Sur le Développement des Fonctions à une seule Variable*

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§1. In my Memoir *Sur les fractions continues* I have shown that if one seeks, according to the given values of the function $F(x)$

$$F(x_1), \quad F(x_2), \dots \quad F(x_n),$$

its approximate expression under the form of a polynomial of any degree, with some coefficients indicated by the *method of least squares*, one arrives to the development of $F(x)$ into series analogous to those of Fourier, and which are ordered according to the denominators of the reductions of the continued fraction resulting from the development of the expression

$$\sum \frac{\theta^2(x_i)}{x - x_i},$$

the probable errors of the given values of $F(x)$

$$F(x_1), \quad F(x_2), \dots \quad F(x_n)$$

being proportionals to

$$\frac{1}{\theta(x_1)}, \quad \frac{1}{\theta(x_2)}, \dots \quad \frac{1}{\theta(x_n)}.$$

According to that, by making the particular hypotheses on the sequence of values

$$x_1, \quad x_2, \dots \quad x_n$$

and the form of the function $\theta(x)$, one obtains, for the development of the functions, many series more or less remarkable.

If one supposes the values

$$x_1, \quad x_2, \dots \quad x_n$$

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. November 3, 2011 equidistant, infinitely near among themselves, and if one makes

$$x_1 = -1, \qquad x_n = +1,$$

$$\theta^2(x) = \frac{x_2 - x_1}{\sqrt{1-x^2}},$$

the expression

$$\sum \frac{\theta^2(x_i)}{x - x_i},$$

will be reduced to

$$\int_{-1}^{+1} \frac{1}{x-u} \frac{du}{\sqrt{1-u^2}} = \frac{\pi}{\sqrt{x^2-1}}.$$

The continued fraction which results from this expression being

$$\cfrac{\pi}{x - \cfrac{1}{2x - \cfrac{1}{2x - \cdots}}}$$

one recognizes easily that its reductions have for denominators entire functions of x which are able to be represented thus:

$$\cos\phi, \quad \cos 2\phi, \quad \cos 3\phi, \ldots$$

where

$$\phi = \arccos x.$$

By virtue of that which we just said, one is led to the known development as of Fourier of $f(x)$ into series ordered according to the *cosinus* of the multiple arcs.

By making the same hypothesis on the values of

$$x_1, \quad x_2, \ldots \quad x_n$$

and by supposing that $\theta^2(x)$ is reduced to a constant

$$x_2 - x_1,$$

one finds that the expression

$$\sum \frac{\theta^2(x_i)}{x - x_i},$$

becomes

$$\int_{-1}^{+1} \frac{du}{x-u} = \log \frac{x+1}{x-1},$$

and as the reductions of this expression have for denominators the functions designated by $X^{(n)}$, there results from it the known series, ordered according to the values of these functions.

In a note read to the Academy in 1858 I have indicated the very simple expression of the denominators of the reductions of

$$\sum \frac{\theta^2(x_i)}{x - x_i},$$

when one has

$$\theta(x) = 1,$$

and that the values

$$x_1, \quad x_2, \ldots \quad x_n$$

are equidistant. This here has furnished to us a new series for the development of the functions, a series all the more remarkable as it leaves nothing to desire for the parabolic interpolation in one of the most ordinary cases of practice.

We are going to indicate at present yet two cases, where the denominators of the reductions of the expression

$$\sum \frac{\theta^2(x_i)}{x - x_i},$$

have a remarkable form, that which, by virtue of our previous researches, again give place to two new series for the development of the functions, a series which, under certain circumstances, will furnish the results with the least error to fear.

§2. If, from $-\infty$ to $+\infty$, the different values of the variable x have the probability $\sqrt{\frac{k}{\pi}}e^{-kx^2}$, and if one seeks for all these values of x the approximate expression of $F(x)$, under the form of a polynomial, with the least error to fear, one will have, according to our Memoir cited above, this formula for the determination of the sought expression of $f(x)$:

