Appendix V[1]

Chr. Huygens
Tome XIV. pages 116-150

1665

§1[2]

July 1665

A plays heads or tails against B; the two players cast turn by turn under the condition that the one who brings forth tails will put each time a ducat, but who casts heads will take all that which is set; and A will cast first, when one has yet set nothing. And it is understood that the game will not end before something has been set, and removed.[3]

[1] This Appendix is taken from pages 63-74 of Manuscript C. These pages were numbered from 19 to 30 by Huygens.

[2] This paragraph contains the solution of the problem on the game of heads and tails, proposed by Huygens to Hudde in a letter of 4 April 1665.

[3] The last phrase is lacking in the enunciation of the problem sent to Hudde.

[117]

Let be supposed that the one who must cast when nothing has yet been set, neither by one, nor the other, loses a, that is to say

$$\text{let the advantage}\left\{\begin{array}{rlcrc} -a & \text{of the one who has set} & 0 & \text{against} & 0 \\ +b & & 0 & & 1 \\ +c & & 1 & & 1 \\ +d & & 1 & & 2 \\ +e & & 2 & & 2 \\ +f & & 2 & & 3 \\ +g & & 3 & & 3 \\ +h & & 3 & & 4 \\ +i & & 4 & & 4 \\ +k & & 4 & & 5 \end{array}\right\}\text{and who must cast.}$$

(It would merit more to propose the question by adding the condition that A and B would have set each a ducat. Then one will find that the advantage of A is worth $\frac{1}{9}$ of a ducat, as we have calculated on the following page.[4])

Now, since one wins precisely as much as the other loses, it follows that the advantage of B, who has set 0 against 0, the other to cast before, is equal to $+a$. And likewise the one

[4] See page 123.

[118]

who has set 1 against 0, the other to cast before, will have $-b$, because the one who casts has $+b$ and thus consecutively.

Now, as for the one who casts first when there is set 0 against 0, on which the advantage was posed $-a$, if he casts heads he has the same advantage as the other has now, that is to say $+a$; but if he casts tails he must set 1 against 0, and the other will cast, that is to say he will have $-b$. Thus therefore $-a$ is equivalent to a chance to have $+a$ and 1 to have $-b$. Consequently $-a = \frac{a-b}{2}$, by the second proposition of our Calculus in the games of chance.[5] Likewise one can easily imagine that $+b$, that is to say the advantage of the one, who has set 0 against 1 and who must himself cast, is equal to 1 chance to receive one ducat, which we will represent by Δ, and one chance to have $-c$. One has therefore $b = \frac{\Delta - c}{2}$.

One has likewise	$c = 1$	to	Δ	and 1 to	$-d$.	Therefore	$c = \dfrac{\Delta - d}{2}$
One has likewise	$d = 1$	to	2Δ	and 1 to	$-e$	Therefore	$d = \dfrac{2\Delta - e}{2}$
One has likewise	$e = 1$	to	2Δ	and 1 to	$-f$	Therefore	$e = \dfrac{2\Delta - f}{2}$
Likewise	$f = 1$	to	3Δ	and 1 to	$-g$	Therefore	$f = \dfrac{3\Delta - g}{2}$
Likewise	$g = 1$	to	3Δ	and 1 to	$-h$	Therefore	$g = \dfrac{3\Delta - h}{2}$

And thus in sequence $h = \dfrac{4\Delta - i}{2}$; $i = \dfrac{4\Delta - k}{2}$.

[5] It is actually a question of the first Proposition.

