Notes to accompany Disquisitiones analyticae de novo problemate coniecturali

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Bernoulli's aim is to show that the infinitesimal calculus will produce the same result as that obtained by combinatorial means when large numbers are involved. The paper is quite easy to understand. His results are for the most part entirely correct. However, he has made several computational errors.

The two urn problem

We are given two urns of which the first contains n white tickets and the second n black tickets. One ticket is drawn at random from the first urn and then placed into the second. Similarly, after mixing, one ticket is drawn from the second urn and placed into the first. This process, which Bernoulli calls a permutation, is performed a total of r times. We seek the expected number of white tickets in the first urn at time r.

Combinatorial solution

Let the expected number of white tickets in the first urn be denoted by $w(r)$ and in the second by $v(r)$. We have always $w(r) + v(r) = n$ and the initial conditions $w(0) = n$, $v(0) = 0$.

At time $r+1$, the first urn will lose a white ticket with probability $w(r)/n$ and gain a white ticket with probability $v(r)/n$. Therefore we have

$$w(r+1) = w(r) - \frac{w(r)}{n} + \frac{v(r)}{n}$$

Upon elimination of $v(r)$, using the condition $w(r) + v(r) = n$, we obtain

$$w(r+1) = w(r)\left(1 - \frac{2}{n}\right) + 1$$

The solution of this recurrence relation is easy and is

$$w(r) = \frac{n}{2}\left(\frac{n-2}{n}\right)^r + \frac{n}{2}$$

Since $0 < \frac{n-2}{n} < 1$, it follows that $w(r)$ converges to $\frac{n}{2}$ as r approaches infinity. Moreover, since $\lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n = \mathrm{e}^x$, when n is very great we may approximate $w(r)$ as

$$w(r) \approx \frac{n}{2}\left(1 + \mathrm{e}^{-2r/n}\right)$$

Infinitesimal solution

If $\Delta r = 1$, then $\Delta w = -1$ with probability $w(r)/n$, occurring when a white ticket is drawn, and $\Delta w = 1$ with probability $1 - w(r)/n$, occurring when a white ticket is inserted, it having been drawn from the second urn. Thus we may write

$$\Delta w = -\frac{w\Delta r}{n} + \frac{(n-w)\Delta r}{n}$$

If we treat w as a continuous function of r, then we may rewrite this as

$$dw = -\frac{wdr}{n} + \frac{(n-w)dr}{n}$$

The resulting differential equation, subject to the initial condition $w(0) = n$, may be solved by the separation of variable technique. Namely,

$$\frac{dw}{2w-n} = -\frac{dr}{n}$$

has solution

$$w(r) = \frac{n}{2} + \frac{n}{2}\mathrm{e}^{-2r/n}$$

Bernoulli offers one comparison. With $n = 200$ and $r = 100$, the combinatorial value of $w(100) = 136.603$ or nearly $136\frac{3}{5}$. The approximate value of $w(100) = 136.788$ or nearly $136\frac{4}{5}$. Bernoulli errors slightly in reporting the value as $136\frac{2}{3}$.

The three urn problem

Combinatorial solution

Let there now be three urns. The first contains n white tickets, the second n black and the third n red. A ticket is extracted at random from the first and placed into the second, likewise a ticket is extracted from the second and placed into the third, and finally a ticket is extracted from the third and placed into the first. This process is continued a total of r times. The expected number of white tickets in each urn is sought.

Let $w(r)$, $v(r)$ and $u(r)$ denote the expected number of white tickets in urn 1, urn 2 and urn 3 respectively after r cycles of extractions and insertions.

We have the recurrence relations

$$\begin{aligned} w(r+1) &= w(r) - \frac{w(r)}{n} + \frac{u(r)}{n} \\ v(r+1) &= v(r) - \frac{v(r)}{n} + \frac{w(r)}{n} \\ u(r+1) &= u(r) - \frac{u(r)}{n} + \frac{v(r)}{n} \end{aligned}$$

for which $w(r)+v(r)+u(r) = n$ for all r and the initial conditions $w(r) = n,\ v(r) = u(r) = 0$.

