1)} + \frac{\lambda_2^2}{\theta^2(x_2)} + \ldots + \frac{\lambda_n^2}{\theta^2(x_n)}.$$

In order to determine, by the condition of minimum of that sum, the quantities λ_0, λ_1, λ_2,... λ_n, linked among them by the equations (1), we take the differential of them, and, according to the ordinary process of the minima and maxima, we will equate it to the sum of the differentials of the equations (1), each multiplied by some arbitraries μ_0, μ_1, μ_2,... μ_m respectively. Equating next the terms which will have for factors $d\lambda_0$, $d\lambda_1$, $d\lambda_2$,... $d\lambda_n$, we find the $(n+1)$ equations.

$$(2) \qquad \begin{cases} \frac{2\lambda_0}{\theta^2(x_0)} = \mu_0 + \mu_1 x_0 + \mu_2 x_0^2 + \ldots + \mu_m x_0^m; \\ \frac{2\lambda_1}{\theta^2(x_1)} = \mu_0 + \mu_1 x_1 + \mu_2 x_1^2 + \ldots + \mu_m x_1^m; \\ \ldots \\ \ldots \\ \frac{2\lambda_n}{\theta^2(x_n)} = \mu_n + \mu_1 x_n + \mu_2 x_n^2 + \ldots + \mu_m x_n^m. \end{cases}$$

Reunited to the equations (1), equations (2) determine completely the $n+1$ unknowns

$$\lambda_0, \quad \lambda_1, \quad \lambda_2, \ldots \quad \lambda_n,$$

and the $m+1$ arbitraries

$$\mu_0, \quad \mu_1, \quad \mu_2, \ldots \quad \mu_n.$$

Thus all the difficulty is restored to the solution of these equations. Here is the particular process that we will employ in order to arrive there.

§2.

By putting

$$\phi(x) = \frac{\mu_0 + \mu_1 x + \mu_2 x^2 + \ldots + \mu_m x^m}{2},$$

we transform the equations (2) into these here:

$$\frac{\lambda_0}{\theta^2(x_0)} = \phi(x_0), \quad \frac{\lambda_1}{\theta^2(x_1)} = \phi(x_1), \ldots \quad \frac{\lambda_n}{\theta^2(x_n)} = \phi(x_n);$$

and we draw from them

$$(3) \qquad \lambda_0 = \theta^2(x_0)\phi(x_0), \quad \lambda_1 = \theta^2(x_1)\phi(x_1), \ldots \quad \lambda_n = \theta^2(x_n)\phi(x_n).$$

Transporting these values into the equations (1), they take the form

$$\begin{gathered} \theta^2(x_0)\phi(x_0) + \theta^2(x_1)\phi(x_1) + \ldots + \theta^2(x_n)\phi(x_n) = 1; \\ \theta^2(x_0)\phi(x_0)x_0 + \theta^2(x_1)\phi(x_1)x_1 + \ldots + \theta^2(x_n)\phi(x_n)x_n = X; \\ \theta^2(x_0)\phi(x_0)x_0^2 + \theta^2(x_1)\phi(x_1)x_1x^2 + \ldots + \theta^2(x_n)\phi(x_n)x_n^2 = X^2; \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \theta^2(x_0)\phi(x_0)x_0^{m-1} + \theta^2(x_1)\phi(x_1)x_1x^{m-1} + \ldots + \theta^2(x_n)\phi(x_n)x_n^{m-1} = X^{m-1}; \\ \theta^2(x_0)\phi(x_0)x_0^m + \theta^2(x_1)\phi(x_1)x_1x^m + \ldots + \theta^2(x_n)\phi(x_n)x_n^m = X^m. \end{gathered}$$

It is not difficult to note, under this form, that the first members are the coefficients of $\frac{1}{x}$, $\frac{1}{x^2}$, $\frac{1}{x^3}$,... $\frac{1}{x^m}$, $\frac{1}{x^{m+1}}$, in the series one obtains by developing, according to the decreasing powers of x, the function

$$\frac{\theta^2(x_0)\phi(x_0)}{x-x_0}+\frac{\theta^2(x_1)\phi(x_1)}{x-x_1}+\ldots+\frac{\theta^2(x_n)\phi(x_n)}{x-x_n}.$$

The second members are likewise the coefficients of the development of

$$\frac{1}{x-X}.$$

Consequently these equations are able to be replaced by the condition imposed on the difference of the two functions

$$\frac{\theta^2(x_0)\phi(x_0)}{x-x_0}+\frac{\theta^2(x_1)\phi(x_1)}{x-x_1}+\ldots+\frac{\theta^2(x_n)\phi(x_n)}{x-x_n}\text{ and }\frac{1}{x-X},$$

by not being contained at all in its development according to the descending powers of x the terms in $\frac{1}{x}$, $\frac{1}{x^2}$, $\frac{1}{x^3}$,... $\frac{1}{x^m}$, $\frac{1}{x^{m+1}}$. If therefore one sets this difference under the form of a function $\frac{M}{N}$, the degree of the denominator N will surpass the degree of the numerator at least by $m+2$. The preceding equations will be reduced therefore to

$$\frac{\theta^2(x_0)\phi(x_0)}{x-x_0}+\frac{\theta^2(x_1)\phi(x_1)}{x-x_1}+\ldots+\frac{\theta^2(x_n)\phi(x_n)}{x-x_n}-\frac{1}{x-X}=\frac{M}{N}.$$

On the other hand by setting, for brevity,

$$(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_n)=f(x),$$

and designating by U the entire function, contained in the fraction $\frac{\theta^2(x)\phi(x)f'(x)}{f(x)}$, one knows, by the theory of the decomposition of rational fractions into simple fractions, that

$$\frac{\theta^2(x)\phi(x)f'(x)}{f(x)}=U+\frac{\theta^2(x_0)\phi(x_0)}{x-x_0}+\frac{\theta^2(x_1)\phi(x_1)}{x-x_1}+\ldots+\frac{\theta^2(x_n)\phi(x_n)}{x-x_n}.$$

The equation formed just now will take therefore the form

$$\frac{\theta^2(x)\phi(x)f'(x)}{f(x)}-U-\frac{1}{x-X}=\frac{M}{N},$$

or else, the equivalent

$$\frac{(x-X)f'(x)\theta^2(x)}{f(x)}-\frac{U(x-X)+1}{\phi(x)}=\frac{(x-X)M}{\phi(x)N}.$$

By being supported on this relation, it is not difficult to find the expression of the function $\phi(x)$.

We note, in fact, that the fraction $\frac{(x-X)M}{\phi(x)N}$ is of a degree inferior to the degree of $\frac{1}{\phi^2(x)}$. For $\phi(x)$ represents the quantity

$$\frac{\mu_0 + \mu_1 x + \mu_2 x^2 + \ldots + \mu_m x^m}{2},$$

and, hence, not able to be of a degree superior to m. At the same time the degree of N surpasses at least by $(m+2)$ the degree of M; thus the fraction $\frac{(x-X)M}{N}$ is of a degree inferior to the one of $\frac{1}{\phi(x)}$.

