Sur l'Interpolation par la Méthode des Moindres Carrés*

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In the Memoir *Sur les fractions continues*[1] I have given the series which presents the definitive result of the parabolic interpolation by the method of *least squares*. As this series furnishes directly the expression of the interpolated function under the form of a polynomial with the most probable coefficients, and unless one fixes in advance the number of its terms, one imagines that, under the theoretic reasoning, it leaves nothing to desire for parabolic interpolation. But in order to render its usage completely practical, there remains to indicate the convenient direction to follow in the evaluation of its terms. It is this that we have done for the simplest case where the values of the variable, corresponding to the known values of the interpolated function, are equidistant. By treating this particular case in the note *Sur une nouvelle formule*,[2] we have indicated a reduction of our series to the formula that is here, very proper to the application:

$$\begin{aligned} u &= \frac{1}{n}\sum u_i.\phi_0(z) \\ &+ \frac{3}{n(n^2-1^2)}\sum \frac{i}{1}\frac{n-i}{1}\Delta u_i.\phi_1(z) \\ &+ \frac{5}{n(n^2-1^2)(n^2-2^2)}\sum \frac{i(i+1)}{1.2}\frac{(n-1)(n-i-1)}{1.2}\Delta^2 u_i.\phi_2(z) \\ &+ \frac{7}{n(n^2-1^2)(n^2-2^2)(n^2-3^2)}\sum \frac{i(i+1)(i+2)}{1.2.3}\frac{(n-1)(n-i-1)(n-i-2)}{1.2.3}\Delta^3 u_i.\phi_3(z) \\ &+ \text{etc.,} \end{aligned}$$

by designating by

$$u_1, \quad u_2, \quad u_3, \ldots u_n$$

the given values of u which correspond to the equidistant values of x

$$x = x_1, \quad x_2, \quad x_3, \ldots x_n,$$

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[1] *Journal de mathématiques pure et appliquées*. IInd Series, T. III, (1858), p. 289–323.

[2] "Sur unde nouvelle série," *Bulletin physico-mathématique de l'Académie Impériale des sciences de St. Pétersbourg*. T. XVII, p. 257–261.

and by making, for brevity,

$$z = \frac{x - \frac{1}{2}(x_1 + x_n)}{x_2 - x_1}.$$

In this series the signs of summation extending to all the values of i, from $i = 1$ to $i = n$, and

$$\phi_0(z), \quad \phi_1(z), \quad \phi_2(z), \quad \phi_3(z), \ldots$$

are the entire functions of z that one draws from the formula

$$\Delta^l \left(z + \frac{n-1}{2}\right)\left(z + \frac{n-3}{2}\right)\cdots\left(z + \frac{n-2l+1}{2}\right)\left(z - \frac{n+1}{2}\right)\left(z - \frac{n+3}{2}\right)\cdots\left(z - \frac{n+2l-1}{2}\right),$$

by adopting for l the values

$$0, \quad 1, \quad 2, \quad 3, \ldots$$

As these functions are linked among them by the equation

$$\phi_l(z) = 2(2l-1)z\phi_{l-1}(z) - (l-1)^2[n^2 - (l-1)^2]\phi_{l-2}(z),$$

and as

$$\begin{aligned}
\phi_0(z) &= \Delta^0 1 = 1, \\
\phi_1(z) &= \Delta \left(z + \frac{n-1}{2}\right)\left(z - \frac{n+1}{2}\right) = 2z,
\end{aligned}$$

one finds immediately

$$\begin{aligned}
\phi_2(z) &= 12z^2 - (n^2 - 1), \\
\phi_3(z) &= 120z^3 - 6(3n^2 - 7)z, \\
\phi_4(z) &= 1680z^4 - 120(3n^2 - 13)z^2 + 9(n^2 - 1)(n^2 - 9), \\
\phi_5(z) &= 30240z^5 - 8400(n^2 - 7)z^3 + 30(15n^4 - 230n^2 + 407)z, \\
&\ldots\ldots\ldots
\end{aligned}$$

This development of u which results from our series, as long as the values

$$x_1, \quad x_2, \quad , x_3, \ldots x_n$$

are equidistant, is very convenient for the evaluation of the expression of u, seeing that its terms, as those of the formula of interpolation of Newton, contain the differences

$$\Delta u_i, \quad \Delta^2 u_i, \quad \Delta^3 u_i, \ldots,$$

of which the orders are increasing, and that these differences, under the signs of summation, are accompanied only with the factors

$$\begin{aligned}
&\frac{i}{n}, \qquad \frac{n-i}{1}, \\
&\frac{i(i+1)}{1.2}, \qquad \frac{(n-i)(n-i-1)}{1.2}, \\
&\frac{i(i+1)(i+2)}{1.2.3}, \qquad \frac{(n-i)(n-i-1)(n-i-2)}{1.2.3}, \\
&\ldots\ldots\ldots
\end{aligned}$$

which according to the known property of the polygonal numbers, are evaluated easily by way of addition. And as this series furnishes us the expression of u with the most probable coefficients, one imagines that it leaves nothing to desire for interpolation in the particular case where the values of the variable which correspond to the known values of the function are equidistant.

But this is not the only part that one is able to draw from our series for the application; its usage is also very useful in all the other cases of parabolic interpolation, as we are going to demonstrate now, by indicating the direction which leads easily to the successive determination of its terms. One will see, according to that, that our series procures a very proper means to evaluate, term by term, the expression of the interpolated function u, and that it gives, at the same time, the sum of the squares of the differences between its known values

$$u_1, \quad u_2, \quad u_3, \ldots u_n,$$

and those which result from the set of terms found for its expression. According to what one will have, at once, the mean error with which the found terms of u represent its given values, and thence one will recognize immediately the one to which one is able to be arrested. Thus, by means of our series one will find all at once and the number of terms of u which are important for the interpolation and their coefficients determined by the method of least squares. In order to comprehend the superiority of this method of interpolation over those of which one is ordinarily served, we note that it will give precisely, in general more easily, the same results, as those that one finds by the resolution of the equations furnished by the method of *least squares* which suppose that the number of terms in the expression of u is fixed in advance. On the other hand, by determining both the number of terms of u that one must calculate and their values prescribed by the method of least squares, it will be, if it is not in certain exceptional cases, more expeditious than the method of interpolation of Cauchy which is far from giving the most probable results resulting from the method of least squares.

§I.

According to that which we have shown in the Memoir cited above, if the given values of the function u

$$u_1, \quad u_2, \quad u_3, \ldots u_n$$

which correspond to

$$x = x_1, \quad x_2, \quad x_3, \ldots x_n$$

are affected of errors of the same nature, and if one seeks its expression, by the method of *least squares*, under the form of a polynomial of any degree, one will have[3]

$$u = K_0\psi_0(x) = K_1\psi_1(x) + K_2\psi_2(x) + \cdots,$$

where

$$K_0, \quad K_1, \quad K_2, \ldots$$

[3]We will borrow from our previous Memoir only the form of this series; but all that which is important for its application will be given in that which follows.

are some constant coefficients, and

$$\psi_0(x), \quad \psi_1(x), \quad \psi_2(x), \dots$$

the denominators of the reductions from the sum

$$\sum \frac{1}{x-x_i} = \frac{1}{x-x_1} + \frac{1}{x-x_2} + \frac{1}{x-x_3} + \dots + \frac{1}{x-x_n},$$

that one finds by its development into continued fraction

$$\cfrac{\alpha_1}{q_1 + \cfrac{\alpha_2}{q_2 + \cfrac{\alpha_3}{q_3 + \dots}}}$$