[119]

One has therefore

$$-a = \cfrac{a-b}{2} \cfrac{-\Delta + c}{4} \cfrac{+\Delta - d}{8} \cfrac{-2\Delta + e}{16} \cfrac{+2\Delta - f}{32} \cfrac{-3\Delta + g}{64} \cfrac{3\Delta - h}{128} \cfrac{-4\Delta + i}{256} \cfrac{+4\Delta - k}{512}, \text{ etc.}$$

whence it follows:

$$-a = \frac{1}{2}a - \frac{1}{4}\Delta + \frac{1}{8}\Delta - \frac{2}{16}\Delta + \frac{2}{32}\Delta - \frac{3}{64}\Delta + \frac{3}{128}\Delta - \frac{4}{256}\Delta + \frac{4}{512}\Delta - \frac{1}{512}k$$

[120]

but this last quantity, represented here by $\frac{1}{512}k$, becomes infinitely small if the sequence continues indefinitely, because, since one does not lose yet so many ducats as one has set, it follows that b is less than one ducat or Δ, and d less than 2Δ, and f less than 3Δ, and h less than 4Δ, and k less than 5Δ, but the denominator of the fraction rises in the double proportion 2, 4, 8, 16, etc. Hence it is necessary, as it has been said, that the last quantity, being here $\frac{1}{512}k$, becomes finally as small as one wishes it: it must therefore be counted for zero.

One has therefore

$$-a = \frac{1}{2}a - \frac{1}{4}\Delta + \frac{1}{8}\Delta - \frac{2}{16}\Delta + \frac{2}{32}\Delta - \frac{3}{64}\Delta + \frac{3}{128}\Delta - \frac{4}{256}\Delta + \frac{4}{512}\Delta - \frac{5}{1024}\Delta \text{ etc.}$$

to infinity.

Next one sees here that the quantities where Δ enters and which are preceded by $+$ are to those preceded by $-$ as 1 is to 2.

But, as for those which are preceded by $-$, their sum can also be written as it follows:

[121]

$$
\begin{aligned}
-\frac{1}{4}\Delta-\frac{1}{16}\Delta-\frac{1}{64}\Delta-\frac{1}{256}\Delta-\frac{1}{1024}\Delta\,\text{etc.}\\
-\frac{1}{16}\Delta-\frac{1}{64}\Delta-\frac{1}{256}\Delta-\frac{1}{1024}\Delta\text{etc.}\\
-\frac{1}{64}\Delta-\frac{1}{256}\Delta-\frac{1}{1024}\Delta\text{etc.}\\
-\frac{1}{256}\Delta-\frac{1}{1024}\Delta\text{etc.}\\
-\frac{1}{1024}\Delta\text{etc.}
\end{aligned}
$$

because all this together is equal to $-\frac{1}{4}\Delta-\frac{2}{16}\Delta-\frac{3}{64}\Delta-\frac{4}{256}\Delta-\frac{5}{1024}\Delta$ etc.

Now, in the superior series, since each quantity which follows is $\frac{1}{4}$ of the one which precedes it, these quantities will be all together $=\frac{4}{3}$ of the first, that is to say $-\frac{1}{4}\Delta$. Their sum will be therefore $-\frac{1}{3}\Delta$.

Likewise the second series is $=\frac{4}{3}$ of its first quantity $-\frac{1}{16}\Delta$. And thus throughout, each series being $=\frac{4}{3}$ of its first quantity. It results from it that all the series

[122]

together are equal to $\frac{4}{3}$ of the slanting series which is formed by all the first quantities and which is identical to the superior series. But this last is equal to $-\frac{1}{3}\Delta$; consequently all the series together are equal to $\frac{4}{3}$ of $-\frac{1}{3}\Delta$, this which makes $-\frac{4}{9}\Delta$. Thus therefore the quantities of the preceding equation, where Δ enters and which are preceded by $-$, have for sum $-\frac{4}{9}\Delta$; but these were to those which are preceded by $+$ as 2 is to 1. It follows that those preceded by $+$ make together $+\frac{2}{9}\Delta$. If one subtracts from those affected by the $-$ sign, one obtains $-\frac{2}{9}\Delta$.

One has therefore the equation $-a = \frac{1}{2}a - \frac{2}{9}\Delta$, or else $-3a = -\frac{4}{9}\Delta$, or finally $a = \frac{4}{27}\Delta$, and it follows that A, who casts first, loses $\frac{4}{27}$ of a ducat.[6]

[6] The problem is therefore resolved. We add that some earlier pages in the same Manuscript (pages 38-42) one encounters some calculations, dated 16 March 1665, by which Huygens, without succeeding to resolve the problem, which he calls "quaestio difficillima," enclose the value a of the advantage of the second player B within some limits more and more narrowed.

