Disquisitiones analyticae de novo problemate coniecturali*

Daniel Bernoulli†

Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae, Bd. XIV 1 p. 3-25. 1769 (1770)

§ 1. Let there be two, three, or many urns, in which one at a time tickets in fixed and in equal number are supposed restored, but the tickets of one and any urn were distinguished from tickets of the remaining urns by its peculiar color at the beginning; next hereafter the tickets successively, nevertheless by lot, are exchanged by this law that with any trial from the urns individually one ticket is extracted, and then is transferred into the urn next in order, moreover those which were extracted from the urn positioned in the last place, are restored into the first; thus with these having been assumed and with the number of permutations, given by the aforementioned method of causes, the number of tickets of any color whatsoever is sought which probably will be contained in any urn whatever, but as often as an extraction had been made from the urns individually simultaneously, and simultaneously with it, by what I said, any transposed ticket whatsoever into the next urn only, I indicate that entire operation with the name of one permutation. Such is the argument, which now I have proposed discussing to myself; indeed it was able to be proposed generally, by assuming the initial number of tickets in the separate urns unequals to whatever degree and they mixed from the beginning in whatever manner, but I have believed to be able to give something in elegant brevity.

§ 2. However much easy the calculation may be for two urns I will apply it yet on account of the connection, which will be considered with the following: therefore let there be in the first urn n white tickets and just as many black in the other urn, there will be according to the well-known rules of combinations and probabilities, after the first permutation, the number of white tickets remaining in the first urn will be held$= n-1$, after the second permutation

$$= \frac{(n-1)(n-2)}{n} + 1,$$

after the third permutation

$$\frac{(n-1)(n-2)^2}{nn} + \frac{n-2}{n} + 1;$$

*Analytical inquiries concerning a new problem in conjecture

†Translated by Richard J. Pulskamp, Department of Mathematics and Computer Science, Xavier University, Cincinnati OH.

after the fourth

$$\frac{(n-1)(n-2)^3}{n^3}+\frac{(n-2)^2}{nn}+\frac{n-2}{n}+1;$$

after the fifth

$$\frac{(n-1)(n-2)^4}{n^4}+\frac{(n-2)^3}{n^3}+\frac{(n-2)^2}{nn}+\frac{n-2}{n}+1;$$

and thus hereafter; therefore there will be deduced, if generally the number of permutations made will have been r and there will be put for the sake of brevity $\frac{n-2}{n}=m$ the number of white tickets remaining in the first urn will be

$$=\frac{1-m^{r-1}}{1-m}+(n-1)m^{r-1}=\frac{1}{2}n(1+m^r).$$

Hence the distribution of the remaining tickets is understood through one another.

§ 3. Because m is always smaller than unity the term m^r vanishes, if r be an exceedingly great number and then the number of white tickets remaining in the first urn is simply $=\frac{1}{2}n$; the situation is asymptotic, toward which, while the permutations happen, it is approaching more and more, unless n will be either equal to unity or to two; since if a single white ticket were introduced of the first urn and a single black ticket of the other urn, there is $m=-1$ and with alternate trials either none or one white ticket will be located in the urn, since certainly our formula is changed into $\frac{1}{2}(1+(-1)^r)$; if in fact there will have been $n=2$ there is $m=0$ and the formula indicates unity whether a mean value or a probability among several possible values. I shall not press these; another is what I intend chiefly of course in order that what is going to be is investigated, since at the same time the number of tickets itself is very great, thus it is able to be considered just as for infinity, for then the number m^r besides is able to be neglected, unless r is as if infinitely greater even than the number n itself. With this supposition made we have fallen into the proof, to be able to be obtained by a single infinitesimal calculation briefly, not reasoned by the method of combinations, I have revealed in Commentationes Volume XII.[1] Therefore I have believed that the agreement of each method, that I made visible by a new argument, especially seeing that this same infinitesimal method in the following, which will be more abstract; I have decided to use equally. Now I return to the path.

§ 4. But if in the same manner the number n is considered as infinite, $n/2$ will be as if infinitely small by equal right, what I shall call α and it will make $m=1-\alpha$; therefore our formula $\frac{1}{2}n(1+m^r)$ expressing the number of white tickets remaining in the first urn will give

$$\frac{1}{2}n(1+(1-\alpha)^r);$$

It is true

$$(1-\alpha)^r=1-r\alpha+\frac{rr\alpha\alpha}{1.2}-\frac{r^3\alpha^3}{1.2.3}+\frac{r^4\alpha^4}{1.2.3.4}-\text{etc.}$$

[1] See "De Usu Algorithmi infinitesimalis in Arte Coniectandi Specimen," *Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae*, XII, 1766-67, pp. 87-98 and "De Duratione media Matrimoniorum, pro quacunque coniugum aetate, aliisque Quaestionibus affinibus," same journal, pp. 99-126.