Thence we conclude that in the relation above the fraction $\frac{U(x-X)+1}{\phi(x)}$ reproduces exactly the function $\frac{(x-X)f'(x)\theta^2(x)}{f(x)}$ at least to the term of degree $\frac{1}{\phi^2(x)}$ inclusively, that is to the term of which the degree will be the one of unity divided by the square of its denominator. But, one knows, this degree of exactitude belongs exclusively to the convergent fractions obtained by the reduction of $\frac{(x-X)f'(x)\theta^2(x)}{f(x)}$ into continued fraction. Besides, in the sequence of these convergent fractions that which will follow $\frac{U(x-X)+1}{\phi(x)}$ will have necessarily a denominator of a degree superior to m. For, without that, the difference

$$\frac{(x-X)f'(x)\theta^2(x)}{f(x)} - \frac{U(x-X)+1}{\phi(x)}$$

will not be of a degree inferior to $\frac{1}{\phi(x)x^m}$, as our relation supposes it

$$\frac{(x-X)f'(x)\theta^2(x)}{f(x)} - \frac{U(x-X)+1}{\phi(x)} = \frac{(x-X)M}{N},$$

where, one has seen, the fraction $\frac{(x-X)M}{N}$ is not able to be of a degree superior to $(-m-1)$.

Thus, the fraction $\frac{U(x-X)+1}{\phi(x)}$ will be found in the number of convergent fractions of which one will form the sequence by the development of $\frac{(x-X)f'(x)\theta^2(x)}{f(x)}$ by continued fraction; and in this sequence the convergent fraction, which will come immediately after, will have a denominator of degree superior to m; so that the fraction $\frac{U(x-X)+1}{\phi(x)}$, of which the denominator is of a degree which does not exceed m, is necessarily the last convergent fraction with denominator of a degree which does not exceed m, in the sequence of convergent fractions resulting from the development of the expression $\frac{(x-X)f'(x)\theta^2(x)}{f(x)}$, into continued fraction.

Seeking therefore this convergent fraction, if we represent it by $\frac{\phi^0(x)}{\phi_0(x)}$, we will have the equation

$$\frac{U(x-X)+1}{\phi(x)} = \frac{\phi^0(x)}{\phi_0(x)};$$

whence

$$U(x-X)+1 = \frac{\phi^0(x)\phi(x)}{\phi_0(x)}.$$

This equation supposes that the product $\phi^0(x)\phi_0(x)$ is divisible by $\phi_0(x)$; and as the properties of the convergent fractions require that $\phi^0(x)$ and $\phi_0(x)$ are first among them, $\phi_0(x)$ could not divide the product without dividing $\phi(x)$. Representing by q the quotient of this division, we will have

$$\phi(x) = q\phi_0(x),$$

and this value, being in the equation which precedes, gives

$$U(x - X) + 1 = q\phi^0(x).$$

In order to draw thence an expression of $\phi(x)$, we will note that $\phi(x)$ is not able to be of a degree superior to m. If therefore the factor $\phi_0(x)$ is of degree m, the factor q is reduced to a constant. It is easy to calculate it, for by putting $x = X$, in the last equation, there results from it

$$1 = q\phi^0(X) \quad \text{and} \quad q = \frac{1}{\phi^0(X)},$$

next finally,

$$\phi(x) = \frac{\phi_0(x)}{\phi^0(X)}.$$

Such is the value of the function $\phi(x)$, when $\phi_0(x)$ is of degree m precisely. In each other case, the degree of $\phi_0(x)$, being less than m, the factor q of the expression

$$\phi(x) = q\phi_0(x),$$

is able to receive for value any entire function of x, provided that the degree of the product $q\phi_0(x)$ not surpass m. Thus, in this case, there will be an infinity of values of the sought function $\phi(x)$. But if one agrees to take among these values that of which the degree is least elevated, one will be anew obliged to take for q a constant, and one will find, as previously, for $\phi(x)$ the value

$$\phi(x) = \frac{\phi_0(x)}{\phi^0(X)}.$$

According to the equations (3), the function thus determined gives

$$\lambda_0 = \theta^2(x_0)\phi(x_0), \quad \lambda_1 = \theta^2(x_1)\phi(x_1), \ldots \quad \lambda_n = \theta^2(x_n)\phi(x_n),$$

and these values are the coefficients of the formula

$$F(X) = \lambda_0 F(x_0) + \lambda_1 F(x_1) + \lambda_2 F(x_2) + \ldots + \lambda_n F(x_n),$$

by which $F(X)$ is expressed by means of the particular values $F(x_0)$, $F(x_1)$, $F(x_2)$,... $F(x_n)$.

Therefore one will have finally for $F(X)$ the expression

$$F(X) = \frac{\theta^2(x_0)\phi_0(x_0)}{\phi^0(x_0)}F(x_0) + \frac{\theta^2(x_1)\phi_0(x_1)}{\phi^0(x_1)}F(x_1) + \ldots + \frac{\theta^2(x_n)\phi_0(x_n)}{\phi^0(x_n)}F(x_n)F(x_n).$$

As for the quantities $\phi_0(x)$, $\phi^0(x)$, one has seen that it suffices, in order to determine them, to reduce to continued fraction the function

$$\frac{(x-X)f'(x)\theta^2(x)}{f(x)}$$

and to take, in the sequence of convergent fractions, the last of those of which the degree of the denominator does not surpass m. The numerator of this last fraction is $\phi^0(x)$ and the denominator $\phi_0(x)$.

The question that we ourselves have proposed at the commencement of the first paragraph is thus resolved.

§3.

In examining the formula that we just found, we are not able to lack convincing us that it must present important simplifications. Effectively, according to the nature of the question, the sought function $F(X)$ must be represented by an entire function of X, while the formula found by us contains the denominator $\phi^0(X)$ and offers a compositions such, that one not perceive how X will vanish from this denominator. This results from that which the functions $\phi^0(x)$, $\phi_0(x)$ determined by the development of the expression $\frac{(x-X)f'(x)\theta^2(x)}{f(x)}$ as continued fraction, contains X in their coefficients.

Finally to bring forth our value of $F(X)$ in a from which permits to see clearly the composition of it, we are going to show in what manner one passes from the convergent fractions of the expression $\frac{f'(x)\theta^2(x)}{f(x)}$ to the convergent fractions of the product $\frac{(x-X)f'(x)\theta^2(x)}{f(x)}$, and hence, to the fraction $\frac{\phi^0(x)}{\phi_0(x)}$.

For more simplicity, we admit that the continued fraction

$$q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cdots}}$$

resulting from the development of $\frac{f'(x)\theta^2(x)}{f(x)}$ contains only the denominators q_1, q_2,... of the first degree in x; and that, hence, the convergent fractions

$$\frac{q_0}{1}, \quad \frac{a_0 q_1 + 1}{q_1}, \quad \frac{q_0 q_1 q_2 + q_2 + q_0}{q_1 q_2 + 1},$$

have for denominators functions of the degrees 0, 1, 2, ... We will represent these convergent fractions respectively by

$$\frac{\pi_0(x)}{\psi_0(x)}, \quad \frac{\pi_1(x)}{\psi_1(x)}, \quad \frac{\pi_2(x)}{\psi_2(x)}, \ldots$$

It is advisable to remark still that in the function $\frac{f'(x)\theta^2(x)}{f(x)}$ the degree of the numerator is able to be less, but by one unit only, than the degree of the denominator; that which excluded certain special cases, depending on particular conditions among the coefficients of the functions $\theta(x)$ and $f(x)$ and giving to the development in continued fraction a form such that many of the denominators q_1, q_2,... could be of second, of third degree, or of superior degrees. Moreover, it is easy to be convinced that this exception could not exist in the fraction

$$q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cdots}}$$

for none of the cases of ordinary interpolation, where x_0, x_1, x_2, $\ldots x_n$, roots of the equation $f(x) = 0$, have some real values completely different from one another, and where the function $\theta(x)$, containing no imaginary coefficient, takes for $x = x_0$, x_1, x_2,... x_n the finite values $\frac{1}{k_0}$, $\frac{1}{k_1}$, $\frac{1}{k_2}$,... $\frac{1}{k_n}$. Under this hypothesis, one has effectively, by being served with the notation[1] of Mr. Cauchy (*Journal de l'École Polytechnique* 25th Cahier),

$$\mathfrak{Z}_{-\infty}^{+\infty}\left(\left(\frac{f'(x)\theta^2(x)}{f(x)}\right)\right) = n + 1;$$

and, according to the process which serves to determine the value of $\mathfrak{Z}_{-\infty}^{+\infty}\left(\left(\frac{f'(x)\theta^2(x)}{f(x)}\right)\right)$, it is clear that for $f(x)$ of degree $(n+1)$ it remains always inferior to $(n+1)$, if in the fraction

$$q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cdots}}$$

resulting from the development of $\frac{f'(x)\theta^2(x)}{f(x)}$, any one of the denominators q_1, q_2, q_3,... is of a degree superior to the first.