Now, because of their confused and incomplete writing, it would be very difficult to reproduce these calculations. We limit ourselves therefore to give a résumé of it.

We note, to this effect, that the equations which one can deduce from the successive expressions for $-a$ can be written:

$$-\frac{3}{2}a = -\frac{1}{2}b = -\frac{1}{4}\Delta + \frac{1}{4}c = \frac{1}{8}\Delta - \frac{1}{8}d = -\frac{1}{4}\Delta + \frac{1}{16}e.$$
$$= -\frac{6}{32}\Delta - \frac{1}{32}f = -\frac{15}{64}\Delta + \frac{1}{64}g = \text{etc.}$$

It follows from it;

$b = 3a$; $c = \Delta - 6a$; $d = 12a - \Delta$; $c = 4\Delta - 24a$; $f = 48a - 6\Delta$; $g = 15\Delta - 96a$; $h = 192a - 27\Delta$; $i = 58\Delta - 384a$; $k = 768a - 112\Delta$; $l = 229\Delta - 1536a$; $m = 3072a - 453\Delta$; $n = 912\Delta - 6144a$; $o = 12288a - 1818\Delta$; $p = 3643\Delta - 24576a$.

In these equations the coefficients of a constitute a geometric sequence and the formation of the coefficients of Δ is easily explicated by the following algorithm:

$$4 \times 2 - 2 = 6$$
$$6 \times 2 + 3 = 15$$
$$15 \times 2 - 3 = 27$$
$$27 \times 2 + 4 = 58, \text{ etc.}$$

We see now in what manner, among others, the two tighter limits have been obtained by Huygens. We consider to this effect first the advantage n of the one who casts first when he has

[123]

$-a$	or	$-\frac{4}{27}\Delta = \frac{2}{27}\Delta - \frac{1}{2}b;$	$b = \frac{4}{9}\Delta$	
$-\frac{1}{2}b$	or	$-\frac{2}{9}\Delta = -\frac{1}{4}\Delta + \frac{1}{4}c;$	$c = \frac{1}{9}\Delta$	advantage of the one who casts when one has set 1 against 1
$\frac{1}{4}c$	or	$\frac{1}{36}\Delta = \frac{1}{8}\Delta - \frac{1}{8}d;$	$d = \frac{7}{9}\Delta$	
$-\frac{1}{8}d$	or	$\frac{7}{27}\Delta = \frac{1}{8}\Delta + \frac{1}{16}e;$	$e = \frac{4}{9}\Delta$	note that $e = b$.

s set into the game 6 ducats against 6 which have been set by the other player. It is clear that one will have $n = 3\Delta + \frac{1}{2}(-o)$, and likewise $o = 3\frac{1}{2}\Delta + \frac{1}{2}(-p)$; hence $n = \frac{5}{4}\Delta + \frac{1}{4}p$. Now, one has evidently $p > n,$ since the stake is greatest in the case in which the advantage p is returned. It results from it $n > \frac{5}{4}\Delta + \frac{1}{4}n$ and consequently $n > \frac{5}{3}\Delta;$ but $o = 6\Delta - 2n,$ therefore $o < \frac{8}{3}\Delta$. One deduces from it $12288a - 1818\Delta < \frac{8}{3}\Delta$, or else $a < \frac{5462}{36864}\Delta$.

Hence from the assumption $q > 0$, one can deduce in the same manner from the equations $o = 3\frac{1}{2}\Delta + \frac{1}{2}(-p)$ and $p = 3\frac{1}{2}\Delta + \frac{1}{2}(-q)$ the relation $p < \frac{7}{3}\Delta,$ whence it follows $3643\Delta - 24576a < \frac{7}{3}\Delta,$ or else, $a > \frac{5461}{36864}\Delta$.

