Appendix IV[1]

Chr. Huygens
Tome XIV pages 108-115.

[1665]

§1.[2]

A and B choose blindly by turn, A always one of 3 [or of $\theta + \lambda$] tokens of which 2 [θ] white and 1 [λ] black, but B one of an unknown number of white and black tokens, under the condition that the one who will draw a white token will have all that which is set; but the one who draws a black token will add each time a ducat to the stake, and A will draw first. One demands when one wants that the chances of A and of B are equivalent, what proportion must exist between the numbers of white and black tokens of B.

[1] The Piece is taken from pages 54-61 (numbered 10-17 by Huygens) of Manuscript C. It contains the solution of the problem posed by Hudde in his letter of 5 May 1665 and of some other problems which are related there.

[2] This paragraph is occupied with the problem of Hudde, where A must choose among 2 white tokens and 1 black, and with the generalization which Huygens has given there by representing the number of whites by θ and the one of the blacks by λ. Huygens treats jointly these two problems, but, in order to avoid the confusion which would result from it here, it has seemed preferable to omit all that which relates to the particular solution excepting the end where the concern is with the application of the general solution to this special case.

[109]

A chose one of 3 [or of $\theta+\lambda$] tokens of which 2 [θ] white, 1 [λ] black. B chose ω tokens of which ϕ white, ψ black. $\omega=\phi+\psi$.

a	is the advantage of	A	when he has set	0	against	0	and when he must choose.
b		B		0		1	
c		A		1		1	
d		B		1		2	
e		A		2		2	
f		B		2		3	
g		A		3		3	
h		B		3		4	
i		A		4		4	
$\aleph$		B		4		5	
$\beth$		A		5		5	

$-k$	is the advantage of	B	when he has set	0	against	0	and when he must choose.
$+l$		A		0		1	
m		B		1		1	
n		A		1		2	
o		B		2		2	
p		A		2		3	
q		B		3		3	
r		A		3		4	
s		B		4		4	
t		A		4		5	

Δ is a ducat which is set into the stake.

$$a=\theta.k \quad \lambda.-b$$
$$a=\frac{\theta k-\lambda b}{\rho}$$

$$b=\phi.\Delta \quad \psi.-c$$
$$-\frac{\lambda}{\rho}b=\frac{-\lambda\phi\Delta+\lambda\psi c}{\rho\omega}$$

$$c=\theta.\Delta \quad \lambda.-d$$
$$\frac{\lambda\psi c}{\rho\omega}=\frac{\lambda\theta\psi\Delta-\psi\lambda\lambda d}{\rho\rho\omega}$$

$$d=\phi.2\Delta \quad \psi.-e$$
$$-\frac{\psi\lambda\lambda}{\rho\rho\omega}d=\frac{-2\phi\psi\lambda^2\Delta+\psi\psi\lambda^2 c}{\rho\rho\omega\omega}$$

[110]

$c = \theta.2\Delta \quad \lambda.-f$

$\frac{\psi\psi\lambda\lambda}{\rho\rho\omega\omega} c = \frac{2\theta\lambda^2\psi\psi\Delta - \lambda\lambda^2\psi\psi f}{\rho^3\omega\omega}$

$f = \phi.3\Delta \quad \psi.-g$

$-\frac{\lambda\lambda^2\psi\psi}{\rho^3\omega\omega} f = \frac{-3\phi\psi\psi\lambda\lambda^2\Delta + \lambda\lambda^2\psi^3 g}{\rho^3\omega^3}$

$g = \theta.3\Delta \quad \lambda.-h$

$\frac{\lambda\lambda\lambda\psi^3}{\rho^3\omega^3} g = \frac{3\theta\lambda\lambda^2\psi^3\Delta - \lambda^4\psi^3 h}{\rho^4\omega^3}$

$h = \phi.4\Delta \quad \psi.-i$

$-\frac{\lambda^4\psi^3}{\rho^4\omega^3} h = \frac{-4\lambda^4\phi\psi^3\Delta + \lambda^4\psi^4 i}{\rho^4\omega^4}$

$i = \theta.4\Delta \quad \lambda.-\aleph$

$\frac{\lambda^4\psi^4}{\rho^4\omega^4} = \frac{4\theta\lambda^4\psi^4\Delta - \lambda^5\psi^4\aleph}{\rho^5\omega^4}$