Convinced by these considerations that the limitations that we have brought to the form of the continued fraction deduced from the function $\frac{f'(x)\theta^2(x)}{f(x)}$, has no particular importance at all, we are able to approach at present the determination of $\frac{\phi^0(x)}{\phi_0(x)}$, that is of the last of the convergent fractions furnished by the development of the expression $(x - X)\frac{f'(x)\theta^2(x)}{f(x)}$, of which the denominators have no degree higher than m. We will demonstrate that this fraction is expressed by the formula

$$\frac{\psi_m(X)\pi_{m+1}(x) - \psi_{m+1}(X)\pi_m(x)}{\frac{1}{x-X}[\psi_m(X)\psi_{m+1}(x) - \psi_m(x)\psi_{m+1}(X)]}$$

in which $\frac{\pi_m(x)}{\psi_m(x)}$, $\frac{\pi_{m+1}(x)}{\psi_{m+1}(x)}$ designate the convergent fractions of the expression $\frac{f'(x)\theta^2(x)}{f(x)}$, of which the denominators are of the degrees m and $m+1$.

In fact, the composition of this formula shows with evidence that its denominator is reduced to an entire function of a degree which does not surpass m. On the other hand, if we take the difference between this same formula and the expression $\frac{(x-X)f'(x)\theta^2(x)}{f(x)}$,

[1] The notation cannot be reproduced exactly. Cauchy indicates by it the contour integral.

we find

$$\frac{\left[\frac{f'(x)\theta^2(x)}{f(x)} - \frac{\pi_{m+1}(x)}{\psi_{m+1}(x)}\right]\psi_m(X)\psi_{m+1}(x) - \left[\frac{f'(x)\theta^2(x)}{f(x)} - \frac{\pi_m(x)}{\psi_m(x)}\right]\psi_{m+1}(X)\psi_m(x)}{\frac{1}{x-X}[\psi_m(X)\psi_{m+1}(x) - \psi_m(x)\psi_{m+1}(X)]}$$

and this difference is not able to be of a degree superior to the one of

$$\frac{1}{\frac{1}{x-X}[\psi_m(X)\psi_{m+1}(x) - \psi_m(x)\psi_{m+1}(X)]} \cdot \frac{1}{x^m}.$$

For, according to the properties of the convergent fractions, the two terms

$$\left(\frac{f'(x)\theta^2(x)}{f(x)} - \frac{\pi_{m+1}(x)}{\psi_{m+1}(x)}\right)\psi_{m+1}(x); \quad \left(\frac{f'(x)\theta^2(x)}{f(x)} - \frac{\pi_m(x)}{\psi_m(x)}\right)\psi_m(x)$$

are of a degree less than $\frac{1}{\phi_m(x)}$, and, hence, than $\frac{1}{x^m}$.

Thus, the fraction

$$\frac{\psi_m(X)\pi_{m+1}(x) - \psi_{m+1}(X)\pi_m(x)}{\frac{1}{x-X}[\psi_m(X)\psi_{m+1}(x) - \psi_m(x)\psi_{m+1}(X)]}$$

which has a denominator of which the degree does not exceed m, will give exactly the terms of the function

$$(x-X)\frac{f'(x)\theta^2(x)}{f(x)}$$

to the term of which the degree is the same as the one of the expression

$$\frac{1}{\frac{1}{x-X}[\psi_m(X)\psi_{m+1}(x) - \psi_m(x)\psi_{m+1}(X)]} \cdot \frac{1}{x^m}.$$

But this function is able to be represented with this exactitude only by the convergent fractions that give its development into continued fraction, and only by those which are followed by other convergent fractions of which the denominators have a degree superior to m. Consequently, our fraction is to the number of these convergent fractions, and as the degree of its denominator not surpass m, it is the last which possesses a denominator of this kind and that we have designated by $\frac{\phi^0(x)}{\phi_0(x)}$.

This conclusion permits us to replace, in the formula of the preceding paragraph

$$F(X) = \frac{\theta^2(x_0)\phi_0(x_0)}{\phi^0(x_0)}F(x_0) + \frac{\theta^2(x_1)\phi_0(x_1)}{\phi^0(x_1)}F(x_1) + \ldots + \frac{\theta^2(x_n)\phi_0(x_n)}{\phi^0(x_n)}F(x_n),$$

the expressions

$$\frac{\phi_0(X)}{\phi^0(X)}, \quad \frac{\phi_0(x_1)}{\phi^0(X)}, \ldots \quad \frac{\phi_0(x_n)}{\phi^0(X)}$$

by these here respectively

$$\frac{\frac{1}{x_0-X}[\psi_m(X)\psi_{m+1}(x_0)-\psi_{m+1}(x)\psi_m(x_0)]}{\psi_m(X)\pi_{m+1}(X)-\psi_{m+1}(X)\pi_m(X)},$$
$$\frac{\frac{1}{x_1-X}[\psi_m(X)\psi_{m+1}(x_1)-\psi_{m+1}(x)\psi_m(x_1)]}{\psi_m(X)\pi_{m+1}(X)-\psi_{m+1}(X)\pi_m(X)},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\frac{\frac{1}{x_n-X}[\psi_m(X)\psi_{m+1}(x_n)-\psi_{m+1}(x)\psi_m(x_n)]}{\psi_m(X)\pi_{m+1}(X)-\psi_{m+1}(X)\pi_m(X)},$$

But the common denominator of all these expressions is reduced to $(-1)^m$ according to the theory of continued fractions. So that the formula which gives $F(X)$ is brought back to the form

$$\begin{aligned} F(X) \quad = \quad & (-1)^m \frac{\psi_m(X)\psi_{m+1}(x_0) - \psi_{m+1}(x)\psi_m(x_0)}{x_0 - X}\theta^2(x_0)F(x_0) + \\ & (-1)^m \frac{\psi_m(X)\psi_{m+1}(x_1) - \psi_{m+1}(x)\psi_m(x_1)}{x_1 - X}\theta^2(x_1)F(x_1) + \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & (-1)^m \frac{\psi_m(X)\psi_{m+1}(x_n) - \psi_{m+1}(x)\psi_m(x_n)}{x_n - X}\theta^2(x_n)F(x_n) \end{aligned}$$

One is able to write it under this brief form:

$$F(X) = (-1)^m \sum_{i=0}^{i=n} \frac{\psi_m(X)\psi_{m+1}(x_i) - \psi_{m+1}(x)\psi_m(x_i)}{x_i - X}\theta^2(x_i)F(x_i).$$

Here is therefore a new formula proper to the determination of $F(X)$ by means of the values of $F(x_0)$, $F(x_1)$, $F(x_2)$,... $F(x_n)$. It is constructed by aid of the functions $\psi_m(x)$, $\psi_{m+1}(x)$, which are the denominators of two of the convergent fractions obtained by the development into continued fraction of the expression $\frac{f'(x)\theta^2(x)}{f(x)}$. From the composition itself of this new form one concludes immediately that it is also an entire function of X.

