SUITE DU MÉMOIRE SUR LE CALCUL DES PROBABILITÉS*

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ARTICLE VI.
Application of the principles of the preceding article
to some questions of criticism
of past events.

I.

I have exposed, in the *preceding article*, a method to express the probability of extraordinary facts,[1] by having regard to that of the testimonies which attest them, & to the proper probability of these facts, which, despite the very great probability which the same testimonies would produce for an ordinary fact, can then become very small & much below $\frac{1}{2}$.

II.

I have observed at the same time that it was not necessary in this case to understand, by the proper probability of a fact, the ratio of the number of combinations where it takes place, with the total number of combinations. For example, if in a deck of ten cards one has drawn one of them, & if a witness says to me that this is such a card in particular, the proper probability of this fact, which it is a question to compare with the probability which is born of testimony, is not the probability to draw this card, which would be $\frac{1}{10}$, but the probability to bring forth this card rather than another such card determined in particular; & as all these probabilities are equal, proper probability is here $\frac{1}{2}$.

This distinction was necessary, & it suffices to explicate the contrarity of opinions between two classes of philosophers. The ones who are able to persuade themselves only the same testimonies can produce, for an extraordinary fact, a probability equal to

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[1] *Translator's note*: I have chosen to render the word *faits* as *facts* rather than as *events* for two reasons. First, Condorcet has used *évenemens* consistently in this latter sense. Second, this permits the reader to identify his choice of language.

that which they produce for an ordinary fact; & that, for example, if I believe a man of good sense who says to me that a woman has birthed a boy, I must believe it equally if he said to me that she has birthed twelve.

The others to the contrary are convinced that the testimonies conserve all their force, for the extraordinary facts & very small probables, & they are struck by this observation, that if one draws a lottery of 100,000 tickets, & if a man, worthy of faith, says that the number 256, for example, has had the first prize, a person will not doubt his testimony, although there are odds 99,999 against 1 that this event has not arrived.

Now, in the manner of the preceding observation, one sees that in the second case the proper probability of the fact being $\frac{1}{2}$, the testimony conserves all its force, instead that in the first, this probability being very small, reduces nearly to nothing that of the testimony.

III.

I have proposed next to take, for the proper probability of the fact, the ratio of the number of combinations which give this fact, or a similar fact to the total number of combinations.

Thus, for example, in the case where one draws a card from a deck of ten cards, the number of the combinations where one draws any determined card whatever is one; the one of the combinations where one draws another determined card is also one; therefore $\frac{1}{2}$ will express the proper probability.

If one says to me that one has drawn twice in sequence the same card, then one will find that there are only ten combinations which give twice a same card, & ninety which give two different cards: the proper probability of the fact is therefore only $\frac{1}{10}$, & that of the testimony begins to become weaker.

But I believe I must abandon this manner to consider the question, 1.° because it appears to me too hypothetical; 2.° because often this comparison of similar events would be difficult to make, or, this which is yet worse, it would be made only after some arbitrary assumptions; 3.° because by applying it to some examples, it leads to some results too extended from those which common reason would give.

IV.

I have therefore sought another, & it has appeared to me more exact to take, for proper probability of an event, the ratio of the probability of this event taken in the ordinary sense, with the mean probability of all the other events.

Thus, in the preceding example, we have ten combinations where one draws two similar cards, & forty-five combinations where one draws two different cards. The probability to draw two different determined cards is $\frac{2}{100}$; that to draw two similar determined cards is $\frac{1}{100}$. The mean probability of another event than the one which brings forth the two similar given cards, will be therefore $\frac{45\cdot\frac{2}{100}+9\cdot\frac{1}{100}}{54} = \frac{99}{54\cdot 100}$; that to bring forth the two similar determined cards will be $\frac{1}{100}$; therefore the proper probability of the event will be $\frac{54}{153}$.[2]

[2] *Translator's note*: $\frac{1}{100}/\left(\frac{99}{5400}+\frac{1}{100}\right) = \frac{54}{153}$. Condorcet defines the *proper probability* so that in this case it would be $\frac{1}{100}/\frac{99}{5400} = \frac{54}{99}$, but computes it in a different manner.

