MÉMOIRE
SUR
LE CALCUL DES PROBABILITÉS*

M. le MARQUIS DE CONDORCET

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FOURTH PART
Reflections on the method to determine the Probability
of future events, after the Observation
of past events.

This part of the Analysis which teaches to determine the probability of future events, according to the order which they have followed the past events of the same kind which one has observed, is susceptible to a great number of useful & curious applications: I have believed in consequence that it would not be unuseful to examine the principles on which this Analysis is founded; such is the object of the following Reflections.

I.

Let two events A & N be that I suppose simply contradictories, that is to say, not able to subsist together & necessarily taking place the one or the other. If A has taken place m times & N n times, & if one demands the probability to have in $p+q$ events, p events A & q events N, the probability demanded will be

$$\frac{p+q\cdots q+1}{1.2\ldots q}\cdot\frac{\int x^{m+p}(1-x)^{n+q}\partial x}{\int x^m(1-x)^n\partial x},$$

this integral being taken from $x=1$ to $x=0$; such is the general rule.

II.

One sees first that this law expresses really the probability only under the two following hypotheses.

1.° If the probability of events A & N remain the same in every sequence of events; this is evident by the same formula which expresses the law.

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2.$^\circ$ In the case where this same probability is variable, but where one would suppose at the same time that the value of the probability, although it can be different for each event, is however taken at random for each, according to a certain general probability x for A, & $1 - x$ for N.

We suppose, for example, a sequence of urns which contain some white & some black balls: it is clear that one can suppose that there exists in each urn a like number of white balls & of black balls; & this is the first hypothesis. One can suppose also that one has refilled the urns, by drawing some balls at random from another urn which contained a certain number of white & black balls. In this last case which represents the second hypothesis, the ratio of the number of white balls to that of the black balls is not necessarily the same in all the urns, but solely the mean value of this ratio is the same in each, & is equal to that of the same ratio for the urn of which all the balls have been drawn. If therefore one supposes the number of balls infinite, the formula above will be acceptable equally in the two hypotheses.

If one applies this method to some real cases, that is to say, to some natural events, one sees first that each event in itself is determined by one law, as the drawing of one ball would be equally; that thus in one & the other case, which we call *probability*, is only the ratio of the number of combinations which bring forth an event to the one of the combinations which do not bring it forth; combinations that our ignorance makes us regard as equally possible.

Thus the first hypothesis consists to suppose that the ratio between these equally possible combinations, remains the same for all these events; & the second consists in regarding this ratio as variable, but determined in a manner that the mean value of the possible ratios is always the same for each event.

The results will be the same, because these two hypotheses differ really only with respect to me, who, in the first case, regards all the events as equally probable, & who, knowing in the second that they must not be, but not knowing the law according to which their probability varies, supposes it dependent similarly on a like general law.

But under every other hypothesis, the formula above can not be regarded as giving some rigorous results; & it is necessary to examine if there is not among these hypotheses & that which supposes all the events independent, some other supposition which is proper to represent the probability, in a manner more true in a part of the questions which one can have to resolve; alternatively it would suffice, instead of employing without restriction the preceding method, to follow that which I have indicated in the *Essai sur la probabilité des decisions, page 178*.

III.

If one examines the same general formula, one will find that the probability will be the same in some order as the m events A & the events N are themselves succeeded.

This equality takes place necessarily all the time that one supposes x constant for all the sequence of events; but it appears at the same time that there must result from it an objection against this hypothesis.

We suppose indeed, that for one hundred thousand events, A has arrived 51,000 times & N 49,000 times: if on each sequence of one hundred consecutive events, on finds that one has had 51 times A & 49 times N, will not one believe authorized to judge that in future events, the number of A will surpass the number of N with more probability, than if in the same number one would have had sometimes the number of N superior to the one of A, & that the order of these events had been more irregular?

We suppose next that, these events being partitioned into sequences of one thousand each, the number of A carry away much in the first, that this superiority diminished little by little, that it becomes nearly null towards the middle, that next towards the end N commences to carry away, in a manner however that the difference for the totality is always of two thousand in favor of A. Would not one be then with some reason tempted to believe that N will carry away on A in the future, if especially one does not consider a very great number of future events?

The results of the hypothesis of x constant are therefore here in contradiction with that which the reason appears to indicate. It is necessary therefore to examine if one must not, at least in many cases, substitute a method where the probability could depend on the order of the events.

IV.