$\aleph = \phi.5\Delta \quad \psi.-\beth$

$-\frac{\lambda^5\psi^4}{\rho^5\omega^4}\aleph = \frac{-5\lambda^5\psi^4\phi\Delta + \lambda^5\psi^5\beth}{\rho^5\omega^5}$

$-k = \phi.-a \quad \psi.-l$

$\frac{\theta}{\rho} k = \frac{\theta\phi a - \theta\psi l}{\rho\omega}$

$l = \theta.\Delta \quad \lambda.-m$

$\frac{\theta\psi}{\rho\omega} l = \frac{\theta\theta\psi\Delta + \theta\psi\lambda m}{\rho\rho\omega}$

$m = \phi.\Delta \quad \psi.-n$

$\frac{\theta\psi\lambda m}{\rho\rho\omega} = \frac{-\lambda\theta\psi\phi\Delta - \lambda\theta\psi\psi n}{\rho\rho\omega\omega}$

$n = \theta.2\Delta \quad \lambda.-o$

$\frac{\lambda\theta\psi\psi}{\rho\rho\omega\omega} n = \frac{2\lambda\theta\theta\psi\psi\Delta - \lambda^2\theta\psi\psi o}{\rho^3\omega\omega}$

$o = \phi.2\Delta \quad \psi.-p$

$-\frac{\lambda^2\theta\psi\psi}{\rho^3\omega\omega} o = \frac{-2\lambda^2\theta\psi\psi\phi\Delta - \lambda^2\theta\psi^3 p}{\rho^3\omega^3}$

$p = \theta.3\Delta \quad \lambda.-q$

$\frac{\lambda^2\theta\psi^3}{\rho^3\omega^3} p = \frac{3\lambda^2\theta\theta\psi^3\Delta - \lambda\lambda^2\theta\psi^3 q}{\rho^4\omega^3}$

$q = \phi.3\Delta \quad \psi.-r$

$-\frac{\lambda\lambda\lambda\theta\psi^3}{\rho^4\omega^3} q = \frac{-3\lambda\lambda^2\theta\phi\psi^3\Delta + \lambda\lambda^2\theta\psi^4 r}{\rho^4\omega^4}$

$r = \theta.4\Delta \quad \lambda.-s$

$\frac{\lambda\lambda\lambda\theta\psi^4}{\rho^4\omega^4} r = \frac{4\theta\theta\lambda^3\psi^4\Delta - \lambda^4\theta\psi^4 s}{\rho^5\omega^4}$

$s = \phi.4\Delta \quad \psi.-t$

$-\frac{\lambda^4\theta\psi^4}{\rho^5\omega^4} s = \frac{-4\lambda^4\theta\psi^4\phi\Delta + \lambda^4\theta\psi^5 t}{\rho^5\omega^5}$

$$a = \overset{\text{A}}{-\frac{\lambda b}{\rho}} \quad \overset{\text{B}}{+\frac{\theta k}{\rho}}$$ [3]

[3] See above page 109. In that which follows, Huygens designates by A the first term $-\frac{\lambda b}{\rho}$, by B the second term $\frac{\theta k}{\rho}$ of the second member of this equation.

[111]

A

$$\frac{\lambda b}{\rho} = \frac{-\lambda\phi\Delta + \lambda\psi c}{\rho\omega}$$

$$\frac{\lambda\theta\psi\Delta - \psi\lambda\lambda d}{\rho\rho\omega}$$ [4]

$$\frac{-2\phi\psi\lambda^2\Delta + \psi\psi\lambda^2 c}{\rho\rho\omega\omega}$$

$$\frac{2\theta\lambda^2\psi\psi\Delta - \lambda\lambda^2\psi\psi f}{\rho^3\omega\omega}$$

$$\frac{-3\phi\psi\psi\lambda\lambda^2\Delta + \lambda\lambda^2\psi^3 g}{\rho^3\omega^3}$$

$$\left[-\frac{\lambda^4\psi^3 h}{\rho^4\omega^3}\right] \qquad \frac{+\,3\theta\lambda^3\psi^3\Delta - \lambda^4\psi^3 h}{\rho^4\omega^3}$$

$$\frac{-4\lambda^4\phi\psi^3\Delta + \lambda^4\psi^4 i}{\rho^4\omega^4}$$

$$\frac{4\theta\lambda^4\psi^4\Delta - \lambda^5\psi^4\aleph}{\rho^5\omega^4}$$

$$\frac{-5\lambda^5\psi^4\phi\Delta + \lambda^5\psi^5\beth}{\rho^5\omega^5}$$

[4] This notation indicates that one can replace $\frac{\lambda\psi c}{\rho\omega}$ by $\frac{\lambda\theta\psi\Delta - \psi\lambda^2 d}{\rho\rho\omega}$.