We suppose next that, in the same example, one seeks the proper probability of the fact, that one has drawn three times in sequence the same card.

We have here three kinds of facts, 1.° those where one has brought forth three different cards; the probability of each determined fact of this kind is $\frac{6}{1000}$, & there are 120 of them: 2.° those where one draws twice the same card; these facts are 90 in number, & the probability of each is $\frac{3}{1000}$: 3.° those where one draws three times the same card; they are 10 in number, & the probability of each is $\frac{1}{1000}$. We have therefore $\frac{1}{1000}$ for the probability of the determined fact, & $\frac{6.120+3.90+9}{1000.219}$ for the mean probability of the other facts. The proper probability of the alledged fact will be therefore $\frac{219}{1218}$.[3]

Thus we suppose, for example, that the probability of the testimony is $\frac{99}{100}$, that is to say, that the witness deceives or wishes to deceive only one time in one hundred, one will have, after his testimony, the probability $\frac{99}{100}$ or $\frac{9{,}900}{10{,}000}$ that one has drawn a determined card; the probability $\frac{9{,}818}{10{,}000}$ that one has drawn twice the same card; & the probability $\frac{9{,}540}{10{,}000}$ that one has drawn it three times.[4]

We suppose further that the observation is constant that, on twenty million men, one alone has lived 120 years, & that the longest life has been 130; that a man says to me that someone just died at 120 years, & that I seek the proper probability of this event: I will regard first as a unique fact, the one of living more than 130 years, a fact that I suppose not to be arrived; I will have therefore 131 different facts, of which the one to die at 120 years is one alone. The probability of this will be $\frac{1}{20{,}000{,}131}$; the mean probability of the 130 others will be $\frac{20{,}000{,}130}{20{,}000{,}131.130}$; therefore the proper probability[5] sought will be $\frac{130}{20{,}000{,}260}$, or around $\frac{1}{15384}$.

V.

This method will be applied equally to the indeterminate events. Thus, by continuing the same example, if the witness has said only that one has twice brought forth the same card, without naming it, then these ten events, each having the probability $\frac{1}{100}$, $\frac{1}{100}$ will express their mean probability; $\frac{2}{100}$ will express likewise that of 45 other events each having the probability $\frac{2}{100}$: thus the proper probability[6] of the event will be $\frac{1}{3}$.

It can appear singular that the proper probability of the event, instead of being the same here as in the case of the determined event, is sensibly less, that it influences differently on the credibility of the witness; that thus the same man is less believable when he says, in general, that he as seen brought forth twice in sequence the same card, that when he says he has seen brought forth twice in sequence such a card in

[3]*Translator's note*: The mean probability of the other events is $\frac{999}{219000}$. As before, the proper probability is computed as $\frac{1}{1000} / \left(\frac{999}{219000} + \frac{1}{1000}\right)$.

[4]*Translator's note*: Todhunter offers the following explanation of these values. Let p denote the proper probability and let t denote the probability of testimony. (Here $t = \frac{99}{100}$.) The probability to be computed is $P = \frac{pt}{pt+(1-p)(1-t)}$. 1.° The proper probability to draw one determined card from ten is $\frac{1}{2}$. Thus $P = \frac{99}{100}$. 2.° To draw the same given card twice has $p = \frac{54}{153}$. Thus $P = \frac{54}{55} = .9818$. 3.° To draw the same given card three times has $p = \frac{219}{1218}$. Thus $P = \frac{803}{840} = .9560$.

[5]*Translator's note*: The proper probability is $\frac{260{,}001{,}703}{400{,}003{,}122{,}003{,}406} = \frac{1}{153846}$.

