Sur l'interpolation des valeurs équidistantes*

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§1. If one seeks by the method of *least squares* the expression of a certain function $f(x)$ under the form of a polynomial, the values of the function $f(x)$ of which one is served in order to determine its expression being

$$f(1), \quad f(2), \ldots \quad f(m-1),$$

the sought polynomial, as one knows, is given[1] by the formula

$$\frac{\sum_1^m \phi_0(x) f(x)}{\sum_1^m \phi_0^2(x)} \phi_0(x) + \frac{\sum_1^m \phi_1(x) f(x)}{\sum_1^m \phi_1^2(x)} \phi_1(x) + \cdots, \tag{1}$$

where

$$\phi_0(x), \quad \phi_1(x), \ldots$$

designating the denominators of the convergent fractions of the sum

$$\sum_{\lambda=1}^{\lambda=m} \frac{1}{x-\lambda} = \frac{1}{x-1} + \frac{1}{x-2} + \cdots + \frac{1}{x-m+1},$$

obtained by its development into continued fraction

$$\cfrac{C_0}{A_0 x + B_0 = \cfrac{C_1}{A_1 x + B_1 + \cfrac{C_2}{A_2 x + B_2 + \cdots}}}$$

The functions

$$\phi_0(x), \quad \phi_1(x), \ldots$$

as I have already indicated, enjoy the same role in *the inverse calculation of the differences* as the functions of Legendre in the integral calculus and, by analogy to these last, are able to be represented by the formula[2]

$$\phi_n(x) = \Delta^n (x-1)(x-2) \ldots (x-n)(m+n-1-x) \ldots (m-x),$$

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[1]See the Memoir under the title: "Sur les fractions continues."

[2]*Sur une nouvells serie.*

by setting aside some constant factors which are evidently suppressed in formula (1).

This expression of the functions

$$\phi_0(x), \quad \phi_1(x), \dots$$

simplifies notably, as one is going to see, the calculation of all the sums which figure in formula (1).

§2. By approaching the deduction of the mentioned expression of the functions

$$\phi_0(x), \quad \phi_1(x), \dots$$

let it be appropriate to designate, for brevity, by $\Phi(x)$ the product

$$(x-1)(x-2)\dots(x-n)(m+n-1-x)(m+n-x-2)\dots(m-x).$$

This product being annulled for

$$x = 1, 2, \dots n,$$

$$x = m+n-1, \quad m+n-2, \dots \quad m,$$

all the quantities

$$\Phi(x), \quad \Phi(x+1), \dots, \quad \Phi(x+n-1)$$

for $x = 1$ and $x = m$ will be equal to zero, and the same thing will hold in regard to the differences

$$\Delta^{n-1}\Phi(x), \quad \Delta^{n-2}\Phi(x+1), \dots, \quad \Phi(x+n-2)$$

which are determined according to the values

$$\Phi(x), \quad \Phi(x+1), \dots, \quad \Phi(x+n-1).$$

This put, it is not difficult to show that the sum

$$\sum_1^m F(x)\Delta^n\Phi(x),$$

whatever be the function $F(x)$, will be reduced to the sum

$$(-1)^n \sum_1^m \Phi(x+n)\Delta^n F(x).$$

In order to convince ourselves of it, let us note that, transforming the sum

$$\sum_1^m F(x)\Delta^n\Phi(x)$$

n times in sequence by aid of the known formula

$$\sum U_x \Delta V_x = U_x V_x - \sum V_{x+1}\Delta U_x,$$

we will find that it is reduced to the following expression

$$F(x)\Delta^{n-1}\Phi(x) - \Delta F(x)\Delta^{n-2}\Phi(x+1) + \Delta^2 F(x)\Delta^{n-3}\Phi(x+2) + \cdots$$
$$+(-1)^{n-1}\Phi(x+n-1)\Delta^{n-1}F(x) + (-1)^n \sum \Phi(x+n)\Delta^n F(x).$$

Now, by virtue of that which one has seen in regard to the function

$$\Phi(x+n-1)$$

and the differences

$$\Delta^{n-1}\Phi(x), \quad \Delta^{n-2}\Phi(x+1), \ldots \quad \Delta\Phi(x+n-2),$$

all the terms of the preceding formula beyond the sign $\sum$ are reduced to zero for $x = 1$ and for $x = m$; therefore one will have within these limits

(2)
$$\sum_1^m F(x)\Delta^n\Phi(x) = (-1)^n \sum_1^m \Phi(x+n)\Delta^n F(x).$$

By making

$$F(x) = \phi_0(x), \quad F(x) = \phi_1(x), \ldots \quad F(x) = \phi_{n-1}(x)$$

and noting that for these values of $F(x)$ one has

$$\Delta^n F(x) = 0,$$

one will find

$$\sum_1^m \phi_0(x)\Delta^n\Phi(x) = 0, \quad \sum_1^m \phi_1(x)\Delta^n\Phi(x) = 0, \ldots \quad \sum_1^m \phi_{n-1}(x)\Delta^n\Phi(x) = 0.$$

Now, these equalities, by virtue of § VII of the Memoir cited on continued fractions, show that in the development of the function $\Delta^n\Phi(x)$ into series

$$A\phi_0(x) + B\phi_1(x) + \cdots + G\phi_{n-1}(x) + H\phi_n(x)$$

the coefficients

$$A, \quad B, \ldots \quad G$$

are reduced to zero, whence it follows that $\Delta^n\Phi(x)$ differ from $\phi_n(x)$ only by a constant factor. This factor, as it has been said, is suppressed in formula (1). By setting aside this factor we take

$$\phi_n(x) = \Delta^n\Phi(x),$$

that which reduces equality (2) to the following

$$\sum_1^m F(x)\phi_n(x) = (-1)^n \sum_1^m \Phi(x+n)\Delta^n F(x),$$

where

$$\Phi(x) = (x-1)(x-2)\dots(x-n)(m+n-x-1)\dots(m-x).$$

§3. Passing to the evaluation of the sums contained in formula (1), we will introduce, in order to shorten the writing, the sign $\Gamma(\mu)$ in order to designate the product 1, 2, 3… $(\mu-1)$.

