The True Valuation of the Risk in Games*

Leonhard Euler†

Opera postuma 1, 1862, p. 315–318

The measuring of risks or expectations is afflicted with many difficulties, which gamblers experience either settling a contested deposit or having been defeated by the victor being obligated to pay a designated sum of money. Without doubt this measure, the fundamentals of which Pascal has laid down and after him Huygens, Jacob Bernoulli and other renowned men have developed carefully and remarkably. Following the way of thinking of these men I do not act imprudently, if I take up a game, where it may happen equally easily, that I would either win or lose 100 Rubles. But if all of my wealth is worth only 100 R., I seem to myself to begin this game about to be played most imprudently; for the gain in respect to the loss, because they can happen equally, is by no means great enough. In the second case I gain 100 R. and therefore in this way I become twice as rich; however in the opposite case I am held to give up my 100 R. to the other, in this way then I arrive at extreme poverty and I become infinitely poorer. Moreover is there anyone rational who is willing to place oneself into extreme poverty and the most detrimental condition, only so that he should be paid back twice richer? But if in fact my resources would be much greater and almost endless, then I would hesitate less to begin such a game, when in the second case I would be made richer by as much as made poorer in the opposite case.

Most of all the truth of this matter is brought about by the following game: There is promised to pay 1 R. to A himself, if the number of points by making a toss with a die will have been even; if the number of points of the second throw will have been even to pay 2 R. in favor of this man; for the third throw, if the number of points likewise should be even, 4 R.; for the fourth throw 8 R. and in this fashion further, until an odd number of points would happen, in which case A receives nothing and the game is finished. The expectation of A himself is sought, or at what other price it would lawful to sell this. Moreover the traditional expectation of A himself is found from the rule to be worth an infinite number of Rubles by the cited authors. In fact, singularly here the most renowned Niklaus Bernoulli[1] asks, who would be so obtuse, as to not prefer to receive 20 R. than the proposed proposition. From this especially the discrepancy

*See commentaries 201, 313, 823.

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[1]Consult Daniel Bernoulli, "Specimen theoriae novae de mensura sortis," *Comment. acad. sc. Petrop.* 5 (1730/1), 1738, p. 175–192. Of special interest is the discussion of p. 187–192. The letter given to cousin Daniel by Nicolaus Bernoulli was lost, however a letter by the same author written 9 Sept. 1713 to P.R. de Montmort is printed in the book by the celebrated P.R. de Montmort, *Essay d'analyse sur les jeux de hazard*, 2nd ed. Paris 1713, p. 401–402,

is manifest between the estimation of the risk according to the rules and this one, as a man of sound mind would make, obviously the rules consider necessary a countless number of Rubles as equivalent to the plan of the game, but in fact this man thinks that he himself is able to be satisfied rightly with 20 Rubles and considers him foolish, who would have paid so much rather than the 20 R for this proposition.

But in this question, as in all the others, it is agreed chiefly to direct the attention toward the wealth of him, whose risk is demanded; for where each one has more, also he will estimate such conditions at a higher price, and for whom his wealth is infinite, this man would be able by calculation to buy the designed game with an infinite amount of money, yet with the infinitesimal part of so much of his own wealth. Therefore the published rules of estimating the risks pertain to the richest men, or if that, which is able to be acquired and lost in the game, has the least ratio to the wealth of the gamblers. If indeed the wealth should be finite for the gamblers and the gains and losses should hold the same limited ratio to the worth, those rules need a correction. For unless this is offered, men are unable to have recourse without danger to those causes of the estimation of their own risk and thus the former would be of absolutely no use. On which account in order to search out the true value of each risk it is necessary to consider beyond the conditions of the game also the resources of those playing and to obtain the conclusions depending on them. Therefore this game does not seem fair to me, by which I am rendered either wealthier or poorer by a, but finally it is counted just, where by turns I am rendered either richer or poorer by a, since each of the two is able to happen equally easily, and in this case for me it is equal whether I should assent to begin or to reject the game indeed to that game I should in no manner agree to, unless I would be richest. But this also is much more certain that he must be considered foolish, who wishes to begin the game with me after the later condition and to pay 100 R. to me having for example 100 R, if I should win, if indeed I should lose, he must be content to receive only 50 R from me. From this therefore the injustice of all games is observed unless they are set up by infinitely wealthy men.

However great are the resources and wealth of each, not only out of the amount of money and resources, but further out of the studies and abilities of him, on account of which they also are able to abound in debt [,it must be computed]. Therefore in this way it is agreed that the resources of any man are determined and at the same time the amount of money is able to be limited equally well. On which account it will be allowed to substitute a certain amount of money for the wealth of anyone, which I will name with the name of his status in the following, and also on that account anyone is said to attain twice better status, whose wealth becomes twice as great, or rather who estimates that he himself is to be made twice as wealthy. For finally he must be estimated as twice as rich, who equally is inclined to ask for twice the sum of money in the case, in which formerly he has not hesitated to pay out only the single amount. Therefore with these having been premised, he who in the game or certain business has two cases opposite (and) equally inclined (to happen), by one of which he is brought into status b, by the other into status c, the status of him must be considered to be worth $\sqrt{bc}$. For this status ought to be so much less than the one b, as it is greater than the other c. In this way one will have promised to place him into the status $\sqrt{bc}$, which is able to grant his own risk to him lawfully. In a similar way, one who has three obvious cases, one of which places itself in status b, the second in status c and the third in status

d, the status of which must be said to be worth $\sqrt[3]{bcd}$ or by the other, so that he grant the proper status to that one, he ought to be placed into this status $\sqrt[3]{bcd}$.