which equal series is simply

$$c^{-r\alpha} \quad \text{or} \quad = \frac{1}{c^{r\alpha}} = \frac{1}{c^{\frac{2r}{n}}};$$

where the letter c denotes the number, of which the hyperbolic logarithm is unity, or approximately the number 2.718. Therefore if the sought number of the white tickets remaining in the first urn after r permutations is indicated by x, there will be had

$$x = \frac{1}{2}n\left(1 + \frac{1}{c^{\frac{2r}{n}}}\right).$$

§ 5. Now certainly it is sought in what way the same value may be able to be discovered by another method, by considering certainly the quantities x and r as if flowing, because certainly they are able to become so, as long as unity is able to be considered as very small with respect to the number of tickets remaining in the urn. Thus I put this by no means with difficulty it is apparent going to be

$$dx = \frac{-x}{n}dr + \frac{n-x}{n}dr,$$

where the first member is due to the extracted ticket; the other for the inserted. Hence there is

$$\frac{dx}{2x-n} = \frac{-dr}{n}$$

or

$$\frac{1}{2}\log\frac{2x-n}{n} = \frac{-r}{n}$$

or

$$\frac{2x-n}{-n} = c^{\frac{-2r}{n}},$$

whence immediately there is

$$x = \frac{1}{2}n\left(1 + \frac{1}{c^{\frac{2r}{n}}}\right).$$

clearly as we had before.

§ 6. Therefore questions of this kind are easier to solve by far, when the number of tickets is able to be considered as infinite, but that generally they change its entire nature and rarely until now they seem on account of the reckoned of the probabilities to be returned to this point, although I may have deduced a second solution out of the pure principles of the art of conjecture; for it is evident, our next solution clearly to be the same, if two vessels are supposed with two small passages communicating between them, through which a continuous transference takes place of the two fluids contained in each of the two vessels and able to be mixed perfectly at once, and if it is sought out of the mixing together for any moment of time whatsoever. I wish however it be observed numbers even moderately large are able to be considered as infinity without sensible error; there may have been, for the sake of an example, $n = 200$ and $r = 100$, the formula in the second paragraph will give the number of white tickets in the first urn probably remaining $= 136\frac{3}{5}$, while the hypothesis of an infinite magnitude produces $136\frac{2}{3}$. On the contrary the numbers should have been able to be selected much smaller.

§ 7. Our Problem takes on another shape when we propose more than two urns, but if there should have been three, immediately we fall into calculations intricate enough; I am amazed at the new nature of the calculus to be employed and thence I have judged the broader to be accessible and besides I had supposed the use of the infinitesimal algorithm in questions of this sort studied. First certainly as I shall penetrate toward an examination concerning the number of tickets as if infinite, I shall set forth the general solution deduced out of more customary principles in order that thus anyone may observe the way, if he had wished to pursue the thing further, that he must trample underfoot.

§ 8. Therefore let there be now three urns, of which the first at the beginning contains n white tickets, the second as many black, while the third as many red, and let be supposed after a given number of permutations accomplished, to remain in the first urn A white tickets, and likewise in the second and third urn the number of white tickets to be B and C. However always there is $A + B + C = n$; Thus with these assumed any one will see easily after a new permutation arriving the number of tickets in the first, second and third urn to be

$$\frac{(n-1)A + C}{n}; \quad \frac{(n-1)B + A}{n} \quad \text{and} \quad \frac{(n-1)C + B}{n};$$

however because of the known distribution of the white tickets all others become known automatically certainly the number of black tickets in the first urn will be $= C$;
in the second $= A$;
in the third $= B$;
and of the red in the first urn $= B$;
in the second urn $= C$;
in the third $= A$;
Therefore I shall investigate with respect to the single distribution of the whites. Because certainly, as we have seen, the position following from the preceding given is recognized, and the initial state was given, each variation is understood from the beginning all the way to any completed permutation. I raise the table, by which the progression of terms shines forth to a greater extent.