We will consider here the hypothesis where one supposes x variable in the two cases, 1.° in a sequence of events which are not linked among them by any law relative to the time & to the order of their production; 2.° of a sequence of linked events among them by a law relative to that order. In the first case, if one knew the law of the probability of these events, the formula which would express it would not be a function of the time or of the place which the event occupies; it would be in the second.

If, for example, I suppose some packs of red & black cards, of which the number is $m+n+p+q$, that I have drawn from $m+n$ of these packs m red cards, & n blacks, & that I seek the probability to draw from the $p+q$ packs remaining, p red cards & q blacks. If I do not know that these packs have been formed by drawing at random some cards from one given pile of red & black cards, it is clear that I have no reason to suppose either that the ratio of the number of red cards to the one of the black cards, is constant for all the packs, or that the mean value of this supposed variable ratio, is the same for all; & at the same time I have no reason to suppose that this ratio varies according to the order in which one ranks the cards, or according to the one where one draws a card from each.

In a sequence to the contrary of natural events, there is presented a great number of cases where one can suppose that the order of the times influences on the production of the events.

V.

Since in the first case the probability can be different for the successive events, & since however it is independent of the order that they follow, it is clear that it can be subject to no other law than that which gives birth to the probability that it will be rather the same than different for the diverse events. Suppose therefore that t expresses the number of events as much past as future, $t' = m+n$ the one of the past events, $t'' = p+q$ the one of the future events, & if $x', x'', x''' \ldots x''^{t}$ express the different probabilities in favor of A; instead of the formula of *paragraph I*, one will have for the probability of p events A, & of q events N, in t'' future events, the function

$$\frac{t''.t''-1\dots p+1}{1.2.3\dots q}\times$$
$$\frac{\int\left[\left(\frac{x'+x''+x'''\dots+x'''t}{t}\right)^{m+p}\cdot\left(1-\frac{x'+x''+x'''\dots+x'''t}{t}\right)^{n+q}\ \partial x'\partial x''\cdots\partial x'''t\right]}{\int\left[\left(\frac{x'+x''+x'''\dots+x'''t}{t}\right)^{m}\cdot\left(1-\frac{x'+x''+x'''\dots+x'''t}{t}\right)^{n}\ \partial x'\partial x''\partial x''\cdots\partial x'''t\right]}$$

the integrals being taken t times successively for each x, from 1 to 0; & one will have

$$\begin{aligned}
&\int\left[(x'+x''+x'''\cdots+x'''t)^m\cdot(1-x'-x''-x'''\cdots-x'''t)^n\,\partial x'\partial x''\cdots\partial x''t\right]\\
=&\frac{1}{m+1.m+2\dots m+t}[-(t-1)^{m+t}t^n\\
&+\binom{t}{2}(t-2)^{m+t}2^0\dots\pm t.1^{m+t}(t-1)^n]\\
&+\frac{n}{m+1.m+2\dots m+t+1}t[-t(t-1)^{m+t+n}\\
&+\binom{t}{2}(t-2)^{m+t+2}2^{n-1}\dots\mp t(t-1)^{n-1}]\\
&+\frac{n.n-1}{m+1.m+2\dots m+t+2}\cdot\frac{(t+1)t}{1.2}[-t(t-1)^{m+t+n}\\
&+\binom{t}{2}(t-2)^{m+t+2}2^{n-2}\dots\pm t(t-1)^{n-2}]\\
&+\cdots\\
&+\frac{n.n-1\dots1}{m+1.m+2\dots m+t+n}\cdot\frac{t+n-1\dots t}{1.2\dots n}[t^{m+n+t}-t(t-1)^{m+n+1}\\
&+\binom{t}{2}\cdot(t-2)^{m+n+t}\cdots],
\end{aligned}$$

a function which we have left under this form in order that it be easier to grasp the law from it, & that if one supposes $t=1$, or one x alone, it is reduced to

$$\frac{n.n-1\dots1}{m+1.m+2\dots m+n+1};$$

this which leads to the same result as the ordinary formula, as this must be.

