Sur les fractions continues algébriques*

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Sir,

Among the diverse applications of algebraic continued fractions that one has made to present, that which one encounters in the *interpolation according to the method of least squares* is distinguished by a total particular character: in this case the continued fractions serve to determine the terms in certain developments of the function in series. This interpolation and all the series which result from it embrace yet only one small part or range of one such use of continued fractions, and which is able to be as vast as the one one ordinary usage of these fractions in analysis. In fact, ordinarily they serve to find the systems of polynomials X, Y, which render the difference $uX - Y$ as close as possible to zero, by supposing well understood that the function u is developable into series ordered according to the integer and decreasing powers of the variable, and that the degree of approximation is determined by its highest power in the remainder. In order to resolve the question concerning the *interpolation according to the method of least squares* (*Journal de Mathématiques pures et appliquées* of Mr. Liouville, 2° series, T. III, p. 235), the concern was to tighten the as much as possible the distance of the form $uX - Y$, not toward zero, but toward a certain function (toward $\frac{1}{x-X}$, according to the notation of the passage cited), and it is thus that one is arrived to a new use of the algebraic continued fractions. Now this particular case of approximation of the expression $uX - Y$ to the given function is not the only one which is presented in analysis and which demands a new use of the algebraic continued fractions: whatever be the given function v, the determination of the polynomials X, Y, which render the expression $uX - Y$ the nearest to v, is resolved also by aid of continued fractions, and by some formulas analogous to those that we find in *the interpolation according to the method of least squares*. This question on the determination of polynomials X, Y in the expression $uX - Y$ is so much more interesting, that by its simplicity it is placed immediately after that which one resolves ordinarily by means of algebraic continued

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fractions, that is where the concern is only to render the expression $uX - Y$ as near to zero as it is possible.

Let

$$q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cdots}}$$

be the continued fraction which results from the development of the function u, and

$$\frac{P_1}{Q_1} = \frac{q_0}{1}, \quad \frac{P_2}{Q_2} = \frac{q_0 q_1 + 1}{q_1}, \quad \frac{P_3}{Q_3} = \frac{P_0 q_2 + P_1}{Q_2 q_2 + Q_1}, \ldots$$

its convergent fractions. If one agrees to designate by E the entire part of a function, the polynomials X, Y, which render the difference $uX - Y$ the nearest to the function v, will be given by the following series:

$$\begin{aligned} X &= \quad (Eq_1 Q_1 v - q_1 EQ_1 v)Q_1 - (Eq_2 Q_2 v - q_2 EQ_2 v)Q_2 + \cdots \\ Y &= -Ev + (Eq_1 Q_1 v - q_1 EQ_1 v)P_1 - (Eq_2 Q_2 v - q_2 EQ_2 v)P_2 + \cdots \end{aligned}$$

These series are finite or infinite at the same time as the series of convergent fractions

$$\frac{P_1}{Q_1}, \quad \frac{P_2}{Q_2}, \quad \frac{P_3}{Q_3}, \ldots$$

and their terms, as it is not difficult to note it, may present polynomials of which the degrees go by increasing. Arrested at the convenient terms, these series furnish for X, Y some integer values and degrees more or less elevated, according as the number of terms that one takes, and in every case *these values of X and Y are those which render the difference $uX - Y$ as near to v as it is possible with some entire functions of the same degrees as X and Y, and also with some functions of degrees more elevated, but inferior to the degrees of the functions that one obtains by taking in the expressions of X and Y a higher term.*

The values of X and Y which enjoy this remarkable property result from the development of the function v according to the values of the functions

$$R_1 = uQ_1 - P_1, \quad R_2 = uQ_2 - P_2, \quad R_3 = uQ_3 - P_3, \ldots$$

of which the degrees are below zero and go by decreasing. One such development of the function v is easy to obtain. If one subtracts from v its entire part Ev, and if one divides the rest by R_1, the new remainder remainder by R_2, and so forth, it is clear that the quotients of these divisions, multiplied respectively by R_1, R_2,..., and added to Ev, will give the same value of the function v exact except for the remainder. Now the development of v that one finds from this manner presents, as it is easy to be assured, the following series:

$$v = Ev + (Eq_1 Q_1 v - q_1 EQ_1 v)R_1 - (Eq_2 Q_2 v - q_2 EQ_2 v)R_2 + \cdots$$

where the terms are certain functions of which the degrees go by decreasing.

