Sur une série qui fournit les valeurs extrêmes des intégrales, lorsque la fonction sous le signe est décomposée en deux facteurs*

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§1. In a Note under the title: *Sur le développement des fonctions à une seule variable*, we have indicated many series for the development of functions, which result from the general formula of interpolation by the method of least squares that we have given in the Memoir under the title: *Sur les fractions continues*. The terms of these series are composed of polynomials determined by the development into continued fraction of an integral of the form

$$\int_a^b \frac{\theta^2(z)}{x-z}dz;$$

the denominators of the reductions which are obtained in one such development are justly those polynomials according to which the functions are developed into series, of which there was question in our Note.

We are going to show now a series of another kind containing the same polynomials. This series does not give the approximate values of the functions under the form of the polynomials, as the ancients did it, but it furnishes some approximate expressions, with some complementary terms, of definite integrals, these expressions being formed of some simpler integrals under a certain regard, namely: for the evaluation of an integral of the form

$$\int_a^b f_0(x)f_1(x)\theta^2(x)\,dx,$$

where figures a product of three functions

$$f_0(x), \quad f_1(x), \quad \theta^2(x),$$

*"On a series which furnishes the extreme values of integrals, when the function under the sign is decomposed into two factors," Translated from the Russian into French by C. A. Possé. Translated from French by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. November 11, 2011

one obtains an approximate expression composed of integrals where figure under the sign of integration separately the functions

$$f_0(x)\theta^2(x), \quad f_1(x)\theta^2(x), \quad \theta^2(x),$$

multiplied by the polynomials mentioned above.

§2. This series, just as its complementary term, are deduced easily by considering the multiple integral

$$T = \int P_0 S_0 S_1 \theta_0^2 \, dx_0 dx_1 \ldots dx_n, \tag{1}$$

the functions

$$\theta_0, \quad P_0, \quad S_0, \quad S_1$$

being determined by the formulas

$$\left\{ \begin{array}{l} \theta_0 = \theta(x_0)\theta(x_1)\ldots\theta(x_n), \\ \phi(x) = (x-x_0)(x-x_1)\ldots(x-x_n), \\ S_0 = \frac{f_0(x_0)}{\phi'(x_0)} + \frac{f_0(x_1)}{\phi'(x_1)} + \cdots + \frac{f_0(x_n)}{\phi'(x_n)}, \\ S_1 = \frac{f_1(x_0)}{\phi'(x_0)} + \frac{f_1(x_1)}{\phi'(x_1)} + \cdots + \frac{f_1(x_n)}{\phi'(x_n)}, \\ P_0 = \phi'(x_0)\phi'(x_1)\ldots\phi'(x_n). \end{array} \right. \tag{2}$$

The last equality, after the substitution of the values of $\phi'(x_0), \phi'(x_1), \ldots \phi'(x-n)$, is reduced to that which follows:

$$P_0 = \pm[(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)\ldots(x_{n-1}-x_n)]^2. \tag{3}$$

The limits of all the variables in the integrals that we will consider are the same, namely: a and b.

Noting according to the structure of the functions S_0, S_1 that their product is equal to a sum of terms of the form:

$$\frac{f_0(x_i)}{\phi'(x_i)} \frac{f_1(x_k)}{\phi'(x_k)},$$

where

$$\begin{array}{rcl} i & = & 0, 1, \ldots n, \\ k & = & 0, 1, \ldots n, \end{array}$$

we conclude that integral (1) is decomposed into a sum of integrals:

$$\sum \int \frac{f_0(x_i)}{\phi'(x_i)} \frac{f_1(x_k)}{\phi'(x_k)} P_0 \theta_0^2 \, dx_0 dx_1 \ldots dx_n.$$

This sum contains terms of two kinds, namely: 1) those in which $i = k$, 2) those in which i differs from k.