[6]*Translator's note*: The proper probability is $\frac{1}{100}/(\frac{1}{100} + \frac{2}{100})$.

particular. This comes from that which, in the second case, there are nine other possible combinations, of which the enunciation would not be more probable than that which he has made, instead that in the first, all the other enunciations that he has not made, are more probable; it is that in the first case it is only the extraordinary fact which he has enunciated; & that in the second, he has enunciated an extraordinary fact, with respect to a part of the possible events, & a common fact with respect to another part. In the first, the question is only of the extraordinary fact; in the second, the question is of the extraordinary fact & of the determined fact, that it is necessary to compare all at once to some more ordinary determined facts, & to some determined facts which are similar to it.

One will follow next the same rule, if the question is of undetermined facts which contain many combinations of a different probability.

We suppose, for example, that one says to me that a player of trictrac has five times in sequence brought forth more than ten points.

As we consider here only the number of points, we have, for each trial, 11 possible events; that is to say, 2 dice, from 2 to 12, of which the probabilities are $\frac{1}{36}$, $\frac{2}{36}$, $\frac{3}{36}$, $\frac{4}{36}$, $\frac{5}{36}$, $\frac{6}{36}$, $\frac{5}{36}$, $\frac{4}{36}$, $\frac{3}{36}$, $\frac{2}{36}$, & $\frac{1}{36}$, & the last two are the only ones which rise above 10. This put, as one considers here only the value of the sum of the points at each trial, there[7] will be only $\frac{15.14.13.12.11}{1.2.3.4.5}$, or 3003 events, of which 6 alone belong to the proposed combination; now, as the probability of these six trials is $\frac{1}{12^5}$, one will have $\frac{1}{12^5.6}$ for their mean probability, & $\frac{11^5}{12^5.2997}$ will be the probability of the other trials: the proper probability of the event will be therefore here $\frac{2997}{11^5.6+2997}$, or $\frac{2997}{969{,}303}$.

VI.

In the same manner as the events, which have a proper probability below $\frac{1}{2}$, become less believable in measure as this probability diminishes, the events similar in themselves, & of which the proper probability is above $\frac{1}{2}$, become more believable in measure as this probability increases, although attested by an equal number of witnesses. It is thus, for example, that an astronomical fact which would be found in accord with the theory of gravitation, would be believed more easily on the assertion of a single Scholar, even by those who would not have verified his calculations; while if the same Scholar had announced a fact contrary to this theory, he would have need of a rather great authority, even in order that one believed a duty to examine reasonably his calculations.

VIII.

If we consider now two classes of events solely A & B, that the probability of the events A are a & b, that of the events B, a being greater than b; that there are m events A, & n different events B, the proper probability of a determined event of the class of B will be expressed by $\frac{b(m+n-1)}{ma+b(m+2n-2)}$: a quantity which will approach $\frac{b}{a+b}$ if m is much greater than n, but could never be below, & will be rigorously equal to it only when $n = 1$. It approaches much to $\frac{2b}{a+b}$, if $m = n$, & if a is incomparably greater

[7] *Translator's note*: This example is a mess. See Todhunter's discussion.

than b; & the value of this proper probability could surpass $\frac{2b}{a+b}$ besides, but only in the case where $n > m$, & it will have $\frac{1}{2}$ for its limit when $a = b$.

Suppose, for example, that a certain event whatever has not arrived to a single individual for one hundred million individuals in the same space. Then $\frac{1}{100{,}000{,}002}$ will express the probability that it will arrive to a given individual & $\frac{100{,}000{,}001}{100{,}000{,}002}$ that it will not arrive.

If, after this, a witness recounts that this event has arrived to a determined individual, then, as it could equally arrive to all those in the same space, one will have for m & n very great equal numbers among them, a will surpass b incomparably; so that one will deviate little from the truth by supposing $\frac{1}{50{,}000{,}000}$ the expression of the proper probability of the event.

But we will treat in the following more in detail an application of this method to the probability that natural events can have, for those who have not at all observed them personally.

VIII.