In this fraction the constants

$$\alpha_1, \quad \alpha_2, \quad \alpha_3, \dots$$

are able to be chosen arbitrarily. In order to fix the ideas, we will suppose that they are chosen in a manner to this that the coefficients of x in the quotients

$$q_1, \quad q_2, \quad , q_3, \dots$$

are equal to 1, and we will designate by

$$a_1, \quad -a_2, \quad -a_3, \dots$$

the values of

$$\alpha_1, \quad \alpha_2, \quad \alpha_3, \dots$$

which fulfill this condition. According to that, and by noting that the denominators

$$q_1, \quad q_2, \quad , q_3, \dots$$

will be some functions of the first degree, one will have, for the determination of the functions

$$\psi_0(x), \quad \psi_1(x), \quad \psi_2(x), \dots$$

this development of

$$\sum \frac{1}{x-x_i}$$

in continued fraction:

$$\sum \frac{1}{x-x_i} = \cfrac{a_1}{x - b_1 - \cfrac{a_2}{x - b_2 - \cfrac{a_3}{x - b_2 - \cdots}}}$$

Whence one draws, for the evaluation of its reductions

$$\frac{\phi_0(x)}{\psi_0(x)}, \quad \frac{\phi_1(x)}{\psi_1(x)}, \quad \frac{\phi_2(x)}{\psi_2(x)}, \quad ,\dots \frac{\phi_\lambda(x)}{\psi_\lambda(x)}, \quad ,\dots,$$

the following formulas:

(1)

$$\begin{cases} \psi_0(x) = 1, & \phi_0(x) = 0, \\ \psi_1(x) = x - b_1, & \phi_1(x) = a_1, \\ \psi_2(x) = (x - b_2)\psi_1(x) - a_2\psi_0(x), & \phi_2(x) = (x - b_2)\phi_1(x) - a_2\phi_0(x), \\ \cdots & \cdots \\ \psi_\lambda(x) = (x - b_\lambda)\psi_{\lambda-1}(x) - a_\lambda\psi_{\lambda-2}(x), & \phi_\lambda(x) = (x - b_\lambda)\phi_{\lambda-1}(x) - a_\lambda\phi_{\lambda-2}(x), \end{cases}$$

and thence, by making

(2)

$$\begin{cases} \psi_0(x)\sum \frac{1}{x-x_i} - \phi_0(x) = R_0, \\ \psi_1(x)\sum \frac{1}{x-x_i} - \phi_1(x) = R_1, \\ \psi_2(x)\sum \frac{1}{x-x_i} - \phi_2(x) = R_2, \\ \cdots\cdots \\ \psi_\lambda(x)\sum \frac{1}{x-x_i} - \phi_\lambda(x) = R_\lambda, \end{cases}$$

one obtains, relative to the functions

$$R_0, \quad R_1, \quad R_2, \dots R_\lambda,$$

this sequence of equations:

(3)

$$\begin{cases} R_0 = \sum \frac{1}{x-x_i}, \\ R_1 = (x - b_1)R_0 - a_1, \\ R_2 = (x - b_2)R_1 - a_2R_0, \\ \cdots\cdots \\ R_\lambda = (x - b_\lambda)R_{\lambda-1} - a_\lambda R_{\lambda-2}. \end{cases}$$

It is by means of these formulas that we will arrive to determine all the quantities which are important for the evaluation of the terms of our series.

§II.

As the reductions

$$\frac{\phi_0(x)}{\psi_0(x)}, \quad \frac{\phi_1(x)}{\psi_1(x)}, \quad \frac{\phi_2(x)}{\psi_2(x)}, \quad \dots \frac{\phi_\mu(x)}{\psi_\mu(x)}, \quad , \frac{\phi_{\mu+1}(x)}{\psi_{\mu+1}(x)} \dots$$

of the continued fraction

$$\cfrac{a_1}{x - b_1 - \cfrac{a_2}{x - b_2 - \cfrac{a_3}{x - b_2 - \cdots}}},$$

which results from the development of

$$\sum \frac{1}{x-x_i},$$

have for denominators the functions

$$\psi_0(x), \quad \psi_1(x), \quad \psi_2(x), \ldots \psi_\mu(x), \quad \psi_{\mu+1}(x), \ldots,$$

respectively of the degrees

$$0, \quad 1, \quad 2, \ldots \mu, \quad \mu+1, \ldots,$$

the fraction

$$\frac{\phi_\mu(x)}{\psi_\mu(x)}$$

will represent the value of

$$\sum \frac{1}{x-x_i}$$

exactly to $\frac{1}{x^{2\mu}}$, and, consequently, the difference

$$\sum \frac{1}{x-x_i} - \frac{\phi_\mu(x)}{\psi_\mu(x)}$$

will be of degree inferior to -2μ. But the function $\psi_\mu(x)$ being of degree μ, this supposes that the expression

$$R_\mu = \psi_\mu(x) \sum \frac{1}{x-x_i} - \phi_\mu(x),$$

is of degree inferior to $-\mu$, and thence one will conclude that its development is not able to contain the terms with some powers of x superior to $x^{-\mu-1}$. Therefore, one will have

$$R_\mu = \frac{(\mu,\mu)}{x^{\mu+1}} + \frac{(\mu,\mu+1)}{x^{\mu+2}} + \frac{(\mu,\mu+2)}{x^{\mu+3}} + \cdots,$$

by designating by

$$(\mu,\mu), \quad (\mu,\mu+1), \quad (\mu,\mu+2), \ldots$$

the coefficients of

$$\frac{1}{x^{\mu+1}}, \quad \frac{1}{x^{\mu+2}}, \quad \frac{1}{x^{\mu+3}}, \ldots$$

in the development of R_μ.

According to this, by adopting for the index μ the values

$$0, \quad 1, \quad 2, \ldots \lambda-2, \quad \lambda-1, \quad \lambda,$$

one finds for the functions

$$R_0, \quad R_1, \quad R_2, \ldots R_{\lambda-1}, \quad R_{\lambda-2}, \quad R_\lambda$$

the following developments

$$
(4) \qquad \begin{cases}
R_0 = \frac{(0,0)}{x} + \frac{(0,1)}{x^2} + \frac{(0,2)}{x^3} + \cdots, \\
R_1 = \frac{(1,1)}{x^2} + \frac{(1,2)}{x^3} + \frac{(1,3)}{x^4} + \cdots, \\
R_2 = \frac{(2,2)}{x^3} + \frac{(2,3)}{x^4} + \frac{(2,4)}{x^5} + \cdots, \\
\cdots\cdots \\
R_{\lambda-2} = \frac{(\lambda-2,\lambda-2)}{x^{\lambda-1}} + \frac{(\lambda-2,\lambda-1)}{x^{\lambda}} + \frac{(\lambda-2,\lambda)}{x^{\lambda+1}} + \cdots, \\
R_{\lambda-1} = \frac{(\lambda-1,\lambda-1)}{x^{\lambda}} + \frac{(\lambda-1,\lambda)}{x^{\lambda+1}} + \frac{(\lambda-1,\lambda+1)}{x^{\lambda+2}} + \cdots, \\
R_{\lambda} = \frac{(\lambda,\lambda)}{x^{\lambda+1}} + \frac{(\lambda,\lambda+1)}{x^{\lambda+2}} + \frac{(\lambda,\lambda+2)}{x^{\lambda+3}} + \cdots [2pt]
\end{cases}
$$

where

$$
\begin{array}{l}
(0,0), \quad (0,1), \quad (0,2), \ldots, \\
(1,1), \quad (1,2), \quad (1,3), \ldots, \\
(2,2), \quad (2,3), \quad (2,4), \ldots, \\
\cdots\cdots \\
(\lambda-2,\lambda-2), \quad (\lambda-2,\lambda-1), \quad (\lambda-2,\lambda), \ldots, \\
(\lambda-1,\lambda-1), \quad (\lambda-1,\lambda), \quad (\lambda-1,\lambda+1), \ldots, \\
(\lambda,\lambda), \quad (\lambda,\lambda+1), \quad (\lambda,\lambda+2), \ldots,
\end{array}
$$

are some constant values which present themselves as auxiliary quantities.