The indices i and k having in this sum all the values from 0 to n, it will have $n+1$ terms of the first kind and $(n+1)n$ terms of the second kind. Now, according to the symmetry with respect to the variables

$$x_0, x_1, \dots x_n,$$

the terms of the first kind will have the common value

$$\int \frac{f_0(x_0)}{\phi'(x_0)} \frac{f_1(x_0)}{\phi'(x_0)} P_0 \theta_0^2 \, dx_0 dx_1 \dots dx_n,$$

representing the term which corresponds to $i=0$, $k=0$, and the terms of the second kind will have the common value

$$\int \frac{f_0(x_0)}{\phi'(x_0)} \frac{f_1(x_1)}{\phi'(x_1)} P_0 \theta_0^2 \, dx_0 dx_1 \dots dx_n,$$

representing the term which corresponds to $i=0$, $k=1$; therefore the integral (1) that we will consider is decomposed in the following manner:

$$(4) \qquad \left\{ \begin{array}{ll} T = & (n+1) \int \frac{f_0(x_0)}{\phi'(x_0)} \frac{f_1(x_0)}{\phi'(x_0)} P_0 \theta_0^2 \, dx_0 dx_1 \dots dx_n \\ & +(n+1)n \int \frac{f_0(x_0)}{\phi'(x_0)} \frac{f_1(x_1)}{\phi'(x_1)} P_0 \theta_0^2 \, dx_0 dx_1 \dots dx_n. \end{array} \right.$$

§3. In order to simplify the first of the integrals which enter into the second member of this equality we will introduce some new functions θ_1, Φ, P_1, by putting

$$(5) \qquad \left\{ \begin{array}{l} \theta(x_1)\theta(x_2)\dots\theta(x_n) = \theta_1, \\ (x-x_1)(x-x_2)\dots(x-x_n) = \Phi(x), \\ \Phi'(x_1)\Phi'(x_2)\dots\Phi'(x_n) = P_1. \end{array} \right.$$

Comparing these equalities to the equalities (2) let us note that

$$(6) \qquad \left\{ \begin{array}{l} \theta_0 = \theta_1 \cdot \theta(x_0), \\ \phi(x) = (x-x_0)\Phi(x). \end{array} \right.$$

If one differentiates the last equality with respect to x, we will have

$$\phi'(x) = \Phi(x) + (x-x_0)\Phi'(x);$$

whence, by making

$$x = x_0, x_1, x_2, \dots x_n,$$

and noting that according to (5) the values $x = x_1$, x_2,$\dots x_n$ annul the function $\Phi(x)$, we deduce

$$(7) \quad \phi'(x_0) = \Phi(x_0), \quad \phi'(x_1) = (x_1 - x_0)\Phi'(x_1), \dots \quad \phi'(x_n) = (x_n - x_0)\Phi'(x_n).$$

By multiplying these equalities, we find:

$$\phi'(x_0)\phi'(x_1)\phi'(x_2)\dots\phi'(x_n) =$$

$$(x_1 - x_0)(x_2 - x_0)\cdots(x_n - x_0)\Phi'(x_0)\Phi'(x_1)\Phi'(x_2)\dots\Phi'(x_n).$$

As one has according to (5)

$$(x_1 - x_0)(x_2 - x_0)\cdots(x_n - x_0) = (-1)^n\Phi(x_0),$$

$$\Phi'(x_1)\Phi'(x_2)\dots\Phi'(x_n) = P_1,$$

and according to (2)

$$\Phi'(x_0)\Phi'(x_1)\Phi'(x_2)\dots\Phi'(x_n) = P_0,$$

the equality obtained gives us

$$P_0 = (-1)^n\Phi^2(x_0)P_1. \tag{8}$$

Substituting the values of P_0, θ_0, $\phi'(x_0)$ drawn from (8), (6), (7) in the integral

$$\int \frac{f_0(x_0)}{\phi'(x_0)}\frac{f_1(x_1)}{\phi'(x_1)}P_0\theta_0^2\,dx_0dx_1\dots dx_n,$$

we note that it is reduced to the following:

$$\int (-1)^n f_0(x_0)f_1(x_0)P_1\theta^2(x_0)\theta_1^2\,dx_0dx_1dx_2\dots dx_n.$$