Number of permutations.	Number of white tickets, which probably will be in the urn. First	Second	Third
0.	n	0	0
1.	$n-1$	1	0
2.	$\frac{(n-1)^2}{n}$	$\frac{2(n-1)}{n}$	$\frac{1}{n}$
3.	$\frac{(n-1)^3-1}{nn}$	$\frac{3(n-1)^2}{nn}$	$\frac{3(n-1)}{nn}$
4.	$\frac{(n-1)^4+4(n-1)}{n^3}$	$\frac{4(n-1)^3+1}{n^3}$	$\frac{6(n-1)^2}{n^3}$
5.	$\frac{(n-1)^5+10(n-1)^2}{n^4}$	$\frac{5(n-1)^4+5(n-1)}{n^4}$	$\frac{10(n-1)^3+1}{n^4}$
6.	$\frac{(n-1)^6+20(n-1)^3+1}{n^5}$	$\frac{6(n-1)^5+15(n-1)^2}{n^5}$	$\frac{15(n-1)^4+6(n-1)}{n^5}$
7.	$\frac{(n-1)^7+35(n-1)^4+7(n-1)}{n^6}$	$\frac{7(n-1)^6+35(n-1)^3+1}{n^6}$	$\frac{21(n-1)^5+21(n-1)^2}{n^6}$
8.	$\frac{(n-1)^8+56(n-1)^5+28(n-1)^2}{n^7}$	$\frac{8(n-1)^7+70(n-1)^4+8(n-1)}{n^7}$	$\frac{28(n-1)^6+56(n-1)^3+1}{n^7}$
9.	$\frac{(n-1)^9+84(n-1)^6+84(n-1)^3+1}{n^8}$	$\frac{9(n-1)^8+26(n-1)^5+36(n-1)^2}{n^8}$	$\frac{36(n-1)^7+126(n-1)^4+9(n-1)}{n^8}$
	etc.	etc.	etc.

§ 9. Whoever will have applied the mind to the previous table, here he will observe easily the law of progression, which is of such kind. The number of completed permutations will be indicated generally by the letter r, and the binomial is assumed of which one member is $n-1$ the other 1, and that binomial is raised to the rank r, so that it is held

$$((n-1)+1)^r;$$

next this quantity is transformed into the series

$$(n-1)^r + r(n-1)^{r-1} + \frac{r.r-1}{1.2}(n-1)^{r-2} + \frac{r.r-1.r-2}{1.2.3}(n-1)^{r-3} \ldots + 1.$$

In the series, which I shall call the generator, those must be added the first, fourth, seventh etc. term, and the sum of them is divided by n^{r-1}, and thus the number of white tickets in the first urn will be had. Next if the second, fifth, eighth etc. term are added and the sum together is divided by n^{r-1} the number of white tickets in the second urn will be had. Finally the sum of the third, sixth, ninth etc. term divided by n^{r-1} will give the number of white tickets in the third urn. Whence it is proven through it, the number of all white tickets, by the power of this construction, to be continually $= n$, just as the nature of the thing demands.

§ 10. Consequently I advance the things, by which order they have been observed to me: Now certainly I have been able to understand easily out of the preceding paragraph, however many urns will have been, the aforementioned generating series always to remain the same, but the first term of it to extend to the first urn, the second term to the second urn, the third to the third and thus further until it arrived to the last urn, and then the following terms in the same order again to be assigned to the first urn, the second etc. until with the second the period is ended, by which beginning with the third made the period starts again from the first urn and thus in succession, all the way until all terms of the series must have been exhausted. Further the individual terms will be divided by n^{r-1}; by this fact the sum of all the terms pertaining to the same urn will indicate the number of all white tickets contained in that urn. And this is the general solution of our problem set forth in the first paragraph.

§ 11. Let us see now whether that general solution is in agreement with the solution, which we gave for two urns at the end § 2. where we have seen the number of white tickets contained in the first urn to be

$$= \frac{1}{2}n(1+m^r)$$

or

$$= \frac{1}{2}n\left(1+\left(\frac{n-2}{n}\right)^r\right),$$

for the sake of brevity there we had put $m = \frac{n-2}{n}$;
Certainly the present general solution gives the number of white tickets in the first urn

$$= \frac{(n-1)^r}{n^{r-1}} + \frac{r.(r-1)(n-1)^{r-2}}{1.2.n^{r-1}} + \frac{r.(r-1)(r-2)(r-3)(n-1)^{r-4}}{1.2.3.4.n^{r-1}} + \text{etc.}$$

therefore it is necessary that the aforementioned equal series must have been summed

$$= \frac{1}{2}n\left(1+\left(\frac{n-2}{n}\right)^r\right),$$

in order that that equality is evident, we shall investigate the aforementioned quantity under this other form

$$= \frac{1}{2}n\left(\left(\frac{(n-1)-1}{n}\right)^r+1\right),$$

or

$$\frac{((n-1)-1)^r+((n-1)+1)^r}{2n^{r-1}};$$

Now certainly if both binomial numerators in the indefinite series are converted it will happen that clearly the same terms alternately either are doubled or are destroyed and thus the general solutions must exhibit the form its very self. But for this particular case at any rate the expression of the second paragraph is to be preferred as by far more advantageous.

If a single urn is considered, the same number of white tickets must remain certainly in that without change, because it itself indicates the general solution equally.

§ 12. Now I approach nearer to it, what I had most resolved of course in order that I may reveal for three urns in what way that work is able to be accomplished by the power of the infinitesimal calculus, if there are as many tickets as possible and the permutations are repeated more frequently so that the numbers n and r are able to be counted as if infinite, of which kind we have made the examination for two urns §§ 4 and 5, I shall make this inquiry with each method, certainly by ordinary analysis and by infinitesimals, that the agreement may shine forth between the two again.