By aid of this notation the products

$$(x-1)(x-2)\dots(x-n),$$

$$(m+n-x-1)(m+n-x-2)\dots(m-x)$$

will be represented by

$$\frac{\Gamma(x)}{\Gamma(x-n)}, \qquad \frac{\Gamma(m+n-x)}{\Gamma(m-x)},$$

this which gives for the expressions of the functions $\Phi(x)$, $\phi_n(x)$

$$\Phi(x) = \frac{\Gamma(x)}{\Gamma(x-n)}\frac{\Gamma(m+n-x)}{\Gamma(m-x)},$$

(3) $$\phi_n(x) = \Delta^n \frac{\Gamma(x)}{\Gamma(x-n)}\frac{\Gamma(m+n-x)}{\Gamma(m-x)},$$

and hence the preceding transformation of the sum is reduced to the following

$$\sum_1^m \phi_n(x)F(x) = (-1)^n \sum_1^m \frac{\Gamma(x+n)}{\Gamma(x)}\frac{\Gamma(m-x)}{\Gamma(m-n-x)}\Delta^n F(x).$$

By making here

$$F(x) = f(x),$$

we will find for the evaluation of the sums

$$\sum_1^m \phi_0(x)f(x), \quad \sum_1^m \phi_1(x)f(x), \quad \sum_1^m \phi_2(x)f(x), \dots$$

the following formula

$$\sum_1^m \phi_n(x)f(x) = (-1)^n \sum_1^m \frac{\Gamma(x+n)}{\Gamma(x)}\frac{\Gamma(m-x)}{\Gamma(m-n-x)}\Delta^n f(x).$$

By making in the same formula

$$F(x) = \phi_n(x),$$

we will have

$$\sum_1^m \phi_n^2(x) = (-1)^n \sum_1^m \frac{\Gamma(x+n)}{\Gamma(x)} \frac{\Gamma(m-x)}{\Gamma(m-n-x)} \Delta^n \phi_n(x).$$

Now according to (3)

$$\Delta^n \phi_n(x) = \Delta^{2n} \frac{\Gamma(x)}{\Gamma(x-n)} \frac{\Gamma(m+n-x)}{\Gamma(m-x)},$$

besides

$$\frac{\Gamma(x)}{\Gamma(x-n)} = (x-1)(x-2)\dots(x-n),$$

$$\frac{\Gamma(m+n-x)}{\Gamma(m-x)} = (m+n-x-1)(m+n-2)\dots(m-x),$$

therefore, in the product

$$\frac{\Gamma(x)}{\Gamma(x-n)} \frac{\Gamma(m+n-x)}{\Gamma(m-x)}$$

the term of the highest degree in x is equal to

$$(-1)^n x^{2n};$$

there results from it that the difference of order $2n$ of this function is reduced to the constant quantity

$$(-1)^n 2n(2n-1)\dots 3.2.1,$$

that which one is able to represent by aid of the sign Γ as it follows:

$$(-1)^n \Gamma(2n+1).$$

One deduces from it that

$$\Delta^n \phi_n(x) = (-1)^n \Gamma(2n+1),$$

and, consequently, the transformation found above of the sum

$$\sum_1^m \phi_n^2(x)$$

gives us

$$\sum_1^m \phi_n^2(x) = \Gamma(2n+1) \sum_1^m \frac{\Gamma(x+n)}{\Gamma(x)} \frac{\Gamma(m-x)}{\Gamma(m-n-x)}.$$

Now, $\Gamma(m-n-x)$ being infinity for

$$x = m-n, \quad m-n+1, \dots \quad m-1,$$

all the elements of the sum

$$\sum_1^m \frac{\Gamma(x+n)}{\Gamma(x)} \frac{\Gamma(m-x)}{\Gamma(m-n-x)}$$

by departing from $x = m - n$ are reduced to zero, and consequently one is able to take for upper limit of this sum $m - n$ instead of m; by virtue of this the preceding equality is reduced to the following

$$\sum_1^m \phi_n^2(x) = \Gamma(2n+1) \sum_1^{m-n} \frac{\Gamma(x+n)}{\Gamma(x)} \frac{\Gamma(m-x)}{\Gamma(m-n-x)}. \tag{4}$$

§4. In order to determine the value of the sum, which figures in the last equality and in some others parallel to that here, that we will encounter later, we are going to find now the expression of all the sums of the form

$$\sum_1^N \frac{\Gamma(x+p-1)}{\Gamma(x)} \frac{\Gamma(N-x+q-1)}{\Gamma(N-x)},$$

that which is easy to do by aid of the binomial of Newton.

We note for this that the developments of the powers

$$(1-t)^{-p}, \quad (1-t)^{-q}, \quad (1-t)^{-p-q}$$

into series by the formula of Newton by aid of the signs $\sum$ and Γ are able to be written in the following manner:

$$\begin{aligned}
(1-t)^{-p} &= \sum \frac{\Gamma(p+\lambda)}{\Gamma(p)\Gamma(\lambda+1)} t^\lambda, \\
(1-t)^{-q} &= \sum \frac{\Gamma(q+\mu)}{\Gamma(q)\Gamma(\mu+1)} t^\mu, \\
(1-t)^{-p-q} &= \sum \frac{\Gamma(p+q+\nu)}{\Gamma(p+q)\Gamma(\nu+1)} t^\nu,
\end{aligned}$$

the summations there being extended over all the integer values of λ, μ, ν from 0 to ∞. By comparing the product of the first two sums to the last we will have the identity

$$\sum \frac{\Gamma(p+\lambda)}{\Gamma(p)\Gamma(\lambda+1)} t^\lambda \cdot \sum \frac{\Gamma(q+\mu)}{\Gamma(q)\Gamma(\mu+1)} t^\mu = \sum \frac{\Gamma(p+q+\nu)}{\Gamma(p+q)\Gamma(\nu+1)} t^\nu,$$

which is able to be represented in another manner under the form

$$\sum\sum \frac{\Gamma(p+\lambda)}{\Gamma(p)\Gamma(\lambda+1)} \cdot \frac{\Gamma(q+\mu)}{\Gamma(q)\Gamma(\mu+1)} t^{\lambda+\mu} = \sum \frac{\Gamma(p+q+\nu)}{\Gamma(p+q)\Gamma(\nu+1)} t^\nu.$$

By determining here the terms having the factor t^{N-2}, we perceive that in the first member of this identity the coefficient of t^{N-2} is equal to the sum of the values of the expression

$$\frac{\Gamma(p+\lambda)}{\Gamma(p)\Gamma(\lambda+1)} \cdot \frac{\Gamma(q+\mu)}{\Gamma(q)\Gamma(\mu+1)},$$

corresponding to all the whole and positive values of λ and μ, of which the sum $\lambda+\mu$ is equal to $N-2$, or, that which reverts to the same, to the sum of all the values of this expression, in which

$$\mu = N-2-\lambda,$$

λ being successively the values 0, 1, 2,… $N-2$. One sees according to this that the first member of the identity considered contains the following term with the power t^{N-2}:

$$\sum_{\lambda=0}^{\lambda=N-1} \frac{\Gamma(p+\lambda)}{\Gamma(p)\Gamma(\lambda+1)} \cdot \frac{\Gamma(q+N-2-\lambda)}{\Gamma(q)\Gamma(N-1-\lambda)} t^{N-2}.$$

Besides, the term with the same power of t in the second member of this identity being

$$\frac{\Gamma(p+q+N-2)}{\Gamma(p+q)\Gamma(N-1)} t^{N-2},$$

we conclude that it involves the following equality:

$$\sum_{\lambda=0}^{\lambda=N-1} \frac{\Gamma(p+\lambda)}{\Gamma(p)\Gamma(\lambda+1)} \cdot \frac{\Gamma(q+N-2-\lambda)}{\Gamma(q)\Gamma(N-1-\lambda)} = \frac{\Gamma(p+q+N-2)}{\Gamma(p+q)\Gamma(N-1)}.$$