The functions P_1, θ_1 not containing according to (5) the variable x_0, this integral is decomposed into the following two factors:

$$\int (-1)^n P_1\theta_1^2\,dx_1dx_2\dots dx_n \cdot \int f_0(x_0)f_1(x_0)\theta^2(x_0)\theta_1^2\,dx_0.$$

By virtue of that, designating by C the value of the integral

$$\int (-1)^n P_1\theta_1^2\,dx_1dx_2\dots dx_n,$$

independent of the functions $f_0(x)f_1(x)$, we obtain for the determination of the first of the integrals contained in equality (4) the formula:

$$\int \frac{f_0(x_0)}{\phi'(x_0)}\frac{f_1(x_1)}{\phi'(x_1)}P_0\theta_0^2\,dx_0dx_1\dots dx_n = C\int f_0(x_0)f_1(x_0)\theta^2(x_0)\theta_1^2\,dx_0. \tag{9}$$

§4. Passing to the simplification of the integral

$$\int \frac{f_0(x_0)}{\phi'(x_0)}\frac{f_1(x_1)}{\phi'(x_1)}P_0\theta_0^2\,dx_0dx_1dx_2dx_3\dots dx_n,$$

we will introduce yet three new functions, by putting

$$\begin{cases} \theta(x_2)\theta(x_3)\dots\theta(x_n) = \theta_2, \\ (x - x_2)(x - x_3)\dots(x - x_n) = \Phi_1(x), \\ \Phi_1'(x_2)\Phi'(x_3)\dots\Phi_1'(x_n) = P_2. \end{cases} \tag{10}$$

Comparing the first two of the equalities (10) to the corresponding equalities (5), one deduces

$$\left\{\begin{array}{l}\theta_1 = \theta(x_1)\theta_2, \\ \Phi(x) = (x - x_1)\Phi_1(x).\end{array}\right. \tag{11}$$

This last equality being differentiated with respect to x, gives

$$\Phi'(x) = \Phi_1(x) + (x - x_1)\Phi_1'(x);$$

whence, by putting

$$x = x_1, x_2, \dots x_n$$

and noting that the function $\Phi_1(x)$ is annulled according to (10) for $x = x_2, x_3, \dots x_n$, we obtain

$$\begin{aligned}\Phi'(x_1) = \Phi_1(x_1),\ \Phi'(x_2) = (x_2 - x_1)\Phi_1'(x_2), \dots,\\ \Phi'(x_n) = (x_n - x_1)\Phi_1'(x_1);\end{aligned} \tag{12}$$

Multiplying these equalities, we find

$$\Phi'(x_1)\Phi'(x_2)\Phi'(x_3)\dots\Phi'(x_n) =$$

$$(x_2 - x_1)(x_2 - x_1)\dots(x_n - x_1)\Phi_1'(x_1)\Phi_1'(x_2)\Phi_1'(x_3)\dots\Phi_1'(x_n);$$

whence one draws, by virtue of the equalities (5), (10), the following expression of the function P_1:

$$P_1 = (-1)^{n-1}\Phi_1^2(x_1)P_2;$$

and consequently equality (8) gives us

$$P_0 = -\Phi^2(x_0)\Phi_1^2(x_1)P_2.$$

By substituting according to (12) $\Phi'(x_1)$ instead of $\Phi_1(x_1)$ and replacing according to (7) the function $\Phi(x_0)$ by $\phi'(x_0)$ and the function $\Phi'(x_1)$ by $\frac{\phi'(x_1)}{x_1 - x_0}$, we obtain

$$P_0 = -\left(\frac{\phi'(x_0)\phi'(x_1)}{x_1 - x_0}\right)^2 P_2.$$

Now by putting this value of P_0 into the integral

$$\int \frac{f_0(x_0)}{\phi'(x_0)}\frac{f_1(x_1)}{\phi'(x_1)}P_0\theta_0^2\, dx_0dx_1dx_2dx_3\dots dx_n$$