After computing the first few values of $w,\ u,$ and v, Bernoulli observes from the pattern that the solution appears to be given by the generating function

$$n^r = ((n-1)+1)^r = \sum_{k=0}^{r} \binom{r}{k} (n-1)^{r-k}$$

Indeed he claims

$$\begin{aligned} w(r)\cdot n^{r-1} &= \binom{r}{0}(n-1)^{r-0} + \binom{r}{3}(n-1)^{r-3} + \binom{r}{6}(n-1)^{r-6} + \cdots \\ v(r)\cdot n^{r-1} &= \binom{r}{1}(n-1)^{r-1} + \binom{r}{4}(n-1)^{r-4} + \binom{r}{7}(n-1)^{r-7} + \cdots \\ w(r)\cdot n^{r-1} &= \binom{r}{2}(n-1)^{r-2} + \binom{r}{5}(n-1)^{r-5} + \binom{r}{8}(n-1)^{r-8} + \cdots \end{aligned}$$

Verification of solution

Certainly,

$$\begin{aligned} w(1)\cdot n^0 &= \binom{1}{0}(n-1)^1 + \binom{1}{3}(n-3)^{-2} + \cdots = n-1, \\ v(1)\cdot n^0 &= \binom{1}{1}(n-1)^0 + \binom{1}{4}(n-3)^{-3} + \cdots = 1 \\ u(1)\cdot n^0 &= \binom{1}{2}(n-1)^{-2} + \binom{1}{5}(n-1)^{-4} + \cdots = 0 \end{aligned}$$

Now the first recurrence relation

$$w(r+1) = w(r) - \frac{w(r)}{n} + \frac{u(r)}{n} \tag{1}$$

may be rewritten as

$$w(r+1) = w(r)\left(\frac{n-1}{n}\right) + \frac{u(r)}{n}$$

Moreover, we may rewrite $w(r)$ as

$$w(r) = \left(\frac{n-1}{n}\right)^r \cdot n + \binom{r}{3}\left(\frac{n-1}{n}\right)^{r-3} \cdot \frac{1}{n^2} + \binom{r}{6}\left(\frac{n-1}{n}\right)^{r-6} \cdot \frac{1}{n^5} + \cdots$$

and $u(r)$ as

$$u(r) = \binom{r}{2}\left(\frac{n-1}{n}\right)^{r-2} \cdot \frac{1}{n} + \binom{r}{5}\left(\frac{n-1}{n}\right)^{r-5} \cdot \frac{1}{n^4} + \binom{r}{8}\left(\frac{n-1}{n}\right)^{r-8} \cdot \frac{1}{n^7} + \cdots$$

Substituting these series into the recurrence relation (1), we have

$$\begin{aligned} w(r+1) =& w(r)\left(\frac{n-1}{n}\right) + \frac{u(r)}{n} \\ &= \left(\frac{n-1}{n}\right)^{r} \cdot n \cdot \left(\frac{n-1}{n}\right) \\ &+ \binom{r}{3}\left(\frac{n-1}{n}\right)^{r-3} \cdot \frac{1}{n^2} \cdot \left(\frac{n-1}{n}\right) + \binom{r}{6}\left(\frac{n-1}{n}\right)^{r-6} \cdot \frac{1}{n^5} \cdot \left(\frac{n-1}{n}\right) + \cdots \\ &+ \binom{r}{2}\left(\frac{n-1}{n}\right)^{r-2} \cdot \frac{1}{n} \cdot \frac{1}{n} + \binom{r}{5}\left(\frac{n-1}{n}\right)^{r-4} \cdot \frac{1}{n^4} \cdot \frac{1}{n} + \cdots \end{aligned}$$