I am going to try now to make to a question of critique the application of the principles which I just established. Newton appears to be the first who has had the idea to apply the calculus of probabilities to the critique of facts. He proposes, in his work on chronology, to employ the knowledge of the mean duration of the generations & of reigns, such as experience gives us, either to fix in a manner at least approximate, some points of chronology quite uncertain, or to judge more or less with confidence that the different imagined systems merit in order to accommodate among them some periods which appear to contradict themselves.

Some philosophers have been served since with this evaluation of the mean duration of the reigns, in order to prove the slight probability of the duration attributed by the ancient historians to certain sequences of Kings, & to show thence how little this part of their history would merit belief. They have thought that the proper probability of these facts must influence on the weight which it is necessary to attribute to the witnesses who attest them, & have concluded from them that, despite the authority of most accredited historians, the facts invariably must be rejected.

The scholar Fréret,[8] who has combatted the principles of the chronology of Newton, would regard as a kind of usurpation the usage which would begin to introduce itself, to employ in the critique of the calculus of probabilities: he has destined one of his Memoirs to try to show the inutility & the danger. This calculus, according to him, must be limited to the theory of games of chance; one knew however then of the applications to the probabilities of human life, to the loans in pensions or in tontines, & even to some questions of Law; but it appears that Fréret, although he had in physics, in mathematics & especially in astronomy, some quite extensive knowledge, knew not the works of Halley, of the Bernoullis, & of Moivre.

He brings for principle motive of his opinion, that, in the games of chance, the number of the possible combinations is finite, or at least given by a rigorous theory: an advantage which one loses necessarily when one wishes to apply the calculus to the probability of natural facts.

[8] *Translator's note*: Nicolas Frerét (1688-1749), French historian.

It is true that then one knew not, as today, a direct method to calculate the probability of future or unknown events according to the observation of past events, or rather to determine the mean value of this probability: but one could use the calculus according to this hypothesis, that the sequence of future events will be similar to that of past events; a hypothesis which one knows already to have a sufficient exactitude, when the number of observations is very great in itself, & with respect to that of the unknown or future events of which one seeks the probability.

Following each appearance, some applications of too hypothetical calculations, based on false, or even very bizarre, principles which had been then a sort of celebrity by their same singularity, had struck the naturally just & sensible mind of this Scholar, & had prejudiced him against some researches for which these first tries were not proper to inspire confidence; but he certainly denied the general principle, that it is necessary to have regard to the proper probability, either physical, or moral, of the events; few critiques have even made a more fortunate use of them, & there has been in this century a scholarly few who had more strongly sensed the utility of the study of the natural sciences, & who are delivered with more zeal & success.

Among those who have adopted the same principle, one must cite Mr. de Voltaire, who perhaps even has sometimes abused it, especially when he has wished to apply it to the moral probability of events, much more difficult to evaluate than their physical probability.

In the number of applications which he has made of this principle, one reproaches him chiefly to have employed the little probability which he supposes to the very long duration of the reigns of the Kings of Rome, in order to cast uncertainty on this part of the Roman History. As this fact is one of those to which it is most easy to apply the calculus, we have chosen it for example: we are going therefore to seek what is the proper probability of this event, in order to see if it is rather small in order to weaken much the testimony of the historians who have reported it.

IX.

We will observe first that these Kings were electives, & instead of using here either the generations of the hereditary Kings, which cannot be applied to it, or those of the elective Kings, who would give us a too small number of observations, we will prefer a hypothesis which must not depart much from the truth, by preventing besides that it is a little too favorable to the long duration of the reigns.

We suppose therefore, 1.° that the elective Kings can be elected, or can begin to rule from the age of 30 years to that of 60; & that it is equally probable that they will be elected in any period taken in this interval. We will suppose, 2.° that, from 30 to 90 years, the mortality is constant, that is to say, as if of 60 men of 30 years there would die of them one per year, one of 45 men of 45 years, one of 30 men of 60 years, &c. an assumption a little too favorable to the duration of life.

This put, the shortest duration of each reign will be one year, the longest of 60; the shortest duration of seven reigns will be seven years, & the longest 420.

