Appendix II[1]

Chr. Huygens
Tome XIV pp. 96-101

[1665]

§1.[2]

A	B	C
x	y	z [3]

A has 4 chances to have a, 8 to have z.

$$\frac{4a + 8z}{12} = x$$

$$\frac{1}{3}a + \frac{2}{3}z = x$$

B has 4 chances to have 0, 8 to have x

$$\frac{0 + 8x}{12} = y$$

$$\frac{2}{3}x = y$$

$$\frac{2}{3}x = \frac{3}{2}z; \quad x = \frac{9}{4}z$$

$$\frac{9}{4}z = \frac{1}{3}a + \frac{2}{3}z; \quad z = \frac{4}{19}a; \quad y = \frac{6}{19}a; \quad x = \frac{9}{19}a$$

C has 4 chances to have 0, 8 to have y

$$\frac{8y}{12} = z$$

$$y = \frac{2}{3}z$$

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[1] This Appendix contains the solutions of Huygens of the second and of the fourth of the Exercises added by him to his Treatise. According to the place which it occupies in pages 42-44 of Manuscript C it must date from 1665. One rediscovers the results of §§ 1 and 2 in the letter of Huygens to Hudde of 4 April 1665.

[2] This paragraph treats the second Exercise.

[3] x, y, and z represent respectively the mathematical expectations of the three players A, B, C at the beginning of the game. a represents the stake.

<div style="text-align:center">§2.</div>

A wagers against B that among 12 tokens, of which 4 white and 8 black, he will take blindly 7 tokens of which 3 will be white, and no more. One demands the ratio of the chance of A to that of B; response: as 35 is to 64.[4]

If A having taken 6 tokens, has 3 of them white and 3 black, he has 1 chance to have 0 and 5 to have a (the stake.) $\frac{0+5a}{6} = \frac{5}{6}a$.[5]

6	[tokens]	2	[white]	4	[black]	$\dfrac{2 \cdot a + 4 \cdot 0}{6} = \dfrac{1}{3}a$
5		3		2		$\dfrac{1 \cdot 0 + 6 \cdot \frac{5}{6}a}{7} = \dfrac{5}{7}a$
5		2		3		$\dfrac{2 \cdot \frac{5}{6}a + 5 \cdot \frac{1}{3}a}{7} = \dfrac{10}{21}[a]$
5		1		4		$\dfrac{3 \cdot \frac{1}{3}[a] + 4 \cdot 0}{7} = \dfrac{1}{7}[a]$
4		3		1		$\dfrac{1 \cdot 0 + 7 \cdot \frac{5}{7}[a]}{8} = \dfrac{5}{8}[a]$
4		2		2		$\dfrac{2 \cdot \frac{5}{7}[a] + 6 \cdot \frac{10}{21}[a]}{8} = \dfrac{15}{28}[a]$

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[4] We see that this problem is identical to the fourth of the Exercises. In consequence of a misunderstanding which had taken place between him and Hudde on the conception of the problem, Huygens has added to the enunciation the words "en niet meer," which indicate that, in order to win, A must take 3 white tokens "and no more." Consult the correspondence between Huygens and Hudde on the misunderstanding in question.

[5] As we will see Huygens supposes that the tokens are taken one after the other and he begins by calculating the mathematical expectation of A after the sixth coup in the only two cases where the one can win at the seventh coup. Next he considers the situation of the game after the fifth coup, and thus consecutively, in order to rise finally to the beginning of the game.

<div style="text-align:center">2</div>

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4	[tokens]	1	[white]	3	[black]	$\dfrac{3 \cdot \frac{10}{21}[a] + 5 \cdot \frac{1}{7}[a]}{8} = \dfrac{15}{56}[a]$
4		0		4		$\dfrac{4 \cdot \frac{1}{7}[a] + 4 \cdot 0}{8} = \dfrac{1}{14}[a]$
3		3		0		$\dfrac{1 \cdot 0 + 8 \cdot \frac{5}{8}[a]}{9} = \dfrac{5}{9}[a]$
3		2		1		$\dfrac{2 \cdot \frac{5}{8}[a] + 7 \cdot \frac{15}{28}[a]}{9} = \dfrac{5}{9}[a]$
3		1		2		$\dfrac{3 \cdot \frac{15}{28}[a] + 6 \cdot \frac{15}{56}[a]}{9} = \dfrac{5}{14}[a]$
3		0		3		$\dfrac{4 \cdot \frac{15}{56}[a] + 5 \cdot \frac{1}{14}[a]}{9} = \dfrac{10}{63}[a]$
2		2		0		$\dfrac{2 \cdot \frac{5}{9}[a] + 8 \cdot \frac{5}{9}[a]}{10} = \dfrac{5}{9}[a]$
2		1		1		$\dfrac{3 \cdot \frac{5}{9}[a] + 7 \cdot \frac{5}{14}[a]}{10} = \dfrac{5}{12}[a]$
2		0		2		$\dfrac{4 \cdot \frac{5}{14}[a] + 6 \cdot \frac{10}{63}[a]}{10} = \dfrac{5}{21}[a]$
1		1		0		$\dfrac{3 \cdot \frac{5}{9}[a] + 8 \cdot \frac{5}{12}[a]}{11} = \dfrac{15}{33}[a]$
1		0		1		$\dfrac{4 \cdot \frac{5}{12}[a] + 7 \cdot \frac{5}{21}[a]}{11} = \dfrac{10}{33}[a]$
0		0		0		$\dfrac{4 \cdot \frac{15}{33}[a] + 8 \cdot \frac{10}{33}[a]}{12} = \dfrac{35}{99}[a]$

Therefore the chance of A is to that of B as 35 is to 64.