§III.

By carrying in the formulas (3) the developments of

$$
R_0, \quad R_1, \quad R_2, \ldots R_{\lambda-2}, \quad R_{\lambda-1}, \quad R_{\lambda},
$$

according to (4), one will obtain this sequence of formulas:

$$
\begin{aligned}
\sum \frac{1}{x-x_i} &= \frac{(0,0)}{x} + \frac{(0,1)}{x^2} + \frac{(0,2)}{x^3} + \cdots, \\
\frac{(1,1)}{x^2} + \frac{(1,2)}{x^3} + \frac{(1,3)}{x^4} + \cdots &= (x-b_1)\left[\frac{(0,0)}{x} + \frac{(0,1)}{x^2} + \frac{(0,2)}{x^3} + \cdots\right] \\
&\quad -a_1, \\
\frac{(2,2)}{x^3} + \frac{(2,3)}{x^4} + \frac{(2,4)}{x^5} + \cdots &= (x-b_2)\left[\frac{(1,1)}{x^2} + \frac{(1,2)}{x^3} + \frac{(1,3)}{x^4} + \cdots\right] \\
&\quad -a_2\left[\frac{(0,0)}{x} + \frac{(0,1)}{x^2} + \frac{(0,2)}{x^3} + \cdots\right], \\
\cdots\cdots &\quad \cdots\cdots \\
\frac{(\lambda,\lambda)}{x^{\lambda+1}} + \frac{(\lambda,\lambda+1)}{x^{\lambda+2}} + \frac{(\lambda,\lambda+2)}{x^{\lambda+3}} + \cdots &= (x-b_\lambda)\left[\frac{(\lambda-1,\lambda-1)}{x^{\lambda}} + \frac{(\lambda-1,\lambda)}{x^{\lambda+1}} + \frac{(\lambda-1,\lambda+1)}{x^{\lambda+2}} + \cdots\right] \\
&\quad -a_\lambda\left[\frac{(\lambda-2,\lambda-2)}{x^{\lambda-1}} + \frac{(\lambda-2,\lambda-1)}{x^{\lambda}} + \frac{(\lambda-2,\lambda)}{x^{\lambda+1}} + \cdots\right].
\end{aligned}
$$

The first of these formulas, according to the development of

$$\sum \frac{1}{x - x_i}$$

into series

$$\frac{\sum x_i^0}{x} + \frac{\sum x_i}{x^2} + \frac{\sum x_i^2}{x^3} + \cdots ,$$

gives us

$$\frac{\sum x_i^0}{x} + \frac{\sum x_i}{x^2} + \frac{\sum x_i^2}{x^3} + \cdots = \frac{(0,0)}{x} + \frac{(0,1)}{x^2} + \frac{(0,2)}{x^3} + \cdots$$

Whence it follows

$$(0,0) = \sum x_i^0, \quad (0,1) = \sum x_i, \quad (0,2) = \sum x_i^2, \ldots$$

By the second one obtains, by equating among them the coefficients of the same powers of x,

$$\begin{aligned} &0 = (0,0) - a_1, \quad 0 = (0,1) - b_1(0,0), \quad (1,1) = (0,2) - b_1(0,1), \\ &(1,2) = (0,3) - b_1(0,2), \quad (1,3) = (0,4) - b_1(0,3), \ldots\ldots, \end{aligned}$$

this which gives us

$$a_1 = (0,0), \quad b_1 = \tfrac{(0,1)}{(0,0)},$$
$$(1,1) = (0,2) - b_1(0,1), \quad (1,2) = (0,3) - b_1(0,2), \quad (1,3) = (0,4) - b_1(0,3), \ldots$$

By treating in the same manner all the other formulas one will recognize that in general, in the case of $\lambda > 1$, the quantities a_λ and b_λ are determined thus:

$$a_\lambda = \frac{(\lambda-1,\lambda-1)}{(\lambda-2,\lambda-2)}, \quad b_\lambda = \frac{(\lambda-1,\lambda)}{(\lambda-1,\lambda-1)} - \frac{(\lambda-2,\lambda-1)}{(\lambda-2,\lambda-2)},$$

and that all the quantities

$$(\lambda,\lambda), \quad (\lambda,\lambda+1), \quad (\lambda,\lambda+2), \ldots,$$

as function of

$$\begin{aligned} &(\lambda-2,\lambda-2), \quad (\lambda-2,\lambda-1), \quad (\lambda-2,\lambda), \ldots, \\ &(\lambda-1,\lambda-1), \quad (\lambda-1,\lambda), \quad (\lambda-1,\lambda+1), \ldots, \end{aligned}$$

is found by this formula:

$$(\lambda,\mu) = (\lambda-1,\mu+1) - b_\lambda(\lambda-1,\mu) - a_\lambda(\lambda-2,\mu).$$

One will find thus successively the quantities

$$\begin{aligned} &a_1, \quad b_1, \\ &a_2, \quad b_2, \\ &a_3, \quad b_3, \\ &\ldots\ldots \end{aligned}$$

and with these quantities, according to (1), one will obtain easily the functions

$$\psi_0(x), \quad \psi_1(x), \quad \psi_2(x), \ldots$$

which enter into the composition of the terms of our series.

§IV.

By passing to the determination of the coefficients of our series, we will show that by virtue of formulas (2) and (4) one will have

(5) $$\sum x_i^\mu \psi_\lambda(x_i) = 0,$$

if $\mu < \lambda$, and

(6) $$\sum x_i^\mu \psi_\lambda(x_i) = (\lambda, \mu),$$

if $\mu =$ or $> \lambda$. In order to arrive there, we note that according to (2)

$$R_\lambda = \sum \frac{\psi_\lambda(x)}{x - x_i} - \phi_\lambda(x),$$

and as the rest of the division of $\psi_\lambda(x)$ by $x - x_i$ is equal to $\psi_\lambda(x_i)$, this formula is reduced to this here:

$$R_\lambda = \sum \left[F(x, x_i) + \frac{\psi_\lambda(x_i)}{x - x_i}\right] - \phi_\lambda(x),$$

where $F(x, x_i)$ is an entire function that one finds as quotient in the division of $\psi_\lambda(x)$ by $x - x_i$. Now if one decomposes the sum

$$\sum \left[F(x, x_i) + \frac{\psi_\lambda(x_i)}{x - x_i}\right]$$

into two parts

$$\sum F(x, x_i), \qquad \sum \frac{\psi_\lambda(x_i)}{x - x_i},$$

and if one develops, in the sum

$$\sum \frac{\psi_\lambda(x_i)}{x - x_i},$$

the fraction

$$\frac{1}{x - x_i}$$

into series

$$\frac{1}{x} + \frac{x_i}{x^2} + \frac{x_i^2}{x^3} + \cdots,$$

this formula will give us

$$R_\lambda = \sum F(x, x_i) - \phi_\lambda(x) + \frac{\sum \psi_\lambda(x_i)}{x} + \frac{\sum x_i \psi_\lambda(x_i)}{x^2} + \frac{\sum x_i^2 \psi_\lambda(x_i)}{x^3} + \cdots,$$

this which supposes, according to (5), the identity of these two series:

$$\sum F(x,x_i) - \phi_\lambda(x) + \frac{\sum \psi_\lambda(x_i)}{x} + \frac{\sum x_i\psi_\lambda(x_i)}{x^2} + \frac{\sum x_i^2\psi_\lambda(x_i)}{x^3} + \cdots,$$

$$\frac{(\lambda,\lambda)}{x^{\lambda+1}} + \frac{(\lambda,\lambda+1)}{x^{\lambda+2}} + \frac{(\lambda,\lambda+2)}{x^{\lambda+3}} + \cdots$$