$$\begin{aligned} F(x) \quad &= \quad \frac{\int_{-\infty}^{+\infty} \sqrt{\frac{k}{\pi}} e^{-kx^2} \psi_0(x) F(x)\, dx}{\int_{-\infty}^{+\infty} \sqrt{\frac{k}{\pi}} e^{-kx^2} \psi_0^2(x)\, dx} \psi_0(x) + \frac{\int_{-\infty}^{+\infty} \sqrt{\frac{k}{\pi}} e^{-kx^2} \psi_1(x) F(x)\, dx}{\int_{-\infty}^{+\infty} \sqrt{\frac{k}{\pi}} e^{-kx^2} \psi_1^2(x)\, dx} \psi_1(x) \\ &\quad + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

where

$$\psi_0(x), \quad \psi_1(x), \ldots,$$

are the denominators of the reductions of the continued fraction which results from

$$\int_{-\infty}^{+\infty} \frac{\sqrt{\frac{k}{\pi}} e^{-ku^2}}{x - u} du$$

Now, this development of $F(x)$ is reduced to a very remarkable form, all the functions

$$\psi_0(x), \quad \psi_1(x), \ldots \quad \psi_2(x), \ldots$$

as it is easy to be assured, being expressible in this very simple manner:

(1) $$\psi_0(x) = e^{kx^2}.e^{-kx^2}, \quad \psi_1(x) = e^{kx^2}.\frac{de^{-kx^2}}{dx}, \ldots \psi_l(x) = e^{kx^2}.\frac{d^l e^{-kx^2}}{dx^l}.$$

In fact, according to the values of the functions

$$\psi_0(x), \quad \psi_1(x), \ldots,$$

one finds in general

$$\begin{aligned}
\int_{-\infty}^{+\infty} \sqrt{\frac{k}{\pi}} e^{-kx^2} \psi_l(x) F(x)\,dx &= \sqrt{\frac{k}{\pi}} \int_{-\infty}^{+\infty} \frac{d^l e^{-kx^2}}{dx^l} F(x)\,dx = (-1)^l \sqrt{\frac{k}{\pi}} \int_{-\infty}^{+\infty} e^{-kx^2} F^{(l)}(x)\,dx, \\
\int_{-\infty}^{+\infty} \sqrt{\frac{k}{\pi}} e^{-kx^2} \psi_l^2(x)\,dx &= \sqrt{\frac{k}{\pi}} \int_{-\infty}^{+\infty} \frac{d^l e^{-kx^2}}{dx^l} \psi_l(x)\,dx = (-1)^l \sqrt{\frac{k}{\pi}} \int_{-\infty}^{+\infty} e^{-kx^2} \psi_l^{(l)}(x)\,dx, \\
&= 1.2.3\ldots(2k)^l,
\end{aligned}$$

by virtue of which the preceding formula becomes

(2)

$$\left\{ \begin{array}{l}
F(x) = \sqrt{\frac{k}{\pi}} \int_{-\infty}^{+\infty} e^{-kx^2} F(x)\,dx.\psi_0(x) \; - \frac{\sqrt{\frac{k}{\pi}}}{2k} \int_{-\infty}^{+\infty} e^{-kx^2} F'(x)\,dx.\psi_1(x) \\
\qquad + \frac{\sqrt{\frac{k}{\pi}}}{2.4k^2} \int_{-\infty}^{+\infty} e^{-kx^2} F''(x)\,dx.\psi_2(x) - \frac{\sqrt{\frac{k}{\pi}}}{2.4.6k^3k} \int_{-\infty}^{+\infty} e^{-kx^2} F'''(x)\,dx.\psi_3(x) \\
\qquad + \cdots\cdots\cdots\cdots\cdots\cdots
\end{array} \right.$$

where

$$\psi_0(x), \quad \psi_1(x), \quad \psi_2(x), \quad \psi_3(x), \ldots$$

are some entire functions of x which, according to (1), have the following values:

$$\begin{gathered}
\psi_0(x) = e^{kx^2}.e^{-x^2} = 1, \\
\psi_1(x) = e^{kx^2} \tfrac{de^{-kx^2}}{dx} = -2kx, \\
\psi_2(x) = e^{kx^2} \tfrac{d^2 e^{-kx^2}}{dx^2} = 4k^2x^2 - 2k, \\
\psi_3(x) = e^{kx^2} \tfrac{d^3 e^{-kx^2}}{dx^3} = -8k^3x^3 + 12k^2x, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{gathered}$$