We suppose that one has had A two times in sequence, & that one demands the probability to have a third time, it will be $\frac{3}{4}$ for the ordinary formula, & by this $\frac{3}{5}$ alone.[1]

[1] *Translator's note*: Here $m=2, p=1,$ and $n=q=0$. The first formula is $\binom{p+q}{p}\frac{\int_0^1 x^{m+p}(1-x)^{n+q}dx}{\int_0^1 x^m(1-x)^n dx}$. This gives $\frac{\int_0^1 x^3dx}{\int_0^1 x^2dx}=\frac{3}{4}$. As for the second formula, since $m+n+p+q=3$, we have 3 variables. Put $u=\frac{1}{3}(x_1+x_2+x_3)$. The probability is thus $\binom{p+q}{p}\frac{\int_0^1\int_0^1\int_0^1 u^{m+p}(1-u)^{n+q}dx_1dx_2dx_3}{\int_0^1\int_0^1\int_0^1 u^m(1-u)^n dx_1dx_2dx_3}=\frac{\int_0^1\int_0^1\int_0^1 u^3dx_1dx_2dx_3}{\int_0^1\int_0^1\int_0^1 u^2dx_1dx_2dx_3}=\frac{3}{5}$.

If one has had A three times, & if one seeks the probability to have a fourth, it will be by the first formula $\frac{4}{5}$, & by the second[2] $\frac{40824}{67200}$, $>\frac{3}{5}$ but$<\frac{4}{5}$.

If one seeks the probability, that in the indefinite sequence of events, the number of events A will surpass that of the events N, it will be expressed by the function

$$\frac{\int\left[(x'+x''\cdots x'''t)^m(t-x'-x''\cdots-x'''t)^n\,\partial x'\partial x''\cdots\partial x'''t\right]^{\frac{1}{n}}}{\int\left[(x'+x''+x'''\cdots+x'''t)^m(t-x'-x''-x'''\cdots-x'''t)^n\,\partial x'\partial x''\cdots\partial x'''t\right]}$$

then integrals always being supposed taken from 1 to 0, with respect to $x',\ x'',\ x'''\ldots x'''t$; but those of the numerator being taken only from

$$x'+x''+x'''\cdots+x'''t=t$$

up to

$$x'+x''+x'''\cdots+x'''t=\frac{t}{2}$$

& making next $t=\frac{1}{0}$.

VI.

If we suppose now that there exists a variation in the probability, which can depend on the order of the events, let x' be the probability of the first A, & $1-x'$ that of the first N; $\frac{x'+x''}{2}$ & $\frac{2-x'-x''}{2}$ can express the probabilities of the second A or of the second N, $\frac{x'+x''+x'''}{3}$ & $\frac{3-x'-x''-x'''}{3}$ that of the third A or of the third N; & those of the r^{th} A or N can be by $\frac{x'+x''+x'''\cdots+x'''r}{r}$ & $\frac{r-x'-x''-x'''\cdots-x'''r}{r}$, where one sees that x' is the probability of A on the first trial, x'' that of A on the second if it is different from that of the first, x''' that of A on the third if it is different from that of the two others, & thus in sequence.

One sees next that as one does not know the law of the order of the events, but that one knows only that there can exist one of them, the method consists, the same as in the *preceding article*, to take only the probability that that of the successive events will be or will not be the same, with this sole difference that here one has regard to the order that the events are themselves followed.

If therefore one has a certain number of events A & N which are themselves succeeded, & if one seeks the probability that in a given number of future events, the A & N follow any given order whatever, one will take successively for each event A or N, the expression which corresponds to it, according to the rank where it is arrived, or the one where one supposes that it must arrive; one will form a product of all these successive values, 1.° for the past events alone, 2.° as much for the past events as for the future events; the first product contains $r,\ x$ if r is the number of past events, & the second contains $r+r'$, x if r' is the number of future events: one will take $r+r'$ times the integral of the second product successively for each x, from 1 to zero; & r times the integral of the first product for each x, from 1 to zero; & the integral of the second product, divided by that of the first, will give the value of the sought probability.

[2] *Translator's note*: Here $m=3,\ p=1$, and $n=q=0$. Since $m+n+p+q=4$, we have four variables. Now put $u=\frac{1}{4}(x_1+x_2+x_3+x_4)$. We have the probability $\binom{0+0}{0}\frac{\int_0^1\int_0^1\int_0^1\int_0^1 u^4dx_1dx_2dx_3}{\int_0^1\int_0^1\int_0^1\int_0^1 u^3dx_1dx_2dx_3}$.

One sees that here the formula varies according to the order of the past events & according to the one of the future events.

We suppose here that one has had two times A, & one seeks the probability[3] to have it again one time, it will be $\frac{25}{42}$, which is smaller than $\frac{3}{4}$ which the ordinary method gives, & that $\frac{3}{5}$ which the hypothesis of *paragraph V* gives.

If one has had three times A, & if one demands the probability to have a fourth, one will find that it[4] is $\frac{1799}{3000}$, instead of $\frac{4}{5}$ or $\frac{40824}{67200}$ which one would have had under the other two hypotheses.