and by replacing there according to (6) the function θ_0 by the product $\theta(x_0)\theta_1$ and the function θ_1 according to (11) by the product $\theta(x_0)\theta_2$, we find that this integral is reduced to the following:

$$-\int f_0(x_0)f_1(x_1)\theta^2(x_0)\theta^2(x_1)\frac{\phi'(x_0)\phi'(x_1)}{(x_1 - x_0)^2}P_2\theta_2^2\, dx_0dx_1\dots dx_n,$$

that which one is able also to represent by

$$-\int f_0(x_0)f_1(x_1)F(x_0,x_1)\theta^2(x_0)\theta^2(x_1)\,dx_0dx_1,$$

where one has designated by

$$F(x_0,x_1)$$

the function of the two variables x_0, x_1, determined by the equality

$$F(x_0,x_1)=\int \frac{\phi'(x_0)\phi'(x_1)}{(x_1-x_0)^2}P_2\theta_2^2\,dx_2dx_3\dots dx_n. \tag{13}$$

§5. By virtue of these transformations of the integrals which enter into the second member of equality (4), it is reduced to this here

$$\left\{\begin{array}{rl} T= & (n+1)C\int f_0(x_0)f_1(x_0)\theta^2(x_0)\,dx_0 \\ & -(n+1)n\int f_0(x_0)f_1(x_1)F(x_0,x_1)\theta^2(x_0)\theta^2(x_1)\,dx_0dx_1, \end{array}\right. \tag{14}$$

T being according to (1) the value of the integral

$$\int P_0S_0S_1\theta_0^2\,dx_0dx_1\dots dx_n.$$

We are going to show now how are obtained the limits between which is found contained the value of this integral.

According to (2) the functions S_0, S_1 are determined by the equalities:

$$\begin{aligned} S_0 &= \frac{f_0(x_0)}{\phi'(x_0)}+\frac{f_0(x_1)}{\phi'(x_1)}+\dots+\frac{f_0(x_n)}{\phi'(x_n)}, \\ S_1 &= \frac{f_1(x_0)}{\phi'(x_0)}+\frac{f_1(x_1)}{\phi'(x_1)}+\dots+\frac{f_1(x_n)}{\phi'(x_n)}. \end{aligned}$$

By putting here according to (7)

$$\Phi(x_0),\quad (x_1-x_0)\Phi'(x_1),\dots\quad (x_n-x_0)\Phi'(x_n)$$

instead of

$$\phi'(x_0),\quad \phi'(x_1),\dots\quad \phi'(x_n),$$

we note that these equalities are able to be represented under the form

$$\left\{\begin{array}{l} S_0=\frac{1}{\Phi(x_0)}\left[f_0(x_0)-\frac{\Phi(x_0)f_0(x_1)}{(x_0-x_1)\Phi'(x_1)}-\dots-\frac{\Phi(x_0)f_0(x_n)}{(x_0-x_n)\Phi'(x_n)}\right], \\ S_1=\frac{1}{\Phi(x_0)}\left[f_1(x_0)-\frac{\Phi(x_0)f_1(x_1)}{(x_0-x_1)\Phi'(x_1)}-\dots-\frac{\Phi(x_0)f_1(x_n)}{(x_0-x_n)\Phi'(x_n)}\right]. \end{array}\right. \tag{15}$$

If one considers the first of these equalities, where according to (5)

$$\Phi(x)=(x-x_1)(x-x_2)\dots(x-x_n),$$

one notes that the expression contained between the brackets [ ] represents the the difference between the value of the function $f_0(x)$ for $x = x_0$ and the value that the formula of interpolation of Lagrange gives for the determination of $f_0(x_0)$ according to the values of $f_0(x)$ for $x = x_1, x_2, \ldots x_n$.