But as

$$\sum F(x,x_i), \qquad \phi_\lambda(x)$$

are some entire functions, that are not able to take place unless the terms with the denominators

$$x, \quad x^2, \quad x^3, \dots x^\lambda, \quad x^{\lambda+1}, \quad x^{\lambda+2}, \dots,$$

in these two sequences, are not respectively equal. Therefore

$$\sum \psi_\lambda(x_i) = 0, \quad \sum x_i\psi_\lambda(x_i) = 0, \quad \sum x_i^2\psi_\lambda(x_i) = 0, \dots \sum x_i^{\lambda-1}\psi_\lambda(x_i) = 0,$$
$$\sum x_i^\lambda\psi_\lambda(x_i) = (\lambda,\lambda), \quad \sum x_i^{\lambda+1}\psi_\lambda(x_i) = (\lambda,\lambda+1), \dots,$$

this which proves the equations (5) and (6).

According to this it is easy to determine the coefficients

$$K_0, \quad K_1, \quad K_2, \dots$$

of the series

$$u = K_0\psi_0(x) + K_1\psi_1(x) + K_2\psi_2(x) + \cdots$$

For this we multiply the series by x_i^μ, where μ is any number, and we sum its terms for all the values of

$$x = x_1, \quad x_2, \quad x_3, \dots x_n.$$

We will obtain thus

$$\sum x_i^\mu u_i = K_0 \sum x_i^\mu \psi_0(x_i) + K_1 \sum x_i^\mu \psi_1(x_i) + K_2 \sum x_i^\mu \psi_2(x_i) + \cdots,$$

where by u_i we designate the value of u which corresponds to $x = x_i$, and as, by virtue of (5) and (6), one will have

$$\sum x_i^\mu\psi_0(x) = (0,\mu), \quad \sum x_i^\mu\psi_1(x_i) = (1,\mu), \dots \sum x_i^\mu\psi_\mu(x_i) = (\mu,\mu),$$
$$\sum x_i^\mu\psi_{\mu+1}(x_i) = 0, \quad \sum x_i^\mu\psi_{\mu+2}(x_i) = 0, \quad \sum x_i^\mu\psi_{\mu+3}(x_i) = 0, \dots,$$

there results from it

$$\sum x_i^\mu u_i = (0,\mu)K_0 + (1,\mu)K_1 + \cdots + (\mu-1,\mu)K_{\mu-1} + (\mu,\mu)K_\mu.$$

Whence, for the determination of the coefficient K_μ, as function of the coefficients K_0, K_1,… $K_{\mu-1}$, one draws this very simple formula:

$$K_\mu = \frac{\sum x_i^\mu u_i - (0,\mu)K_0 - (1,\mu)K_1 - \cdots - (\mu-1,\mu)K_{\mu-1}}{(\mu,\mu)}.$$

By adopting here for the index μ the values 0, 1, 2, 3,..., one obtains, for the successive determination of the coefficients

$$K_0, \quad K_1, \quad K_2, \quad K_3, \ldots,$$

this sequence of equations:

$$\begin{aligned}
K_0 &= \frac{\sum u_i}{(0,0)}, \\
K_1 &= \frac{\sum x_i u_i - (0,1)K_0}{(1,1)}, \\
K_2 &= \frac{\sum x_i^2 u_i - (0,2)K_0 - (1,2)K_1}{(2,2)}, \\
K_3 &= \frac{\sum x_i^3 u_i - (0,3)K_0 - (1,3)K_1 - (2,3)K_2}{(3,3)}, \\
& \ldots\ldots\ldots
\end{aligned}$$

§V.

There remains to show us how one will arrive in an easy manner to find the sum of the squares of the differences among the given values of u

$$u_1, \quad u_2, \quad u_3, \ldots u_n,$$

corresponding to

$$x = x_1, \quad x_2, \quad x_3, \ldots x_n,$$

and those which, for the same values of x, result from our series arrested at the term $K_\lambda \psi_\lambda(x)$, λ being any number.

In order to arrive there, we are going to show that one will have

$$\sum \psi_\mu(x_i)\psi_\nu(x_i) = 0, \tag{7}$$

as long as $\nu < \mu$, and

$$\sum \psi_\mu(x_i)\psi_\nu(x_i) = (\mu, \mu), \tag{8}$$

in the case of $\mu = \nu$.

In fact, according to (1), the function $\psi_\nu(x)$ will be of the form

$$x^\nu + A_1 x^{\nu-1} + A_2 x^{\nu-2} + \cdots,$$

and consequently one will have

$$\sum \psi_\mu(x_i)\psi_\nu(x_i) = \sum x_i^\nu \psi_\mu(x_i) + A_1 \sum x_i^{\nu-1}\psi_\mu(x_i) + A_2 \sum x_i^{\nu-2}\psi_\mu(x_i) + \cdots \tag{9}$$

But by virtue of (5), in the case of $\nu < \mu$, all the sums

$$\sum x_i^\nu \psi_\mu(x_i), \quad \sum x_i^{\nu-1}\psi_\mu(x_i), \quad \sum x_i^{\nu-2}\psi_\mu(x_i), \ldots$$

are reduced to zero, and thence, according to the preceding formula, one will find

$$\sum \psi_\mu(x_i)\psi_\nu(x_i) = 0,$$

this which proves equation (7).

Likewise, in the case of

$$\mu = \nu,$$

one finds, according to (5) and (6), that the sum

$$\sum x_i^\nu \psi_\mu(x_i)$$

is equal to (μ, μ), and that the sums

$$\sum x_i^{\nu-1}\psi_\mu(x_i), \quad \sum x_i^{\nu-2}\psi_\mu(x_i), \dots$$

are annulled. By virtue of which, for $\mu = \nu$, formula (9) gives us equation (8)

$$\sum \psi_\mu(x_i)\psi_\nu(x_i) = (\mu, \mu).$$

By means of equations (7) and (8), that we just proved, it is easy to show that one will have always

$$\sum u_i \psi_\mu(x_i) = (\mu, \mu)K_\mu. \tag{10}$$

In order to be assured of it, we observe that our series

$$u = K_0\psi_0(x) + K_1\psi_1(x) + K_2\psi_2(x) + \cdots ,$$

prolonged to the last term, represents exactly all the given values of u

$$u_1, \quad u_2, \quad u_3, \dots u_n,$$

and thence one will have

$$\sum u_i\psi_\mu(x_i) = K_0 \sum \psi_0(x_i)\psi_\mu(x_i) + K_1 \sum \psi_1(x_i)\psi_\mu(x_i) + K_2 \sum \psi_2(x_i)\psi_\mu(x_i) + \cdots$$

But according to (7) the sums

$$\sum \psi_0(x_i)\psi_\mu(x_i), \quad \sum \psi_1(x_i)\psi_\mu(x_i), \dots \sum \psi_{\mu-1}(x_i)\psi_\mu(x_i), \quad \sum \psi_{\mu+1}(x_i)\psi_\mu(x_i), \dots$$

are annulled, and according to (8) one finds

$$\sum \psi_\mu(x_i)\psi_\mu(x_i) = (\mu, \mu).$$

Therefore the preceding development of $\sum u_i\psi(x_i)$ will be reduced to a term

$$(\mu, \mu)K_\mu,$$

this which gives equation (10).