As here one must have a different function, following the order that one has supposed, either to the past events, or to the future events, it is easy to see that it will be necessary to make new calculations for each disposition of the events, this which, if the numbers r and r' are quite great, would render impossible the use of this method: one must therefore, in the case where this hypothesis would appear able to be admitted, to seek to determine in the sequence of events, such as it is offered, a constant order which has always been observed; this order one time supposed known, one will regard its constancy as a unique event which is repeated without ever missing; one will seek next the probability that it continue to have in the sequence the same constancy, & it will be in this new hypothesis that one will apply the calculus.

Thus let n be the number of events arrived constantly, & p the one of the future events: the probability that this event will take place, or that this law will be observed during the interval of these p revolutions, will be expressed by

$$\frac{\int\left[\left(x' \cdot \frac{x'+x''}{2} \cdot \frac{x'+x''+x'''}{3} \cdots \frac{x'+x''+x'''\cdots+x'''^{n+p}}{n+p}\right) \partial x' \partial x'' \cdots \partial x'''^{n+p}\right]}{\int\left[\left(x' \cdot \frac{x'+x''}{2} \cdot \frac{x'+x''+x'''}{3} \cdots \frac{x'+x''+x'''\cdots+x'''^{n}}{n}\right) \partial x' \partial x'' \cdots \partial x'''^{n}\right]}$$

In order to determine next the value of

$$\int \left(x \cdot' x' + x'' \cdots x' + x'' \cdots + x'''^n . \partial x' \partial x'' \cdots \partial x'''^n\right),$$

one will take a series of terms z', z'', z''', $z''''\ldots z'''^n$, such that

$$\begin{aligned}
z' &= 1, \\
\Delta z'' &= \frac{n-1}{2}, \\
\Delta\, z''' &= \frac{n-2}{2} z'' + \frac{n-1.n-2}{2.3}, \\
\Delta z'''' &= \frac{n-3}{1.2} z''' + \frac{n-2.n-3}{1.2.3} z'' + \frac{n-1.n-2.n-3}{1.2.3.4}, \\
&\cdots
\end{aligned}$$

[3] *Translator's note*: Here $n = 2$ and $p = 1$. The denominator has n variables and the numerator will have $n + p = 3$ variables. Put $u = x_1 \cdot \frac{x_1+x_2}{2}$ and $v = x_1 \cdot \frac{x_1+x_2}{2} \cdot \frac{x_1+x_2+x_3}{3}$. The probability will be $\frac{\int_0^1 \int_0^1 \int_0^1 v\, dx_1 dx_2 dx_3}{\int_0^1 \int_0^1 \int_0^1 u\, dx_1 dx_2 dx_3}$.

[4] *Translator's note*: Here $n = 3$ and $p = 1$. Put $u = x_1 \cdot \frac{x_1+x_2}{2} \cdot \frac{x_1+x_2+x_3}{3}$ and $v = x_1 \cdot \frac{x_1+x_2}{2} \cdot \frac{x_1+x_2+x_3}{3} \cdot \frac{x_1+x_2+x_3+x_4}{4}$. The probability will be $\frac{\int_0^1 \int_0^1 \int_0^1 \int_0^1 v\, dx_1 dx_2 dx_3}{\int_0^1 \int_0^1 \int_0^1 \int_0^1 u\, dx_1 dx_2 dx_3}$.

& the value sought will be

$$\frac{z'}{n+1} + \frac{z''}{n} + \frac{z'''}{n-1} + \frac{z''''}{n-2} + \text{\&c.},$$

or

$$\int \left[\left(z'x^n + z''x^{n-1} + z'''x^{n-2} + z''''x^{n-3} + \text{\&c.} \right) \partial x \right],$$

the integral being taken from $x = 1$ to $x = 0$.

One can observe that it is not necessary to know the value of this formula in order to be assured, 1.° that the more n will be great, the more p remains the same, one will have a great probability to have these future events subject to the same law; so that for a given number, one can take n great enough in order that this probability be as great as one will wish; 2.° that n remaining the same, the more p will increase, the more this probability will diminish; so that whatever be n, it will become necessarily too small for a certain limit of the values of p; & thus under this hypothesis, the probability that an observed law will continue to take place, is necessarily decreasing, & consequently one can count at each period on the constancy of a law, only for a number p of determined events, or for a given time. It is true that if in this given time & for these p events, the law continues to be observed, one will have in this period either an equal probability, that the law will be yet constant for a greater number p' of events greater & for a longer time, or a probability greater for a second sequence of p events, or for a time equal to the first. This diminution in the probability that the same law will embrace a greater number of future events, & will be observed in the more extended time, agrees with this that the reason indicates to us.