B

$$\frac{\theta k}{\rho} = \frac{\theta\phi a}{\rho\omega} + \frac{+\theta\psi l}{\rho\omega}$$

$$\frac{\theta\theta\psi\Delta - \theta\lambda\psi m}{\rho\rho\omega}$$

$$\frac{-\lambda\theta\psi\phi\Delta + \lambda\theta\psi\psi n}{\rho\rho\omega\omega}$$

$$\frac{+2\lambda\theta\theta\psi\psi\Delta - \lambda\lambda\theta\psi\psi o}{\rho^3\omega\omega}$$

$$\frac{-2\lambda\lambda\theta\psi\psi\phi\Delta + \lambda\lambda\theta\psi^3 p}{\rho^3\omega^3}$$

$$\left[\frac{-\lambda^3\theta\psi^3 q}{\rho^4\omega^3}\right] \qquad \frac{+3\lambda\lambda\theta\theta\psi^3\Delta - \lambda^3\theta\psi^3 q}{\rho^4\omega^3}$$

$$\frac{-3\lambda^3\theta\phi\psi^3\Delta + \lambda^3\theta\psi^4 r}{\rho^4\omega^4}$$

$$\frac{+4\theta\theta\lambda^3\psi^4\Delta - \lambda^4\theta\psi^4 s}{\rho^5\omega^4}$$

$$\frac{-4\lambda^4\theta\psi^4\phi\Delta + \lambda^4\theta\psi^5 t}{\rho^5\omega^5}$$

It would be necessary to demonstrate the proportion in which following one another the $+$ and the $-$ of A and of B, this which is not uncomfortable by regarding their origin in the calculations pages 10 and 11.[5]

[5] It concerns from the pages which we have borrowed that which precedes the last equation of page 110.

[112]
Note that the last quantities $\frac{\lambda^5\psi^5\beth}{\rho^5\omega^5}$ and $\frac{\lambda^4\theta\psi^5 t}{\rho^5\omega^5}$ become infinitely small, since the $+$ of A and of B and also the $-$ diminish always according to the proportion of $\rho\omega$ to $\psi\lambda$, (being $\rho\omega$ greater than $\lambda\psi$ because $\rho = \theta + \lambda$ and $\omega = \phi + \psi$, and that the quantities r, t or q, s and i, $\beth$ or h, $\aleph$ increase only by unity as one sees by the hypothesis page 10.[6]

let $\lambda =$ the $+$ of A[7]
therefore $\theta =$ the $+$ of B

$\theta\psi \quad \phi\rho \quad \lambda$ (the $+$ [of] A) $\quad \dfrac{\lambda\phi\rho}{\theta\psi}$[8] $=$ the $-$ of A

$\theta\omega \quad \lambda\phi \quad \theta$ (the $+$ of B) $\quad \dfrac{\lambda\phi}{\omega} =$ the $-$ of B

[6] By "the hypothesis" Huygens intends the definitions of the quantities a, $-b$, etc. given at the beginning of this Piece on page 109. Now, each of the quantities, which enter into one same series, can be considered as the sum of three parts of which the first and the second are proportionals, respectively, to the stakes of players A and B, which increase each time by unity, and of which the third does not vary. In the first and in the second part the reasoning of Huygens is applied; as for the third, his product with the indicated coefficients, approaches *a fortiori* to zero.

[7] Since the solution sought depends exclusively on the *ratios* which exist between the sums of the $+$ and the $-$ of A and of B, one can replace these sums by some quantities which are their proportionals, and choose for one of them an arbitrary value. In this manner it is not necessary to determine the sum of one of the series formed by the quanties $+$ or $-$ of A or of B, as Huygens had done for the $+$ of B on the occasion of the problem which precedes, where, evidently, this summation can not be avoided; see page 106.