which may be simplified first to

$$\begin{aligned} w(r+1) = &\left(\frac{n-1}{n}\right)^{r+1} \cdot n \\ &+ \binom{r}{3}\left(\frac{n-1}{n}\right)^{r-2} \cdot \frac{1}{n^2} + \binom{r}{6}\left(\frac{n-1}{n}\right)^{r-5} \cdot \frac{1}{n^5} + \cdots \\ &+ \binom{r}{2}\left(\frac{n-1}{n}\right)^{r-2} \cdot \frac{1}{n^2} + \binom{r}{5}\left(\frac{n-1}{n}\right)^{r-4} \cdot \frac{1}{n^5} + \cdots \end{aligned}$$

and then to

$$\begin{aligned} w(r+1) = &\left(\frac{n-1}{n}\right)^{r+1} \cdot n \\ &+ \left[\binom{r}{3} + \binom{r}{2}\right]\left(\frac{n-1}{n}\right)^{r-2} \cdot \frac{1}{n^2} + \cdots \\ &+ \left[\binom{r}{6} + \binom{r}{5}\right]\left(\frac{n-1}{n}\right)^{r-4} \cdot \frac{1}{n^5} + \cdots \end{aligned}$$

This finally is

$$\left(\frac{n-1}{n}\right)^{r+1} \cdot n + \binom{r+1}{3}\left(\frac{n-1}{n}\right)^{r-2} \cdot \frac{1}{n^2} + \binom{r+1}{6}\left(\frac{n-1}{n}\right)^{r-4} \cdot \frac{1}{n^5} + \cdots$$

which is immediately recognized as the series representing $w(r+1)$.

The verification of the other two solutions is handled similarly.

Approximation of solution

We examine $w(r)$ in the case where n and r are extremely large. As before

$$w(r) = \left(\frac{n-1}{n}\right)^{r} \cdot n + \binom{r}{3}\left(\frac{n-1}{n}\right)^{r-3} \cdot \frac{1}{n^2} + \binom{r}{6}\left(\frac{n-1}{n}\right)^{r-6} \cdot \frac{1}{n^5} + \cdots$$

The following approximations hold

1. For large n, $\left(1-\frac{1}{n}\right)^r \approx e^{-r/n}$

2. As long as k is not too great with respect to r, $\binom{r}{k} \approx \frac{r^k}{k!}$

3. For large n and k small with respect to n, $\left(\frac{n-1}{n}\right)^{r-k} \approx \left(1-\frac{1}{n}\right)^r$

Hence, we have

$$w(r) \approx n \cdot e^{\frac{-r}{n}} \left(1 + \frac{r^3}{3!n^3} + \frac{r^6}{6!n^6} + \cdots\right)$$

To find a closed form for the series, we put

$$S = 1 + \frac{r^3}{3!n^3} + \frac{r^6}{6!n^6} + \cdots$$

and note that $\frac{d^3S}{dr^3} = \frac{S}{n^3}$ subject to the initial conditions that $S(0) = 1$, $S'(0) = S''(0) = 0$. This gives

$$S = \frac{1}{3}\mathrm{e}^{r/n} + \frac{2}{3}\mathrm{e}^{-r/(2n)} \cos\left(\frac{r\sqrt{3}}{2n}\right)$$

so that

$$w(r) = \frac{n}{3}\left(1 + 2\mathrm{e}^{-3r/(2n)} \cos\left(\frac{r\sqrt{3}}{2n}\right)\right)$$

For the second urn, the initial conditions must be $S(0) = S''(0) = 0$, $S'(0) = 1/n$.

Infinitesimal method – Construction of the differential equation

Let x, y and z be the number of white tickets in the first, second and third urn respectively. As usual r denotes the number of cycles of extraction and insertion of tickets. Formulas for x and y in terms of r are to be found.

To this end, we note that

$$dx = -\frac{xdr}{n} + \frac{(n-x-y)dr}{n}$$
$$dy = -\frac{ydr}{n} + \frac{xdr}{n}$$

so that we may write

$$dr = \frac{ndx}{n-2x-y} \text{ and } dr = \frac{ndy}{x-y}$$

and further equate the two.

