Sur une nouvelle série*

Pafnuty Chebyshev

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In my Memoir on continued fractions, presented to the Academy in 1855 and published in its Memoirs (Volume III), I am arrived to a formula which, according to the given values of a function, affected with any errors, furnishes directly its value under the form of a polynomial with coefficients indicated by the *method of least squares*. This formula comprehends, as particular cases, the known developments of the functions following the *cosinus* of the multiple arcs and following the values of certain functions designated by $X^{(n)}$. One draws from our formula many other series, by making different particular hypotheses on the series of known values of the sought function. Under the simplest hypothesis, where one supposes these values equidistant, such as

$$u_1 = f(h), \quad u_2 = f(2h), \ldots \quad u_n = f(nh),$$

and their probable errors equal, our formula furnishes the development of $u = f(x)$ according to the denominators of the continued fraction which results from the development of the expression

$$\frac{1}{x-h} + \frac{1}{x-2h} + \ldots + \frac{1}{x-nh}.$$

But as one finds that these denominators, except for a constant factor and by taking $\Delta x = h$, are expressed by

$$\Delta^l (x-h)(x-2h)\cdots(x-lh)(x-nh-h)(x-nh-2h)\cdots(n-nh-lh),$$

there results from it, by virtue of a very simple transformation of the sums, this remark-

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. November 3, 2011

able series:

$$\begin{aligned}
= \quad & \frac{1}{n}\sum u_i + \frac{3\sum i(n-i)\Delta u_i}{1^2.n(n^2-1^2)h^2}\Delta(x-h)(x-nh-h) \\
& + \frac{5\sum i(i+1)(n-i)(n-i-1)\Delta^2 u_i}{1^2.2^2.n(n^2-1^2)(n^2-2^2)h^4}\Delta^2(x-h)(x-2h)(x-nh-h)(x-nh-2h) \\
& + \frac{7\sum i(i+1)(i+2)(n-i)(n-i-1)(n-i-2)\Delta^3 u_i}{1^2.2^2.3^2.n(n^2-1^2)(n^2-2^2)(n^2-3^2)h^6} \times \\
& \qquad \Delta^3(x-h)(x-2h)(x-3h)(x-nh-h)(x-nh-2h)(x-nh-3h) \\
& + \dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

in which the signs of summation are extended to all the values from $i = 1$ to $i = n$.

Moreover, one finds that the functions

$$\begin{gathered}
\Delta(x-h)(x-nh-h), \\
\Delta^2(x-h)(x-2h)(x-nh-h)(x-nh-2h), \\
\Delta^3(x-h)(x-2h)(x-3h)(x-nh-h)(x-nh-2h)(x-nh-3h, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{gathered}$$

that we will designate, for brevity, by

$$\Delta^1, \quad \Delta 2, \quad \Delta^3, \dots,$$

are linked among them by the equation

$$\Delta^l = (2l-l)h(2x-nh-h)\Delta^{l-1} - (l-1)^2[n^2-(l-1)^2]h^4\Delta^{l-2},$$

whence one draws easily the values of all these functions:

$$\begin{gathered}
\Delta^1 = h(2x-nh-h), \\
\Delta^2 = 3h^2(2x-nh-h)^2 - (n^2-1)h^4, \\
\Delta^3 = 15h^3(2x-nh-h)^3 - 3(3n^2-7)h^5(2x-nh-h), \\
\Delta^4 = 105h^4(2x-nh-h)^4 - 30(3n^2-13)h^6(2x-nh-h)^2 + 9(n^2-1)(n^9-9)h^8, \\
\Delta^5 = 945h^5(2x-nh-h)^5 - 1050(n^2-7)h^7(2x-nh-h)^3 \\
+15(15n^4-230n^2+407)h^9(2x-nh-h), \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{gathered}$$

and one obtains immediately the development of the expression

$$\frac{1}{x-h} + \frac{1}{x-2h} + \dots + \frac{1}{x-nh}$$

as continued fraction

$$\cfrac{2n}{2x-nh-h-\cfrac{1^2(n^2-1^2)h^2}{3(2x-nh-h)-\cfrac{2^2(n^2-22)h^2}{5(2x-nh-h)-\cfrac{3^2(n^2-3^2)h^2}{7(2x-nh-h)-\cdots}}}}$$

The series that we just obtained, for the evaluation of u according to its equidistant values, leaves nothing to desire for parabolic interpolation of such values, seeing that in this series all the terms are calculated very easily according to the consecutive differences of the given values. In the case of

$$h = \frac{1}{n}$$

and n infinitely great, our series is reduced to a series ordered according to the values of the functions $X^{(n)}$. In the case of

$$h = \frac{1}{n^2}$$

and n infinitely great, it is reduced to the series of Maclaurin. On the other hand by multiplying its terms by u_i and summing from $i = 1$ to $i = n$, one draws from it this formula

$$\begin{aligned}\sum u_i^2 = \frac{(\sum u_i)^2}{n} \quad &+ \quad \frac{3[\sum i(n-i)\Delta u_i]^2}{1^2.n(n^2-1^2)h^2} + \frac{5[\sum i(i+1)(n-i)(n-i-1)\Delta^2 u_i]^2}{1^2.2^2.n(n^2-1^2)(n^2-2^2)h^4} \\ &+ \quad \frac{7[\sum i(i+1)(i+2)(n-i)(n-i-1)(n-i-2)\Delta^3 u_i]^2}{1^2.2^2.3^2.n(n^2-1^2)(n^2-2^2)(n^2-3^2)h^6} \\ &+ \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\end{aligned}$$

which, int its turn, in the case of

$$h = \frac{1}{n}$$

and n infinitely great, becomes

$$\begin{aligned}\int_0^1 u^2\,dx \quad &= \quad \left(\int_0^1 u\,dx\right)^2 + \frac{3}{1^2}\left(\int_0^1 x(1-x)\frac{du}{dx}\,dx\right)^2 \\ &+ \frac{5}{1^2.2^2}\left(\int_0^1 x^2(1-x)^2\frac{d^2u}{dx^2}\,dx\right)^2 \\ &+ \frac{7}{1^2.2^2.3^2}\left(\int_0^1 x^3(1-x)^3\frac{d^3u}{dx^3}\,dx\right)^2 + \dots\end{aligned}$$

We note yet that the functions

$$\begin{gathered}\Delta(x-h)(x-nh-h), \\ \Delta^2(x-h)(x-2h)(x-nh-h)(x-nh-2h), \\ \Delta^3(x-h)(x-2h)(x-3h)(x-nh-h)(x-nh-2h)(x-nh-3h), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\end{gathered}$$

which enter into our series, are very remarkable by some properties analogous to those of the functions of Legendre $X^{(n)}$.

These functions, besides, furnish some approximate expressions of the sum

$$\sum_{1}^{n} F(ih)$$

which enjoy the same important property as those which have been given by Gauss for quadratures.

In one of our later Memoirs one will see all the necessary details on the series that we just gave and the remarkable functions of which its terms are composed.