One finds therefore at last:

$$\frac{5462}{36864}\Delta > a > \frac{5461}{36864}\Delta.$$

It is true, that on the last of the pages cited from the Manuscript, Huygens verified if, in effect:

$$\frac{5462}{36864} > \frac{4}{27} > \frac{5461}{36864};$$

but this little calculation has probably been added after the value $\frac{4}{27}$ had been found by another method.

Moreover Huygens has calculated the superior limit $\frac{5462}{36864}\Delta > a$ again by another method, that is to say by employing directly the relation $p > n,$ which can be written $3643\Delta - 24576a > 912\Delta - 6144a.$

[124]

§2[7]

By the aid of the calculation which precedes one will understand easily also the calculation of the question posed on page 1.[8] But this which follows will serve to explicate more in detail this last calculation: when we write "$1 - b$ and $2 + k$",[9] this expresses briefly that one has "one chance to have $-b$ and two chances to have $+k$." And thus hence. In this calculation δ represents a ducat, or else that which is set each time.

[7] This paragraph contains the explication of the calculations which we have reproduced in Appendix III.

[8] See page 102.

[9] See page 103.

[125]

The calculation is continued pages 2 and 3.[10] At the beginning of page 2 one finds $a = \frac{2}{9}a$ increased by all the quantities of the two descending series where δ enters.[11] P. 3 contains in first place the continuation of the previous assumptions which begin on page 1 and for which there was not enough to place on this page. Next the quantities of the series B of page 2, where δ enters and which are affected of the $+$ sign are added to page 3 by the aid of the theorem which one finds at the top of this page.[12] Now, in noticing the proportion of these quantities to those indicated — of the same series B and to those affected with the $-$ sign and with the $+$ sign of the other series A, one determines also the sums of these last quantities. It results from it that one can replace the equation of page 2 by: $\frac{2}{9}a - \frac{23}{49}\delta = a$ and finally by $-a = \frac{207}{343}\delta$. This which was just said that $-a$, that is to say the advantage of J. who casts first when there is yet nothing in the game, is equal to $\frac{207}{343}\delta$. Thus, although this advantage was supposed to be negative, that is that it would be J. who would lose, one finds nevertheless that he wins $\frac{207}{343}$ of a ducat.

[10] See pages 104-107.

[11] See on page 104 the equation $a = \frac{1}{3}b - \frac{2}{3}k$ and on page 105 the expressions for $\frac{1}{3}b$ and $-\frac{2}{3}k$.

[12] See page 106.

[126]

§3[13]

On page 10[14] the number of tokens of B was posed $=\omega$, of which ϕ white and ψ black; so that $\phi+\psi=\omega$. This question is resolved first on pages 10, 11 and 12[15] according to that which was proposed.[16] The proportion of ϕ to ψ is determined by the equation $\phi=\frac{1}{6}\psi+\frac{1}{6}\sqrt{37\psi\psi}$,[17] so that it can not be defined by some numbers.[18] In the equation which begins at the top of page 11, a is equal to all the quantities of the two descending series, where Δ enters, increased by the expression $\frac{2\phi a}{3\omega}$ which one finds above of the series marked B.[19] But because one desires that the chances of the two players

[13] Explication of the calculations which one finds in Appendix IV, pages 108-115 of T. 14.

[14] See page 109.

[15] That is to say following the manner in which the problem was proposed by Hudde who supposed that the player A must choose among 2 white and 1 black tokens.

[16] See §1 (pages 108-113) of Appendix IV, where one must substitute everywhere $\theta=2,\lambda=1$ in order to find again the expressions which are related to the problem such as it was posed by Hudde.

[17] Read: $\frac{1}{6}\psi+\frac{1}{6}\sqrt{73\psi\psi}$ and compare page 113.

[18] One discovers this manner to indicate the incommensurable of a ratio in the work of Euclid, See Prop. 7 of Book X, where one reads: "Incommensurabiles magnitudines inter se proportionem non habent, quam numerus ad numerum." (page 217 of the 1607 edition of the Elements by Clavius.)