By multiplying it by $\Gamma(p).\Gamma(q)$ and putting

$$\lambda+1 = x,$$

we will reduce it to the form

$$\sum_{1}^{N} \frac{\Gamma(x+p-1)}{\Gamma(x)} \frac{\Gamma(N-x+q-1)}{\Gamma(N-x)} = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \frac{\Gamma(p+q+N-2)}{\Gamma(N-1)}. \tag{5}$$

By setting here

$$N = m-n, \quad p = n+1, \quad q = n+1,$$

we will have, in order to determine the sum, contained in formula (4),

$$\sum_{1}^{m-n} \frac{\Gamma(x+n)}{\Gamma(x)} \frac{\Gamma(m-x)}{\Gamma(m-n-x)} = \frac{\Gamma^2(n+1)}{\Gamma(2n+2)} \frac{\Gamma(m+n)}{\Gamma(m-n-1)},$$

whence there results

$$\sum_{1}^{m} \phi_n^2(x) = \frac{\Gamma(2n+1)}{\Gamma(2n+2)} \frac{\Gamma^2(n+1)\Gamma(m+n)}{\Gamma(m-n-1)}.$$

Replacing further $\Gamma(2n+2)$ by $(2n+1)\Gamma(2n+1)$ one will have

$$\sum_{1}^{m} \phi_n^2(x) = \frac{\Gamma^2(n+1)\Gamma(m+n)}{\Gamma(m-n-1)}.$$

§5. In order to determine the functions

$$\phi_0(x), \quad \phi_1(x), \quad \phi_2(x), \ldots,$$

by the aid of formula (3), we are going to deduce beforehand the expressions of the differences of the functions

$$\frac{\Gamma(x)}{\Gamma(x-p)}, \qquad \frac{\Gamma(q-x)}{\Gamma(q-p-x)}.$$

Calculating the differences of the first order, one will have

$$\Delta\frac{\Gamma(x)}{\Gamma(x-p)} = \frac{\Gamma(x+1)}{\Gamma(x-p+1)} - \frac{\Gamma(x)}{\Gamma(x-p)};$$

$$\Delta\frac{\Gamma(q-x)}{\Gamma(q-p-x)} = \frac{\Gamma(q-x-1)}{\Gamma(q-p-x-1)} - \frac{\Gamma(q-x)}{\Gamma(q-p-x)}.$$

Replacing

$$\Gamma(x+1), \quad \Gamma(x-p), \quad \Gamma(q-p-x-1), \quad \Gamma(q-x)$$

by the equivalent quantities

$$x\Gamma(x), \quad \frac{\Gamma(x-p+1)}{x-p}, \quad \frac{\Gamma(q-p-x)}{q-p-x-1}, \quad (q-x-1)\Gamma(q-x-1),$$

we will have, after some reductions,

$$\begin{aligned}\Delta\frac{\Gamma(x)}{\Gamma(x-p)} &= p\frac{\Gamma(x)}{\Gamma(x-p+1)},\\ \Delta\frac{\Gamma(q-x)}{\Gamma(q-p-x)} &= -p\frac{\Gamma(q-x-1)}{\Gamma(q-p-x)}.\end{aligned}$$

By applying these formulas to the determination of the differences of the 2nd, 3rd etc. orders, we find in general

$$\Delta^l\frac{\Gamma(x)}{\Gamma(x-p)} = p(p-1)\ldots(p-l+1)\frac{\Gamma(x)}{\Gamma(x-p+l)},$$

$$\Delta^l\frac{\Gamma(q-x)}{\Gamma(q-p-x)} = (-1)^l p(p-1)\ldots(p-l+1)\frac{\Gamma(q-x-l)}{\Gamma(q-p-x)}.$$

Replacing the product

$$p(p-1)\ldots(p-l+1)$$

by

$$\frac{\Gamma(p+1)}{\Gamma(p-l+1)},$$

one obtains

$$
(6) \qquad \begin{cases} \triangle^l \frac{\Gamma(x)}{\Gamma(x-p)} = \frac{\Gamma(p+1)}{\Gamma(p-l+1)} \frac{\Gamma(x)}{\Gamma(x-p+l)}, \\ \triangle^l \frac{\Gamma(q-x)}{\Gamma(q-p-x)} = (-1)^l \frac{\Gamma(p+1)}{\Gamma(p-l+1)} \frac{\Gamma(q-x-l)}{\Gamma(q-p-x)}. \end{cases}
$$

Passing to the determination of the function $\phi_n(x)$ according to the formula

$$
\phi_n(x) = \triangle^n \frac{\Gamma(x)}{\Gamma(x-n)} \frac{\Gamma(m+n-x)}{\Gamma(m-x)},
$$

we are going to develop the function

$$
\frac{\Gamma(m+n-x)}{\Gamma(m-x)}
$$

into series by aid of the formula

$$
F(x) = F(n+1) + \frac{x-n-1}{1} \Delta F(n+1) + \frac{(x-n-1)(x-n-2)}{1.2} \Delta^2 F(n+1) + \cdots,
$$

which one is able to represent under the following form, by means of the signs $\sum$ and Γ,

$$
F(x) = \sum \frac{\Gamma(x-n)}{\Gamma(x-n-\lambda)} \frac{\Delta^\lambda F(n+1)}{\Gamma(\lambda+1)},
$$

the sum being extended over all the whole and positive values of λ and $\Delta^0 F(n+1)$ coinciding with the initial function $F(n+1)$.

By putting into this formula

$$
F(x) = \frac{\Gamma(m+n-x)}{\Gamma(m-x)}
$$

and noting that according to (6) one has in this case

$$
\triangle^\lambda F(x) = (-1)^\lambda \frac{\Gamma(n+1)}{\Gamma(n-\lambda+1)} \frac{\Gamma(m+n-\lambda-x)}{\Gamma(m-x)},
$$

we will have

$$
\frac{\Gamma(m+n-x)}{\Gamma(m-x)} = \sum (-1)^\lambda \frac{\Gamma(n+1)\Gamma(m-\lambda-1)\Gamma(x-n)}{\Gamma(\lambda+1)\Gamma(n-\lambda+1)\Gamma(m-n-1)\Gamma(x-n-\lambda)},
$$

this which, being multiplied by

$$
\frac{\Gamma(x)}{\Gamma(x-n)}
$$

is reduced to that which follows:

$$
\frac{\Gamma(x)}{\Gamma(x-n)} \frac{\Gamma(m+n-x)}{\Gamma(m-x)} = \sum (-1)^\lambda \frac{\Gamma(n+1)\Gamma(m-\lambda-1)}{\Gamma(\lambda+1)\Gamma(n-\lambda+1)\Gamma(m-n-1)} \frac{\Gamma(x)}{\Gamma(x-n-\lambda)}.
$$