If the discussion may concern the first urn immediately, we shall have, by the power of the ninth paragraph, the number of white tickets contained in that thing, for the present with the altered form and by neglecting the neglected terms thus the expression,

$$n.\left(1-\frac{1}{n}\right)^r\left(1+\frac{r^3}{1.2.3n^3}+\frac{r^6}{1.2.3.4.5.6n^6}+\text{etc.}\right).$$

It is true through the fourth paragraph the quantity

$$\left(1-\frac{1}{n}\right)^r = \frac{1}{c^{\frac{r}{n}}},$$

which quantity is able to be substituted hence; presently by a like method the number of white tickets in the second urn and finally in the third is able to be defined, we have discovered by this method the number of white tickets.
In urn I.

$$= \frac{n}{c^{\frac{r}{n}}}\left(1+\frac{r^3}{1.2.3n^3}+\frac{r^6}{1.2.3.4.5.6n^6}+\text{etc.}\right)$$

In urn II.

$$= \frac{n}{c^{\frac{r}{n}}}\left(\frac{r}{n}+\frac{r^4}{1.2.3.4n^4}+\frac{r^7}{1.2.3.4.5.6.7n^7}+\text{etc.}\right)$$

In urn III.

$$= \frac{n}{c^{\frac{r}{n}}}\left(\frac{rr}{1.2.n.n}+\frac{r^5}{1.2.3.4.5n^5}+\frac{r^8}{1.2.3.4.5.6.7.8n^8}+\text{etc.}\right)$$

There remains to be performed here also that we try to reduce the latter factors, expressed as infinite series in these formulas, to a concisely expressed quantity with finite terms which at least I have not mentioned with regard to the thing, anything had been fulfilled by others now; whatever it be, by no means I suppose it is by my thing, if all likewise I will expose for view.

§ 13. I shall begin with the first infinite series

$$1+\frac{r^3}{1.2.3n^3}+\frac{r^6}{1.2.3.4.5.6n^6}+\text{etc.}$$

which sum S is sought, and thus we shall have

$$1+\frac{r^3}{1.2.3n^3}+\frac{r^6}{1.2.3.4.5.6n^6}+\text{etc.} = S$$

Of this equation the differential element dr of the third order is assumed to be set constant and the quantity r to be considered as if flowing; thus we shall return to the proposed series, multiplied by $\frac{dr^3}{n^3}$,
and hence we will have

$$\frac{Sdr^3}{n^3} = d^3S.$$

The completed integration of this equation necessarily will contain three constant quantities, truly easily it is apparent the equation to be able to be assumed

$$S = \alpha c^{\frac{mr}{n}},$$

if there is put $m^3 = 1$;
thus m will have three roots, namely $m = 1$;

$$m = (-1 + \sqrt{-3}) : 2$$

and

$$m = (-1 - \sqrt{-3}) : 2,$$

on account of which now we are able to assume the equation

$$S = \alpha c^{\frac{r}{n}} + \beta c^{(-1+\sqrt{-3})\frac{r}{2n}} + \gamma c^{(-1-\sqrt{-3})\frac{r}{2n}},$$

which thus takes delight in the three constant arbitrary quantities α, β and γ, of which it is able to be satisfied by help of individual circumstances, which are established with regard to it, because the number of tickets initially in any urn is arbitrary, if the most general question is posed; whence our argument is proven, if even ten urns are supposed unless it is not able to be explained correctly by a differential equation of the tenth order, nevertheless any such always will admit complete integration. But it is known, the imaginary exponential quantities to be able to be converted into sines and cosines; there is namely

$$c^{\frac{r\sqrt{-3}}{2n}} = \sin\frac{r\sqrt{3}}{2n} \quad \text{and} \quad c^{\frac{-r\sqrt{-3}}{2n}} = \cos\frac{r\sqrt{3}}{2n}$$

and with these substitutions completed we shall obtain the equation expressed with the pure real terms, namely

$$S = \alpha\, c^{\frac{r}{2n}} + \beta c^{\frac{-r}{2n}} \sin\frac{r\sqrt{3}}{2n} + \gamma c^{\frac{-r}{2n}} \cos\frac{r\sqrt{3}}{2n}.$$

Now no other remains than that the constant quantities α, β and γ are determined; it certainly must be desired next in order, because $S = 1$, $dS = 0$ must happen by the fact $r = 0$, and $ddS = 0$ hence it follows $\alpha + \gamma = 1$; next

$$\frac{\alpha}{n} + \frac{\beta\sqrt{3}}{2n} - \frac{\gamma}{2n} = 0$$

and

$$\frac{\alpha}{nn} - \frac{\beta\sqrt{3}}{2nn} - \frac{\gamma}{2nn} = 0.$$

Out of this equation it is deduced $\alpha = \frac{1}{3}$; $\beta = 0$ and $\gamma = \frac{2}{3}$, and thus there is

$$S = \frac{1}{3}c^{\frac{r}{n}} + \frac{2}{3}c^{\frac{-r}{2n}} \cos\frac{r\sqrt{3}}{2n}.$$