[19] See on page 111 the first term of the second member of the series for $\frac{\theta k}{\rho}$, where it is always necessary to substitute $\theta=2,\lambda=1,\rho=3$.

[127]

are equal, one has therefore $a = 0$. And, hence, one has also $\frac{2\phi a}{3\omega} = 0$. This is why this expression is supposed equal to zero in the equation which one finds at the base of page 11.[20] And one will note that this equation contains only the sums of the quantities, where Δ enters, of the descending series A and B. And because one had $a = -\frac{1}{3}b + \frac{2}{3}k$[21] and that a must be $= 0$, it follows that one must have also $-\frac{1}{3}b + \frac{2}{3}k = 0$, or else $\frac{1}{3}b = \frac{2}{3}k$. Consequently, all the $+$ and the $-$ quantities, containing Δ, of the series A must be equal after changing the sign to $+$ and to $-$ of the quantities, containing Δ, of the series B, as we have posed in the equation at the base of page 11, which equation is reprised on page 12.[22]

On the same pages 10 and 11 we have begun the Calculation of the general question, after which the number of tokens, namely 3, from which A draws his, was posed $= \rho$, and the number of white tokens (being 2) $= \theta$, and of black (being 1) $= \lambda$; so that $\theta + \lambda = \rho$. And to this calculation are related the quantities that we have surrounded by

[20] It concerns the equation $-\lambda + \frac{\lambda\phi\rho}{\theta\psi} = \theta - \frac{\lambda\phi}{\omega}$ (page 112), where rather from this which one deduces by posing $\theta = 2, \lambda = 1$.

[21] See the last line of page 110.

[22] See on pages 112-113 the successive reductions of the equation in question.

[128]

the circles on pages 10 and 11.[23] And the calculation is continued on pages 14 and 15, where the Rule is deduced which gives the general solution of this question, namely: to make so that the chances become equals, that is to say $\phi\phi = -\psi\phi + \frac{\theta\theta\psi\phi}{\lambda\lambda+\lambda\theta} + \frac{\theta\psi\psi}{\lambda}$.[24]

Next I have wished to calculate also,[25] for the general case, what are the chances of two players when the number of white and black tokens of each of the 2 players is given, *ceteris positis ut prius*, that is to say: what is the advantage or the disadvantage of A who casts first. This calculation can easily be deduced from the preceding equation of page 14,[26] but one must nevertheless seek then, for one of the descending series A or B, the sum of the quantities where Δ enters which are affected with the $+$ sign or with the $-$ sign. Thus I have determined in the present case the $+$ of the series B. This calculation begins towards the end of page 15[27] and continues to page 16, where this sum is found by the theorem which one encounters at the top of page 3; [28] one knows then also the sums of the $+$ and of the $-$ of the other quantities of the series A and B, since the proportions which the ones to the others have are known and indicated at the right side

[23] These are those quantities which we have reproduced on pages 109-114 to the exclusion of those which are related to the particular solution.

[24] See page 112 after the first line.

[25] It concerns §2 of Appendix IV, pages 113-115.

[26] See the last line of page 110.

[27] See the beginning of §2 on page 113.

[28] See the Theorem on page 106.

[129]

of page 14.[29] Whence one deduces the rule from page 16 towards the end,[30] where the quantity $-\frac{\theta\phi a}{\rho\omega}$ proceeding from the equation at the top of page 14.[31] Indeed, here this quantity is no longer equal to zero, since it is not thus for a. Perhaps the equation which summarizes this rule towards the end of page 16,[32] could it be divided, if one replaced throughout ρ by $\theta + \lambda$ and ω by $\phi + \psi$; quantities which are equal to them according to that which we have posed above.[33]

[29] See page 112 after the first line.

[30] See the first Rule on page 114.

[31] The equation $a = -\frac{\lambda b}{\rho} + \frac{\theta k}{\rho}$, where $\frac{\theta k}{\rho} = \frac{\theta\phi a}{\rho\omega} + \frac{\theta\psi l}{\rho\omega}$; see pages 110-111.

[32] See the first Rule on page 114.