The ratio of this difference to the value of

$$\frac{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)}{1.2.\ldots n},$$

do not exit, as one knows, from the limits in which the derivative remains contained

$$\frac{d^n f_0(x)}{dx^n}$$

for $x = x_0, x_1, x_2, \ldots x_n$ and for the intermediate values.

Noting besides, according to that which has been admitted in regard to the limits of integr ation, that all these values are found between a and b, we conclude that this ratio is equal to a certain quantity M_0, mean between the values of the derivative

$$\frac{d^n f_0(x)}{dx^n}$$

within the limits a and b. Consequently, according to equality (15), we will have

$$S_0 = \frac{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)}{1.2.\ldots n}\frac{M_0}{\Phi(x_0)}.$$

By replacing here according to (5) the product

$$(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)$$

by $\Phi(x_0)$, we find the expression of S_0 which, being simplified, is reduced to the following:

$$S_0 = \frac{M_0}{1.2\ldots n}.$$

One finds in an analogous manner

$$S_1 = \frac{M_1}{1.2\ldots n},$$

M_1 being a mean among the values of the derivative

$$\frac{d^n f_1(x)}{dx^n}$$

in the interval $x = a$, $x = b$.

By virtue of the equalities that we have deduced in regard to the values of the functions S_0, S_1 in the integral

$$T = \int S_0 S_1 P_0 \theta_0^2 \, dx_0 dx_1 \ldots dx_n,$$

and noting that the factor $P_0\theta_0^2$ according to (3) does not change sign, we conclude that for certain values M_0, M_1 not exiting from the limits between which the derivatives remain

$$\frac{d^n f_0(x)}{dx^n}, \quad \frac{d^n f_1(x)}{dx^n}$$

in the interval from $x = a$ to $x = b$, the following equality will take place:

$$T = \frac{M_0 M_1}{(1.2\ldots n)^2}\int P_0\theta_0^2\,dx_0dx_1\ldots dx_n. \tag{16}$$

§6. Substituting this value of T into equality (14) we obtain the equation

$$\frac{M_0M_1}{(1.2\ldots n)^2}\int P_0\theta_0^2\,dx_0dx_1\ldots dx_n = (n+1)C\int f_0(x_0)f_1(x_0)\theta^2(x_0)\,dx_0$$
$$-(n+1)n\int f_0(x_0)f_1(x_1)F(x_0,x_1)\theta^2(x_0)\theta^2(x_1)\,dx_0dx_1,$$

whence there results the following formula for the evaluation of the integral $\int f_0(x_0)f_1(x_0)\theta^2(x_0)\,dx_0$:

$$\begin{aligned}\int f_0(x_0)f_1(x_0)\theta^2(x_0)\,dx_0 = \\ \int f_0(x_0)f_1(x_1)\frac{nF(x_0,x_1)}{C}\theta^2(x_0)\theta^2(x_1)\,dx_0dx_1 \\ +\frac{M_0M_1\int P_0\theta_0^2\,dx_0dx_1\ldots dx_n}{(1.2\ldots n)^2(n+1)C}.\end{aligned} \tag{17}$$

It is from this formula that we draw the series which makes the object of this Note, by developing the function

$$\frac{n}{C}F(x_0,x_1)$$

following the terms composed of the polynomials which are obtained, as we have said in §1, by means of the development of the integral

$$\int\frac{\theta^2(z)\,dz}{x-z}$$

into continued fraction and that we will designate by

$$\psi_0(x), \quad \psi_1(x), \quad \psi_2(x), \ldots$$

According to the known property of these polynomials, by virtue of formula (17), it is easy to make one such development from the function

$$\frac{n}{C}F(x_0,x_1)$$

without that it be necessary to determine the value of the integral

$$\int\frac{\phi'(x_0)\phi'(x_1)}{(x_0-x_1)^2}P_2\theta_2^2\,dx_2dx_3\ldots dx_n,$$

which gives the expression of the function

$$F(x_0, x_1)$$

according to equation (13).