By virtue of the demonstrated equations, it is easy to find the sum

$$\sum[u_i - K_0\psi_0(x_i) - K_1\psi_1(x_i) - \cdots - K_\lambda\psi_\lambda(x_i)]^2,$$

where

$$u_i,$$

for $i = 1, 2, 3, \ldots n$, designates the given values of u

$$u_1, \quad u_2, \quad u_3, \ldots u_n,$$

and the expression

$$K_0\psi_0(x_i) + K_1\psi_1(x_i) + K_2\psi_2(x_i) + \cdots + K_\lambda\psi_\lambda(x_i)$$

their approximate values, obtained by our series, arrested at the term $K_\lambda\psi_\lambda(x)$.

For that we set the square

$$[u_i - K_0\psi_0(x_i) - K_1\psi_1(x_i) - K_2\psi_2(x_i) - \cdots - K_\lambda\psi_\lambda(x_i)]^2$$

under the form

$$\begin{aligned}
u_i^2 \quad - \quad & 2u_i[K_0\psi_0(x_i) + K_1\psi_1(x_i) + K_2\psi_2(x_i) + \cdots + K_\lambda\psi_\lambda(x_i)] \\
+ \quad & K_0\psi_0(x_i)[K_0\psi_0(x_i) + K_1\psi_1(x_i) + K_2\psi_2(x_i) + \cdots + K_\lambda\psi_\lambda(x_i)] \\
+ \quad & K_1\psi_1(x_i)[K_0\psi_0(x_i) + K_1\psi_1(x_i) + K_2\psi_2(x_i) + \cdots + K_\lambda\psi_\lambda(x_i)] \\
+ \quad & \cdots\cdots\cdots \\
+ \quad & K_\lambda\psi_\lambda(x_i)[K_0\psi_0(x_i) + K_1\psi_1(x_i) + K_2\psi_2(x_i) + \cdots + K_\lambda\psi_\lambda(x_i)],
\end{aligned}$$

this which gives

$$\begin{aligned}
&\sum[u_i - K_0\psi_0(x_i) - K_1\psi_1(x_i) - K_2\psi_2(x_i) - \cdots - K_\lambda\psi_\lambda(x_i)]^2 \\
&= \sum u_i^2 - 2K_0\sum u_i\psi_0(x_i) - 2K_1\sum u_i\psi_1(x_i) - 2K_2\sum u_i\psi_2(x_i) - \cdots - 2K_\lambda\sum u_i\psi_\lambda(x_i) \\
&+K_0^2\sum\psi_0(x_i)\psi_0(x_i) + K_0K_1\sum\psi_0(x_i)\psi_1(x_i) + K_0K_2\sum\psi_0(x_i)\psi_2(x_i) + \cdots + K_0K_\lambda\sum\psi_0(x_i)\psi_\lambda(x_i) \\
&+K_1K_0\sum\psi_1(x_i)\psi_0(x_i) + K_1^2\sum\psi_1(x_i)\psi_1(x_i) + K_1K_2\sum\psi_1(x_i)\psi_2(x_i) + \cdots + K_1K_\lambda\sum\psi_1(x_i)\psi_\lambda(x_i) \\
&+\cdots\cdots\cdots \\
&+K_\lambda K_0\sum\psi_\lambda(x_i)\psi_0(x_i) + K_\lambda K_1\sum\psi_\lambda(x_i)\psi_1(x_i) + K_\lambda K_2\sum\psi_\lambda(x_i)\psi_2(x_i) + \cdots + K_\lambda^2\sum\psi_\lambda(x_i)\psi_\lambda(x_i)
\end{aligned}$$

But according to (10) we will have

$$\sum u_i\psi_0(x_i) = (0,0)K_0, \quad \sum u_i\psi_1(x_i) = (1,1)K_1, \quad \sum u_i\psi_2(x_i) = (2,2)K_2, \ldots,$$

and according to (8) and (9)

$$\begin{array}{llll}
\sum\psi_0(x_i)\psi_0(x_i) = (0,0), & \sum\psi_1(x_i)\psi_1(x_i) = (1,1), & \sum\psi_2(x_i)\psi_2(x_i) = (2,2), & \ldots, \\
\sum\psi_1(x_i)\psi_0(x_i) = 0, & \sum\psi_2(x_i)\psi_0(x_i) = 0, & \ldots, & \\
\sum\psi_0(x_i)\psi_1(x_i) = 0, & \sum\psi_2(x_i)\psi_1(x_i) = 0, & \ldots, & \\
\sum\psi_0(x_i)\psi_2(x_i) = 0, & \sum\psi_1(x_i)\psi_2(x_i) = 0, & \ldots, & \\
\cdots\cdots,
\end{array}$$

By virtue of what the preceding formula becomes

$$\begin{aligned}&\sum[u_i - K_0\psi_0(x_i) - K_1\psi_1(x_i) - K_2\psi_2(x_i) - \cdots - K_\lambda\psi_\lambda(x_i)]^2\\ &= \sum u_i^2 - 2(0,0)K_0^2 - 2(1,1)K_1^2 - 2(2,2)K_2^2 - \cdots - 2(\lambda,\lambda)K_\lambda^2\\ &\quad +(0,0)K_0^2 + (1,1)K_1^2 + (2,2)K_2^2 + \cdots + (\lambda,\lambda)K_\lambda^2,\end{aligned}$$

and is reduced to this here:

$$\begin{aligned}&\sum[u_i - K_0\psi_0(x_i) - K_1\psi_1(x_i) - K_2\psi_2(x_i) - \cdots - K_\lambda\psi_\lambda(x_i)]^2\\ &= \sum u_i^2 - (0,0)K_0^2 - (1,1)K_1^2 - (2,2)K_2^2 - \cdots - (\lambda,\lambda)K_\lambda^2.\end{aligned}$$

Such is the formula giving the sum of the squares of the differences which exist between the given values of u and their representations by the series

$$u = K_0\psi_0(x) + K_1\psi_1(x) + K_2\psi_2(x) + \cdots,$$

arrested at the term $K_\lambda\psi_\lambda(x)$. By designating, for brevity, this sum by

$$\sum d_\lambda^2,$$

we will have

$$\sum d_\lambda^2 = \sum u_i^2 - (0,0)K_0^2 - (1,1)K_1^2 - (2,2)K_2^2 - \cdots - (\lambda,\lambda)K_\lambda^2.$$

Whence, for the successive determination of the sums

$$\sum d_0^2, \quad \sum d_1^2, \quad \sum d_2^2, \ldots$$

which correspond respectively to the case where our series is arrested at the terms 1, 2, 3, ..., results this sequence of equations:

$$\begin{aligned}\sum d_0^2 &= \sum u_1^2 - (0,0)K_0^2,\\ \sum d_1^2 &= \sum d_0^2 - (1,1)K_1^2,\\ \sum d_2^2 &= \sum d_1^2 - (2,2)K_2^2,\\ &\cdots\cdots\end{aligned}$$

§VI.

We are going now to summarize the definitive formulas by which one will arrive to calculate, term by term, the expression of u according to the series

$$u = K_0\psi_0(x) + K_1\psi_1(x) + K_2\psi_2(x) + \cdots,$$

and one will understand, at the same time, the sum of the squares of the errors committed in the representation of the given values of u, by being arrested at the terms 1, 2, 3, $\ldots\lambda$.