[33] See page 109. We remark that in the case where one such division would be possible the numerator of the fraction which is equal to $a - \frac{\theta\phi a}{\delta\omega}$ according to the first Rule of page 114 must contain a rational factor, but that it is easy to establish that it is not so.

Indeed, if one supposes $a = 0$, it is necessary that this numerator be equal to zero, that is to say that one has:

$$\theta\theta\omega\psi - \lambda\rho\omega\phi + \theta\lambda\omega\psi - \lambda\theta\phi\psi = 0.$$

This equation must therefore bring forth, beyond the eventual solutions which would depend on the suspected factor, the irrational solution of the problem treated in §1 of Appendix IV (pages 108-113). Now, this last solution is expressed by the equation which summarizes the Rule of page 112 and which can be written:

$$\lambda\rho\phi\phi + \lambda\rho\phi\psi - \theta\theta\phi\psi - \theta\rho\psi\psi = 0.$$

The equality of the degree of the first members of these equations with respect to the quantities $\theta, \lambda, \rho, \phi, \psi, \omega$ prove already that the numerator in question does not contain a supplementary factor. Besides the complete identity of the two equations is easily verified by substituting into the first $\phi + \psi$ for ω.

[130]

§4.[34]

In the question of heads or tails page 21,[35] one wishes to know how much each player must set at the beginning (each the same sum), so that A, who casts first, had a chance as good as B. It comes: each $\frac{2}{3}$ of a ducat.

Let z be that which each set since the beginning.

A, who casts first, has 1 chance to obtain z, and 1 chance to put Δ[36] in addition to his stake z and to let the other cast; this which is worth to him $-c$. Consequently, this is worth c to B. And one will have $c = 1$ chance to obtain $\Delta + z$ and 1 chance to set $\Delta + z$ against $\Delta + z$ and to let the other cast; this which is worth $-d$ to B. And thus consecutively:

[34] This paragraph, taken from page 63 of Manuscript C (numbered 19 by Huygens), contains the solution of a problem posed by Huygens in his letter to Hudde of 10 May 1665.

[35] See page 116.

[36] That is to say, a ducat.

[131]

$$a = \frac{z-c}{2}$$

$$\frac{-\Delta - z + d}{4}$$ [37]

$$\frac{\Delta + z - e}{8}$$

$$\frac{-2\Delta - z + f}{16}$$

$$\frac{2\Delta + z - g}{32}$$

$$\frac{-3\Delta - z + h}{64}$$

$$\frac{3\Delta + z - i}{128}$$

$$-\frac{1}{4} - \frac{1}{16}\Delta - \frac{1}{64}[\Delta]$$

$$-\frac{1}{16}\Delta - \frac{1}{64}[\Delta]$$

$$-\frac{1}{64}[\Delta]$$ [38]

$$\left.\begin{array}{r} \frac{4}{9}\Delta = \text{the } - \ \Delta^{39} \\ \frac{2}{9}\Delta = \text{the } + \ \Delta \end{array}\right\}\Gamma$$

$$-\frac{2}{9}\Delta = \text{the } + \text{ and } - \ \Delta$$

$$\left.\begin{array}{r} \frac{3}{4} \text{ of } \frac{1}{4}[z] = \frac{1}{3}z = \text{the } - \ z \\ \frac{1}{6}z = \text{the } + \ z \end{array}\right\}\Gamma$$

$$-\frac{1}{6}z \text{ the } + \text{ and } - z$$

[37] By this notation Huygens indicates that one can replace $-\frac{c}{2}$ by $\frac{-\Delta-z+d}{4}$

[38] The sum of these quantities represent the series “of $-\Delta$,” that is to say of the negative terms of the expression for $-a$.

[39] Compare pages 121-122.