§4.

Now we are going to show how the series, of which we have spoken in the Note presented last year to the Academy, is deduced from this formula; and it will serve us also to the explanation of some properties of the functions $\psi_0(x)$, $\psi_1(x)$, $\psi_2(x)\ldots$, determined by the development of $\frac{f'(x)\theta^2(x)}{f(x)}$ in continued fraction.

The formula that we just found gives $F(X)$ under the hypothesis of the form

$$F(x) = a + bx + cx^2 + \ldots + gx^{m-1} + hx^m.$$

We will represent this value of $F(X)$ by Y_m, and by Y_{m-1}, the value of $F(x)$, which will be deduced from the hypothesis, where $F(X)$ will be expressed by

$$F(x) = a + bx + cx^2 + \ldots + gx^{m-1}.$$

The new formula will furnish the following two values:

$$
(4)\quad \begin{aligned} Y_m &= (-1)^m \sum_{i=0}^{i=n} \frac{\psi_m(X)\psi_{m+1}(x_i) - \psi_{m+1}(X)\psi_m(x_i)}{x_i - X}\theta^2(x_i)F(x_i); \\ &= (-1)^{m-1} \sum_{i=0}^{i=n} \frac{\psi_{m-1}(X)\psi_m(x_i) - \psi_m(X)\psi_{m-1}(x_i)}{x_i - X}\theta^2(x_i)F(x_i). \end{aligned}
$$

Taking the difference of these values, one finds

$$
Y_m - Y_{m-1} =
$$

$$
(-1)^m \sum_{i=0}^{i=n} \frac{\psi_m(X)[\psi_{m+1}(x_i) - \psi_{m-1}(x_i)] - \psi_m(x_i)[\psi_{m+1}(X) - \psi_{m-1}(X)]}{x_i - X}\theta^2(x_i)F(x_i)
$$

The properties of the functions $\psi_{m+1}(x)$, $\psi_m(x)$, $\psi_{m-1}(x)$ permit simplifying notably this difference. These functions are, in fact, the denominators of convergent factors resulting from the development of the expression $\frac{f'(x)\theta^2(x)}{f(x)}$ into a continued fraction.

$$
q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cdots \cfrac{1}{q_m + \cfrac{1}{q_{m+1} + \cdots}}}}
$$

in which the denominators

$$
q_1, q_2, \ldots q_m, q_{m+1}, \ldots
$$

must be, by hypothesis, linear functions of the variable x. One has therefore consequently

$$
\begin{aligned}
q_1 &= A_1 x + B_1, \\
q_2 &= A_2 x + B_2, \\
&\ldots \\
&\ldots \\
q_{m+1} &= A_{m+1} x + B_{m+1}, \\
&\ldots
\end{aligned}
$$

Hence, the general rule for the formation of convergent fractions gives

$$
\begin{aligned}
\psi_{m+1}(x) &= q_{m+1}\psi_m(x) + \psi_{m-1}(x) \\
&= (A_{m+1}x + B_{m+1})\psi_m(x) + \psi_{m-1}(x);
\end{aligned}
$$

and thence

$$
\psi_{m+1}(x) - \psi_{m-1}(x) = (A_{m+1}x + B_{m+1})\psi_m(x).
$$

Changing x into x_i and into X, there results from it

$$\begin{aligned}\psi_{m+1}(x_i) - \psi_{m-1}(x_i) &= (A_{m+1}x_i + B_{m+1})\psi_m(x_i),\\ \psi_{m+1}(X) - \psi_{m-1}(X) &= (A_{m+1}X + B_{m+1})\psi_m(X).\end{aligned}$$

If one transports these values into that of the difference $Y_m - Y_{m-1}$, one obtains
Taking the difference of these values, one finds

$$Y_m - Y_{m-1} =$$

$$(-1)^m \sum_{i=0}^{i=n} \frac{\psi_m(X)\psi_{m+1}(x_i)[A_{m+1}x_i + B_{m+1}] - \psi_m(x_i)\psi_m(X)[A_{m+1}X + B_{m+1}]}{x_i - X}\theta^2(x_i)F(x_i)$$

or, by reducing,

$$Y_m - Y_{m-1} = (-1)^m A_{m+1}\psi_m(X)\sum_{i=0}^{i=n}\psi_m(x_i)\theta^2(x_i)F(x_i).$$

We put into this relation $m = 1, 2, 3, \ldots, (m-1), m$ successively, we will have

$$\begin{aligned}Y_1 - Y_0 &= -A_2\psi_1(X)\sum_{i=0}^{i=n}\psi_1(x_i)\theta^2(x_i)F(x_i);\\ Y_2 - Y_1 &= A_3\psi_2(X)\sum_{i=0}^{i=n}\psi_2(x_i)\theta^2(x_i)F(x_i);\\ & \quad \cdots\cdots\cdots\cdots\cdots\cdots\\ & \quad \cdots\cdots\cdots\cdots\cdots\cdots\\ Y_{m-1} - Y_{m-2} &= (-1)^{m-1}A_m\psi_{m-1}(X)\sum_{i=0}^{i=n}\psi_{m-1}(x_i)\theta^2(x_i)F(x_i);\\ Y_m - Y_{m-1} &= (-1)^m A_{m+1}\psi_m(X)\sum_{i=0}^{i=n}\psi_m(x_i)\theta^2(x_i)F(x_i);\end{aligned}$$

and the sum of these equations will give

$$\begin{aligned}Y_m - Y_0 &= -A_2\psi_1(X)\sum_{i=0}^{i=n}\psi_1(x_i)\theta^2(x_i)F(x_i) + A_3\psi_2(X)\sum_{i=0}^{i=n}\psi_2(x_i)\theta^2(x_i)F(x_i)\\ &\quad + \ldots + (-1)^{m-1}A_m\psi_{m-1}(X)\sum_{i=0}^{i=n}\psi_{m-1}(x_i)\theta^2(x_i)F(x_i)\\ &\quad + (-1)^m A_{m+1}\psi_m(X)\sum_{i=0}^{i=n}\psi_{m-1}(x_i)\theta^2(x_i)F(x_i).\end{aligned}$$

Y_m will have for value

$$Y_m = Y_0 - A_2\psi_1(X)\sum_{i=0}^{i=n}\psi_1(x_i)\theta^2(x_i)F(x_i) + A_3\psi_2(X)\sum_{i=0}^{i=n}\psi_2(x_i)\theta^2(x_i)F(x_i)$$
$$\cdots\cdots\cdots\cdots + (-1)^m A_{m+1}\psi_m(X)\sum_{i=0}^{i=n}\psi_{m-1}(x_i)\theta^2(x_i)F(x_i).$$

In order to determine the quantity Y_0, we make $m = 0$, in formula (4), we find

$$Y_0 = \sum_{i=0}^{i=n}\frac{\psi_0(X)\psi_1(x_i) - \psi_1(X)\psi_0(x_i)}{x_i - X}\theta^2(x_i)F(x_i)$$

$\psi_0(x)$, $\psi_1(x)$ designate the denominators of the first two convergent fractions of the expression $\frac{f'(x)\theta^2(x)}{f(x)}$, of which the development in continued fraction has received the forms

$$q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cdots}} = q_0 + \cfrac{1}{A_1x + B_1 + \cfrac{1}{A_2x + +B_2 + \cdots}}$$

There results from it

$$\psi_0(x) = 1, \quad \psi_1(x) = A_1x + B_1;$$

and the function Y_0 becomes

$$Y_0 = \sum_{i=0}^{i=n}\frac{A_1x_i + B1 - A_1X - B_1}{x_i - X}\theta^2(x_i)F(x_i)$$
$$= A_1\sum_{i=0}^{i=n}\theta^2(x_i)F(x_i),$$

that one is able to write

$$Y_0 = A_1\psi_0(X)\sum_{i=0}^{i=n}\psi_0(x_i)\theta^2(x_i)F(x_i),$$

provided that one is reminded that $\psi_0(x) = 1$.