In determining according to his formula the difference

$$\triangle^n \frac{\Gamma(x)}{\Gamma(x-n)} \frac{\Gamma(m+n-x)}{\Gamma(m-x)},$$

we find that it is expressed by

$$\sum(-1)^\lambda \frac{\Gamma(n+1)\Gamma(m-\lambda-1)}{\Gamma(\lambda+1)\Gamma(n-\lambda+1)\Gamma(m-n-1)} \triangle^n \frac{\Gamma(x)}{\Gamma(x-n-\lambda)};$$

Now this difference being equal to the function

$$\phi_n(x),$$

and formula (6) giving

$$\triangle^n \frac{\Gamma(x)}{\Gamma(x-n-\lambda)} = \frac{\Gamma(n+\lambda+1)}{\Gamma(\lambda+1)} \cdot \frac{\Gamma(x)}{\Gamma(x-\lambda)},$$

we obtain the following expression of the function $\phi_n(x)$:

$$\phi_n(x) = \sum(-1)^n \frac{\Gamma(n+1)\Gamma(m-\lambda-1)\Gamma(n+\lambda+1)}{\Gamma^2(\lambda+1)\Gamma(n-\lambda+1)\Gamma(m-n-1)} \frac{\Gamma(x)}{\Gamma(x-\lambda)},$$

where the summation, as one has seen, is extended over all the whole and positive values of λ.

Besides, the divisor $\Gamma(n-\lambda+1)$ being reduced to ∞ by departing from $\lambda = n+1$ and the corresponding terms being reduced consequently to zero, one will be able to take for upper limit of the sum $\lambda = n+1$.

§6. Substituting the values of the sums

$$\sum_1^m \phi_n(x) f(x), \qquad \sum_1^m \phi_n^2(x)$$

found above into formula (1) and denoting generally by

$$C_{\alpha,\beta}$$

the expression

$$\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)} = \frac{(\alpha+\beta)(\alpha+\beta-1)\dots(\alpha+1)}{1.2\dots\beta},$$

which represents the coefficient of $a^\alpha b^\beta$ in the development of $(a+b)^{\alpha+\beta}$, we find that the general term of this formula is able to be represented as it follows:

$$(-1)^n \frac{2n+1}{m-1} \frac{\Gamma(m-n-1)}{\Gamma(m-1)\Gamma(n+1)} \frac{\sum_1^m C_{x-1,n} C_{m-n-x-1,n} \Delta^n f(x)}{C_{m-1,n}} \phi_n(x)$$

or

$$\frac{\Gamma(m-n-1) F_n}{\Gamma(n+1)\Gamma(m-1)} \phi_n(x),$$

where one has put for brevity

$$F_n = (-1)^n \frac{2n+1}{m-1} \frac{\sum_1^m C_{x-1,n} C_{m-n-x-1,n} \Delta^n f(x)}{C_{m-1,n}},$$

There results from it that the development of the function $f(x)$ according to formula (1), arrested at the $(l+1)$st term, will be represented by the sum:

$$\sum_{n=0}^{n=l+1} \frac{\Gamma(m-n-1)F_n}{\Gamma(n+1)\Gamma(m-1)} \phi_n(x).$$

Designating this approximate value of $f(x)$ by $F(x)$, we will have:

$$F(x) = \sum_{n=0}^{n=l+1} \frac{\Gamma(m-n-1)F_n}{\Gamma(n+1)\Gamma(m-1)} \phi_n(x),$$

where, as one has seen, the function $\phi_n(x)$ has the following value:

$$\phi_n(x) = \sum_{n=0}^{n=l+1} (-1)^\lambda \frac{\Gamma(n+1)\Gamma(m-\lambda-1)\Gamma(n+\lambda+1)}{\Gamma^2(\lambda+1)\Gamma(n-\lambda+1)\Gamma(m-n-1)} \frac{\Gamma(x)}{\Gamma(x-\lambda)}.$$

Developing the sums contained in these formulas we obtain $F(x)$ under the form of a polynomial of degree l; this expression of $F(x)$ differs only by the notations of that which we have given in the note mentioned above. On the other hand, by aid of these formulas, serving for the determination of $F(x)$, it is easy to calculate the values

$$F(1), \quad \Delta F(1), \quad \Delta^2 F(1), \ldots,$$

as we are going to see immediately.

By determining according to these formulas the differences

$$\Delta^\mu F(x), \quad \Delta^\mu \phi_n(x),$$

μ being any number we will have:

$$(7) \quad \left\{ \begin{array}{c} \Delta^\mu F(x) = \sum_{\lambda=0}^{\lambda=n+1} \frac{\Gamma(m-n-1)F_n}{\Gamma(n+1)\Gamma(m-1)} \Delta^\mu \phi_n(x), \\ \Delta^\mu \phi_n(x) = \sum_{\lambda=0}^{\lambda=n+1} (-1)^\lambda \frac{\Gamma(n+1)\Gamma(m-\lambda-1)\Gamma(n+\lambda+1)}{\Gamma^2(\lambda+1)\Gamma(n-\lambda+1)\Gamma(m-n-1)} \Delta^\mu \frac{\Gamma(x)}{\Gamma(x-\lambda)}. \end{array} \right.$$

Now according to (6)

$$\Delta^\mu \frac{\Gamma(x)}{\Gamma(x-\lambda)} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)} \frac{\Gamma(x)}{x-\lambda+\mu)},$$

this which gives, for $x=1$,

$$\Delta^\mu \frac{\Gamma(1)}{\Gamma(1-\lambda)} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)} \frac{\Gamma(1)}{1-\lambda+\mu)}.$$

The divisor $\Gamma(\lambda-\mu+1)$ being infinite for $\lambda-\mu \leq -1$ and the divisor $\Gamma(1-\lambda+\mu)$ for $\lambda-\mu \geq 1$, the difference

$$\triangle^{\mu}\frac{\Gamma(1)}{\Gamma(1-\lambda)}$$

differs from 0 only for $\lambda-\mu=0$, that is for $\lambda=\mu$; therefore, in the sum which represents the value of $\Delta^{\mu}\phi_n(x)$, for $x=1$, there remains only a single term, corresponding to $\lambda=\mu$ and in this term the difference

$$\triangle^{\mu}\frac{\Gamma(x)}{\Gamma(x-\lambda)},$$

according to the formula that we just found, by virtue of the equality $\lambda=\mu$, is reduced to $\Gamma(\mu+1)$. According to that, by virtue of (7), for $x=1$, there results

$$\begin{aligned}\triangle^{\mu}\phi_n(1) &= (-1)^{\mu}\frac{\Gamma(n+1)\Gamma(m-\mu-1)\Gamma(n+\mu+1)}{\Gamma(\mu+1)\Gamma(n-\mu+1)\Gamma(m-n-1)},\\ \triangle^{\mu}F(1) &= \sum_{n=0}^{n=l+1}(-1)^{\mu}\frac{\Gamma(m-\mu-1)\Gamma(n+\mu+1)F_n}{\Gamma(\mu+1)\Gamma(m-1)\Gamma(n-\mu+1)}.\end{aligned}$$