Since, in the limit, x and y will converge to $n/3$, Bernoulli makes the substitutions of $x = \frac{n}{3} + p$ and $y = \frac{n}{3} - q$ for which $dx = dp$ and $dy = -dq$. This gives

$$\frac{ndp}{2p - q} = \frac{ndq}{p + q}$$

Now Bernoulli needs to put this into a form which is integrable. To this end he puts $q = tp$ so that $dq = tdp + pdt$. This gives

$$\frac{ndp}{2p - tp} = \frac{n(tdp + pdt)}{tp + p}$$

or, after separating the variables,

$$\frac{dp}{p} = \frac{(t-2)dt}{t - t^2 - 1}$$

We now return to dr. Substituting successively for x, y and q, we have

$$\begin{aligned} dr &= \frac{ndy}{x - y} = \frac{-ndq}{p + q} = \frac{-n(tdp + pdt)}{tp + p} = \frac{-n}{1 + t}\left(t\frac{dp}{p} - dt\right) \\ &= \frac{ndt}{t - t^2 - 1} \end{aligned}$$

Since $t = \frac{1}{2}$ when $r = 0$, Bernoulli puts $t = s + \frac{1}{2}$, then since $dt = ds$ we have

$$\frac{dp}{p} = \frac{(t-2)dt}{t - t^2 - 1} = \frac{(3 - 2s)ds}{2\left(s^2 + \frac{3}{4}\right)}$$

and

$$dr = \frac{-nds}{s^2 + \frac{3}{4}}.$$

Finally, Bernoulli writes

$$\frac{dp}{p} = -\frac{3}{2n}dr - \frac{sds}{s^2 + \frac{3}{4}}. \tag{2}$$

Solution of the differential equation (2)

When $r = 0$, we have $x = n$, $y = 0$, $p = \frac{2}{3}n$, $q = \frac{n}{3}$, $t = \frac{1}{2}$ and $s = 0$. Integrating the differential equation (2)

$$\frac{dp}{p} = -\frac{3}{2n}dr - \frac{sds}{s^2 + \frac{3}{4}}.$$

gives

$$\ln(p) = -\frac{3r}{2n} - \frac{1}{2}\ln(3 + 4s^2) + C$$

where C is an arbitrary constant. From the initial conditions we obtain $C = \ln\left(\frac{2n}{\sqrt{3}}\right)$ and therefore

$$p = \frac{n\mathrm{e}^{-3r/(2n)}}{\sqrt{3}\sqrt{s^2 + \frac{3}{4}}} \tag{3}$$

Likewise, since

$$dr = -\frac{nds}{\sqrt{s^2 + \frac{3}{4}}}$$

we have

$$r = -\frac{2n}{\sqrt{3}} \arctan\left(\frac{2s}{\sqrt{3}}\right) + C$$

where C is an arbitrary constant. The initial conditions dictate that $C = 0$, hence

$$r = -\frac{2n}{\sqrt{3}} \arctan\left(\frac{2s}{\sqrt{3}}\right) \tag{4}$$

A bit of algebra applied to (4) permits us to write

$$\sqrt{s^2 + \frac{3}{4}} = \frac{\sqrt{3}}{2} \sec\left(\frac{r\sqrt{3}}{2n}\right)$$

Substituting this expression into the formula for p, equation (3), then gives

$$p = \frac{2}{3}\mathrm{e}^{-3r/(2n)} \cos\left(\frac{r\sqrt{3}}{2n}\right) = x - \frac{n}{3} \tag{5}$$

Similarly, since $t = s + \frac{1}{2}$, and since equation 4 can be used to express s in terms of r, we may write

$$t = \frac{1}{2} - \frac{\sqrt{3}}{2} \tan\left(\frac{r\sqrt{3}}{2n}\right)$$

and further since $y = \frac{n}{3} - q = \frac{n}{3} - tp$, we have

$$y = \frac{n}{3} - \frac{n}{3}\mathrm{e}^{-3r/(2n)} \cos\left(\frac{r\sqrt{3}}{2n}\right) \times \left[1 - \sqrt{3}\tan\left(\frac{r\sqrt{3}}{2n}\right)\right] \tag{6}$$

The number of white tickets in the third urn will be $z = n - x - y$.