We ourselves dare to be answerable for only the most regular law which we observe in the phenomena, is conserved without any modification during an indefinite time. We suppose in truth that there can exist a more complicated constant law, which during a time seems the same to our eyes as that which one has first established, & which next deviates in a sensible manner from it; but it is easy to see that this is precisely the case where the first observed law, ceasing to be constant, one substitutes into it another which embraces at the time the phenomena in which the first law responded to it & those which appear to escape.

VII.

We have therefore here three different hypotheses; 1.° that where the probability is constant, that is to say, where one supposes each event equally probable, or at least the mean probability for each, determined in a similar manner; 2.° that where one supposes this probability variable, but independent of the times where the events are arrived, & of the order in which they have been observed; 3.° that where one supposes them independent, or rather able to depend on this order.

This last hypothesis is the most general, & even it is that to which one must stop as often as one has no motive to believe that the one of the first two must be preferred: indeed, in the one one supposes the probability constant; in the second, one supposes it independent of the order of the events; assumptions which can not be rigorously legitimate; instead that in the third one makes properly no assumption: the case of the constant probability & the one of the probability independent of the order of the events, enter likewise each with the kind of probability that the observation can give to the one or to the other hypothesis: thus as often as one will wish to know, after the events, an observed law in Nature, one will begin first to determine, after examination of these events, some constant law to which all these events have been subject, or according to which they can be classified & reduced to some more general events which take place constantly. One will seek the most simple law as is possible, that which, for the same number of observed events, gives the greatest number of events subject to the law, & which one can regard as brought forth constantly; next one will seek the probability that this law will be observed for future times.

VIII.

The probability of the constancy of an observed law, such as one could deduce it from the third hypothesis, & even that which one could determine after the first, diminishes so promptly, that unless the number of observed events is very great, one could have only for some very short times a probability so great that this law will continue to hold; & however in order to have a just motive to believe this law is constant, it is necessary as often as this probability is very great, & that it subsists such for a very long time.

But we will remark that, if the question is of natural events of which each, however subject to some different laws, has always appeared constantly subject each to its particular law, this observed constancy in all these events must increase for each the probability of that which will take place in the sequence of future events; & we are going to seek the expression of the probability under this new hypothesis.

For this, 1.° we will designate by $A', A'', A''' \ldots A'''^{n}$, m events which we will suppose to have had place constantly, & we will make the mean probability of the first of each of these events equal to $\frac{x'+x''+x'''\cdots+x'''^{n}}{m}$.

As these events are supposed independent from one another, it would appear that one must express the probability of each by some different quantities; but it is necessary to observe that here these are not at all the probabilities of the particular events A', A'', $\ldots A'''^{n}$ which one examines, but that of the event which takes place in general, rather than the contradictory event, that is to say, of the event which, by the nature of the things, arrives constantly, while the contradictory event does not arrive, & that thus one can suppose to them an equal mean probability, as one has supposed for the similar events *paragraph 5*.

2.° Let $n'+1$ be the number of times that the event A' is arrived, $n''+1$ the number of times that the event A'' is arrived, & $n''^{m}+1$ the number of times that one has had the event A''^{m}, & let one seek the probability that this event A''^{m} will arrive p times more.

We will form the products

$$
\begin{aligned}
P' =& \frac{x' + x'' \cdots + x_1''^{m+1}}{m+1} \cdot \frac{x' + x'' \cdots + x_1''^{m+2}}{m+2} \cdot \frac{x' + x'' \cdots + x_1''^{m+3}}{m+3} \\
& \cdots \frac{x' + x'' \cdots + x_1''^{m+n}}{m+n'}, \\
P'' =& \frac{x' + x'' \cdots + x_2''^{m+1}}{m+1} \cdot \frac{x' + x'' \cdots + x_2''^{m+2}}{m+2} \cdot \frac{x' + x'' \cdots + x_2''^{m+3}}{m+3} \\
& \cdots \frac{x' + x'' \cdots + x_2''^{m+n}}{m+n''} \\
& \cdots \\
P''^m =& \frac{x' + x'' \cdots + x_m''^{m+1}}{m+1} \cdot \frac{x' + x'' \cdots + x_m''^{m+2}}{m+2} \cdot \frac{x' + x'' \cdots + x_m''^{m+3}}{m+3} \\
& \cdots \frac{x' + x'' \cdots + x_m''^{m+n''^m}}{m+n^m}; \\
Q =& \frac{x' + x'' \cdots + x_m''^{m+1}}{m+1} \cdot \frac{x' + x'' \cdots + x_m''^{m+2}}{m+2} \cdot \frac{x' + x'' \cdots + x_m''^{m+3}}{m+3} \\
& \cdots \frac{x' + x'' \cdots + x_m''^{m+n''^m} + p}{m+n''^m+p}
\end{aligned}
$$