By means of this value, the preceding expression of Y_m, or of the value of $F(X)$ under the hypothesis

$$F(x) = a + bx + cx^2 + \ldots + gx^{m-1} + hx^m,$$

takes the symmetric form

$$Y_m = A_1\psi_0(X)\sum_{i=0}^{i=n}\psi_0(x_i)\theta^2(x_i)F(x_i) - A_2\psi_1(X)\sum_{i=0}^{i=n}\psi_1(x_i)\theta^2(x_i)F(x_i) + \ldots$$
$$\ldots + (-1)^m A_{m+1}\psi_m(X)\sum_{i=0}^{i=n}\psi_m(x_i)\theta^2(x_i)F(x_i).$$

In this expression the functions $\psi_0(x)$, $\psi_1(x)$, $\psi_2(x)$,... and the constants A_1, A_2, A_3... are determined by the development of the function $\frac{f'(x)\theta^2(x)}{f(x)}$ in a continued fraction of the form

$$q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \cdots}}}$$

The functions $\psi_0(x)$, $\psi_1(x)$, $\psi_2(x)$,... are the denominators of the convergent fractions that one deduces from this continued fraction; and the constants A_1, A_2, A_3... are the coefficients of x in the denominators q_1, q_2, q_3,...

In the particular case for which the law of errors is the same for all the quantities $F(x_0)$, $F(x_1)$, ... one is able, conformably to §1, to take all the values k_0, k_1, k_2,... equal to 1, and hence the function $\theta(x)$, determined by the equations

$$\theta(x_0) = \frac{1}{k_0}, \quad \theta(x_1) = \frac{1}{k_1}, \quad \theta(x_2) = \frac{1}{k_2}, \ldots$$

is itself reduced to unity. The formula found above takes therefore then the form

$$Y_m = A_1\psi_0(X)\sum_{i=0}^{i=n}\psi_0(x_i)F(x_i) - A_2\psi_1(X)\sum_{i=0}^{i=n}\psi_1(x_i)F(x_i) + \ldots$$
$$\ldots + (-1)^m A_{m+1}\psi_m(X)\sum_{i=0}^{i=n}\psi_m(x_i)F(x_i).$$

Here $\psi_0(x)$, $\psi_1(x)$, $\psi_2(x)$, ..., A_1, A_2, A_3,... are determined by the continued fraction that gives the function $\frac{f'(x)}{f(x)}$. It is of this series that we have spoken in the Note already mentioned.[2] But at present we ourselves will not be limited to this particular hypothesis, which reduces the function $\theta(x)$ to unity, and we will consider the series in its general form. We will thus be led to some curious propositions on the functions $\psi_0(x)$, $\psi_1(x)$, $\psi_2(x)$, ...

§5.

It is not difficult to see that if the quantities

$$F(x_0), \quad F(x_1), \quad F(x_2), \ldots \quad F(x_n)$$

are determined exactly by the formula

$$F(x) = a + bx + cx^2 + \ldots + gx^{m-1} + hx^m,$$

[2]The Note of Mr. Chebyshev, with date of 20 October (1 November 1854), contains effectively only this particular formula. The two pages of this Note are reproduced in its entirety, with the exception of the corollary that is here:

"In the particular case of $x_0 = \frac{n}{n}$, $x_1 = \frac{n-2}{n}$, $x_2 = \frac{n-4}{n}$, ..., $x_n = \frac{-n}{n}$, and of n infinitely great, this formula furnishes the development of $F(x)$ according to the values of certain functions that Legendre has designated by X^m (*Exerc*, part. V, §10) and which are determined by the reduction of the expression $\log\frac{x+1}{x-1}$ in continued fraction." [Note of Bienaymé.]

our series will give the exact expression of this function, whatever is able to be the function $\theta(x)$. This is that which becomes evident, if one notes that the series results from the formula

$$F(X) = \lambda_0 F(x_0) + \lambda_1 F(x_1) + \ldots + \lambda_n F(x_n),$$

and that according to one of the conditions which fix the values of the factors λ_0, λ_1, $\lambda_2, \ldots \lambda_n$, the following equations must be satisfied:

$$\begin{array}{lllll}
\lambda_0 & +\lambda_1 & +\lambda_2 & +\ldots+\lambda_n & = 1, \\
\lambda_0 x_0 & +\lambda_1 x_1 & +\lambda_2 x_2 & +\ldots+\lambda_n x_n & = X, \\
\lambda_0 x_0 x^2 & +\lambda_1 x_1 x^2 & +\lambda_2 x_2 x^2 & +\ldots+\lambda_n x_n x^2 & = X^2, \\
\ldots & & & & \\
\ldots & & & & \\
\lambda_0 x_0^m & +\lambda_1 x_1^m & +\lambda_2 x_2^m & +\ldots+\lambda_n x_n^m & = X^m.
\end{array}$$

Now, by virtue of these equations, the sum

$$\lambda_0 F(x_0) + \lambda_1 F(x_1) + \ldots + \lambda_n F(x_n),$$

when one replaces the quantities $F(x_0)$, $F(x_1)$, $F(x_2)$,... $F(x_n)$ by their value drawn from the equation

$$F(x) = a + bx + cx^2 + \ldots + gx^{m-1} + hx^m,$$

is reduced to

$$a + bX + cX^2 + \ldots + gX^{m-1} + hX^m,$$

an exact expression of $F(X)$, after the hypothesis itself

$$F(x) = a + bx + cx^2 + \ldots + gx^{m-1} + hx^m.$$

When therefore if the concern is of an entire function $F(X)$, the formula of the preceding paragraph permits to represent it thus:

$$F(X) = A_1\psi_0(X)\sum_{i=0}^{i=n}\psi_0(x_i)\theta^2(x_i)F(x_i) - A_2\psi_1(X)\sum_{i=0}^{i=n}\psi_1(x_i)\theta^2(x_i)F(x_i)$$
$$+ \ldots + (-1)^m A_{m+1}\psi_m(X)\sum_{i=0}^{i=n}\psi_m(x_i)\theta^2(x_i)F(x_i).$$

If we make

$$F(x) = \psi_m(x),$$

we will find

$$\psi_m(X) = A_1\psi_0(X)\sum_{i=0}^{i=n}\psi_0(x_i)\theta^2(x_i)\psi_m(x_i) - A_2\psi_1(X)\sum_{i=0}^{i=n}\psi_1(x_i)\theta^2(x_i)\psi_m(x_i)$$
$$+ \ldots + (-1)^m A_{m+1}\psi_m(X)\sum_{i=0}^{i=n}\psi_m(x_i)\theta^2(x_i).$$

or else, by setting all the terms under one member alone,

$$A_1\psi_0(X)\sum_{i=0}^{i=n}\psi_0(x_i)\psi_m(x_i)\theta^2(x_i) - A_2\psi_1(X)\sum_{i=0}^{i=n}\psi_1(x_i)\psi_m(x_i)\theta^2(x_i) + \dots$$
$$\dots + \psi_m(X)[(-1)^m A_{m+1}\sum_{i=0}^{i=n}\psi_m^2(x_i)\theta^2(x_i) - 1] = 0.$$