It had been more exact to diminish the reigns each by a half-year, in a manner that the shortest duration was 3 years $\frac{1}{2}$, & the longest $416\frac{1}{2}$, or to increase by 3 years $\frac{1}{2}$ the duration of the seven reigns; but the difference is not here very considerable, & we give again this advantage to the favorable opinion of the long duration of these reigns.

X.

Let in general n be the greatest duration expressed in years, for the one who is elected the youngest, m for the one who is elected the least young, & p the number of the reigns. If one takes the formula

$$\frac{[(n-m+1).x(1-x)-x^{m+1}+x^{n+2}]^p}{(1-x)^{2p}.\left[\frac{(n+m).(n-m+1)}{2}\right]^p},$$

the coefficient of x^r in this formula, developed into series, will express the probability that the p reigns will endure r years.

Now here, $n=60$, $m=30$, $p=7$, & as it is necessary to count the reigns from the foundation to the death of Tarquin, $r=257$.

It will be necessary therefore to seek the coefficient of x^{257} in the formula

$$\frac{[31x(1-x)-(x^{31}-x^{62})]^7}{(1-x)^{14}.45^7.31^7}.$$

XI.

If one calls P that coefficient which expresses the probability that the seven reigns have endured 257 years, & if one wishes to seek the proper probability that this duration has taken place, one will observe that there are here 414 events, since the reigns can endure from 7 years to 420 years; that the probability of the determined event being P, that of the mean probability of the 413 other events will be $\frac{1-P}{413}$, & that thus, the proper probability will be $\frac{413P}{1+412P}$: it is therefore P which it remains to us to seek.

For this, it will suffice to develop the numerator of the function above, which, by having regard in it only to the terms where the coefficient of x will not surpass 257, will give 24 terms; & as one knows that in general, the coefficient of x^n in $\frac{1}{(1-x)^m}$ is $\frac{n+m-1...n+1}{1.2.3...m-1}$, one will have easily each of the 24 terms & the value of P,[9] which will be $\frac{792}{1{,}000{,}000}$.

We will have therefore $P=\frac{792}{1{,}000{,}000}$, & the proper probability of the fact will be $\frac{246{,}169}{1{,}000{,}000}$, or, very little from a thing near, $\frac{1}{4}$.

XII.

If, instead of this fact, we examine the one of the Augurer *Accius Naevius*,[10] reported also by the writers of the Roman History; as until now no razor has since cut of stone, by supposing only a million of facts contrary to that story, that we can regard as certain, it is of this that we have said, *no. VII*, that the proper probability of this unexpected event would be $\frac{2}{1{,}000{,}000}$ very nearly.

[9] *Translator's note*: The exact value of P is $\frac{7467086351990577494}{10280623468896528046875}=.000726$. The proper probability is then 0.23088.

[10] *Translator's note*: According to Livy I.36, King Tarquinius Priscus, in order to ridicule the art of divination, asked Attus Naevius if what he, the king, had in mind could be done or not. Attus replied that it could. What the king had in mind was that a razor slice through a whetstone. When the instruments were procured, Attus was able to do so.

We suppose now that, in order to believe a fact, to place it in the class of those later which one can be permitted to argue, one requires a probability $\frac{9{,}999}{10{,}000}$, we have need only to attribute a probability $\frac{29{,}997}{29{,}998}$ to the report of the historians who have spoken of the duration of the reign of the seven Kings of Rome; instead that, in order to have the same probability, it would be necessary to attribute one of $\frac{449{,}949{,}001}{449{,}949{,}002}$ to the historian who has reported the fact.

One sees, in the first case, an extraordinary fact which, while a common fact would require only, in order to have the same degree of belief, either one or many historians are mistaken only one time in ten thousand, would require that they not be mistaken either one time in 29,998, or nearly 30,000. In the second, one sees a fact so prodigious, that the most excessive credibility could not suppose to the historians the necessary authority in order to give a sufficient motive to believe it.

XIII.