The rule from these is held as this: all the statuses, which are able to result from each of the cases, would be multiplied in relation to themselves by turns and from this result the root of the rank of such degree, as many as there are cases, would be extracted; this will be the value of the status equivalent to the expectation. Following the usual method to this point one ought to combine together all statuses, which are able to happen in each of the cases, into one sum and divide this (sum) by the number of cases. Thus the difference between these two methods consists in this, in so far as ours uses multiplication, when the other addition, likewise the exponent, when this uses multiplication itself; if we determine the operations geometrically, those men in fact (determine them) arithmetically, thus as these men adjust those operations on the statuses themselves, we would transfer the same (operations) onto the logarithms of the statuses, and the same, which it brings forth, the number corresponding to the logarithm of this, indicates to us the sought status of him playing. Let there be m cases, in which I am placed in status a, n cases, in which in b, p cases, in which in c. My mean status will be or representing my expectation

$$\sqrt[m+n+p]{a^m b^n c^p};$$

for $m+n+p$ is the number of all cases and $a^m b^n c^p$ is the amount of all statuses, which are able to happen in each of the cases. Indeed the mean status is from the rules of Huygens

$$\frac{ma+nb+pc}{m+n+p}$$

for which our formula by the logarithm is similar

$$\frac{m\ln a+n\ln b+p\ln c}{m+n+p}$$

Let my status be worth A and let m cases be presented to me, in which I gain a or in which I am placed in status $A+a$, indeed there are n cases, in which I gain b or I arrive at status $A+b$, and p cases, in which I gain c and therefore I obtain status $A+c$; my expected status will be

$$\sqrt[m+n+p]{(A+a)^m(A+b)^n(A+c)^p};$$

therefore I must estimate that I will profit

$$\sqrt[m+n+p]{(A+a)^m(A+b)^n(A+c)^p}-A.$$

If A is set to be infinitely more than a, b and c, it will be

$$(A+a)^{\frac{m}{m+n+p}}=A^{\frac{m}{m+n+p}}+\frac{mA^{\frac{-n-p}{m+n+p}}a}{m+n+p},$$

$$(A+b)^{\frac{n}{m+n+p}}=A^{\frac{n}{m+n+p}}+\frac{mA^{\frac{-m-p}{m+n+p}}b}{m+n+p},$$

$$(A+c)^{\frac{p}{m+n+p}}=A^{\frac{p}{m+n+p}}+\frac{mA^{\frac{-m-n}{m+n+p}}c}{m+n+p};$$

the amount of these is

$$A + \frac{pc + nb + ma}{p + n + m};$$

from which if A would be removed, it will be held as

$$\frac{ma + nb + pc}{m + n + p},$$

which is the value of my gain, and also the formula is the same, if I should have deduced the gain of my expectation from the rules of Huygens. From this it is observed, which in the beginning I have noted, if the statuses of those gambling should be infinitely great, that the reported rules provide the true expectation of each. In fact in our formula it is observed easily that the expected status ought to permit the $-$ sign to be prefixed to these, if the letters a, b or c should indicate loss in place of the gain.

Let there be one case, in which I having the resources A obtain a, and one, in which I lose b; my expected status will be

$$= \sqrt{(A + a)(A - b)};$$

wherefrom the value if it will have been greater than A, I expect to gain and this game must be estimated to amount to me better status; therefore I do not grant freely this condition of the other, but from this I claim, so that he would pay to me

$$\sqrt{(A + a)(A - b)} - A,$$

by which I would be placed into the expected status. However opposite that, if $\sqrt{(A + a)(A - b)}$ will have been less than A, the game is directed towards my loss and likewise I would opt to abandon the game or to assign another to my place, to him, granted that he would accept, also

$$A - \sqrt{(A + a)(A - b)}$$

I would pay, not in fact the greater sum, because it would be in like manner either to pay this or to keep myself in the game. When indeed

$$\sqrt{(A + a)(A - b)} = A,$$

then the game for me in a word is indifferent neither do I hesitate to accept it nor do I hesitate to bequeath it to another. Indeed this corresponds, when there is

$$Aa = Ab + ab \quad \text{or} \quad A = \frac{ab}{a - b},$$

i. e. if the excess of the gain beyond the loss is to the loss as the gain to my status. Therefore such a game must be regarded by me as fair, not that, in which it is $a = b$. For if the gain a is equal to the loss b equally inclined (to happen), unless my resources would be infinite, I undertake the game always to my loss and an undertaking of the game must be compared to the loss so much

$$A - \sqrt{(A^2 - a^2)} = \frac{1}{2} \cdot \frac{a^2}{A} + \frac{1}{8} \cdot \frac{a^4}{A^3} + \frac{1}{16} \cdot \frac{a^6}{A^5} + \frac{5}{128} \cdot \frac{a^8}{A^7} + \text{etc.}$$