And that value, expressed by limited and likewise real terms, is able to be substituted into the series continued to infinity

$$1 + \frac{r^3}{1.2.3n^3} + \frac{r^6}{1.2.3.4.5.6n^6} + \text{etc.}$$

let be for the sake of example

$$\frac{r\sqrt{3}}{2n} = \text{to the quadrant of a circle of which unity expresses the radius} = \pi = \text{approximately } \frac{11}{7},$$

it will happen

$$S = \frac{1}{3}c^{\frac{2\pi}{\sqrt{3}}}.$$

This to the first urn.

In the same way clearly the value of the second series is determined which pertains to the second urn; naturally likewise that value is expressed by the general equation with this single distinction insofar as the coefficients α, β and γ now acquire another value, seeing that by the fact $r = 0$ this second series must become $= 0$, or, if the series is indicated by S', it is necessary there be $S' = 0$; next

$$\frac{dS'}{dr} = \frac{1}{n} \quad \text{and} \quad \frac{ddS'}{dr^2} = 0,$$

whence

$$\alpha + \gamma = 0;$$
$$\alpha + \frac{1}{2}\beta\sqrt{3} - \frac{1}{2}\gamma = 1$$

and

$$\alpha - \frac{1}{2}\beta\sqrt{3} - \frac{1}{2}\gamma = 0$$

and thus there is

$$\alpha = \frac{1}{3}; \qquad \beta = \frac{1}{\sqrt{3}} \quad \text{and} \quad \gamma = -\frac{1}{3};$$

Therefore

$$S' = \frac{1}{3}c^{\frac{r}{n}} + \frac{1}{\sqrt{3}}c^{\frac{-r}{2n}} \sin\frac{r\sqrt{3}}{2n} - \frac{1}{3}c^{\frac{-r}{2n}} \cos\frac{r\sqrt{3}}{2n}.$$

Finally if we indicate the sum of the series pertaining to the third urn by S'', we shall discover with little change and with them completely ready at hand.

$$S'' = \frac{1}{3}c^{\frac{r}{n}} - \frac{1}{\sqrt{3}}c^{\frac{-r}{2n}} \sin\frac{r\sqrt{3}}{2n} - \frac{1}{3}c^{\frac{-r}{2n}} \cos\frac{r\sqrt{3}}{2n}.$$

§ 14. But if now in paragraph 12 we substitute the values just now discovered in place of the series we shall discover the number of white tickets for each urn, namely In the first urn

$$= \frac{1}{3}n + \frac{2}{3}n.c^{\frac{-3r}{2n}} \cos\frac{r\sqrt{3}}{2n}.$$

In the second urn

$$= \frac{1}{3}n + \frac{n}{\sqrt{3}}c^{\frac{-3r}{2n}} \sin\frac{r\sqrt{3}}{2n} - \frac{n}{3}c^{\frac{-3r}{2n}} \cos\frac{r\sqrt{3}}{2n}$$

In the third urn

$$= \frac{1}{3}n - \frac{n}{\sqrt{3}}c^{\frac{-3r}{2n}} \sin\frac{r\sqrt{3}}{2n} - \frac{n}{3}c^{\frac{-3r}{2n}} \cos\frac{r\sqrt{3}}{2n}.$$

Next the black tickets will have themselves in the second urn, the third as well as the first and likewise the red tickets in the third, first and second urn, just as the white tickets have themselves in the first, second and third urn, thus as the distributions of the tickets, all and one at a time, in individual urns, now have been determined exactly according to the rules of probability under the hypothesis that the numbers n and r are thus great, in order that they are able to be held as if for infinite. And all such are expressed with finite terms, which formerly were not able to be determined in any other way than through indefinite series. It should be observed further I should wish, each and every to have been deduced from the principles of the art of conjecture by common analysis; since § 13. they have been said not so much for sake of the thing itself, than I have exposed such for the sake of a more advantageous and more elegant calculation, moreover yet in order that I may reveal a conspicuous agreement between the common methods and the new method with a remarkable example, which use by single infinitesimal calculations now I have employed everywhere for questions of this sort: I admit for my part also that new method to have its own thorns and troubles; but, unless I am deceived, by so much more it will respond to our scope and by its own excellence and novelty likewise it will be commendable. That other method itself requires that the numbers n and r are very great, which same we have assumed in the last three paragraphs continuously.