This gives us ultimately this remarkable series:

$$\begin{aligned}
F(x) \quad &= \quad \sqrt{\frac{k}{\pi}} \int_{-\infty}^{+\infty} e^{-kx^2} F(x)\,dx + \sqrt{\frac{k}{\pi}} \int_{-\infty}^{+\infty} e^{-kx^2} F'(x)\,dx.\frac{x}{1} \\
&+ \quad \sqrt{\frac{k}{\pi}} \int_{-\infty}^{+\infty} e^{-kx^2} F''(x)\,dx.\frac{x^2 - \frac{1}{2k}}{1.2} + \sqrt{\frac{k}{\pi}} \int_{-\infty}^{+\infty} e^{-kx^2} F''(x)\,dx'.\frac{x^3 - \frac{3}{2k}x}{1.2.3} \\
&+ \quad \text{etc.}
\end{aligned}$$

which, under form of polynomial, furnishes the approximate expressions of $F(x)$ with the least error to fear for all the values of x, between $x = -\infty$ and $x = +\infty$, as long as their probabilities are expressed by the formula $\sqrt{\frac{k}{\pi}} e^{-kx^2}$. If one makes $k = \infty$, this series is reduced to that of Maclaurin which gives the expression of $F(x)$ with the least error, as long as the concern is only with values of x in the neighborhood of $x = 0$. Now it is this which one could envisage, seeing that the function $\sqrt{\frac{k}{\pi}} e^{-kx^2}$, that we have taken in order to express the probabilities of the different values of x, in the case of $k = \infty$, ceases to cancel itself only for x equal to zero.

According to the development of $F(x)$ that we just obtained, one finds many interesting identities. Thus, by seeking, according to (2), the value of the integral

$$\int_{-\infty}^{+\infty} e^{-kx^2} F^2(x)\, dx,$$

one arrives to this formula:

$$\begin{aligned}\sqrt{\frac{k}{\pi}} \int_{-\infty}^{+\infty} e^{-kx^2} F^2(x)\, dx &= \left(\int_{-\infty}^{+\infty} e^{-kx^2} F(x)\, dx\right)^2 + \frac{1}{2k}\left(\int_{-\infty}^{+\infty} e^{-kx^2} F'(x)\, dx\right)^2 \\ &+ \frac{1}{2.4.k^2}\left(\int_{-\infty}^{+\infty} e^{-kx^2} F''(x)\, dx\right)^2 + \frac{1}{2.4.6.k^3}\left(\int_{-\infty}^{+\infty} e^{-kx^2} F'''(x)\, dx\right)^2 \\ &+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\end{aligned}$$

On the other hand, by having regard to the values of (1) of the functions

$$\psi_0(x), \quad \psi_1(x), \ldots \quad \psi_2(x), \ldots$$

one finds that they are linked among them by the equation

$$\psi_l(x) = -2kx\psi_{l-1}(x) - 2(l-1)k\psi_{l-2}(x).$$

Thence one draws easily the values of these functions, and one finds immediately this development of the integral

$$\int_{-\infty}^{+\infty} \frac{e^{-ku^2}}{x-u} du$$

as continued fraction

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{e^{-ku^2}}{x-u} du &= \cfrac{-2\sqrt{k\pi}}{-2k - \cfrac{2k}{-2kx - \cfrac{4k}{-2kx - \cfrac{6k}{2kx - \cdots}}}} \\ &= \cfrac{\sqrt{2\pi}}{\sqrt{2}kx - \cfrac{1}{\sqrt{2}kx - \cfrac{2}{\sqrt{2}kx - \cfrac{3}{\sqrt{2}kx - \cdots}}}}\end{aligned}$$

§3. In passing to the other case, we will suppose that the values of x are comprehended between 0 and $+\infty$,and that ke^{-kx} designates the law of their probability. By seeking, under this assumption, and under form of a polynomial, the expression of $F(x)$ with the least error to fear, one will have, conformably to that which we have shown in the Memoir cited,

$$F(x) = \frac{\int_0^{+\infty} ke^{-kx}\psi_0(x)F(x)\,dx}{\int_0^{+\infty} ke^{-kx}\psi_0^2(x)\,dx}\psi_0(x) + \frac{\int_0^{+\infty} ke^{-kx}\psi_1(x)F(x)\,dx}{\int_0^{+\infty} ke^{-kx}\psi_1^2(x)\,dx}\psi_1(x) + \cdots$$