In these formulas, following the notation employed, the given values of the function u and of the variable x are represented by

$$u_1, \quad u_2, \quad u_3, \ldots u_n,$$
$$x_1, \quad x_2, \quad x_3, \ldots x_n.$$

The summations extending to all the values of the index i, from $i = 1$ to $i = n$, and $\sum d_\lambda^2$ designates the sum of the squares of the errors in the representation of the values given of u by our series, arrested at the term $K_\lambda \psi_\lambda(x)$, a sum according to which one will find the mean error by the formula

$$E = \sqrt{\frac{1}{n} \sum d_\lambda^2}.$$

Formulas relative to the determination of the term $K_0 \psi_0(x)$.

$$(0,0) = \sum x_i^0 = n,$$
$$K_0 = \tfrac{\sum u_i}{(0,0)},$$
$$\psi_0(x) = 1,$$
$$\sum d_0^2 = \sum u_i^2 - (0,0) K_0^2.$$

Formulas relative to the determination of the term $K_1 \psi_1(x)$.

$$(0,1) = \sum x_i, \qquad (0,2) = \sum x_i^2,$$
$$a_1 = (0,0)$$
$$b_1 = \tfrac{(0,1)}{(0,0)}, \qquad (1,1) = (0,2) - b_1 (0,1),$$
$$K_1 = \tfrac{\sum x_i u_i - (0,1) K_0}{(1,1)},$$
$$\psi_1(x) = x - b_1,$$
$$\sum d_1^2 = \sum d_0^2 - (1,1) K_1^2.$$

Formulas relative to the determination of the term $K_2 \psi_2(x)$.

$$(0,3) = \sum x_i^2, \qquad (0,4) = \sum x_i^4,$$
$$(1,2) = (0,3) - b_1 (0,2), \qquad (1,3) = (0,4) - b_1 (0,3),$$
$$a_2 = \tfrac{(1,1)}{(0,0)},$$
$$b_2 = \tfrac{(1,2)}{(1,1)} - \tfrac{(0,1)}{(0,0)}, \qquad (2,2) = (1,3) - b_2 (1,2) - a_2 (0,2),$$
$$K_2 = \tfrac{\sum x_i^2 u_i - (0,2) K_0 - (1,2) K_1}{(2,2)},$$
$$\psi_2(x) = (x - b_2) \psi_1(x) - a_2 \psi_0(x),$$
$$\sum d_2^2 = \sum d_1^2 - (2,2) K_2^2.$$
$$\cdots$$
$$\cdots$$

Formulas relative to the determination of the term $K_\lambda \psi_\lambda(x)$.

$$(0, 2\lambda-1) = \sum x_i^{2\lambda-1}, \qquad (0, 2\lambda) = \sum x_i^{2\lambda},$$
$$(1, 2\lambda-2) = (0, 2\lambda-1) - b_1(0, 2\lambda-2), \quad (1, 2\lambda-1) = (0, 2\lambda) - b_1(0, 2\lambda-1),$$
$$(2, 2\lambda-3) = (1, 2\lambda-2) - b_2(1, 2\lambda-3) - a_2(0, 2\lambda-3), \quad (2, 2\lambda-2) = (1, 2\lambda-1) - b_2(1, 2\lambda-2) - a_2(0, 2\lambda-2),$$
$$(3, 2\lambda-4) = (2, 2\lambda-3) - b_3(2, 2\lambda-4) - a_3(1, 2\lambda-4), \quad (3, 2\lambda-3) = (2, 2\lambda-2) - b_3(2, 2\lambda-3) - a_3(1, 2\lambda-3),$$
$$\cdots\cdots\cdots$$
$$(\lambda-1, \lambda) = (\lambda-2, \lambda+1) - b_{\lambda-1}(\lambda-2, \lambda) - a_{\lambda-1}(\lambda-3, \lambda),$$
$$(\lambda-1, \lambda+1) = (\lambda-2, \lambda+2) - b_{\lambda-1}(\lambda-2, \lambda+1) - a_{\lambda-1}(\lambda-3, \lambda+1),$$
$$a_\lambda = \frac{(\lambda-1, \lambda-1)}{(\lambda-2, \lambda-2)}$$
$$b_\lambda = \frac{(\lambda-1, \lambda)}{(\lambda-1, \lambda-1)} - \frac{(\lambda-2, \lambda-1)}{(\lambda-2, \lambda-2)}, \quad (\lambda, \lambda) = (\lambda-1, \lambda+1) - b_\lambda(\lambda-1, \lambda) - a_\lambda(\lambda-2, \lambda),$$
$$K_\lambda = \frac{\sum x_i^\lambda u_i - (0,\lambda)K_0 - (1,\lambda)K_1 - (2,\lambda)K_2 - \cdots - (\lambda-1,\lambda)K_{\lambda-1}}{(\lambda,\lambda)},$$
$$\psi_\lambda(x) = (x - b_\lambda)\psi_{\lambda-1}(x) - a_\lambda \psi_{\lambda-2}(x),$$
$$\sum d_\lambda^2 = \sum d_{\lambda-1}^2 - (\lambda, \lambda)K_\lambda^2.$$

§VII.

The formulas that we just gave to determine successively the terms

$$K_0\psi_0(x), \quad K_1\psi_1(x), \quad K_2\psi_2(x), \ldots K_\lambda\psi_\lambda(x)$$

in the development of u according to our series, and in order to evaluate, at the same time, the sum of the squares of the errors with which the found terms of u represent all its given values, furnish us a method of parabolic interpolation, important for more than one reason. By virtue of the remarkable property of our series, this method gives the expression of u under form of a polynomial with the most probable coefficients. Without fixing in advance the number of its terms, by this method, one will find them successively the one after the other, and one will encounter immediately the one to which one is able to be arrested according to the sum of the squares of the errors with which the found terms of u represent its given values, a sum which gives immediately the mean error of their representation. Moreover, it is easy to see by the composition of our formulas that when the number of given values of u and the one of the terms of its expression are considerable, in our method of interpolation the calculations are less prolix than in those now in use.

This prolixity of the calculations is due nearly entirely to the differennt *multiplications* and *divisions* of which the number increases more or less rapidly, with those of the given values of u and of the terms in its expression. It is for this reason that we are going to show the advantage of our method of interpolation, by leaving aside the *additions* and the *subtractions* which, in the work of these calculations, enter only for quite little of the thing, and for which one is able also to well manifest the advantage of our method.

In order to find by our formulas the expression of u with $\lambda+1$ terms, one must evaluate $3\lambda+1$ sums

$$\sum x_i, \quad \sum x_i^2, \quad \sum x_i^3, \ldots \sum x_i^{2\lambda},$$
$$\sum u_i, \quad \sum x_i u_i, \quad \sum x_i^2 u_i, \ldots \sum x_i^\lambda u_i,$$

and by means of these sums, by seeking the terms

$$K_0\psi_0(x), \quad K_1\psi_1(x), \quad K_2\psi_2(x), \dots K_\lambda\psi_\lambda(x),$$

by that which we have seen, and by reducing them to the definitive form

$$A + Bx + Cx^2 + \cdots,$$

one will have by making some *multiplications* or *divisions* only in number $4\lambda^2 + 2$.