§ 15. Therefore let there be, with the other names retained, the number of white tickets in the first urn $= x$
and in the second urn $= y$;
thus that number for the third urn will be $= n - x - y$;
if now we should consider the quantities x, y, and z as if flowing, there will be

$$dx = \frac{-x}{n}dr + \frac{n-x-y}{n}dr,$$

where the first member is responsible for the extraction from the first urn, the other for the transportation out of the third urn into the first; next

$$dr = \frac{ndx}{n-2x-y};$$

similarly there will be

$$dy = -\frac{ydr}{n} + \frac{xdr}{n}$$

or

$$dr = \frac{ndy}{x-y},$$

next

$$\frac{dx}{n-2x-y} = \frac{dy}{x-y}$$

or

$$xdx - ydx = ndy - 2xdy - ydy;$$

this equation will become a little simpler if there is put $x = \frac{1}{3}n + p$
and $y = \frac{1}{3}n - q$,
thus indeed it becomes

$$2pdq - qdq = pdp + qdp.$$

This equation since it is not yet able to be integrated on account of the mixing of the indeterminates, I shall put
$$q = tp$$
and
$$dq = tdp + pdt;$$
thus it happens, if the calculus is set correctly

$$\frac{dp}{p} = \frac{t-2}{t-tt-1}dt,$$

which last equation thus is to be integrated, when from the beginning there is $x = n$ and $y = 0$, or when there is $p = \frac{2}{3}n$ and $t = \frac{1}{2}$; but the relationship between p and t will be considered with the equation to be integrated, and thence the relationship between x and y will be deduced so that y may be able to be determined through x, by which fact finally it will be returned to the elementary equation

$$dr = \frac{ndy}{x-y}$$

and the integration of it to be attempted, in order that thus the relationship between r and x may be had. But certainly that method, which first offers itself, becomes very complex and clearly useless; therefore forced to enter upon this road I have attacked the thing.

We have obtained above
$$\frac{dp}{p} = \frac{t-2}{t-tt-1}dt$$
together with this other equation

$$\frac{dr}{n} = \frac{dy}{x-y},$$

which with the substitutions made appropriately, which we have assumed, gives

$$\frac{dr}{n} = \frac{dt}{t-tt-1};$$

and each equation now is able to be integrated and thus the value of the quantity p to be determined just as also the value of the quantity t through the functions of the quantity r; thus the entire work will be absolved most agreeably.

Let there be put
$$t = s + \tfrac{1}{2}$$
and now there will be obtained

$$\frac{dp}{p} = \frac{\frac{3}{2}ds}{ss+\frac{3}{4}} - \frac{sds}{ss+\frac{3}{4}}$$

and likewise
$$\frac{dr}{n} = \frac{-ds}{ss+\frac{3}{4}};$$

Hence
$$\frac{dp}{p} = -\frac{3dr}{2n} - \frac{sds}{ss+\frac{3}{4}},$$

which integral is

$$\log \frac{p}{\frac{2}{3}n} = -\frac{3r}{2n} - \frac{1}{2}\log \frac{ss+\frac{3}{4}}{\frac{3}{4}}$$

or

$$p = \frac{2nc^{\frac{-3r}{2n}}}{3\sqrt{\left(1+\frac{4}{3}ss\right)}}.$$

Again the equation

$$\frac{dr}{n} = \frac{-ds}{ss+\frac{3}{4}}$$

gives

$$\frac{r}{n} = \frac{2}{\sqrt{3}}\arctan\frac{-2s}{\sqrt{3}}$$

or

$$\frac{r}{n} = \frac{2}{\sqrt{3}}\,\text{arcsec}\,\sqrt{1+\frac{4}{3}ss},$$

which if inverted, gives

$$\sqrt{\left(1+\frac{4}{3}ss\right)} = \sec\text{arc}.\frac{r\sqrt{3}}{2n};$$

let that value be substituted and thus there will be had

$$p = \frac{2nc^{\frac{-3r}{2n}}}{3\sec\text{arc}\ \frac{r\sqrt{3}}{2n}} \quad \text{or} \quad p = \frac{2n\cos\text{arc}\ \frac{r\sqrt{3}}{2n}}{3c^{\frac{3r}{2n}}}.$$

Seeing that in fact it will become

$$\frac{r}{n} = \frac{2}{\sqrt{3}}\arctan -\frac{2s}{\sqrt{3}},$$

there will be

$$-\frac{2s}{\sqrt{3}} = \tan\text{arc}\ \frac{r\sqrt{3}}{2n}$$

or

$$s = t - \frac{1}{2} = -\frac{\sqrt{3}}{2}\tan\ \text{arc}\ \frac{r\sqrt{3}}{2n}.$$

whence

$$t = \frac{1}{2} - \frac{\sqrt{3}}{2}\tan\ \text{arc}\ \frac{r\sqrt{3}}{2n}.$$

or

$$t = \frac{1}{2} - \frac{\sqrt{3}\times\sin\frac{r\sqrt{3}}{2n}}{2\cos\frac{r\sqrt{3}}{2n}}.$$