Bernoulli's numerical computations

We know that x, y and z will converge to $n/3$ as $r \to \infty$, but the presence of the trigonometric functions shows that the decrease in the number of white tickets in the first urn is not uniform but rather has a damped oscillatory pattern. This leads to the question of when, for example, the first urn will hold precisely $n/3$ tickets. Inspection of equation (5) shows that this occurs when the cosine vanishes, that is, where its argument is an odd multiple of $\pi/2$.

If $n = 3000$, x is reduced to 1000 when $r = 5441$. Substitution of this value of r into equation (6) yields $y = 1114$. Of course, $z = 886$. Tripling the number of permutations to $r = 16332$ gives $x = 1000$, $y = 1000$, $z = 1000$.

Note that due to symmetry, the number of black tickets in the second urn and the number of red tickets in the third urn must each be reduced to 1000 when $r = 1551$. In fact,

	Number of colored tickets remaining		
$r = 1551$	White	Black	Red
Urn 1	1000	886	1114
Urn 2	1114	1000	886
Urn 3	886	1114	1000

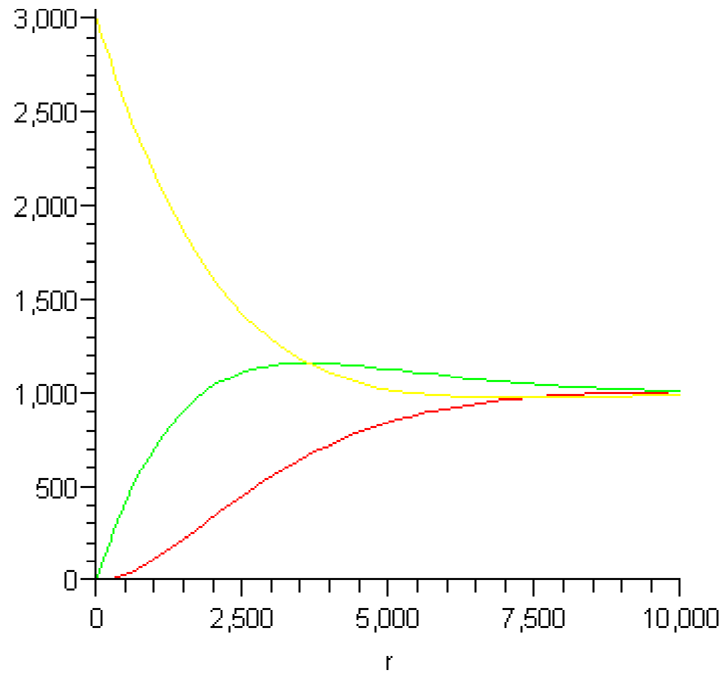
The number of white tickets in the first urn, for example, will be an extremum when the number of white tickets in that urn equals the number of tickets in the third, that is, when $x = z$, for then the probability of a white ticket exiting must equal the probability of a white ticket entering. This occurs for the first time when $r = 7255$. Here $x = 973$, $y = 1053$, $z = 973$.

For the second urn, the extrema occur where $x = y$. This occurs for the first time when $r = 3628$. We have $x = 1163$, $y = 1163$, $z = 674$.

For the third urn, the extrema occur where $y = z$. This occurs for the first time when $r = 10883$. We have $x = 991$, $y = 1004$, $z = 1004$.

Because the damping is exponential, the deviations from the limiting distribution quickly become negligible. We end with a plot of the number of white tickets in each of the three urns when $n = 3000$.

Since the distribution of each color behaves in the same manner, it is clear that by $r = 10000$, for all practical purposes the system has reached a steadystate.



3,000
2,500
2,000
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500
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7,500
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r