But as the functions $\psi(x_0)$, $\psi(x_1)$, $\psi(x_2)$,..., are respectively of the degrees 0, 1, 2, 3, ..., the preceding identity supposes that each of its terms vanish separately. One has therefore, from all necessity,

$$\begin{array}{l} (-1)^m A_{m+1}\sum_{i=0}^{i=n}\psi_m^2(x_i)\theta^2(x_i) - 1 = 0, \\ A_m\sum_{i=0}^{i=n}\psi_{m-1}(x_i)\psi_m(x_i)\theta^2(x_i) = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ A_2\sum_{i=0}^{i=n}\psi_1(x_i)\psi_m(x_i)\theta^2(x_i) = 0, \\ A_1\sum_{i=0}^{i=n}\psi_0(x_i)\psi_m(x_i)\theta^2(x_i) = 0. \end{array}$$

The first of these relations gives us

$$\sum_{i=0}^{i=n}\psi_m^2(x_i)\theta^2(x_i) = \frac{(-1)^m}{A_{m+1}};$$

and, by observing that the coefficients A_m, ... A_2, A_1 each differ from zero, the other relations make conclude that

$$(5)\qquad \left\{\begin{array}{l} \sum_{i=0}^{i=n}\psi_{m-1}(x_i)\psi_m(x_i)\theta^2(x_i) = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \sum_{i=0}^{i=n}\psi_1(x_i)\psi_m(x_i)\theta^2(x_i) = 0, \\ \sum_{i=0}^{i=n}\psi_0(x_i)\psi_m(x_i)\theta^2(x_i) = 0. \end{array}\right.$$

It is thence manifest that, for m' different from m, the sum $\sum_{i=0}^{i=n}\psi_m(x_i)\psi_{m'}(x_i)\theta^2(x_i)$ is equal to zero. If on the contrary $m' = m$, this sum is equal to $\frac{(-1)^m}{A_{m+1}}$ as one has seen just now. One has therefore for the coefficient A_{m+1} the value

$$A_{m+1} = \frac{(-1)^m}{\sum_{i=0}^{i=n}\psi_m^2(x_i)\theta^2(x_i)}.$$

One deduces from it for all the other coefficients A

$$\begin{array}{r} A_1 = \frac{1}{\sum_{i=0}^{i=n}\psi_0^2(x_i)\theta^2(x_i)}, \\ A_2 = -\frac{1}{\sum_{i=0}^{i=n}\psi_1^2(x_i)\theta^2(x_i)}, \\ \dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots \\ A_{m+1} = \frac{(-1)^m}{\sum_{i=0}^{i=n}\psi_m^2(x_i)\theta^2(x_i)}. \end{array}$$

If one introduces these values into the formula

$$F(X) = A_1\psi_0(X)\sum_{i=0}^{i=n}\psi_0(x_i)\theta^2(x_i)F(x_i) - A_2\psi_1(X)\sum_{i=0}^{i=n}\psi_1(x_i)\theta^2(x_i)F(x_i)$$
$$+\ldots+(-1)^m A_{m+1}\psi_m(X)\sum_{i=0}^{i=n}\psi_m(x_i)\theta^2(x_i)F(x_i),$$

it takes the form

(6)
$$\begin{cases} F(X) = \frac{\sum_{i=0}^{i=n}\psi_0(x_i)\theta^2(x_i)F(x_i)}{\sum_{i=0}^{i=n}\psi_0^2(x_i)\theta^2(x_i)}\psi_0(X) + \frac{\sum_{i=0}^{i=n}\psi_1(x_i)\theta^2(x_i)F(x_i)}{\sum_{i=0}^{i=n}\psi_1^2(x_i)\theta^2(x_i)}\psi_1(X)+\ldots \\ \ldots + \frac{\sum_{i=0}^{i=n}\psi_m(x_i)\theta^2(x_i)F(x_i)}{\sum_{i=0}^{i=n}\psi_m^2(x_i)\theta^2(x_i)}\psi_m(X). \end{cases}$$

The composition of this formula shows that it does not change in value, when one introduces some arbitrary constant factors into the functions $\psi_0(x)$, $\psi_1(x)$, $\psi_2(x)$, etc. It will be therefore possible to take in order to determine these functions the development of $\frac{f'(x)\theta(x)}{f(x)}$ in a continued fraction of the form

$$q_0 + \cfrac{L'}{q_1 + \cfrac{L''}{q_2+\cdots}}$$

whatever are able to be the constants L', L'', etc. One knows effectively that the terms of the convergent fractions deduced from any one expression by the development of this expression in continued fraction from one of the two forms

$$q_0 + \cfrac{L'}{q_1 + \cfrac{L''}{q_2+\cdots}} \qquad q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2+\cdots}}$$

differ only by the constant factors.

In precisely the same manner the equations (5) remain completely exact for the functions $\psi_0(x)$, $\psi_1(x)$, $\psi_2(x)$, etc., determined by the development of $\frac{f'(x)\theta^2(x)}{f(x)}$ into a continued fraction of the form

$$q_0 + \cfrac{L'}{q_1 + \cfrac{L''}{q_2+\cdots}}$$

for they will be not at all altered by the introduction of any constant factors in the functions $\psi_0(x)$, $\psi_1(x)$, $\psi_2(x)$, etc. Thus by proceeding actually to the applications of formula (6) and of equations (5), we will not be arrested at all by the supposition made first in the preceding paragraphs and according to which the numerators L', L'', etc. must be equal to unity in the continued fraction

$$q_0 + \cfrac{L'}{q_1 + \cfrac{L''}{q_2+\cdots}}$$

which serve to construct the functions $\psi_0(x)$, $\psi_1(x)$, $\psi_2(x)$, etc.

By virtue of these equations (5), there exist yet some remarkable relations among the functions $\psi_0(x)$, $\psi_1(x)$, $\psi_2(x)$, etc. and one arrives without pain by aid of formula (6), by comparing it for $m = n$ with the formula of interpolation of Lagrange.

For $m = n$, in fact, formula (6) gives to the expression of a function of the n^{th} degree, by the values that it receives for some values of the variable $x = x_0$, x_1, x_2,… x_n the form that is here:

$$\begin{cases} F(X) = \frac{\sum_{i=0}^{i=n} \psi_0(x_i)\theta^2(x_i)F(x_i)}{\sum_{i=0}^{i=n} \psi_0^2(x_i)\theta^2(x_i)}\psi_0(X) + \frac{\sum_{i=0}^{i=n} \psi_1(x_i)\theta^2(x_i)F(x_i)}{\sum_{i=0}^{i=n} \psi_1^2(x_i)\theta^2(x_i)}\psi_1(X) + \dots \\ \qquad \dots + \frac{\sum_{i=0}^{i=n} \psi_n(x_i)\theta^2(x_i)F(x_i)}{\sum_{i=0}^{i=n} \psi_n^2(x_i)\theta^2(x_i)}\psi_n(X). \end{cases}$$

The formula of Lagrange expresses the same function by the form

$$\sum_{i=0}^{i=n} \frac{(X-x_1)(X-x_2)\dots(X-x_{i-1})(X-x_{i+1})\dots}{(x_i-x_1)(x_i-x_2)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots}F(x_i).$$