Because finally we have put above $x = \frac{1}{3}n + p$
and $y = \frac{1}{3}n - q = \frac{1}{3}n - tp,$

it is proper to substitute the discovered values of the quantities p and t, and thus we shall discover the sought numbers x and y, which denote the numbers of white tickets contained in the first and second urns, which two together if they are subtracted from the number n, the number of white tickets in the third urn will be had. Thus with all these facts, we obtain the number of white tickets

In the first urn

$$= \frac{1}{3}n + \frac{2}{3}n \cdot c^{\frac{-3r}{2n}} \cdot \cos \text{arc} \frac{r\sqrt{3}}{2n}$$

In the second urn

$$= \frac{1}{3}n + \frac{n}{\sqrt{3}} \cdot c^{\frac{-3r}{2n}} \cdot \sin \text{arc} \frac{r\sqrt{3}}{2n} - \frac{n}{3} \cdot \cos \text{arc} \frac{r\sqrt{3}}{2n}$$

In the third urn

$$= \frac{1}{3}n - \frac{n}{\sqrt{3}} \cdot c^{\frac{-3r}{2n}} \cdot \sin \text{arc} \frac{r\sqrt{3}}{2n} - \frac{n}{3} \cdot \cos \text{arc} \frac{r\sqrt{3}}{2n}.$$

And those formulas clearly are the same with the ones which we have exposed § 14, and it is not so much from its value but even itself from the expression; the agreement between my new method and the one constructed over familiar principles has appeared to me remarkable.

§ 16. I shall add a little concerning the special character of the argument; and in the first place certainly it is apparent immediately the entire system to incline toward a lasting and asymptotic state, which finally it may achieve after permutations infinite in number; at that time certainly all tickets have been mixed thoroughly in the separate urns with equal parts; nor was it difficult to foresee; the true manner by which continuously the approach to the steady state happens, clearly was unexpected to me; namely I mistrusted it to be that the number of white tickets in the first urn must decrease indefinitely, in the second and third must increase in turn; but now I see the infinite limits to become the state beyond and nearer in any urn, which will be lasting, transitions and variations with the continuous undulatory movement decreasing, to be diminished and finally to vanish. But the analytic formulas deserve to be noted by which the arrivals and retreats of this sort and continuous diminutions of the same sort are revealed: the arrivals and retreats follow the nature of the sine and of the cosine and the diminutions of them must be from the common exponential factor $c^{-3r/2n}$. But expressions of this kind with regard to various physico-mechanical questions, or at least similar ones, I remember to have obtained by myself many times. Now let us inquire into particular Cases.

§ 17. We shall consider first all those Cases in which

$$\cos \text{ arc} \frac{r\sqrt{3}}{2n} = 0,$$

that is, in which $\frac{r\sqrt{3}}{2n}$ is either equal to the quadrant of the circle, of which the radius is unity, or to the triple either to the quintuple or to the septuple etc. quadrant. Let the quadrant of that circle

be namely $= q$
and let us make $\frac{r\sqrt{3}}{2n} = q$
or $= 3q$
or $= 5q$
or $= 7q$ etc.
and so of the following either $\frac{r}{n} = \frac{2q}{\sqrt{3}}$
or $= \frac{6q}{\sqrt{3}}$
or $= \frac{10q}{\sqrt{3}}$
or $= \frac{14q}{\sqrt{3}}$ etc.
In all those cases, with the number to infinity, it becomes

$$\cos \text{ arc } \frac{r\sqrt{3}}{2n} = 0$$

and

$$\sin \text{ arc } \frac{r\sqrt{3}}{2n} = 1$$

and so the number of white tickets in the first urn $= \frac{1}{3}n$; certainly in the second urn will be that number of the following either

$$= \frac{1}{3}n + \frac{n}{\sqrt{3}}c^{-q\sqrt{3}}$$

or

$$\frac{1}{3}n + \frac{n}{\sqrt{3}}c^{-3q\sqrt{3}}$$

or

$$\frac{1}{3}n + \frac{n}{\sqrt{3}}c^{-5q\sqrt{3}}$$

or

$$\frac{1}{3}n + \frac{n}{\sqrt{3}}c^{-7q\sqrt{3}} \quad \text{etc.}$$

Finally if we accept all those negative exponential quantities we will have successively the numbers of white tickets in the third urn. Let there be, for the sake of an example, $n = 3000$ and let us put $q = \frac{11}{7}$ and $c = 2.718$, there will be after the first 5443 permutations the number of white tickets in the first urn $= 1000$, in the second urn $= 1146$ and in the third urn $= 854$; then if 10886 new permutations should arrive, the similar numbers 1000, 1001, and 999 will happen. Now therefore all will have been reduced nearly to the permanent state. Thence it is apparent, what must happen in order that the permanent state not be, even if the number of white tickets in the first urn will have been reduced to the third part, and indeed with the first interchange it happens, the number of black tickets in the same first urn will be $= 854$ and the number of red tickets $= 1146$, therefore the permanent state is unable to be.