The divisor $\Gamma(n-\mu+1)$ being reduced to ∞ for

$$n=0,1,2,\ldots,\mu-1,$$

all the elements of the sum of $n=0$ to $n=\mu$ are reduced to zero, and hence one will be able to take for its lower limit $n=\mu$, by virtue of which, the formula deduced will be written as it follows:

$$\triangle^{\mu}F(1)=(-1)^{\mu}\sum_{n=0}^{n=l+1}\frac{\Gamma(m-\mu-1)\Gamma(n+\mu+1)F_n}{\Gamma(\mu+1)\Gamma(m-1)\Gamma(n-\mu+1)}.$$

§7. In the formula which expresses the value of the difference

$$\triangle^{\mu}F(1),$$

the expression

$$(-1)^{\mu}\frac{\Gamma(m-\mu-1)\Gamma(n+\mu+1)}{\Gamma(\mu+1)\Gamma(m-1)\Gamma(n-\mu+1)}$$

is reduced to 1 for

$$\mu=0,$$

this which corresponds to the case where $\Delta^{\mu}F(1)$ is reduced to $F(1)$; hence, the value of $F(1)$ will be given by the formula

$$F(1)=\sum_{n=0}^{n=l+1}F_n=F_0+F_1+\cdots+F_l.$$

Passing to the research of the differences $\Delta F(1)$, $\Delta^2 F(1)$,... we will represent for brevity the expression

$$(-1)^\mu \frac{\Gamma(m-\mu-1)\Gamma(n+\mu+1)}{\Gamma(\mu+1)\Gamma(m-1)\Gamma(n-\mu+1)}$$

by

$$(\mu, n);$$

according to that the formula that we have found for the evaluation of $\Delta^\mu F(1)$ will take the form

$$\triangle^\mu F(1) = \sum_{n=\mu}^{n=l+1} (\mu, n) F_n.$$

Applying the equality

$$(\mu, n) = (-1)^\mu \frac{\Gamma(m-\mu-1)\Gamma(n+\mu+1)}{\Gamma(m-1)\Gamma(\mu+1)\Gamma(n-\mu+1)},$$

to the determination of the ratio

$$\frac{(\mu+1, n)}{(\mu, n)}$$

we find

$$\frac{(\mu+1, n)}{(\mu, n)} = -\frac{\Gamma(\mu+1)}{\Gamma(\mu+2)} \frac{\Gamma(m-\mu-2)}{\Gamma(m-\mu-1)} \frac{\Gamma(n+\mu+2)}{\Gamma(n+\mu+1)} \frac{\Gamma(n-\mu+1)}{\Gamma(n-\mu)},$$

this which is reduced, because of the equalities

$$\Gamma(\mu+2) = (\mu+1)\Gamma(\mu+1), \quad \Gamma(m-\mu-1) = (m-\mu-2)\Gamma(m-\mu-2),$$

$$\Gamma(n+\mu+2) = (n+\mu+1)\Gamma(n+\mu+1), \quad \Gamma(n-\mu+1) = (n-\mu)\Gamma(n-\mu),$$

to that which follows

$$\frac{(\mu+1, n)}{(\mu, n)} = -\frac{(n+\mu+1)(n-\mu)}{(\mu+1)(m-\mu-2)} = -\frac{n(n+1)-\mu(\mu+1)}{(\mu+1)(m-\mu-2)};$$

whence there results

$$(\mu+1, n) = -\frac{n(n+1)-\mu(\mu+1)}{(\mu+1)(m-\mu-2)}(\mu, n).$$

This relation between the quantities

$$(\mu+1, n), \quad (\mu, n),$$

joins the equality remarked above

$$(0, n) = 1,$$

gives the means to calculate easily all the values of the factor (μ, n) in the formula

$$\triangle^{\mu} F(1) = \sum_{n=\mu}^{n=l+1} (\mu, n) F_n.$$

By virtue of that which precedes we conclude that the quantities

$$F(1), \quad \Delta F(1), \quad \Delta^2 F(1), \quad \Delta^3 F(1), \dots$$

in the case, where the series (1) is arrested at the $(l+1)$st term, will be found by aid of the formulas:

$$(8) \qquad \begin{cases} F(1) & = F_0 + F_1 \quad + \quad F_2 \quad + \quad F_3 \quad + \dots + \quad F_l \\ \Delta F(1) & = (1,1)F_1 \quad +(1,2)F_2 \quad +(1,3)F_3 \quad + \dots + (1,l)F_l, \\ \Delta^2 F(1) & = \quad +(2,2)F_2 \quad +(2,3)F_3 \quad + \dots + (2,l)F_l, \\ \Delta^3 F(1) & = \quad +(3,3)F_3 \quad + \dots + (3,l)F_l, \\ \dots & \\ \dots & \end{cases}$$

the quantities

$$F_0, \quad F_1, \quad F_2, \dots \quad F_l$$

being given by the formulas

$$F_n = (-1)^n \frac{2n+1}{m-1} \frac{\sum_1^m C_{x-1,n} C_{m-n-x-1,n} \Delta^n f(x)}{C_{m-1,n}},$$

$$C_{\alpha,\beta} = \frac{(\alpha+\beta)(\alpha+\beta-1)\dots(\alpha+1)}{1.2\dots\beta},$$

and the factors

$$\begin{array}{r} (1,1), (1,2), (1,3), \dots (1,l), \\ (2,2), (2,3), \dots (2,l), \\ (3,3), \dots (3,l), \\ \dots\dots\dots\dots \end{array}$$

being deduced successively by aid of the equality

$$(9) \qquad (\mu+1, n) = -\frac{n(n+1) - \mu(\mu+1)}{(\mu+1)(m-\mu-2)} (\mu, n),$$

where one puts

$$(0,1) = 1, \quad (0,2) = 1, \quad (0,3) = 1, \dots \quad (0,l) = 1.$$

§8. In order to show by an example the use of our formulas, we are going to apply them to the example which in the Ballistique of N. Majevsky has been calculated according to our old formulas. In this example the values of the variable x and the corresponding values of the function $f(x)$ are the following

$$
\begin{aligned}
x = 1; &\quad f(x) = 0.2020,\\
x = 2; &\quad f(x) = 0.1885,\\
x = 3; &\quad f(x) = 0.1645,\\
x = 4; &\quad f(x) = 0.1434,\\
x = 5; &\quad f(x) = 0.1176,\\
x = 6; &\quad f(x) = 0.0897,\\
x = 7; &\quad f(x) = 0.0581,\\
x = 8; &\quad f(x) = 0.0244,\\
x = 9; &\quad f(x) = -0.0122,
\end{aligned}
$$