But if one seeks this expression of u, ordinarily, by the method of *least squares*, one is led to calculate the same sums

$$\begin{gathered} \sum x_i, \quad \sum x_i^2, \quad \sum x_i^3, \dots \sum x_i^{2\lambda}, \\ \sum u_i, \quad \sum x_i u_i, \quad \sum x_i^2 u_i, \dots \sum x_i^\lambda u_i, \end{gathered}$$

for the composition of the equations determining $\lambda + 1$ coefficients of u, and by resolving these equations in $\lambda + 1$ unknowns, one finds out of these *multiplications* and the *divisions* of which the number, with the growth of λ, increases, as one knows, much more rapidly than $4\lambda^2 + 2$.

According to the method of Cauchy, by seeking, in the development of u, the terms

$$A + Bx + Cx^2 + \cdots + Hx^\lambda,$$

one must, for $x = x_1, x_2, x_3, \dots x_n$, to evaluate many functions, of which the degrees climb to λ, and to compose by their means the sums that one names *subordinate*. Now this requires, evidently, many more *multiplications* than it is necessary in order to calculate the sums

$$\begin{gathered} \sum x_i, \quad \sum x_i^2, \quad \sum x_i^3, \dots \sum x_i^{2\lambda}, \\ \sum u_i, \quad \sum x_i u_i, \quad \sum x_i^2 u_i, \dots \sum x_i^\lambda u_i, \end{gathered}$$

which present themselves in the evaluation of $\lambda + 1$ terms of our series, and also in order to find this here:

$$u_i^2,$$

which enters into the determination of the sums

$$\sum d_0^2, \quad \sum d_1^2, \quad \sum d_2^2, \dots,$$

by which, in our method, one will encounter the number of important terms for the interpolation.

On the other hand, in order to find the functions, comprehended in the *subordinated* sums, and in order to evaluate by them the coefficients A, B, C, $\dots H$ of the expression of

$$u = A + Bx + Cx^2 + \cdots + Hx^\lambda,$$

in the method of Cauchy, it is important to make many *multiplications* and *divisions* of which the total number, with the growth of λ, increases more rapidly than $4\lambda^2 + \lambda + 3$, a number of like observations which present themselves when, by our method, according to the values of

$$\sum x_i, \quad \sum x_i^2, \quad \sum x_i^3, \dots \sum x_i^{2\lambda},$$
$$\sum u_i, \quad \sum x_i u_i, \quad \sum x_i^2 u_i, \dots \sum x_i^{\lambda} u_i, \quad \sum u_i^2,$$

one seeks $\lambda + 1$ terms and one determines successively the sums

$$\sum d_0^2, \quad \sum d_1^2, \quad \sum d_2^2, \dots \sum d_\lambda^2.$$

Thence it is cerain that, because of the number of these operations, the method of Cauchy is far from being as simple as that which results from our series. But as many of these operations, in the method of Cauchy, simplify themselves more and more in measure as the convergence of the series

$$u = A + Bx + Cx^2 + \cdots + Hx^\lambda,$$

is increased, there is no doubt that one encounters in some particular cases where it becomes more expeditious than ours.

§VIII.

In order to show by an example the use of our method of interpolation, we are going to apply it to this sequence of values of x and u:[4]

$x_1 = 0.15411$	$u_1 = 19.47$
$x_2 = 0.19516$	$u_2 = 21.83$
$x_3 = 0.22143$	$u_3 = 23.11$
$x_4 = 0.28802$	$u_4 = 26.11$
$x_5 = 0.32808$	$u_5 = 27.60$
$x_6 = 0.38183$	$u_6 = 28.89$
$x_7 = 0.45517$	$u_7 = 33.17$
$x_8 = 0.57012$	$u_8 = 33.38$
$x_9 = 0.75930$	$u_9 = 32.31$
$x_{10} = 0.91075$	$u_{10} = 31.88$
$x_{11} = 1.13895$	$u_{11} = 25.46$

In seeking to express u by a single term

$$K_0 \psi_0(x),$$

[4]These values represent the results of the first series of observations of Mr. Marié-Davy on the resistance in the changing of conductor which he gives in his Memoir, entitled: *Recherches expérimentales sur l'électricité voltaïque* (Annales de chimie et de physique, series III, tome 19). — By x we designate the inversie of the intensity of the current, reduced to its hundredth part, and by u the resistance. *Translator's note*: This is for the year 1847, pp. 401–444.

one will take

$$(0,0) = \sum x_i^0 = 11, \qquad \begin{array}{rl} u_1 = & 19.47 \\ u_2 = & 21.33 \\ u_3 = & 23.11 \\ u_4 = & 26.11 \\ u_5 = & 27.60 \\ u_6 = & 28.89 \\ u_7 = & 33.17 \\ u_8 = & 33.38 \\ u_9 = & 32.31 \\ u_{10} = & 31.88 \\ u_{11} = & 25.46 \\ \hline \sum u_i = & 303.21 \\ K_0 = \frac{\sum u_i}{(0,0)} = & 27.5645, \\ \psi_0(x) = & 1, \end{array}$$

this which gives, exactly to 0.001,

$$K_0\psi_0(x) = 27.564.$$

In order to find the sum of the squares of the errors with which the found term represents the given values, one will make the following calculations:

$$\begin{array}{rr} u_1^2 = & 379.08 \\ u_2^2 = & 476.55 \\ u_3^2 = & 534.07 \\ u_4^2 = & 681.73 \\ u_5^2 = & 761.76 \\ u_6^2 = & 834.63 \\ u_7^2 = & 1100.25 \\ u_8^2 = & 1114.22 \\ u_9^2 = & 1043.94 \\ u_{10}^2 = & 1016.33 \\ u_{11}^2 = & 648.21 \\ \hline \sum u_i^2 = & 8590.77 \\ -(0,0)K_0^2 = & -8357.84 \\ \hline \sum d_0^2 = \sum u_i^2 - (0,0)K_0^2 = & 232.93 \end{array}$$

this which gives for the mean error

$$E = \sqrt{\frac{1}{n}\sum d_0^2} = \sqrt{\frac{232.93}{11}} = 4.6.$$

By noting according to this the insufficiency of the expression of u by a single term

$$K_o\psi_0(x),$$

one will seek the second term

$$K_1\psi_1(x),$$

and for that one will calculate successively

$$\begin{gathered}(0,1)=\sum x_i, \qquad (0,2)=\sum x_i^2,\\ a_1=(0,0), \quad b_1=\tfrac{(0,1)}{(0,0)},\\ (1,1)=(0,2)-b_1(0,1),\\ \sum x_iu_i, \quad \sum x_iu_i-(0,1)K_0,\\ K_1=\tfrac{\sum x_iu_i-(0,1)K_0}{(1,1)}\psi_1(x)\end{gathered}$$

thus there follows:

$$\begin{array}{rrrr}
x_1 = & 0.15411 & x_1^2 = & 0.02375\\
x_2 = & 0.19516 & x_2^2 = & 0.03809\\
x_3 = & 0.22143 & x_3^2 = & 0.04903\\
x_4 = & 0.28802 & x_4^2 = & 0.08295\\
x_5 = & 0.32808 & x_5^2 = & 0.10764\\
x_6 = & 0.38183 & x_6^2 = & 0.14579\\
x_7 = & 0.45517 & x_7^2 = & 0.20718\\
x_8 = & 0.57012 & x_8^2 = & 0.32504\\
x_9 = & 0.75930 & x_9^2 = & 0.57654\\
x_{10} = & 0.91075 & x_{10}^2 = & 0.82947\\
x_{11} = & 1.13895 & x_{11}^2 = & 1.29721\\ \hline
(0,1)=\sum x_i = & 5.40292 & (0,2)=\sum x_i^2 = & 3.68269\\
a_1=(0,0) = & 11. & -b_1(0,1) = & -2.65378\\ \hline
b_1=\frac{(0,1)}{(0,0)} = & 0.49117 & (1,1)=(0,2)-b_1(0,1) = & 1.02891
\end{array}$$