We make note only according to this equation that $F(x_0, x_1)$ is an entire function of degree inferior to n, as much with respect to x_0 as with respect to x_1, as that results from this that according to (2) the product $\phi'(x_0)\phi'(x_1)$ is divisible by $(x_0 - x_1)^2$ and contains some powers neither of x_0, nor of x_1 superior to $n+1$. According to the property of polynomials

$$\psi_0(x), \quad \psi_1(x), \quad \psi_2(x), \ldots$$

each power of x inferior to n being able to be represented by the sum

$$k_0\psi_0(x) + k_1\psi_1(x) + \cdots + k_{n-1}\psi_{n-1}(x),$$

the function

$$\frac{n}{C}F(x_0, x_1),$$

according to that which one has noted above in regard to the function $F(x_0, x_1)$, will be able to be represented by the sum

$$\sum C_{\lambda,\mu}\psi_\lambda(x_0)\psi_\mu(x_1),$$

λ and μ remaining less than n.

By putting this sum in place of

$$\frac{n}{C}F(x_0, x_1)$$

in formula (17), we obtain the equality

$$\begin{aligned}\int f_0(x_0)f_1(x_0)\theta^2(x_0)\,dx_0 \\ = \int f_0(x_0)f_1(x_1)\sum C_{\lambda,\mu}\psi_\lambda(x_0)\psi_\mu(x_1)\theta^2(x_0)\theta^2(x_1)\,dx_0dx_1 \\ + \frac{M_0M_1\int P_0\theta_0^2\,dx_0dx_1\ldots dx_n}{(1.2\ldots n)^2(n+1)C},\end{aligned}$$

which, as it is easy to understand, is able to be represented as follows:

$$\begin{aligned}\int f_0(x)f_1(x)\theta^2(x)\,dx \\ = \sum C_{\lambda,\mu}\int f_0(x)\psi_\lambda(x)\theta^2(x)\,dx\int f_1(x)\psi_\mu(x)\theta^2(x)\,dx \\ + \frac{M_0M_1\int P_0\theta_0^2\,dx_0dx_1\ldots dx_n}{(1.2\ldots n)^2(n+1)C},\end{aligned} \tag{18}$$

By virtue of the known property of the polynomials

$$\psi_0(x), \quad \psi_1(x), \quad \psi_2(x), \dots$$

to satisfy the equation

$$\int \psi_i(x)\psi_k(x)\theta^2(x)\,dx = 0 \tag{19}$$

for i different from k, it is not difficult to find the value of the coefficients

$$C_{\lambda,\mu},$$

which figure in this formula. In fact, if one puts

$$f_0(x) = \psi_l(x), \quad f_1(x) = \psi_m(x),$$

where

$$l < n, \quad m < n,$$

the derivatives

$$\frac{d^n f_0(x)}{dx^n} = \frac{d^n \psi_l(x)}{dx^n}, \qquad \frac{d^n f_1(x)}{dx^n} = \frac{d^n \psi_m(x)}{dx^n}$$

will be equal to zero. Hence, according to that which has been said in §5 in regard to the quantities M_0, M_1, these here will also be equal to zero; therefore, for such values of the functions $f_0(x)$, $f_1(x)$, formula (18) will be reduced to the equality

$$\int \psi_l(x)\psi_m(x)\theta^2(x)\,dx = \sum C_{\lambda,\mu} \int \psi_l(x)\psi_\lambda(x)\theta^2(x)\,dx \int \psi_m(x)\psi_\mu(x)\theta^2(x)\,dx.$$