By calculating according to these values of $f(x)$ its differences of diverse orders, we form the following table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	0.2020	-0.0135	-0.0105	$+0.0134$
2	0.1885	-0.0240	$+0.0029$	-0.0076
3	0.1645	-0.0211	-0.0047	$+0.0026$
4	0.1434	-0.0258	-0.0021	-0.0016
5	0.1176	-0.0279	-0.0037	$+0.0016$
6	0.0897	-0.0316	-0.0021	-0.0008
7	0.0581	-0.0337	-0.0029	
8	0.0244	-0.0366		
9	-0.0122			

Having $m = 10$, in this example, we will have for $n = 0$

$$C_{x-1,0}.C_{9-x,0} = 1; \quad C_{9,0} = 1; \quad F_0 = \frac{1}{9}\sum_1^{10} f(x);$$

for $n = 1$

$$C_{x-1,1}.C_{8-x,1} = \frac{x(9-x)}{1.1}; \qquad C_{9,1} = \frac{10}{1},$$

$$F_1 = -\frac{3}{9.10}\sum_1^{10}\frac{x(9-x)}{1.1}\Delta f(x);$$

for $n = 2$

$$C_{x-1,2}.C_{7-x,2} = \frac{(x+1)x}{1.2}\cdot\frac{(9-x)(8-x)}{1.2}, \quad C_{9,2} = \frac{11.10}{1.2},$$

$$F_2 = -\frac{5.2}{9.11.10}\sum\frac{(x+1)x}{1.2}\frac{(9-x)(8-x)}{1.2}\Delta^2 f(x);$$

for $n = 3$

$$C_{x-1,3}.C_{6-x,3} = \frac{(x+2)(x+1)}{1.2.3}\cdot\frac{(9-x)(8-x)(7-x)}{1.2.3}, \quad C_{9,3} = \frac{12.11.10}{1.2.3},$$

$$F_3 = -\frac{7.2.3}{9.12.11.10}\sum_1^{10}\frac{(x+2)(x+1)x(9-x)(8-x)(7-x)}{1.2.3.1.2.3}\Delta^3 f(x).$$

Determining according to the values given above of $f(x)$ and of its differences $\Delta f(x)$, $\Delta^2 f(x)$, $\Delta^3 f(x)$ the sums contained in the quantities

$$F_0, \quad F_1, \quad F_2, \quad F_3,$$

we find

$$\sum f(x) = 0.9760;$$

$$\sum \frac{x}{1} \cdot \frac{9-x}{1} \Delta f(x) = -3.2312;$$

$$\sum \frac{(x+1)x}{1.2} \cdot \frac{(9-x)(8-x)}{1.2} \Delta^2 f(x) = -1.2908;$$

$$\sum \frac{(x+2)(x+1)x}{1.2.3} \cdot \frac{(9-x)(8-x)(7-x)}{1.2.3} \Delta^3 f(x) = 0.0656;$$

this which gives after the substitution into the expressions of the quantities

$$F_0, \quad F_1, \quad F_2, \quad F_3,$$

the following values:

$$F_0 = \frac{1}{9} \cdot 0.9760 = 0.1084$$

$$F_1 = -\frac{3}{9.10} \cdot -3.2312 = 0.1077,$$

$$F_2 = \frac{5.2}{9.11.10} \cdot -1.2908 = -0.0130,$$

$$F_3 = -\frac{7.2.3}{9.12.11.10} \cdot 0.0656 = -0.0002.$$

By adding these quantities, we have according to (8) the value of $F(1)$ corresponding to the case where one is arrested at the 4th term in the expression (1) of the sought function or, that which reverts to the same, when one supposes its fourth differences equal to zero. Thus, one finds for the value of $F(1)$

$$F(1) = 0.1084 + 0.1077 - 0.0130 - 0.0002.$$

According to the same values of

$$F_0, \quad F_1, \quad F_2, \quad F_3,$$

we obtain by virtue of formulas (8) the differences

$$\Delta F(1), \quad \Delta^2 F(1), \quad \Delta^3 F(1)$$

by aid of the factors

$$\begin{array}{r} (1,1), (1,2), (1,3), \\ (2,2), (2,3), \\ (3,3). \end{array}$$

Now according to (9) for

$$m = 10, \quad \mu = 0, \quad n = 1, 2, 3,$$

by putting

$$(0,1) = 1, \quad (0,2) = 1, \quad (0,3) = 1,$$

we find

$$\begin{aligned}
(1,1) &= -\frac{1.2 - 0.1}{1.(10-2)} \cdot 1 = -\frac{1}{4}, \\
(1,2) &= -\frac{2.3 - 0.1}{1.(10-2)} \cdot 1 = -\frac{3}{4}, \\
(1,3) &= -\frac{3.4 - 0.1}{1.(10-2)} \cdot 1 = -\frac{3}{2}.
\end{aligned}$$

Therefore, according to (8), one passes from the value found above of $F(1)$ to the value of $\Delta F(1)$, by omitting the first term and multiplying the others respectively by

$$(1,1) = -\frac{1}{4}, \quad (1,2) = -\frac{3}{4}, \quad (1,3) = -\frac{3}{2}.$$

Thus, we find

$$\begin{aligned}
\Delta F(1) &= -\frac{1}{4} \cdot 0.1077 + \frac{3}{4} \cdot 0.0130 + \frac{3}{2} \cdot 0.0002 = \\
&= -0.0269 + 0.0097 + 0.0003.
\end{aligned}$$

In order to pass from this value of the first difference of $F(1)$ to the second, it is necessary for us to omit according to (8) the first term and to multiply the following respectively by the ratios

$$\frac{(2,2)}{(1,2)}, \qquad \frac{(2,3)}{(1,3)}.$$

Now, having according to (9) for $m = 10$

$$\frac{(2,2)}{(1,2)} = -\frac{2.3 - 1.2}{2(10-3)} = -\frac{2}{7}, \quad \frac{(2,3)}{(1,3)} = -\frac{3.4 - 1.2}{2(10-3)} = -\frac{5}{7},$$

we will have by virtue of that which precedes for the value of $\Delta^2 F(1)$

$$\Delta^2 F(1) = -\frac{2}{7} \cdot 0.0097 - \frac{5}{7} \cdot 0.0003 = -0.0028 - 0.0002.$$

By omitting there the first term and multiplying the second by the ratio

$$\frac{(3,3)}{(2,3)},$$

equal, by virtue of (9), to

$$-\frac{3.4 - 2.3}{3(10-4)} = -\frac{6}{3.6} = -\frac{1}{3},$$

we find

$$\Delta^3 F(1) = -\frac{1}{3} \cdot -0.0002 = 0.0001.$$

Thus, in order to evaluate the sought function, the fourth difference being supposed null, we will have

$$\begin{aligned}
F(1) &= 0.1084 + 0.1077 - 0.0130 - 0.0002 = 0.2029,\\
\Delta F(1) &= -0.0269 + 0.0097 + 0.0003 = -0.0169,\\
\Delta^2 F(1) &= -0.0028 - 0.0002 = -0.0030,\\
\Delta F(1) &= 0.0001.
\end{aligned}$$

§9. The formulas that we have deduced correspond to the case where all the factors

$$f(1), \quad f(2), \ldots, \quad f(m-1)$$

are supposed *equally good*, or that which reverts to the same, when their mean *quadratic* errors are equal.