One would have to be able, instead of the method which we have followed, to suppose to each King of Rome the age that the historians give to him at his advent, & to employ, instead of the hypothesis of *Moivre*, that of *Lambert*,[11] who is much more exact, & leads also to some summable series. One would have had then a proper probability very sensibly below $\frac{1}{4}$, but it would not have been small enough in order to place this duration of the reigns to the number of events which is necessary to reject, & it would be next in the class of those which require only some testimonies stronger & much stronger than the ordinary events require from them.

XIV.

We suppose now that there have been two classes of historians, of whom the first have carried the duration of the seven reigns to 257 years, & the second to 140 years only. In following the same reasonings, one will find that, if these last were the only ones, the probability that seven Kings have reigned 140 years, is $\frac{8{,}887}{1{,}000{,}000}$; & consequently the proper probability of this fact, if the history was transmitted alone, would be $\frac{3{,}661{,}444}{4{,}652{,}557}$, that is to say, greater than $\frac{1}{2}$; & in order to have a probability $\frac{9{,}999}{10{,}000}$, according to the testimony of the historians, it would suffice that that of this testimony was $\frac{9{,}994}{10{,}000}$; that is to say, that instead that it is necessary, for the duration of 257 years, the testimony of a historian who is mistaken only one time in 29,998, it would suffice, for the duration of 140 years, from the testimony of a historian who is mistaken only one time in 1666.

But we suppose some testimonies in favor of the two durations, & according to the theory exposed above, the proper probabilities of the two events, & of another indeterminate event whatever which would have to be able to take place, will be,

For the duration of 140 years	$\frac{3,652,557}{4,968,390}$,
For the duration of 257 years	$\frac{325,512}{4,968,390}$,
For the indeterminate event, non-testimony	$\frac{990,321}{4,968,390}$.

[11] *Translator's note*: This likely refers to Johann Lambert's *Neue Organon*, 1764.

Designating these three probabilities by a, b, c; that of the testimony in favor of the duration of the 140 years by x; that of the testimony in favor of the duration of 257 years by y, we will have, for the probability resulting from the testimony for the first event, $\frac{x(1-y)}{1-xy}$; for the second, $\frac{y(1-x)}{1-xy}$; for that neither the one nor the other takes place $\frac{(1-x)(1-y)}{1-xy}$; & consequently,

$$\frac{a.x.(1-y)}{c+(a-c).x+(b-c).y+(c-a-b).xy}$$
$$\frac{b.y.(1-x)}{c+(a-c).x+(b-c).y+(c-a-b).xy}$$
$$\frac{c.(1-x).(1-y)}{c+(a-c).x+(b-c).y+(c-a-b).xy}$$

for the probabilities of the two witnessed events, & for this that none of the two has taken place. In order that the probability was equal between the two events, it would be necessary that one had $x = \frac{by}{a+(b-a).y}$, & in the proposed example, $x = \frac{325{,}512.y}{3{,}652{,}557-3{,}327{,}045.y}$. We suppose $y = \frac{9{,}999}{10{,}000}$, we will have $x = \frac{9{,}987}{10{,}000}$; that is to say, that a witness who would be mistaken one time on 770, must be believed on the duration of 140 years, rather than the witness who would be mistaken only one time in 10,000, on the duration of 257 years.

We suppose finally x alone equal to $\frac{9}{10}$, & we see what value y must have, in order that the event to which he himself reports has a probability equal to $\frac{9{,}999}{10{,}000}$. In this case one will have the equation $y = \frac{338{,}589{,}476{,}666}{338{,}589{,}802{,}178}$; that is to say, that a witness would be necessary who is mistaken less than one time in one million thirty thousand; thus, in order to have a probability $\frac{9{,}999}{10{,}000}$ of the duration of 257 years, it will suffice to be able to give to the historians who have reported it, a probability such that they are mistaken only one time in around thirty thousand; but if at the same time other historians who are mistaken one time in ten, have fixed to these same reigns a duration of 140 years, it will be necessary, in order to have the same probability, to be able to suppose to the testimonies of the first one such, that they are mistaken only one time in around one million thirty thousand.