§3.[6]

The preceding chance of A is necessarily as great as in the case where among the 12 mentioned tokens he must take 5 of which one white.[7]

4	[tokens]	1	[white]	3	[black]	$\dfrac{3\cdot 0 + 5\cdot a}{8}[=]\dfrac{5}{8}a$
4		0		4		$\dfrac{4\cdot a + 4\cdot 0}{8}[=]\dfrac{1}{2}a$
3		1		2		$\dfrac{3\cdot 0 + 6\cdot \frac{5}{8}[a]}{9}[=]\dfrac{5}{12}[a]$
3		0		3		$\dfrac{4\cdot \frac{5}{8}[a] + 5\cdot \frac{1}{2}[a]}{9}[=]\dfrac{5}{9}[a]$
2		1		1		$\dfrac{3\cdot 0 + 7\cdot \frac{5}{12}[a]}{10}[=]\dfrac{7}{24}[a]$
2		0		2		$\dfrac{4\cdot \frac{5}{12}[a] + 6\cdot \frac{5}{9}[a]}{10}[=]\dfrac{1}{2}[a]$
1		1		0		$\dfrac{3\cdot 0 + 8\cdot \frac{7}{24}[a]}{11}[=]\dfrac{7}{33}[a]$
1		0		1		$\dfrac{4\cdot \frac{7}{24}[a] + 7\cdot \frac{1}{2}[a]}{11}[=]\dfrac{14}{33}[a]$
0		0		0		$\dfrac{4\cdot \frac{7}{33}[a] + 8\cdot \frac{14}{33}[a]}{12}[=]\dfrac{35}{99}[a]$. Good.

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[6] This paragraph contains a verification of the solution obtained in the preceding paragraph.

[7] We remark that the question in this last case of the tokens which, according to the first assumption, have not been taken, of which 4 are black and 1 white. We will call the problem which is related there: the complementary problem.

§4.[8]

[Solution of the preceding problem] under the same assumptions, but with this difference that it is necessary that there be *at least* three white among the 7 tokens which one has taken.

4 [tokens]	1 [white]	3 [black]	$\dfrac{3 \cdot 0 + 5 \cdot a}{8}[=]\dfrac{5}{8}a$ [9]
3	1	2	$\dfrac{3 \cdot 0 + 6 \cdot \frac{5}{8}[a]}{9}[=]\dfrac{5}{12}[a]$
3	0	3	$\dfrac{4 \cdot \frac{5}{8}[a] + 5 \cdot a}{9}[=]\dfrac{5}{6}[a]$
2	1	1	$\dfrac{3 \cdot 0 + 7 \cdot \frac{5}{12}[a]}{10}[=]\dfrac{7}{24}[a]$
2	0	2	$\dfrac{4 \cdot \frac{5}{12}[a] + 6 \cdot \frac{5}{6}[a]}{10}[=]\dfrac{2}{3}[a]$
1	1	0	$\dfrac{3 \cdot 0 + 8 \cdot \frac{7}{24}[a]}{[11]}[=]\dfrac{7}{33}[a]$

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[8] This paragraph contains the solution of the fourth Exercise proposed by Huygens if one gives to this problem the interpretation that Hudde had attached there.

[9] In order to simplify the calculation Huygens begins by replacing the problem in question by the complementary problem according to which A takes 5 tokens of which four, at least, must be black. We remark that it is not necessary to discuss the case of 4 tokens of which 0 white and 4 black, because then the fifth coup will be always winning to A.

[101]

1 [tokens] 0 [white] 1 [black] $$\frac{4 \cdot \frac{7}{24}[a] + 7 \cdot \frac{2}{3}[a]}{[11]}[=]\frac{35}{66}[a]$$

0   0   0 $$\frac{4 \cdot \frac{7}{33}[a] + 8 \cdot \frac{35}{66}[a]}{[12]}[=]\frac{42}{99}[a].$$

42 [to] 57
14 [to] 19. Good.[10]

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[10] This is, indeed, the result obtained by Hudde. We add that a little higher in Manuscript C, on pages 36-37, one encounters another calculation which refers to the same problem and which must have preceded the one which is reproduced here. This calculation was again later and Huygens marked it as "misrekent" (miscalculated). In examining it, one perceives first that Huygens has followed, as in this §4, the interpretation of Hudde, since he considers the game as won when A, already in possession of 2 white tokens and of 3 black tokens, takes next 1 white token. In this manner he finds in this case of 2 white tokens and 3 black tokens for the mathematical expectation of A: $\frac{2a+5\cdot\frac{1}{3}a}{7} = \frac{11}{21}a$, instead of $\frac{10}{21}a$, as in §2 of this Appendix. He must therefore arrive to the result obtained by Hudde, but in the case of 1 white token and 4 black tokens he commits an error of calculation in writing $\frac{3\cdot a+0}{7} = \frac{3}{7}a$ instead of $\frac{3\cdot\frac{1}{3}a+0}{7} = \frac{1}{7}a$; this which explains the inexactitude of the final result to which he arrives.

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