V. ON THE CALCULUS OF PROBABILITIES

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V. On the calculus of probabilities

1. I am quite flattered that my doubts on the calculus of probabilities, exposed in the second volume of my *Opuscules*, & more recently in the fifth volume of my *Mélanges de Philosophie*, have appeared to you worthy of some attention; the more I think on this matter, the more I am persuaded, that there is some case where the ordinary theory is absolutely in error, & which one is able to resolve only by some means similar to those which I have proposed. We take for example the game of heads & tails which I have cited, & which has so much embarrassed the Geometers, as one is able to see in Book V of the Memoirs of Petersburg; according to the ordinary theory, the expected sum on each throw, of which the rank is n, is equal to 2^{n-1}, & the probability of winning is $\frac{1}{2^n}$; whence it follows that the expectation on each throw (according to the ordinary theory) is $\frac{1}{2}$, that also the total expectation is infinite, & that consequently the wager must be infinite, which is absurd. But if instead of supposing the probability of winning $= \frac{1}{2^n}$, one would suppose, for example, $\frac{1}{2^n(1+\zeta nn)}$, ζ being a constant number taken at will; we make (Figure 6) $CA = \sqrt{\left(\frac{1}{\zeta}\right)}$; & having described from ray CA the quarter circle AeG of which the indefinite tangent is AF, we take $AE = n$, $AF = n+1$; we will find easily that $\frac{1}{1+\zeta nn}$, is $=$ to the product of the sine of the arc ef by $\frac{CF}{CE}$, this product being divided by the constant $EF = 1$; now thence it is easy to see that the sum of these products will be infinite only in the case where the ray $CA = \infty$, that is to say where $\zeta = 0$; & it will be accordingly smaller as ζ will be greater: such that if ζ for example, were $= 1$, & consequently $EF = CA$, the sought sum could be very nearly equal to $\frac{1}{2} \times \frac{AeG}{CA} = \frac{1}{2} \times \frac{90^\circ}{57^\circ 17' 44''}$ very nearly; if $\zeta = \frac{1}{16}$, the sum would become about quadruple, octuple if ζ were$= \frac{1}{64}$. I could hold myself rather to this last assumption; because then the expected sum, & consequently that which it would be necessary to put to the game, would be six to seven écus, and this is, I believe, all that which one could reasonably risk.

2. Does one wish a hypothesis yet more simple? There is only to suppose that the probability instead of being $\frac{1}{2^n}$ is $= \frac{1}{2^{n+\alpha n}}$, α being a number such as one would wish; the expected sum will be represented by $\frac{1}{2}$ multiplied by the sum of a decreasing geometric progression, of which the first term is $\frac{1}{2^n}$ (n being $= 1$) & of which the sum will be equal to the square of $\frac{1}{2^\alpha}$ divided by $\frac{1}{2^\alpha} - \frac{1}{2^{2\alpha}}$; that is to say, equal to $\frac{1}{2^\alpha}$ divided by $1 - \frac{1}{2^\alpha}$; whence one could deduce easily the value of α; we suppose, for example, that the greatest sum which one can sacrifice, is ten ecus, one will have $10 = \frac{1}{2^{1+\alpha}} : (1 - \frac{1}{2^\alpha})$; whence one deduces $\frac{1}{2^\alpha} = \frac{10}{11}$ & $\alpha = \frac{\log \frac{11}{10}}{\log 2}$ =very nearly $\frac{21}{300}$ or $\frac{7}{100}$.

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Translated by Richard J. Pulskamp, Department of Mathematics and Computer Science, Xavier University, Cincinnati, OH .