in which one sees that the first m x which belong to the first m events of each class, are the same for all, but that the others x which belong to the subsequent events of each class, are different for each.

This put, one will have the sought probability, that is to say, that of having p times in sequence the future event A''^m expressed by the function

$$\frac{\int (P'P'' \cdots P''^{m-1} Q \, \partial x')}{\int (P'P'' \cdots P''^{m-1} \, \partial x')}.$$

If one supposes that the x are the same, then one will have for this same probability $\frac{x'+x''\cdots+x''^m+1}{x'+x''\cdots x''^m+p+1}$, instead of $\frac{x'+x''\cdots+x''^m+m+1}{x'+x''\cdots x''^m+m+p+1}$ that one will have had if there had been really only a single event.

This formula suffices to show how a natural fact[5] observed a single time, provided that it had been well observed & analyzed in a manner to not be confounded with another, can be regarded as a constant fact with a very great probability; this very great probability is then the effect of the constancy observed in a great number of facts, which render probable the existence of a similar constancy in another fact.

IX.

We will end this article with a last remark, we suppose that one has observed two sequences S & S' of events A & N; that in the first the number of A is m, & n the

[5] *Translator's note*: Condorcet has previously used the word "événemen" to denote an event. He now uses the word "fait" to denote an actual event. We render this as "act" in order to maintain its distinction from the previous.

one of N; that in the second one has had m' A & n' N, that the ratio of m to n differs sufficiently from the one of m' to n', in order that one can suppose that in these two sequences the probability of A is not the same, & one demands in this case the probability to have p times A & q times N in the $p+q$ future events. Let x be the probability of A in the sequence S, x' that probability in the sequence S'; $1-x=z$, $1-x'=z'$ the probabilities of the event N; let finally $X=x^m\cdot(1-x^n)$, & $X'=x'^{m'}\cdot(1-x'^{n'})$, we take first in $(x+z+x'+z')^{p+q}$ the sequence of all the terms where the sum of the exponents of x & of x' equal p, & where that of the exponents of z & z' equal q. Let $Ax^a x'^b$, $z^{a'}z'^{b'}$ be one of these terms, the probability that results from it will be

$$A\cdot\frac{\int Xx^a z^{a'}\partial x\cdot\int X'x'^b z'^{b'}\partial x'}{\int X\partial x\cdot\int X'\partial x'},$$

& the probability sought will be equal to the sum of all these terms thus formed, provided that one supposes that it is equally probable that a future event belongs to the sequence S or to the sequence S'.

If to the contrary, one supposes that this same probability depends on the order observed in these two sequences, then in order to have the probability, one will multiply the term

$$A\cdot\frac{\int Xx^a z^{a'}\partial x\cdot\int X'x'^b z'^{b'}\partial x'}{\int X\partial x\cdot\int X'\partial x'},$$

by $\left(\frac{p+q}{a+a'}\right)\int(X\partial x)^{a+a'}\cdot\int(X'\partial x')^{b+b'}$, & the sought probability will be equal to the sum of all the terms divided by $\int(X\partial x+\int X'\partial x')^{p+q}$. Finally one can suppose this probability regulated according to the number of terms of each sequence, & then it will be necessary to multiply the same term by $\frac{p+q}{a+a'}\int x''^{m+m'+a+b}(1-x'')^{m+n'+a'+b'}\partial x''$, to take the sum of all these terms, & to divide it by $\int x''^{m+m}(1-x'')^{m+n'}\partial x''$.

That which we have said for two sequences S & S', applies easily to any number of similar sequences.

One can choose still other hypotheses, each of which must be preferred according to the nature of the questions which one treats; because in general in this part of the calculus of probabilities where the question is especially to find the mean values, it is necessary to employ it only for the quantities of which the reasoning can teach us neither the value nor the limits, & only as a supplement to a direct knowledge to which we have not reached, & to seek to tighten, as much as it is possible, the number of combinations which our ignorance alone makes us regard as unimportant between them.