Noting according to (19) that the integrals

$$\int \psi_l(x)\psi_\lambda(x)\theta^2(x)\,dx,$$

$$\int \psi_m(x)\psi_\mu(x)\theta^2(x)\,dx$$

are reduced to zero when $\lambda \gtrless l$, $\mu \gtrless m$, we conclude that in the sum

$$\sum C_{\lambda,\mu} \int \psi_l(x)\psi_\lambda(x)\theta^2(x)\,dx \int \psi_m(x)\psi_\mu(x)\theta^2(x)\,dx$$

all the terms, with the exception of the one which corresponds to

$$\lambda = l, \qquad \mu = m,$$

vanish; by virtue of which the equality deduced gives us

$$\int \psi_\lambda(x)\psi_\mu(x)\theta^2(x)\,dx = \sum C_{\lambda,\mu} \int \psi_\lambda^2(x)\theta^2(x)\,dx \cdot \int \psi_\mu^2(x)\theta^2(x)\,dx;$$

whence there results for the determination of the coefficient $C_{\lambda,\mu}$ the following formula

$$C_{\lambda,\mu} = \frac{\int \psi_\lambda(x)\psi_\mu(x)\theta^2(x)\,dx}{\int \psi_\lambda^2(x)\theta^2(x)\,dx \cdot \int \psi_\mu^2(x)\theta^2(x)\,dx}.$$

By making $\lambda = \mu$, we find

$$C_{\lambda,\lambda} = \frac{1}{\int \psi_\lambda^2(x)\theta^2(x)\,dx};$$

while for $\lambda \begin{smallmatrix} > \\ < \end{smallmatrix} l$ according to equality (19) one sees that

$$C_{\lambda,\mu} = 0;$$

It follows that the sum

$$\sum C_{\lambda,\mu} \int f_0(x)\psi_\lambda(x)\theta^2(x)\,dx \int f_1(x)\psi_\mu(x)\theta^2(x)\,dx$$

contains only the terms in which $\lambda = \mu$ and that in these terms the coefficient $C_{\lambda,\mu}$ has the value

$$\frac{1}{\int \psi_\lambda^2(x))\theta^2(x)\,dx};$$

consequently equality (18) is reduced to the following

$$\begin{aligned} \int f_0(x)f_1(x)\theta^2(x)\,dx \\ = \sum \frac{\int f_0(x)\psi_\lambda(x)\theta^2(x)\,dx \int f_1(x)\psi_\lambda(x)\theta^2(x)\,dx}{\int \psi_\lambda^2(x))\theta^2(x)\,dx} \\ + \frac{M_0M_1 \int P_0\theta_0^2\,dx_0dx_1\ldots dx_n}{(1.2\ldots n)^2(n+1)C}, \end{aligned} \tag{20}$$

where the summation extends, according to that which we have noted above, to the following values of λ:

$$\lambda = 0, 1, 2, \ldots n-1.$$

§7. One is able also to find without difficulty the value of the expression

$$\frac{P_0\theta_0^2\,dx_0dx_1dx_2\ldots dx_n}{(1.2\ldots n)^2(n+1)C},$$

contains in the last term of the equality deduced.

One arrives there by putting there

$$f_0(x) = \psi_n(x), \quad f_1(x) = \psi_n(x).$$

The derivatives

$$\frac{d^n f_0(x)}{dx^n}, \qquad \frac{d^n f_1(x)}{dx^n}$$

being reduced into this case by the constant quantity, equal to $\psi_n^{(n)}(0)$, the quantities M_0, M_1 according to that which we have said in §5 will be equal also to $\psi_n^{(n)}(0)$; consequently for

$$f_0(x) = \psi_n(x), \qquad f_1(x) = \psi_n(x)$$

formula (20) will give

$$\int \psi_n^2(x)\theta^2(x)\,dx = \sum \frac{[\int \psi_n(x)\psi_\lambda(x)\theta^2(x)\,dx]^2}{\int \psi_\lambda^2(x))\theta^2(x)\,dx} + \frac{[\psi_n^{(n)}(0)]^2 \int P_0\theta_0^2\,dx_0 dx_1 \dots dx_n}{(1.2\dots n)^2(n+1)C},$$