[8] This notation indicates that $\theta\psi$ is to $\phi\rho$ as λ is to $\frac{\lambda\phi\rho}{\theta\psi}$.

$$
\begin{aligned}
&-\lambda + \frac{\lambda\phi\rho}{\theta\psi} = \theta - \frac{\lambda\phi}{\omega}\text{[9]} \\
&\frac{\omega\rho\lambda\phi}{\theta\psi} = \omega\lambda + \omega\theta - \lambda\phi \qquad\qquad \text{but } \lambda + \theta = \rho \\
&\frac{\omega\rho\lambda\phi}{\theta\psi} = \omega\rho - \lambda\phi \\
&\omega\rho\lambda\phi = \omega\rho\theta\psi - \lambda\phi\theta\psi \qquad\qquad \text{but } \omega = \phi + \psi \text{ and } \rho = \theta + \lambda \\
&\phi\phi\lambda\rho = \phi\psi\theta\theta - \phi\psi\lambda\rho + \theta\rho\psi\psi \\
\text{or } &\phi\phi = -\psi\phi + \frac{\psi\theta\theta}{\lambda} + \frac{\theta\psi\psi}{\lambda} \text{good.} \quad \text{Rule.[10]}
\end{aligned}
$$

[9] Since the chances of the two players must be equivalent at the beginning of the game, one must have $a = 0$ and, hence, $-A = B$; see the last equation of page 110 and note 3.

[10] By this rule the problem can be considered as resolved. Indeed, in order to determine the desired proportion between the numbers ϕ and ψ of the white and black tokens, there is no more concern than to resolve a quadratic equation to roots always real and of contrary signs, of which the positive root is the only one which satisfies the conditions of the problem.

[113]

$$\phi\phi = \frac{1}{3}\phi\psi + 2\psi\psi$$ [11]

$$\phi = \frac{1}{6}\psi + \sqrt{2\frac{1}{36}\psi\psi} \text{ or } \phi = \frac{1}{6}\psi + \frac{1}{6}\sqrt{73\psi\psi}$$

ϕ nearly $= 1\frac{4}{7}\psi$; ϕ to ψ nearly as 11 to 7 nearer as 1193 to 750.[12]

Let $\theta = 10; \lambda = 1,$[13] $[\phi = \frac{89}{22}\psi +]\sqrt{\frac{12761}{484}}[\psi]$ $\frac{113}{22}$ more near

$$\frac{89}{22}$$

$$\frac{202}{22}$$ more near

more near $101\psi = 11\phi$.

§2.[14]

The $+$ of B has this proportion

$$\begin{array}{ccc} \frac{\theta\theta\psi\Delta}{\rho\rho\omega} & \frac{2\lambda\theta\theta\psi\psi\Delta}{\rho^3\omega\omega} & \frac{3\lambda\lambda\theta\theta\psi^3\Delta}{\rho^4\omega^3} \\ \rho^2\omega & 2\lambda\psi & \\ & 2\omega\rho & 3\lambda\psi \end{array}$$ [15]

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[11] Application of the Rule to the problem posed by Hudde where $\theta = 2$, $\lambda = 1$, $\rho = 3$.

[12] We do not see how the first approximation has been obtained, but some small calculations in the margin of the Manuscript permit establishing that the second has been found by calculating the square root of 73000000 which is equal to 8544, whence it follows $\phi = \frac{9544}{6000}\psi = \frac{1193}{750}\psi$.

[13] See apropos of this proportion of 10 to 1, that one finds again many times in the correspondence between Huygens and Hudde.

[14] In this paragraph Huygens determines the advantage of the players A for some given values of θ, λ, ϕ and ψ; a problem which he has resolved a particular case in Appendix III (pages 102-107). To this effect he must seek the sum of one of the four series of which it is question in the earlier note 7, of which he chooses the one of " $+$ of B."

[15] See the first three positive terms of the series B on page 111.

