Extrait d'un Mémoire sur les fractions continues*

Pafnuty Chebyshev

Bulletin physico-mathématique de de l'Académie Impériale de St.-Pétersbourg, T. XIII , p. 287–288.
(Read 12 January 1855)

In a Note, read 20 October of the past year, Mr. Chebyshev has presented a certain formula of interpolation which has the advantage of giving the expression of the function sought with the coefficients indicated by the *method of least squares*. In the present Memoir, he treats this subject in its most general form, namely: by supposing that the values of the sought function are affected with errors of which the laws of probability are different, and, under this hypothesis, he shows how, by aid of the development of a certain expression as continued fraction, one arrives to find the approximate value of the function sought with the least error to fear. For the determination of this value he gives three different formulas, of which the one comprehends, as particular case, that which has been the object of his Note, cited above. Ultimately, he shows according to these formulas the remarkable properties of the expressions determined by the development of certain rational functions as continued fraction.

Thus, in designating by

$$\psi_0(x), \quad \psi_1(x), \quad \psi_2(x), \ldots$$

the denominators of the convergent fractions, resulting from the development of the expression

$$\left(\frac{1}{x-x_0}+\frac{1}{x-x_1}+\frac{1}{x-x_2}+\cdots+\frac{1}{x-x^n}\right)\theta^2(x),$$

as continued fraction, where x_0, x_1, x_2, $\ldots x_n$ are some real values, different among themselves, and $\theta(x)$ an entire function which is not annulled for $x = x_0, x_1, x_2, \ldots x_n$, he shows that the functions $\psi_1(x)$, $\psi_2(x)$,..., among all those of the same degree, which would have the same coefficient of the highest power of x, are distinguished from the others by the least value of the sums

$$\int_{i=0}^{i=n} \psi_1^2(x_i)\theta^2(x_i), \qquad \int_{i=0}^{i=n} \psi_2^2(x_i)\theta^2(x_i), \ldots$$

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. November 3, 2011

On the other hand, if one denotes by $\phi_m(x)$ the function

$$\frac{\psi_m(x)\theta(x)}{\sqrt{\sum_{i=0}^{i=n} \psi_m^2(x_i)\theta^2(x_i)}}$$

and if by taking its values for

$$x = x_0, x_1, x_2, \dots, x_n, \qquad m = 0, 1, 2, \dots, n,$$

one figures the square

$$\begin{array}{llll}
\phi_0(x_0), & \phi_0(x_1), & \phi_0(x_2), \dots & , \phi_0(x_n), \\
\phi_1(x_0), & \phi_1(x_1), & \phi_1(x_2), \dots & , \phi_1(x_n), \\
\phi_2(x_0), & \phi_2(x_1), & \phi_2(x_2), \dots & , \phi_2(x_n), \\
\multicolumn{4}{l}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\
\multicolumn{4}{l}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\
\phi_n(x_0), & \phi_n(x_1), & \phi_n(x_2), \dots & , \phi_n(x_n),
\end{array}$$

one will find that this square verifies the following conditions:

1) the sum of the squares of the terms of any vertical or horizontal line is equal to 1; 2) the sum of the products of the corresponding terms of any two lines, either vertical or horizontal, is equal to 0.

Therefore, this function furnishes the solution of the problem, which has been the object of the researches of Euler in his Memoir: *Problema algebraicum ad affectiones prorsus singulares memorabile*. N. Comm. T. XV.

The Memoir of Mr. Chebyshev, written in Russian, will be printed in the Ученыя Запнски. (*Learned Notes*).