[132]

$-a = \frac{1}{2}z - \frac{1}{6}z - \frac{2}{9}\Delta = 0$[40] because we suppose that $-a$ is zero, so that the chance is equal for the two players.

$$z = \frac{2}{3}\Delta$$

§ 5.[41]

15 July 1665.

On pages 19 and 20[42] we have attempted the solution of the following question, which will be resolved here:

A and B cast in turn heads or tails, under the condition that the one who casts tails will set each time one ducat into the stake, but the one who casts heads will receive each time one ducat if something has been staked. And A will cast the first when there is yet nothing in the stake, and the game will not end before something has been staked, and one will play until that which all has been removed. One demands what is the advantage of A. Answer $\frac{1}{6}$ of a ducat.

[40] That is to say by supposing that the last term, as here $\frac{i}{128}$, approaches indefinitely to zero; compare page 120.

[41] This paragraph contains the solution of a problem of which Huygens and Hudde are occupied in the months of July and August 1665.

[42] These pages contain the calculations which correspond in part with those which are going to follow, but which have not been terminated.

[133]

$$\text{Let}\left\{\begin{array}{llccc} -a & \text{be the advantage of the one who has set} & 0 & \text{against} & 0\\ +b & & 0 & & 1\\ +c & & 1 & & 1\\ +d & & 1 & & 2\\ +e & & 2 & & 2\\ +f & & 2 & & 3\\ +g & & 3 & & 3\\ +h & & 3 & & 4\\ +i & & 4 & & 4\\ +k & & 4 & & 5 \end{array}\right\}\text{and}$$

who must cast.[43]

Hence, the one who does not cast has the advantage $+a$, because that which the one wins, the other loses it, and who must cast.

Let $\Delta =$ a ducat.

$-a = 1$ chance to $+a$ and 1 chance to $-b$ [44] $b = 1$ to Δ and 1 to $-c$ [45]

Hence $-a = \dfrac{a-b}{2}$; $3a = b$ Hence $b = \dfrac{\Delta - c}{2}$

[43] In order to define from the first the sense attached by Huygens to the quantities b, c, d, etc. we call x_m the part due to the one who must cast first, when the concern is to share, without terminating the game, a stake $m\Delta$ ($\Delta =$ one ducat), which is formed during the game. One will have then: $b = x_1$, $c = x_2 - \Delta$, $d = x_3 - \Delta$, $e = x_4 - 2\Delta$, $f = x_5 - 2\Delta$, $g = x_6 - 3\Delta$, $h = x_7 - 3\Delta$, $i = x_8 - 4\Delta$, $k = x_9 - 4\Delta$. On the contrary the advantages (taken in the sense of Huygens) of the other player will be respectively $b' = (\Delta - x_1) - \Delta$ (since one must subtract the ducat set by himself) $= -b$; $c' = (2\Delta - x_2) - \Delta = -c$; $d' = (3\Delta - x_3) - 2\Delta = -d$; $e' = (4\Delta - x_4) - 2\Delta = -e$, etc. Here and in the notes which follow we will accept this hypothesis without which the reasonings of Huygens lose their validity.

[44] That is to say, but $b' = -b$.

[45] Since one has $c' = -c$.

[134]

Next c is equal to a chance to have $\Delta - b$ and 1 to have $-d$, for the reason that the one who casts heads when there is 1 against 1 in the game can be assessed to have won the ducat which the other has set and to leave his proper ducat to the stake: that which is 1 against 0, the other before casting, and this is worth $-b$ to the one who must not cast; so that there is therefore 1 chance to have $\Delta - b$, and 1 to have $-d$, that is to say: to set 2 against 1 and to leave the other to cast. This is why $c = \frac{\Delta - b - d}{2}$.