FIFTH PART

On the probability of extraordinary facts.[6]

I.

[6]*Translator's note*: Here again we encounter the word "fait." In this circumstance, I have rendered it as "fact" since Condorcet is discussing the testimony of witnesses.

If one were to procure a list of extraordinary facts, of which the truth has been attested by an eyewitness, & if one knew moreover, which of these facts had been recognized for true, or found false eventually by the effect of a deeper examination, one could deduce from it by the calculus the probability of a testimony on extraordinary facts; & if one prepared this list according to the different orders of improbability of these facts, one could evaluate for each class the credibility of the testimony.

But independently of the difficulty of procuring such lists, & forming exactly these classifications, one senses how often one would find obstacles even to distinguish those of the extraordinary facts which one must regard as true or as false.

II.

In default of this direct method we propose one which, in truth, is indirect, but of which one can make some very useful applications.

We suppose that u designates the probability of an event A, & e that of an event N, that u' & e' designates the probabilities of two other events A' & N'; $\frac{uu'}{uu'+ee'}$ will express the probability of the combination of the events A, A'; & $\frac{ee'}{uu'+ee'}$ the probability of that of the events N, N'.

Now it is easy to see that these two combinations A, A', & N, N' can designate two events contradictory between them, provided that the production of these events is dependent on two conditions.

We suppose, for example, that one has $u+e$ urns, that u of these urns contain u' tokens of gold, & e' tokens of ivory; & that e of these urns contain e' tokens of silver & u' tokens of wood.

The probability, to have a piece of gold rather than a piece of silver will be $\frac{uu'}{uu'+ee'}$, that to have a piece of silver rather than a piece of gold will be $\frac{ee'}{uu'+ee'}$; so that, if one supposes the metallic tokens equal in weight to one another, likewise of the tokens of ivory & of wood, & if without seeing the one which one has been drawn, one recognizes in the weights that it is metallic, the probabilities that it will be gold or silver will be

$$\frac{uu'}{uu'+ee'}, \quad \frac{ee'}{uu'+ee'}.$$

We suppose now that u & e represent the probabilities of the truth of an extraordinary event & of the falsity of the same event, & that at the same time u' & e' express the probability that a testimony will be either conformed or not to the truth, & that a witness is assured of the truth of this event. One sees that the extraordinary event declared true, the token of gold represents here, that the extraordinary event false & declared true the silver token represents, that one is here in the case precisely where one knows in advance that the token is of metal, & that thus the probability that the extraordinary event declared true is really, will be $\frac{uu'}{uu'+ee'}$, & that it is false $\frac{ee'}{uu'+ee'}$.

We suppose for example, $e = \frac{999999}{1000000}$, & $u = \frac{1}{1000000}$; $n' = \frac{999}{1000}$, $e' = \frac{1}{1000}$, we will have $\frac{uu'}{uu'+ee'} = \frac{999}{1000998}$, & $\frac{ee'}{uu'+ee'} = \frac{999999}{1000998}$. One sees by this example, that a testimony of which, for a ordinary event or of which the probability is $\frac{1}{2}$, there would result a probability $\frac{999}{1000}$ of the event affirmed, would give however, for a very extraordinary event, only a probability less than $\frac{1}{1000}$.

III.

The probability u of the event designates here this probability taken in itself, & such that it exists, when independently of all proof relative to this individual event, one demands what is the probability that it has taken place rather than the contradictory event.

But one must observe that this probability must be that of a determined event, compared to the probability of another determined event, which can not subsist with the first, & not to that of any event whatever of the sum of the possible events.

Thus, for example, if one says that in a lottery of 100000 tickets, it is the number 99 which is drawn first: as the exiting of this ticket is as probable as that of each other determined ticket, I must regard the exiting also as probable as that of a contradictory event. I will make therefore in this case $u = \frac{1}{2}$, & the testimony which assures me of the exiting of this ticket must lose nothing of its force. It is not the same if the testimony has preceded the exiting of the ticket; then this proposition: *the number 99 will exit*, is equivalent to this one: *I have guessed in advance the number which must exit*, the probability u to have guessed just, must be only $\frac{1}{100000}$; & if a witness, of whom the probability is $\frac{999}{1000}$, announces that he has seen himself to realize the prediction of a parallel event, the probability which will result for the truth of his assertion, will be $\frac{999}{100998}$.