The identity of these two expressions, whatever are able to be the values of $F(x_0)$, $F(x_1)$, $F(x_2)$, …, $F(x_n)$, requires that the terms which have these functions for factors are the same in one another. If therefore one compares the terms which multiply $F(x_i)$ one will have the relation

$$\sum_{i=0}^{i=n} \frac{(X-x_1)(X-x_2)\dots(X-x_{i-1})(X-x_{i+1})\dots}{(x_i-x_1)(x_i-x_2)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots} =$$
$$\frac{\psi_0(x_i)\theta^2(x_i)}{\sum_{i=0}^{i=n} \psi_0^2(x_i)\theta^2(x_i)}\psi_0(X) + \frac{\psi_1(x_i)\theta^2(x_i)}{\sum_{i=0}^{i=n} \psi_1^2(x_i)\theta^2(x_i)}\psi_1(X) + \dots\dots + \frac{\psi_n(x_i)\theta^2(x_i)}{\sum_{i=0}^{i=n} \psi_n^2(x_i)\theta^2(x_i)}\psi_n(X).$$

If one makes $X = x_\eta$, provided that η is not equal to i, one obtains

$$0 = \frac{\psi_0(x_i)\theta^2(x_i)}{\sum_{i=0}^{i=n} \psi_0^2(x_i)\theta^2(x_i)}\psi_0(x_\eta) + \frac{\psi_1(x_i)\theta^2(x_i)}{\sum_{i=0}^{i=n} \psi_1^2(x_i)\theta^2(x_i)}\psi_1(x_\eta) + \dots\dots + \frac{\psi_n(x_i)\theta^2(x_i)}{\sum_{i=0}^{i=n} \psi_n^2(x_i)\theta^2(x_i)}\psi_n(x_\eta)$$

By the introduction of the factor $\frac{\theta(x_\eta)}{\theta(x_i)}$ one is able to write this expression in the following manner:

$$0 = \frac{\psi_0(x_i)\theta(x_i)\psi_0(x_\eta)\theta(x_\eta)}{\sum_{i=0}^{i=n} \psi_0^2(x_i)\theta^2(x_i)} + \frac{\psi_1(x_i)\theta(x_i)\psi_1(x_\eta)\theta(x_\eta)}{\sum_{i=0}^{i=n} \psi_1^2(x_i)\theta^2(x_i)} + \dots\dots + \frac{\psi_n(x_i)\theta(x_i)\psi_n(x_\eta)\theta(x_\eta)}{\sum_{i=0}^{i=n} \psi_n^2(x_i)\theta^2(x_i)}$$

Making on the contrary $X = x_i$, we will have

$$1 = \frac{\psi_0^2(x_i)\theta^2(x_i)}{\sum_{i=0}^{i=n} \psi_0^2(x_i)\theta(x_i)} + \frac{\psi_1^2(x_i)\theta^2(x_i)}{\sum_{i=0}^{i=n} \psi_1^2(x_i)\theta^2(x_i)} + \dots\dots + \frac{\psi_n^2(x_i)\theta^2(x_i)}{\sum_{i=0}^{i=n} \psi_n^2(x_i)\theta^2(x_i)}.$$

§6.

These equations, reunited to the equations (5), establish a remarkable propriety with the functions determined by the formula

$$\frac{\psi_m(x)\theta(x)}{\sqrt{\sum_{i=0}^{i=n} \psi_m^2(x_i)\theta^2(x_i)}}.$$

We designate these functions by $\Phi_m(x)$; the equations constructed just now will give to us

$$\sum_{m=0}^{m=n} \Phi_m(x_i)\Phi_m(x_\eta) = 0,$$

as long as η differs from i, and

$$\sum_{m=0}^{m=n} \Phi_m(x_i)\Phi_m(x_\eta) = 1,$$

for $\eta = i$.

According to the form of the function $\Phi_m(x)$ and the equations (5), it is easy to note that

$$\sum_{i=0}^{i=n} \Phi_m(x_i)\Phi_{m\prime}(x_i) = 0, \text{ or } 1,$$

according as $m\prime$ will differ from m or will be equal to m. For the sum of which there is concern must, by the substitution of the values of $\Phi_m(x)$, $\Phi_{m\prime}(x)$,

$$\frac{\sum_{i=0}^{i=n} \Phi_m(x_i)\Phi_{m\prime}(x_i)\theta^2(x_i)}{\sqrt{\sum_{i=0}^{i=n} \psi_m^2(x_i)\theta^2(x_i)}\sqrt{\sum_{i=0}^{i=n} \psi_{m\prime}^2(x_i)\theta^2(x_i)}}$$

Now, according to equations (5), the numerator is annulled if $m\prime$ is not equal to m; and if $m\prime = m$, it must be equal to the denominator, that which reduces the fraction to unity.

These properties lead to another which the function yet possesses

$$\Phi_m(x) = \frac{\psi_m(x)\theta(x)}{\sqrt{\sum_{i=0}^{i=n} \psi_m^2(x_i)\theta^2(x_i)}},$$

composed with the functions $\psi_0(x)$, $\psi_1(x)$, $\psi_2(x)$, etc., which serve as denominators to the convergent fractions deduced from the development of the function $\frac{f'(x)\theta^2(x)}{f(x)}$ into a continued fraction of the form

$$q_0 + \cfrac{L'}{q_1 + \cfrac{L''}{q_2 + \cdots}}$$

If from all the values of the function $\Phi_m(x)$ obtained by making $m = 0$, 1, 2, ..., n and $x = x_0$, x_1, x_2,... x_n one composes the square

$$\begin{array}{llll}
\Phi_0(x_0), & \Phi_0(x_1), & \Phi_0(x_2),\ldots & \Phi_0(x_n), \\
\Phi_1(x_0), & \Phi_1(x_1), & \Phi_1(x_2),\ldots & \Phi_1(x_n), \\
\Phi_2(x_0), & \Phi_2(x_1), & \Phi_2(x_2),\ldots & \Phi_2(x_n), \\
\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\
\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\
\Phi_n(x_0), & \Phi_n(x_1), & \Phi_n(x_2),\ldots & \Phi_n(x_n),
\end{array}$$

the sum of the squares of the terms of any rank, horizontal or vertical, will be equal to unity; the sum of the products of the corresponding terms of two horizontal or vertical ranks will be equal to zero.

The construction of squares of this kind makes the subject of a Memoir of Euler entitled: *Problema algebraicum ob affectiones prorsus singulares memorabile* (N. Comm., t. XV).

§7.

Equations (5) demonstrate yet easily a particular property in the functions

$$\psi_1(x), \quad \psi_2(x), \quad \psi_3(x), \ldots$$

compared to all the functions of same degree and of same coefficient of the highest power of x: *for these functions the sums*

$$\sum_{i=0}^{i=n} \psi_1^2(x_i)\theta^2(x_i), \quad \sum_{i=0}^{i=n} \psi_2^2(x_i)\theta^2(x_i), \quad \sum_{i=0}^{i=n} \psi_3^2(x_i)\theta^2(x_i), \ldots$$

have the smallest value possible.