where

$$\psi_0(x), \quad \psi_1(x), \ldots,$$

are the denominators of the reductions of the continued fraction which results from the development of the formula

$$\int_0^{\infty} \frac{ke^{-ku}}{x-u}du.$$

In this new case one finds also the very simple expressions of the functions $\psi_0(x)$, $\psi_1(x), \ldots$, that are here:

$$\text{(3)} \qquad \psi_0(x) = e^{kx}.e^{-kx}, \quad \psi_1(x) = e^{kx}.\frac{dxe^{-kx}}{dx}, \ldots \quad \psi_l(x) = e^{kx}.\frac{d^l x^l e^{-kx}}{dx^l},$$

Whence it follows in general

$$\begin{aligned}
\int_0^{\infty} ke^{-kx}F(x)\,dx &= \int_0^{\infty} k\frac{d^l x^l e^{-kx}}{dx^l}F(x)\,dx \\
&= (-1)^l k\int_0^{\infty} x^l e^{-kx}F^{(l)}(x)\,dx, \\
\int_0^{\infty} ke^{-kx}\psi_l^2(x) &= \int_0^{\infty} k\frac{d^l x^l e^{-kx}}{dx^l}\psi_l(x)\,dx \\
&= (-1)^l k\int_0^{\infty} x^l e^{-kx}\psi_l^{(l)}(x)\,dx, \\
&= 1^2.2^2\ldots l^2.
\end{aligned}$$

The preceding series will take thus this form:

$$\begin{aligned}
F(x) &= \int_0^{\infty} ke^{-kx}F(x)\,dx.\psi_0(x) - \frac{1}{1^2}\int_0^{\infty} kxe^{-kx}F'(x)\,dx.\psi_1(x) \\
&+ \frac{1}{1^2.2^2}\int_0^{\infty} kx^2e^{-kx}F''(x)\,dx.\psi_2(x) - \frac{1}{1^2.2^2.3^2}\int_0^{\infty} kx^3e^{-kx}F'''(x)\,dx.\psi_3(x) \\
&+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

whence one will have, according to (3),

$$\begin{aligned}
\psi_0(x) &= e^{kx}.e^{-kx} = 1, \\
\psi_1(x) &= e^{kx}.\frac{dxe^{-kx}}{dx} = -kx + 1, \\
\psi_2(x) &= e^{kx}.\frac{d^2x^2e^{-kx}}{dx^2} = k^2x^2 - 4x + 2, \\
\psi_3(x) &= e^{kx}.\frac{d^3x^3e^{-kx}}{dx^3} = -k^3x^3 + 9k^2x^2 - 18kx + 6, \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

This new series comprehends also that of Maclaurin as particular case, corresponding to $k = \infty$. By seeking, according to this series, the value of

$$\int_0^\infty e^{-kx} F^2(x)\, dx,$$

one obtains this identity:

$$\begin{aligned}
\frac{1}{k}\int_0^\infty e^{-kx}F^2(x) &= \left(\int_0^\infty e^{-kx}F(x)\,dx\right)^2 + \frac{1}{1^2}\left(\int_0^\infty xe^{-kx}F'(x)\,dx\right)^2 \\
&+ \frac{1}{1^2.2^2}\left(\int_0^\infty x^2e^{-kx}F''(x)\,dx\right)^2 + \frac{1}{1^2.2^2.3^2}\left(\int_0^\infty x^3e^{-kx}F'''(x)\,dx\right)^2 \\
&+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

and, according to formulas (3), one finds that the functions

$$\psi_0(x), \quad \psi_1(x), \quad \psi_2(x), \ldots$$

are linked among them by the equation

$$\psi_l(x) = -(kx - 2l + 1)\psi_{l-1}(x) - (l-1)^2\psi_{l-2}(x).$$

thence results this development of the integral

$$\int_0^\infty \frac{e^{-ku}}{x-u} du$$

into continued fraction

$$\int_0^\infty \frac{e^{-ku}}{x-u} du = \cfrac{1}{kx - 1 - \cfrac{1^2}{kx - 3 - \cfrac{2^2}{kx - 5 - \cfrac{3^2}{kx - 7 - \cdots}}}}$$