§ 18. But if now we wish to know the number of permutations r, after which the number of white tickets from the first by interchange into the second urn will have been reduced to one-third, it is necessary the two terms

$$\frac{n}{\sqrt{3}} \cdot c^{\frac{-3r}{2n}} \cdot \sin .\text{arc.}\, \frac{r\sqrt{3}}{2n} - \frac{n}{3} \cdot c^{\frac{-3r}{2n}} \cdot \cos .\text{arc.}\, \frac{r\sqrt{3}}{2n} \text{ to become } = 0,$$

but this condition is obtained when there is

$$\text{arc.}\, \frac{r\sqrt{3}}{2n} = \frac{1}{3}q$$

or

$$\frac{r}{n} = \frac{2q}{3\sqrt{3}},$$

which number produces only the third part of it which was required for the first urn; for it is between 1814 and 1815. In like manner this question is determined for the third urn, where now there will be

$$\frac{r}{n} = \frac{10q}{3\sqrt{3}},$$

or five times more than for the second urn and so $r = 9072$ approximately.

But as often as the number of white tickets be $= \frac{1}{3}n$, in whatever urn it may happen, then from that value it recedes again sometimes in one sometimes in another part and later for a second time it approaches and these maximum retreats happen somewhere for any period whatsoever; yet after the first period they all do not vanish so much.

§ 19. Finally let us ask, what is about to be according to the laws of probabilities, for whatever urn the minimum or maximum number of tickets of whatever color, which must be presumed at some time and how many permutations must be required that it may arrive to that place. I shall begin with the white tickets in the first urn: however there are two ways, in which those maxima or minima are able to be defined; the one has been posited in this, that the differential of the numbers of the tickets becomes $= 0$, the other that that number in the first urn becomes equal to the number in the third urn, for then an extraction from the first urn is equal to the transposition from the third urn into the first and each method returns to the same, because the very one confirms our formulas so much more. Thus we understand to be

$$\cos .\text{arc.}\, \frac{r\sqrt{3}}{2n} = \frac{-1}{\sqrt{3}} \sin .\text{arc.} \frac{r\sqrt{3}}{2n}$$

or

$$\tan .\text{arc.}\, \frac{r\sqrt{3}}{2n} = -\sqrt{3};$$

how often it satisfies to this condition, but is able to be satisfied in infinite ways, some minimum or maximum is obtained, but the first will be among the least minima. But the tangent $-\sqrt{3}$ indicates the arc 120°, it is therefore

$$\frac{r\sqrt{3}}{2n} = \frac{4}{3}q$$

or

$$\frac{r}{n} = \frac{8}{3\sqrt{3}}q,$$

thus $r = 7257$; therefore after 7257 permutations the number of white tickets in the first urn is, as great as it is ever to become, and the minimum descends from the initial number 3000 to 953, and then a second time it increases but very little beyond 1000 and in turn it decreases but it descends scarcely below 1000; but seeing that in these cases the numbers of white tickets in the first urn and the third are the same, we will have equally after 7257 permutations 953 white tickets in the third urn and exactly 1094 in the second urn. Thus after the same 7257 permutations there will be in the second urn 953 black tickets, in the third 1094 and in the first again 953; finally there will be likewise in the third urn 953 red tickets, in the first 1094 and in the second 953; in this way the individual tickets for any urn are determined for the case where the white tickets in the first urn will have been reduced to the minimum number.

§ 20. But also again let us expose by our calculation, to what maximum number the white tickets are able to rise in the second urn and the third and how many permutations are required in order that it happen. However it is clear for the second urn to be satisfied to this equation

$$\cos.\text{arc.}\frac{r\sqrt{3}}{2n} = \frac{1}{\sqrt{3}}\sin.\text{arc.}\frac{r\sqrt{3}}{2n},$$

or

$$\tan.\text{arc.}\frac{r\sqrt{3}}{2n} = \sqrt{3};$$

thus

$$\frac{r\sqrt{3}}{2n} = \frac{2}{3}q$$

or

$$\frac{r}{n} = \frac{4q}{3\sqrt{3}} = 1.210;$$

thus $r = 3630$ and then the number of white tickets in the second urn becomes 1163, which hence is the maximum number, to what white tickets in the second urn are able to grow and it is obtained after 3630 permutations and consequently at the time the first urn equally will contain 1163 white tickets, the third urn certainly only 674. Finally the maximum number of white tickets may be in the third urn, when there is put

$$\frac{r\sqrt{3}}{2n} = 2q$$

or $r = 10890$ and then that number is 1004 which likewise prevails for the second urn.

Which particular movements have been reported for the white tickets, these alone will be able to be applied by transposition of the urns to the black and red tickets. The delicate argument through itself was nevertheless with various distinctions, if I am not deceived, useful, but especially I research what I have considered convenient.