In fact, as the functions $\psi_0(x)$, $\psi_1(x)$, $\psi_2(x)$,..., $\psi_m(x)$ are respectively of degrees 0, 1, 2..., m, each entire function V of the degree m is able to be expressed thus:

$$V = A\psi_0(x) + B\psi_1(x) + C\psi_2(x) + \ldots + H\psi_m(x).$$

But here it is necessary to take $H = 1$, since one supposes that the coefficient of x^m is the same in V and in $\psi_m(x)$. One will have therefore under this hypothesis

$$V = A\psi_0(x) + B\psi_1(x) + C\psi_2(x) + \ldots + \psi_m(x).$$

The concern is to find the values of the coefficients A, B, C, etc., which renders a minimum the sum

$$\sum_{i=0}^{i=n} V^2\theta^2(x_i) = \sum_{i=0}^{i=n} [A\psi_0(x_i) + B\psi_1(x_i) + C\psi_2(x_i) + \ldots + \psi_m(x_i)]^2\theta^2(x_i)$$

The known process of the differential calculus gives us the following equations:

$$\begin{aligned}
2\sum_{i=0}^{i=n}[A\psi_0(x_i) + B\psi_1(x_i) + C\psi_2(x_i) + \ldots + \psi_m(x_i)]\psi_0(x_i)\theta^2(x_i) &= 0,\\
2\sum_{i=0}^{i=n}[A\psi_0(x_i) + B\psi_1(x_i) + C\psi_2(x_i) + \ldots + \psi_m(x_i)]\psi_1(x_i)\theta^2(x_i) &= 0,\\
2\sum_{i=0}^{i=n}[A\psi_0(x_i) + B\psi_1(x_i) + C\psi_2(x_i) + \ldots + \psi_m(x_i)]\psi_2(x_i)\theta^2(x_i) &= 0,\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

Equations (5) reduce them to a single term

$$\begin{aligned} 2A\sum_{i=0}^{i=n}\psi_0^2(x_i)\theta^2(x_i) &= 0, \\ 2B\sum_{i=0}^{i=n}\psi_1^2(x_i)\theta^2(x_i) &= 0, \\ 2C\sum_{i=0}^{i=n}\psi_2^2(x_i)\theta^2(x_i) &= 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots & \end{aligned}$$

whence one draws

$$A = 0, \quad B = 0, \quad C = 0, \ldots$$

Thus the conditions of the minimum of the sum $\sum_{i=0}^{i=n} V^2\theta^2(x_i)$, when V is of the form

$$A\psi_0(x) + B\psi_1(x) + C\psi_2(x) + \ldots + \psi_m(x),$$

are

$$A = 0, \quad B = 0, \quad C = 0, \ldots$$

and, hence,

$$V = \psi_m(x).$$

One demonstrates still without difficulty that if one employs formula (6)

$$\frac{\sum_{i=0}^{i=n}\psi_0(x_i)\theta^2(x_i)F(x_i)}{\sum_{i=0}^{i=n}\psi_0^2(x_i)\theta^2(x_i)}\psi_0(X) + \frac{\sum_{i=0}^{i=n}\psi_1(x_i)\theta^2(x_i)F(x_i)}{\sum_{i=0}^{i=n}\psi_1^2(x_i)\theta^2(x_i)}\psi_1(X) + \ldots$$
$$\ldots + \frac{\sum_{i=0}^{i=n}\psi_m(x_i)\theta^2(x_i)F(x_i)}{\sum_{i=0}^{i=n}\psi_m^2(x_i)\theta^2(x_i)}\psi_m(X)$$

to determine by approximation any function $F(X)$, one will obtain in order to express it an entire function of degree m such, that the sum of the squares of the differences between the values of this entire function and the corresponding values of $F(X)$ for $X = x_0, x_1, x_2, \ldots, x_n$, each multiplied by $\theta^2(x_0)$, $\theta^2(x_1)$, $\theta^2(x_2)$, etc., respectively, will be a minimum.

We represent effectively the function sought under the form

$$A\psi_0(X) + B\psi_1(X) + C\psi_2(X) + \ldots + H\psi_m(X)$$

and we choose the values of the coefficients A, B, C, $\ldots$, H for which the sum

$$\sum_{i=0}^{i=n}[F(x_i) - A\psi_0(x_i) - B\psi_1(x_i) - C\psi_2(x_i) - \ldots - H\psi_m(x_i)]^2\theta^2(x_i)$$

will be a minimum. We will find the equations

$$
\begin{aligned}
2\sum_{i=0}^{i=n}[F(x_i) - A\psi_0(x_i) - B\psi_1(x_i) - C\psi_2(x_i) - \ldots - H\psi_m(x_i)]^2\psi_0(x_i)\theta^2(x_i) &= 0,\\
2\sum_{i=0}^{i=n}[F(x_i) - A\psi_0(x_i) - B\psi_1(x_i) - C\psi_2(x_i) - \ldots - H\psi_m(x_i)]^2\psi_1(x_i)\theta^2(x_i) &= 0,\\
2\sum_{i=0}^{i=n}[F(x_i) - A\psi_0(x_i) - B\psi_1(x_i) - C\psi_2(x_i) - \ldots - H\psi_m(x_i)]^2\psi_2(x_i)\theta^2(x_i) &= 0,\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
2\sum_{i=0}^{i=n}[F(x_i) - A\psi_0(x_i) - B\psi_1(x_i) - C\psi_2(x_i) - \ldots - H\psi_m(x_i)]^2\psi_m(x_i)\theta^2(x_i) &= 0.
\end{aligned}
$$

By virtue of the relations (5), these equations are reduced to the form

$$
\begin{aligned}
2\sum_{i=0}^{i=n} F(x_i)\psi_0(x_i)\theta^2(x_i) - 2A\sum_{i=0}^{i=n}\psi_0^2(x_i)\theta^2(x_i) &= 0,\\
2\sum_{i=0}^{i=n} F(x_i)\psi_1(x_i)\theta^2(x_i) - 2B\sum_{i=0}^{i=n}\psi_1^2(x_i)\theta^2(x_i) &= 0,\\
2\sum_{i=0}^{i=n} F(x_i)\psi_2(x_i)\theta^2(x_i) - 2C\sum_{i=0}^{i=n}\psi_2^2(x_i)\theta^2(x_i) &= 0,\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
2\sum_{i=0}^{i=n} F(x_i)\psi_m(x_i)\theta^2(x_i) - 2H\sum_{i=0}^{i=n}\psi_m^2(x_i)\theta^2(x_i) &= 0,
\end{aligned}
$$

whence

$$
\begin{aligned}
A &= \frac{\sum_{i=0}^{i=n}\psi_0(x_i)\theta^2(x_i)F(x_i)}{\sum_{i=0}^{i=n}\psi_0^2(x_i)\theta^2(x_i)},\\
B &= \frac{\sum_{i=0}^{i=n}\psi_1(x_i)\theta^2(x_i)F(x_i)}{\sum_{i=0}^{i=n}\psi_1^2(x_i)\theta^2(x_i)},\\
C &= \frac{\sum_{i=0}^{i=n}\psi_2(x_i)\theta^2(x_i)F(x_i)}{\sum_{i=0}^{i=n}\psi_2^2(x_i)\theta^2(x_i)},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
H &= \frac{\sum_{i=0}^{i=n}\psi_m(x_i)\theta^2(x_i)F(x_i)}{\sum_{i=0}^{i=n}\psi_m^2(x_i)\theta^2(x_i)}.
\end{aligned}
$$

In reporting these values in the expression

$$A\psi_0(X) + B\psi_1(X) + C\psi_2(X) + \ldots + H\psi_m(X),$$

we find, conformably to that which has been advanced, that the formula sought for $F(X)$ is precisely

$$\frac{\sum_{i=0}^{i=n} \psi_0(x_i)\theta^2(x_i)F(x_i)}{\sum_{i=0}^{i=n} \psi_0^2(x_i)\theta^2(x_i)}\psi_0(X) + \frac{\sum_{i=0}^{i=n} \psi_1(x_i)\theta^2(x_i)F(x_i)}{\sum_{i=0}^{i=n} \psi_1^2(x_i)\theta^2(x_i)}\psi_1(X) +$$

$$\frac{\sum_{i=0}^{i=n} \psi_2(x_i)\theta^2(x_i)F(x_i)}{\sum_{i=0}^{i=n} \psi_2^2(x_i)\theta^2(x_i)}\psi_2(X) + \ldots + \frac{\sum_{i=0}^{i=n} \psi_m(x_i)\theta^2(x_i)F(x_i)}{\sum_{i=0}^{i=n} \psi_m^2(x_i)\theta^2(x_i)}\psi_m(X).$$