We suppose next that one has a pack of forty cards, & that one considers the probability to have drawn two times in sequence a determined card, as the king of spades; it is clear that the probability to draw two times the king of spades, is in itself $\frac{1}{1600}$: but if one seeks it relatively to the one who would announce this event as having arrived, one will observe that one must regard as equally possible all the events where one will have brought forth twice the same card. One must therefore regard as the event contradictory to the event which has arrived, only the one to bring forth a similar card, after having brought for the first; & consequently $\frac{1}{40}$ will be the value of u. The determination of that which one must regard as the fact contradictory to the one of which one wishes to know the probability, can have some difficulties in the application, but one will succeed always to raise them by means of the principle that we just exposed.

IV.

As for the quantities u' & e' which designate here the probability of a testimony, it is necessary to observe that the question is only of the probability of the testimony in itself, that is to say, of the probability to see the objects good or bad, & to render that which one has seen with truth; & it is necessary to consider here independently of the degree of possibility which presents the fact considered in itself, & consequently u' & e' expresses the probability of the testimony such that one could know it according to the ordinary facts which have a probability equal to that of the contradictory fact. Indeed, in this case $u = \frac{1}{2}$ & $e = \frac{1}{2}$;therefore $\frac{uu'}{uu'+ee'} = u'$, & $\frac{ee'}{uu'+ee'} = e'$. However this probability of the testimony is not the same for all the facts; it depends, 1.$^\circ$ of the difficulty of observing them well, 2.$^\circ$ of the causes of the errors which can have an influence more or less great on the testimonies, 3.$^\circ$ of the complication of the fact in itself.

As for this last object, it is necessary to remark, 1.$^\circ$ that one must not understand this expression, a simple fact, in a rigorous & metaphysical sense; but in this sense, that a simple fact, is the one of which a man of an ordinary capacity can grasp the whole & the detail in a single glance without too great effort of attention.

2.$^\circ$ That for a complicated fact, one must not understand two isolated facts, but a combination of two facts whence a consequence results which is legitimate only when the two facts are true at the same time.

This put, let u' be the probability of a testimony for a simple fact, it will be $u'u'$ for a fact composed of two simple facts, u'^3 for a fact composed of three simple facts, &c. Thus supposing in general an equal justice & an equal good faith to the witnesses, it is clear that the one who can by a single glance see a totality of three facts of which each would require all the attention of another, will have for this fact a probability u', while each of the others will have only a probability u'^3.

It will happen also very often that some slightly enlightened witnesses, & not knowing that the truth of a complicated fact supposes that of many simple facts, will believe themselves certain of the reality of this fact, although they have seen or believe to see only some ones & sometimes even a single one of the facts which form this combination; then if the number of these facts which they have seen is m, & n the one of the facts which he will have need to observe moreover in order to have seen the whole of the fact of which they affirm the truth, the probability of their testimony, instead of being u'^{m+n}, will be more than $\frac{u'^n}{2^n}$.

Let therefore u' & e' be the probability of the truth & of the falsity produced by the testimonies, u'' that of a testimony on a simple fact, one will have therefore for the probability of a testimony $u' = \frac{u''^m}{2^m}$, & $e' = 1 - \frac{u''^m}{2^m}$; & if the number of concordant witnesses is p, one will have $\frac{u'^p}{u'^p+e'^p}$ & $\frac{e'^p}{u'^p+e'^p}$, which it is necessary to substitute to u' & e' in the formula above.

As most of the extraordinary facts are some complicated facts, & since always they owe this character only to the reunion of many circumstances of which each in particular would be only an ordinary fact, one senses how the preceding consideration must weaken still the force of the witnesses.

V.

If now one seeks the probability that the extraordinary fact declared false by a witness, will be true, it will be expressed by $\frac{ue'}{ue'+eu'}$, & that this extraordinary fact was declared false, is really false, will be expressed by $\frac{eu'}{ue'+eu'}$.

We suppose finally p witnesses who attestent the truth of the extraordinary fact, & q witnesses who deny it, the probability of the truth of this fact will be expressed by $\frac{uu'^pe'^q}{uu'^pe'^q+eu'^qe'^p}$, & that of the falsity by $\frac{ee'^pu'^q}{uu^pe'^q+eu'^qe^p}$; if $q = p$, these two probabilities become $\frac{u}{u+e}$ & $\frac{e}{u+e}$, as this must be, since the testimonies add & subtract nothing here to the probability of the fact.

Greater details would be superfluous here; it suffices us to have shown how one can explicate by the calculus the impairment which testimonies suffer when they fall on some extraordinary facts.