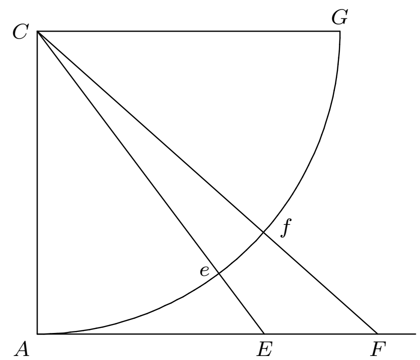

FIGURE 6

3. If one wanted to express the probability by a formula which became $= 0$ when n would be $=$ to a certain number, or greater, it would be necessary to take, for example, instead of $\frac{1}{2^n}$, $\frac{1}{2^n(1+\frac{B}{\sqrt{K-n^q}})}$, q being any positive number, or $\frac{1}{2^n\left(1+\frac{B}{(K-n)^{\frac{q}{2}}}\right)}$, q being an odd whole number. We put the even number 2 in the denominator of the exponent, so that when one comes to number n which gives the probability equal to zero, one does not find the probability negative, by making n greater than this number, which could be shocking; because one never knows the probability to be below zero. It is true that in making n greater than the number in question, it becomes imaginary; but this inconvenience appears to me less than that of becoming negative; & besides it is impossible, (by the imperfection of the algebraic expressions) to express otherwise than we have done, a quantity which becomes $= 0$ at a certain term, & which past this term, does not become real again.

4. I do not know what you will think of this solution of the problem proposed in Book V of the Memoirs of Petersburg; but I believe at least you will find it more simple, more natural & more direct than the solutions of the same problem, proposed in these Memoirs, & which all revolve on some strange considerations to the question, on the state & the fortune of the Players. Also these solutions contradict themselves & destroy one another.

5. In order to sense, by a very simple example, the little utility of these considerations in the solution which one seeks, we suppose that Pierre plays with Paul at *heads* & *tails*, on a single throw, & that he must give an ecu to Paul, if it is *tails* which comes; it is certain, (persons at least do not disown of it) that Paul must give a half-ecu to Pierre for his stake. Now it is not less certain that Paul, in giving this half-ecu, will risk accordingly more as he will be more poor; & that if he has only, for example, this half-ecu for entire possessions, his risk would be even infinite. Therefore; since in the solution of this so simple question, one has no regard to the fortune & to the state of Pierre, because one envisions the question mathematically, it is certain that one must no longer have any regard to the fortune of Pierre in the solution of the problem of the Memoirs of Petersburg. It is not that I do not believe it very reasonable to have regard for the fortune of the Players in the solution of these kinds of problems; I am likewise persuaded that the Mathematicians have quite neglected this object; but I say that the mathematical solution of the proposed question must be independent of this consideration.

6. I forget to say to you (because I make you part of my ideas on this matter to examine as they reach me and inspire the intellect) that instead of supposing the probability $\frac{1}{2^{n+\alpha n}}$

in the problem of Petersburg, it could be perhaps more correct yet to suppose $= \frac{1}{2^{n+\alpha(n-1)}}$. By this means, on the first throw where it is equally probable that one will bring about *heads* or *tails*, the probability (by making $n = 1$) will be exactly $\frac{1}{2}$, as it must be on the first throw; & the expectation of one of the Players, which must be equal to his wager, could be $\frac{1}{2} \times 1$ divided by $1 - \frac{1}{2^n}$; so that if, for example, the greatest wager is supposed ten écus as above, one will have $\frac{1}{2^\alpha} = \frac{19}{20}$, & $\alpha = \frac{\log \frac{20}{19}}{\log 2}$ =very nearly $\frac{23}{300}$.

7. According to this formula, the probability that *heads*, for example, will happen only on the second throw, will be found, (by making $n = 2$) $\frac{1}{2^{2+\frac{23}{300}}}$ instead of $\frac{1}{2^2}$, as one supposes ordinarily; & this result is nothing, it seems to me, than natural; because I ask if it is not a little more probable (physically speaking) that in two throws *tails & heads* both will happen, than there is only *tails* or *heads* will happen twice in sequence. One sees also as n is greater, the more our expression of the probability $\frac{1}{2^{n+\frac{23}{300}(n-1)}}$ or in general $\frac{1}{2^{n+\alpha(n-1)}}$ diminishes with respect to the ordinary expression $\frac{1}{2^n}$, which must be in fact in our principles; so that if, for example, $\alpha(n-1) = 1$ or $n = 1 + \frac{1}{\alpha}$, the probability will be under our principles only the half of that which one supposes it.