$$
\begin{aligned}
x_1\ u_1 &= \quad 3.00052\\
x_2\ u_2 &= \quad 4.26034\\
x_3\ u_3 &= \quad 5.11725\\
x_4\ u_4 &= \quad 7.52020\\
x_5\ u_5 &= \quad 9.05501\\
x_6\ u_6 &= \quad 11.03105\\
x_7\ u_7 &= \quad 15.09799\\
x_8\ u_8 &= \quad 19.03060\\
x_9\ u_9 &= \quad 24.53298\\
x_{10}u_{10} &= \quad 29.03417\\
x_{11}i_{11} &= \quad 28.99767\\
\sum x_i u_i &= \quad 156.67832\\
-(0,1)K_0 &= -148.92903\\
\sum x_i u_i - (0,1)K_0 &= \quad 7.74929\\
K_1 = \tfrac{\sum x_i u_i - (0,1)K_0}{(1,1)} &= \quad 7.5315,\\
\psi_1(x) = x - b_1 &= x - 0.49117.
\end{aligned}
$$

Therefore,

$$K_1\psi_1(x) = 7.5315(x - 0.19117) = 7.532x - 3.699.$$

In passing to the determination of $\sum d_1^2$, one will take

$$
\begin{aligned}
\sum d_0^2 &= \ 232.93\\
-(1,1)K_1^2 &= -58.37\\
\sum d_1^2 = \sum d_0^2 - (1,1)K_1^2 &= \ 174.58,
\end{aligned}
$$

whence, for the mean error of the representation of the given values of u by its two found terms, results

$$E = \sqrt{\frac{1}{n}\sum d_1^2} = \sqrt{\frac{174.56}{11}} = 3.98.$$

An mean error so considerable not being admissible, one will seek the third term

$$K_2\psi_2(x),$$

and for that one will determine successively the quantities

$$
\begin{gathered}
(0,3) = \sum x_i^2, \quad (0,4) = \sum x_i^4.\\
(1,2) = (0,3) - b_1(0,2), \quad (1,3) = (0,4) - b_1(0,3),\\
a_2 = \tfrac{(1,1)}{(0,0)}, \quad b_2 = \tfrac{(1,2)}{(1,1)} - \tfrac{(0,1)}{(0,0)},\\
(2,2) = (1,3) - b_2(1,2) - a_2(0,2),\\
\sum x_i^2 u_i, \quad \sum x_i^2 u_i - (0,2)K_0 - (1,2)K_1,\\
K_2 = \tfrac{\sum x_i^2 u_i - (0,2)K_0 - (1,2)K_1}{(2,2)}
\end{gathered}
$$

and the function $\psi_2(x)$ in the following manner:

$$
\begin{array}{rr}
x_1^3 = & 0.00367 \\
x_2^3 = & 0.00743 \\
x_3^3 = & 0.01086 \\
x_4^3 = & 0.02389 \\
x_5^3 = & 0.03531 \\
x_6^3 = & 0.05567 \\
x_7^3 = & 0.09430 \\
x_8^3 = & 0.18531 \\
x_9^3 = & 0.43776 \\
x_{10}^3 = & 0.75544 \\
x_{11}^3 = & 1.47745 \\
\hline
(0,3) = \sum x_i^3 = & 3.08709 \\
-b_1(0,2) \quad = & -1.80884 \\
\hline
(1,2) = (0,3) - b_1(0,2) = & 1.27825 \\
a_2 = \frac{(1,1)}{(0,0)} = & 0.09354 \\
\frac{(1,2)}{(1,1)} = & 1.24235 \\
-\frac{(0,1)}{(0,0)} = & -0.49117 \\
\hline
b_2 = \frac{(1,2)}{(1,1)} - \frac{(0,1)}{(0,0)} = & 0.75118
\end{array}
$$

$$
\begin{array}{rr}
x_1^4 = & 0.00056 \\
x_2^4 = & 0.00145 \\
x_3^4 = & 0.00240 \\
x_4^4 = & 0.00688 \\
x_5^4 = & 0.01158 \\
x_6^4 = & 0.02126 \\
x_7^4 = & 0.04292 \\
x_8^4 = & 0.10565 \\
x_9^4 = & 0.33240 \\
x_{10}^4 = & 0.68801 \\
x_{11}^4 = & 1.68275 \\
\hline
(0,4) = \sum x_i^4 = & 2.89586 \\
-b_1(0,3) \quad = & -1.51630 \\
\hline
(1,3) = (0,4) - b_1(0,3) = & 1.37956 \\
-b_2(1,2) = & -0.96020 \\
-a_2(0,2) = & -0.34446 \\
\hline
(2,2) = (1,3) - b_2(1,2) - a_2(0,2) = & 0.07490
\end{array}
$$

$$
\begin{array}{rr}
x_1^2\ u_1 = & 0.43241 \\
x_2^2\ u_2 = & 0.83145 \\
x_3^2\ u_3 = & 1.13311 \\
x_4^2\ u_4 = & 2.16596 \\
x_5^2\ u_5 = & 2.97075 \\
x_6^2\ u_6 = & 4.21199 \\
x_7^2\ u_7 = & 6.87215 \\
x_8^2\ u_8 = & 10.84949 \\
x_9^2\ u_9 = & 18.62790 \\
x_{10}^2 u_{10} = & 26.44337 \\
x_{11}^2 i_{11} = & 33.02691 \\
\hline
\sum x_i^2 u_i = & 107.59549 \\
-(0,2)K_0 = & -101.51151 \\
-(1,2)K_1 = & -9.62778 \\
\hline
\sum x_i^2 u_i - (0,2)K_0 - (1,2)K_1 = & -3.54380 \\
K_2 = \frac{\sum x_i^2 u_i - (0,2)K_0 - (1,2)K_1}{(2,2)} = & -47.313
\end{array}
$$

$$\begin{aligned}\psi_2(x) =&(x-b_2)\psi_1(x)-a_2=(x-0.75118)(x-0.49117)-0.09354\\ =&x^2-1.24235x+0.27542.\end{aligned}$$

Whence it follows

$$\begin{aligned}K_2\psi_2(x)=&-47.313(x^2-1.24235x+0.27542)\\ =&-47.313x^2+58.779x-13.031;\end{aligned}$$

and as

$$\begin{array}{rr} \sum d_1^2= & 174.56, \\ -(2,2)K_2^2= & -167.64, \\ \hline \sum d_2^2=\sum d_1^2-(2,2)K_2^2= & 6.92, \end{array}$$

one finds for the mean error

$$E=\sqrt{\frac{1}{n}\sum d_2^2}=\sqrt{\frac{6.92}{11}}=0.79.$$

By proceeding so, one will obtain the expression of u term by term, and thence the mean error in the representation of the given values of u will approach more and more to zero. But if one finds sufficing to reduce this error to 0.79, one will be arrested at the found terms

$$\begin{array}{rcl} K_0\psi_0(x) & = & 27.564 \\ K_1\psi_1(x) & = & 7.532x-3.699 \\ K_2\psi_2(x) & = & -47.313x^2+58.779x-13.031, \end{array}$$

and thence, for the sought expression of u, one will have

$$\begin{array}{l} +27.564 \\ -\ 3.699+\ 7.532x \\ -13.031+58.779x-47.313x^2 \\ \hline u=10.834+66.311x-47.313x^2. \end{array}$$