In the case where these errors of quantities

$$f(1), \quad f(2), \ldots, \quad f(m-1)$$

are unequal or inversely proportional to the quantities

$$\theta(1), \quad \theta(2), \ldots \quad \theta(m-1),$$

formula (1) must be replaced, as one knows, by the following

$$f(x) = \frac{\sum_1^m \phi_0(x)\theta^2(x)f(x)}{\sum_1^m \phi_0^2(x)\theta^2(x)}\phi_0(x) + \frac{\sum_1^m \phi_1(x)\theta^2(x)f(x)}{\sum_1^m \phi_1^2(x)\theta^2(x)}\phi_1(x) + \cdots,$$

where

$$\phi_0(x), \quad \phi_1(x), \ldots$$

designating the denominators of the reductions of the sum

$$\sum \frac{\theta^2(\lambda)}{x-\lambda} = \frac{\theta^2(1)}{x-1} + \frac{\theta^2(2)}{x-2} + \cdots + \frac{\theta^2(m-1)}{x-m+1},$$

obtained by its development into continued fraction

$$\cfrac{C_0}{A_0x + B_0 + \cfrac{C_1}{A_1x + B_1 + \cfrac{C_2}{A_2x + B_2 + \cdots}}}$$

It is not with difficulty to show that the case where

$$\theta^2(x) = \frac{\Gamma(x+\alpha)}{\Gamma(x)}\frac{\Gamma(m-x+\beta+n)}{\Gamma(m-x)},$$

α, β being any constants, is able to be treated by aid of the formulas analogous to those that one has deduced for the case of $\theta(x) = 1$.

We note this that the fraction

$$\frac{\Delta^n \frac{\Gamma(x+\alpha)}{\Gamma(x-n)} \frac{\Gamma(m-x+\beta+n)}{\Gamma(m-x)}}{\frac{\Gamma(x+\alpha)}{\Gamma(x)} \frac{\Gamma(m-x+\beta)}{\Gamma(m-x)}}$$

is reduced to an entire function[3] of degree n and that by virtue of (2) by putting

$$\Phi(x) = \frac{\Gamma(x+\alpha)}{\Gamma(x-n)} \frac{\Gamma(m-x+\beta+n)}{\Gamma(m-x)}$$

one finds

$$\begin{aligned} &\sum_1^m F(x) \Delta^n \frac{\Gamma(x+\alpha)}{\Gamma(x-n)} \frac{\Gamma(m-x+\beta+n)}{\Gamma(m-x)} \\ &\qquad = (-1)^n \sum_1^m \frac{\Gamma(x+\alpha)}{\Gamma(x)} \frac{\Gamma(m-x+\beta)}{\Gamma(m-x)} \Delta^n F(x). \end{aligned} \tag{10}$$

By making here $F(x)$ successively equal to

$$\phi_0(x), \quad \phi_1(x), \dots \quad \phi_{n-l}(x)$$

and noting that for these values of $F(x)$ the difference

$$\Delta^n F(x)$$

is reduced to zero, we conclude that all the sums

$$\begin{aligned} &\sum_1^m \phi_0(x) \Delta^n \frac{\Gamma(x+\alpha)}{\Gamma(x-n)} \frac{\Gamma(m-x+\beta+n)}{\Gamma(m-x)}, \\ &\sum_1^m \phi_1(x) \Delta^n \frac{\Gamma(x+\alpha)}{\Gamma(x-n)} \frac{\Gamma(m-x+\beta+n)}{\Gamma(m-x)}, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &\sum_1^m \phi_{n-1}(x) \Delta^n \frac{\Gamma(x+\alpha)}{\Gamma(x-n)} \frac{\Gamma(m-x+\beta+n)}{\Gamma(m-x)} \end{aligned}$$

are equals to zero; therefore, by taking

$$\theta^2(x) = \frac{\Gamma(x+\alpha)}{\Gamma(x)} \frac{\Gamma(m-x+\beta)}{\Gamma(m-x)}$$

and putting for brevity

$$\frac{\Delta^n \frac{\Gamma(x+\alpha)}{\Gamma(x)} \frac{\Gamma(m-x+\beta+n)}{\Gamma(m-x)}}{\frac{\Gamma(x+\alpha)}{\Gamma(x)} \frac{\Gamma(m-x+\beta)}{\Gamma(m-x)}} = W_x,$$

[3] The expression of this function is indicated below.

we will have

$$\sum_1^m \phi_0(x) W_x \theta^2(x) = 0, \quad \sum_1^m \phi_1(x) W_x \theta^2(x) = 0, \dots \quad \sum_1^m \phi_{n-1}(x) W_x \theta^2(x) = 0.$$

Now, from these equalities, by proceedng as in §2 in the case of $\theta(x) = 1$, one concludes that

$$\phi_n(x) = W_x,$$

and consequently, according to our notations

$$\phi_n(x) = \frac{\Delta^n \frac{\Gamma(x+\alpha)}{\Gamma(x-n)} \frac{\Gamma(m-x+\beta+n)}{\Gamma(m-x)}}{\frac{\Gamma(x+\alpha)}{\Gamma(x)} \frac{\Gamma(m-x+\beta)}{\Gamma(m-x)}}. \tag{11}$$

Passing to the determination of the sums

$$\sum \phi_0(x)\theta^2(x) f(x), \quad \sum \phi_1(x)\theta^2(x) f(x), \quad \sum \phi_2(x)\theta^2(x) f(x), \dots,$$

$$\sum \phi_0^2(x)\theta^2(x), \quad \sum \phi_1^2(x)\theta^2(x), \quad \sum \phi_2^2(x)\theta^2(x), \dots,$$

we note that equality (10), after having substituted the value of

$$\Delta^n \frac{\Gamma(x+\alpha)}{\Gamma(x-n)} \frac{\Gamma(m-x+\beta+n)}{\Gamma(m-x)}$$

drawn from (11) and replaced

$$\frac{\Gamma(x+\alpha)}{\Gamma(x)} \frac{\Gamma(m-x+\beta)}{\Gamma(m-x)}$$

by

$$\theta^2(x),$$

gives

$$\sum_1^m F(x)\phi_n(x)\theta^2(x) = (-1)^n \sum_1^m \frac{\Gamma(x+\alpha+n)}{\Gamma(x)} \frac{\Gamma(m-x+\beta)}{\Gamma(m-x-n)} \Delta^n F(x).$$