In the most ordinary case, where the denominators q_1, q_2, q_3,... of the continued fraction

$$q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cdots}}$$

are all of the same degree, this series is much simplified; all the factors which accompany the functions R_1, R_2, R_3,... become constants, and their values are determined very easily: these factors are reduced to the products

$$A_1 L_1, \quad A_2 L_2, \quad A_3 L_3, \ldots$$

where A_1, A_2, A_3, ... designate the coefficients of x in the denominators q_1, q_2, q_3,... and L_1, L_2, L_3,... the coefficients of $\frac{1}{x}$ in the products

$$Q_1 v, \quad Q_2 v, \quad Q_3 v, \ldots$$

Whence results in the case in question the following series, for the development of the function v:

$$v = Ev + A_1 L_1 R_1 - A_2 L_2 R_2 + \cdots ,$$

and these values of X and Y, in order to render the difference $uX - Y$ as close as possible to v:

$$\begin{aligned} X &= A_1 L_1 Q_1 - A_2 L_2 Q_2 + \cdots , \\ Y &= -Ev + A_1 L_1 P_1 - A_2 L_2 P_2 + \cdots , \end{aligned}$$

In the case where the function v is able to be represented by the sum

$$\Phi(x) + \frac{K_1}{x - x_1} + \frac{K_2}{x - x_2} + \frac{K_3}{x - x_3} + \cdots ,$$

$\Phi(x)$ being an entire function, and K_1, K_2, K_3,..., x_1, x_2, x_3, ... some constants, one finds, by putting $Q_1 = \psi_1(x)$, $Q_2 = \psi_2(x)$, $Q_3 = \psi_3(x)$ ..., the following expressions of the quantities L_1, L_2, L_3,...:

$$\begin{aligned} L_1 &= K_1\psi_1(x_1) + K_2\psi_1(x_2) + K_3\psi_1(x_3) + \cdots = \sum K_i\psi_1(x_i), \\ L_2 &= K_1\psi_2(x_1) + K_2\psi_2(x_2) + K_3\psi_2(x_3) + \cdots = \sum K_i\psi_2(x_i), \\ L_3 &= K_1\psi_3(x_1) + K_2\psi_3(x_2) + K_3\psi_3(x_3) + \cdots = \sum K_i\psi_3(x_i), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

For these values of L_1, L_2, L_3, ... the preceding series become

$$\begin{aligned} v &= Ev + A_1 \sum K_i\psi_1(x_i) R_1 - A_2 \sum K_i\psi_2(x_i) R_2 + A_3 \sum K_i\psi_3(x_i) R_3 - \cdots , \\ X &= A_1 \sum K_i\psi_1(x_i) Q_1 - A_2 \sum K_i\psi_2(x_i) Q_2 + A_3 \sum K_i\psi_3(x_i) Q_3 - \cdots , \\ Y &= -Ev + A_1 \sum K_i\psi_1(x_i) P_1 - A_2 \sum K_i\psi_2(x_i) P_2 + A_3 \sum K_i\psi_3(x_i) P_3 - \cdots , \end{aligned}$$

where one has

$$\psi_1(x) = Q_1, \quad \psi_2(x) = Q_2, \quad \psi_3(x) = Q_3, \ldots,$$

In the particular case where the function v is reduced to a single term $\frac{1}{x-a}$ one finds, according to the first of these formulas, the development according to $\frac{1}{x-a}$,

$$\frac{1}{x-a} = A_1\psi_1(a)R_1 - A_2\psi_2(a)R_2 + A_3\psi_3(a)R_3 - \cdots ,$$

where each term presents the product of a function of a by a function of x, as that takes place in the series which results form the development of $(x-a)^{-1}$ according to the formula of Newton.

By making in the preceding formulas

$$u = \frac{1}{x-x_1} + \frac{1}{x-x_2} + \cdots + \frac{1}{x-x_n} = \sum \frac{1}{x-x_i},$$
$$u = \frac{F(x_1)}{x-x_1} + \frac{F(x_2)}{x-x_2} + \cdots + \frac{F(x_n)}{x-x_i} = \sum \frac{F(x_i)}{x-x_i},$$

one finds the values of the polynomials X and Y by which the difference

$$\sum \frac{1}{x-x_i} \cdot X - Y$$

approaches as close as possible $\sum \frac{F(x_n)}{x-x_i}$, and as such a closeness constitutes the necessary and sufficient condition for which the polynomial X reduces to the *minimum* the sum

$$\sum (X - F(x_i))^2$$

(this which is not difficult to show), it follows that the expression of X, determined in this manner, presents the formula of *interpolation according to the method of least squares*.

I will no longer insist on the part that one is able to draw from the development into series according to the functions determined by the mean of the algebraic continued fractions; that which I just said suffices to show all the interest that the subject presents toward which your lessons and your precious conversation has carried me.

Approve the assurance of a profound respect, etc.

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