The number λ being less than n, according to that which one has seen in the preceding §, the integrals of the form

$$\int \psi_n(x)\psi_\lambda(x)\theta^2(x)\,dx,$$

which figure under the sign of the sum are reduced to zero by virtue of (19) and we find

$$\int \psi_n^2(x)\theta^2(x)\,dx = \frac{[\psi_n^{(n)}(0)]^2 \int P_0\theta_0^2\,dx_0 dx_1 \dots dx_n}{(1.2\dots n)^2(n+1)C}.$$

This equality gives us

$$\frac{\int P_0\theta_0^2\,dx_0 dx_1 \dots dx_n}{(1.2\dots n)^2(n+1)C} = \frac{\int \psi_n(x)\psi_\lambda(x)\theta^2(x)\,dx}{[\psi_n^{(n)}(0)]^2},$$

this which, being substituted into formula (20), reduces it to the following

$$\int f_0(x)f_1(x)\theta^2(x)\,dx = \sum \frac{\int f_0(x)\psi_\lambda(x)\theta^2(x)\,dx \int f_1(x)\psi_\lambda(x)\theta^2(x)\,dx}{\int \psi_\lambda^2(x))\theta^2(x)\,dx} + \frac{M_0M_1}{[\psi_n^{(n)}(0)]^2}\int \psi_n^2(x_0)\theta^2(x_0)\,dx_0,$$

where the sum represents n terms of the series

$$\begin{array}{ll} & \dfrac{\int f_0(x)\psi_0(x)\theta^2(x)\,dx \cdot \int f_1(x)\psi_0(x)\theta^2(x)\,dx}{\int \psi_0^2(x)\theta^2(x)\,dx} \\ + & \dfrac{\int f_0(x)\psi_1(x)\theta^2(x)\,dx \cdot \int f_1(x)\psi_1(x)\theta^2(x)\,dx}{\int \psi_1^2(x)\theta^2(x)\,dx} \\ & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ + & \dfrac{\int f_0(x)\psi_{n-1}(x)\theta^2(x)\,dx \cdot \int f_1(x)\psi_{n-1}(x)\theta^2(x)\,dx}{\int \psi_{n-1}^2(x)\theta^2(x)\,dx} \end{array}$$

which gives the approximate value of the integral

$$\int f_0(x)f_1(x)\theta^2(x)\,dx,$$

and the expression

$$\frac{M_0 M_1}{[\psi_n^{(n)}(0)]^2} \int \psi_n^2(x_0)\theta^2(x_0)\, dx_0$$

represents its complementary term.

Noting according to that which one has seen in §5 that the numerical value of the quantities M_0, M_1 does not surpass the greatest numerical value of the derivatives

$$\frac{d^n f_0(x)}{dx^n}, \qquad \frac{d^n f_1(x)}{dx^n}$$

within the limits of integration and designating these numerical values maxima by A and B, we conclude that the numerical value of the complementary term

$$\frac{M_0 M_1}{[\psi_n^{(n)}(0)]^2} \int \psi_n^2(x_0)\theta^2(x_0)\, dx_0$$

not surpass the value of

$$\frac{AB}{[\psi_n^{(n)}(0)]^2} \int \psi_n^2(x_0)\theta^2(x_0)\, dx_0.$$

As for the sign of the complementary term it is determined easily in the case where the derivatives

$$\frac{d^n f_0(x)}{dx^n}, \qquad \frac{d^n f_1(x)}{dx^n}$$

do not change in sign within the limits of integration.

In this case, according to §5, the quantities M_0, M_1 will have the signs of the derivatives

$$\frac{d^n f_0(x)}{dx^n}, \qquad \frac{d^n f_1(x)}{dx^n}$$

within the limits of integration, and hence the complementary term

$$\frac{M_0 M_1}{[\psi_n^{(n)}(0)]^2} \int \psi_n^2(x_0)\theta^2(x_0)\, dx_0,$$

seeing that the quantities

$$\int \psi_n^2(x_0)\theta^2(x_0)\, dx_0, \qquad [\psi_n^{(n)}(0)]^2,$$

are evidently positive, will have the sign $+$ or $-$ according as these derivatives conserve some signs equal or contrary within the limits of integration.