By making here $F(x) = f(x)$, we find

$$\sum_1^m f(x)\phi_n(x)\theta^2(x) = (-1)^n \sum_1^m \frac{\Gamma(x+\alpha+n)}{\Gamma(x)} \frac{\Gamma(m-x+\beta)}{\Gamma(m-x-n)} \Delta^n f(x),$$

this which serves to facilitate the calculation of the sums

$$\sum_1^m f(x)\phi_0(x)\theta^2(x), \quad \sum_1^m f(x)\phi_1(x)\theta^2(x), \dots$$

In order to evaluate the sum

$$\sum_{1}^{m} \phi_n^2(x)\theta^2(x)$$

we will put into this equality

$$F(x) = \phi_n(x),$$

and formula (11) giving for one such value of $F(x)$

$$\Delta^n F(x) = \Delta^n \phi_n(x) = (-1)^n \frac{\Gamma(n+1)\Gamma(2n+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+1)},$$

we will have

$$\sum_{1}^{m} \phi_n^2(x)\theta^2(x) = \frac{\Gamma(n+1)\Gamma(2n+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+1)} \sum_{1}^{m} \frac{\Gamma(x+\alpha+n)\Gamma(m-x+\beta)}{\Gamma(x)\Gamma(m-x+n)}.$$

The function

$$\Gamma(m-x-n)$$

becoming infinite for

$$x = m-n, \quad m-n-1, \ldots, \quad m-1,$$

by virtue of which all the elements of the sum

$$\sum_{1}^{m} \frac{\Gamma(x+\alpha+n)}{\Gamma(x)} \frac{\Gamma(m-x+\beta)}{\Gamma(m-x-n)},$$

in departing from $x = m-n$ are reduced to zero, one will be able to take $m-n$ for upper limit of this sum, and it will be written as follows:

$$\sum_{1}^{m-n} \frac{\Gamma(x+\alpha+n)}{\Gamma(x)} \frac{\Gamma(m-x+\beta)}{\Gamma(m-x-n)}.$$

Now this sum is equal, according to (5), to

$$\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)} \frac{\Gamma(m+n+\alpha+\beta)}{\Gamma(m-n-1)}.$$

By substituting this value of the sum

$$\sum_{1}^{m-n} \frac{\Gamma(x+\alpha+n)}{\Gamma(x)} \frac{\Gamma(m-x+\beta)}{\Gamma(m-x-n)}$$

in the preceding formula, we find that the sum

$$\sum_{1}^{m} \phi_n^2(x)\theta^2(x)$$

is expressed in the following manner

$$\frac{\Gamma(n+1)\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(m+n+\alpha+\beta)}{(2n+\alpha+\beta+1)\Gamma(m-n-1)\Gamma(n+\alpha+\beta+1)}$$

§10. Passing to the determination of the functions

$$\phi_0(x), \quad \phi_1(x), \ldots$$

according to the formula (11), we note that in general the difference

$$\Delta^n U_x V_x$$

is reduced to the sum

$$V_{x+n}\Delta^n U_x + \frac{n}{1}\Delta V_{x+n-1}.\Delta^{n-1}U_x + \frac{n(n-1)}{1.2}\Delta^3 V_{x+n-2}.\Delta^{n-2}U_x + \cdots$$

that one is able further to represent under the form

$$\sum_{\lambda=0}^{\lambda=n+1}\frac{\Gamma(n+1)}{\Gamma(\lambda+1)\Gamma(n-\lambda+1)}\Delta^\lambda V_{x+n-\lambda}.\Delta^{n-1}U_x.$$

By putting

$$U_x = \frac{\Gamma(x+\alpha)}{\Gamma(x-n)}, \qquad V_x = \frac{\Gamma(m-x+\beta+n)}{\Gamma(m-x)},$$

we find that the difference

$$\Delta^n \frac{\Gamma(x+\alpha)}{\Gamma(x-n)}\frac{\Gamma(m-x+\beta+n)}{\Gamma(m-x)}$$

is equal to

$$\sum_{\lambda=0}^{\lambda=n+1}\frac{\Gamma(n+1)}{\Gamma(\lambda+1)\Gamma(n-\lambda+1)}\Delta^\lambda \frac{\Gamma(m+\beta+\lambda-x)}{\Gamma(m-n+\lambda-x)}\Delta^{n-\lambda}\frac{\Gamma(x+\alpha)}{\Gamma(x-n)},$$

and, having according to (6)

$$\Delta^\lambda \frac{\Gamma(m+\beta+\lambda-x)}{\Gamma(m-n+\lambda-x)} = (-1)^\lambda \frac{\Gamma(n+\beta+1)}{\Gamma(n+\beta-\lambda+1)}\frac{\Gamma(m+\beta-x)}{\Gamma(m-n+\lambda-x)},$$

$$\Delta^{n-\lambda}\frac{\Gamma(x+\alpha)}{\Gamma(x-n)} = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+\lambda+1)}\frac{\Gamma(x+\alpha)}{\Gamma(x-\lambda)};$$

one will have the value of the difference

$$\Delta^n \frac{\Gamma(x+\alpha)}{\Gamma(x-n)}\frac{\Gamma(m-x+\beta+n)}{\Gamma(m-x)}$$

the following expression

$$\sum_{\lambda=0}^{\lambda=n+1} \frac{(-1)^\lambda \Gamma(n+1)\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(m+\beta-x)\Gamma(x+\alpha)}{\Gamma(\lambda+1)\Gamma(n-\lambda+1)\Gamma(\alpha+\lambda+1)\Gamma(n+\beta-\lambda+1)\Gamma(m-n+\lambda-x)\Gamma(x-\lambda)}.$$

Dividing this value of the difference

$$\Delta^n \frac{\Gamma(x+\alpha)}{\Gamma(x-n)} \frac{\Gamma(m-x+\beta+n)}{\Gamma(m-x)}$$

by

$$\frac{\Gamma(x+\alpha)}{\Gamma(x)} \frac{\Gamma(m-x+\beta)}{\Gamma(m-x)},$$

one has the quotient

$$\sum_{\lambda=0}^{\lambda=n+1} \frac{(-1)^\lambda \Gamma(n+1)\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(\lambda+1)\Gamma(n-\lambda+1)\Gamma(\alpha+\lambda+1)\Gamma(n+\beta-\lambda+1)} \frac{\Gamma(m-x)}{\Gamma(m-n+\lambda-x)} \frac{\Gamma(x)}{\Gamma(x-\lambda)}.$$

Now, according to (11), this expression is equal to the function $\phi_n(x)$.
Besides, having

$$\frac{\Gamma(m-x)}{\Gamma(m-n+\lambda-x)} = (m-x-1)(m-x-2)\dots(m-n+\lambda-x),$$

$$\frac{\Gamma(x)}{\Gamma(x-\lambda)} = (x-1)(x-2)\dots(x-\lambda),$$

all the terms of this sum represent some polynomials of degree n.