[114]

ratio of first to second[16]

$$\begin{array}{cccccccc} \Xi & \omega\rho & \lambda\psi & \cdot & \cdot & \cdot & \cdot & \Pi \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot & \cdot & \\ & & & & \cdot & \cdot & \cdot & \\ & & & & & \cdot & \cdot & \\ & & & & & & \cdot & \Sigma \end{array}$$

as excess of greatest beyond 2nd ($\omega\rho - \lambda\psi$) to greatest ($\omega\rho$) [therefore] first $\frac{\theta\theta\psi\Delta}{\rho\rho\omega}$ [to] $\frac{\theta\theta\psi\Delta}{\rho\rho\omega - \rho\lambda\psi}$

series $\Xi\Pi$ or $\Xi\Sigma$

$$\omega\rho - \lambda\psi \text{ [to] } \omega\rho \text{ [therefore] } \frac{\theta\theta\psi\Delta}{\omega\rho\rho - \rho\lambda\psi} \text{ [to] } \frac{\omega\theta\theta\psi\Delta}{\omega\omega\rho\rho - 2\omega\rho\lambda\psi + \lambda\lambda\psi\psi}$$

sum of the whole series or the $+$ of B

$$a - \frac{\theta\phi a}{\rho\omega} = \frac{\overset{+\mathrm{B}}{\omega\theta\theta\psi\Delta} - \overset{-\mathrm{A}}{\omega\lambda\phi\rho\Delta} + \overset{+\mathrm{A}}{\omega\theta\lambda\psi\Delta} - \overset{-\mathrm{B}}{\lambda\phi\theta\psi\Delta}}{\omega\omega\rho\rho - 2\omega\rho\lambda\psi + \lambda\lambda\psi\psi}$$ [17] Rule.

$$\text{Let} \quad \frac{\omega\theta\theta\psi}{\omega\omega\rho\rho - 2\omega\rho\lambda\psi + \lambda\lambda\psi\psi} = \xi$$

$$a - \frac{\theta\phi a}{\rho\omega} = \overset{+\mathrm{B}}{\xi\Delta} - \overset{-\mathrm{A}}{\frac{\lambda\phi\rho\xi\Delta}{\theta\theta\psi}} + \overset{+\mathrm{A}}{\frac{\lambda\xi\Delta}{\theta}} - \overset{-\mathrm{B}}{\frac{\lambda\phi\xi\Delta}{\theta\omega}}.$$ Good. Rule likewise.

$$a - \tfrac{1}{4}a = \tfrac{2}{9} - \tfrac{4}{9} + \tfrac{2}{9} - \tfrac{1}{9} = \tfrac{1}{9}\Delta \qquad a = -\tfrac{4}{27}\Delta$$ [18]

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[16] In order to understand the calculations which follow, it will suffice to consider the Theorem of page 106 and the algorithm of which Huygens is himself served on this page for the summation of the series of the "$+$ of B."

[17] Compare the equation $a = \frac{-\lambda b}{\rho} + \frac{\theta k}{\rho}$ of page 110.

[18] Application to the problem of heads or tails which one will find formulated at the beginning of Appendix V, page 116. It is clear that the solution of this problem can be obtained by means of the preceding general rule by putting $\theta = \lambda = \phi = \psi = 1$, and hence $\rho = \omega = 2$.

[115]

$\frac{7}{9}a = \frac{24}{49}\Delta - \frac{9}{49}\Delta + \frac{12}{49}\Delta - \frac{4}{49}\Delta = \frac{23}{49}\Delta \qquad a = \frac{207}{343}\Delta$[19]

$$\lambda = 1;\ \theta = 10;\ \phi = 10;\ \psi = 11;\ \rho = 11 = \lambda + \theta;\ \omega = 21 = \phi + \psi$$

$$a = \frac{105}{131}\Delta$$[20]

$$\theta = 2;\ \lambda = 1;\ \phi = 2;\ \psi = 3;\ \rho = 3;\ \omega = 5$$

$$a = \frac{5}{16}\Delta$$[21]

[19] Application to the problem treated in Appendix III, p. 102, where $\theta = 2$, $\lambda = 1$, $\phi = 1$, $\psi = 2$, therefore $\rho = \omega = 3$.

[20] See on this problem a letter of Huygens to Hudde of 7 July 1665.

[21] Read $\frac{5}{11}\Delta$. One does not encounter this problem in the letters exchanged between Huygens and Hudde, nor besides in the correspondence of Huygens.