Likewise $d = 1$ to $\Delta - c$ and 1 to $-e$; thus $d = \frac{\Delta - c - e}{2}$ and in the same manner one finds next[46]

$$e = \frac{\Delta - d - f}{2}$$
$$f = \frac{\Delta - e - g}{2}$$
$$g = \frac{\Delta - f - h}{2}$$
$$h = \frac{\Delta - g - i}{2}$$
$$i = \frac{\Delta - h - k}{2}$$
$$k = \frac{\Delta - i - l}{2}$$

[46] Under the reservation which we have formulated in the note 43 from page 133, the reasoning which leads to the mentioned equations is correct. In order to apply it it suffices to suppose that the player who casts heads takes always one of the ducats which belong to the stake of the other player. However this assumption can seem a little artificial. It is therefore perhaps not useless to remark that all these equations, when one substitutes the values of b, c, d, etc. indicated in the same note on page 133, are reduced to some particular cases of the equation $x_m = m\Delta - \frac{1}{2}x_{m-1} - \frac{1}{2}x_{m+1}$; an equation which results from the definition of x_m such as we have given in the note cited, since one has evidently hence from this definition $x_m = \frac{1}{2}\Delta + \frac{1}{2}\{(m-1)\Delta - x_{m-1}\} - \frac{1}{2}\Delta + \frac{1}{2}\{(m+1)\Delta - x_{m+1}\}$.

[135]

One has therefore $-a = \frac{a-b}{2}$

$$\frac{-\Delta + c}{4};\ \text{one has found above } -\frac{b}{2} = \frac{-\Delta + c}{4}$$ [47]

$$\frac{+\Delta - b - a}{8}$$

$$\frac{-\Delta + c + e}{16}$$

$$\frac{+\Delta - d - f}{32}$$

$$\frac{-\Delta + e + g}{64}$$

$$\frac{+\Delta - f - h}{128}$$

$$\frac{-\Delta + g + i}{256}$$

$$\frac{+\Delta - h - k}{512}$$

[47] See page 133.

[136]

that is to say $-a = \frac{1}{2}a - \frac{1}{8}\Delta - \frac{1}{8}b - \frac{1}{8}d$

but $d = \dfrac{\Delta - c - e}{2}$ and $c = \dfrac{\Delta - b - d}{2}$

Therefore $d = \frac{1}{2}\Delta - \frac{1}{4}\Delta + \frac{1}{4}b + \frac{1}{4}d - \frac{1}{2}e$

$$-\frac{1}{8}d = -\frac{1}{24}\Delta - \frac{1}{24}b + \frac{1}{12}e$$

likewise $e = \dfrac{\Delta - d - f}{2}$ and $d = \frac{1}{3}\Delta + \frac{1}{3}b - \frac{2}{3}e$

Therefore $e = \frac{1}{2}\Delta - \frac{1}{6}\Delta - \frac{1}{6}b + \frac{1}{3}e - \frac{1}{2}f$

$$\frac{1}{12}e = \frac{1}{24}\Delta - \frac{1}{48}b - \frac{1}{16}f$$

$f = \dfrac{\Delta - e - g}{2}$ and $e = \frac{1}{2}\Delta - \frac{1}{4}b - \frac{3}{4}f$

Therefore $f = \frac{1}{2}\Delta - \frac{1}{4}\Delta + \frac{1}{8}b + \frac{3}{8}f - \frac{1}{2}g$

$$-\frac{1}{16}f = -\frac{1}{40}\Delta - \frac{1}{80}b + \frac{1}{20}g$$

[137]

$$g = \frac{\Delta - f - h}{2} \text{ and } f = \frac{2}{5}\Delta + \frac{1}{5}b - \frac{4}{5}g$$

$$\text{Therefore } g = \frac{1}{2}\Delta - \frac{1}{5}\Delta - \frac{1}{10}b + \frac{2}{5}g - \frac{1}{2}h$$

$$\frac{1}{20}g = \frac{1}{40}\Delta - \frac{1}{120}b - \frac{1}{24}h$$

$$h = \frac{\Delta - g - i}{2} \text{ and } g = \frac{1}{2}\Delta - \frac{1}{6}b - \frac{5}{6}h$$

$$\text{Therefore } h = \frac{1}{2}\Delta - \frac{1}{4}\Delta + \frac{1}{12}b + \frac{5}{12}h - \frac{1}{2}i$$

$$-\frac{1}{24}h = -\frac{1}{56}\Delta - \frac{1}{168}b + \frac{1}{28}i$$

$$i = \frac{\Delta - h - k}{2} \text{ and } h = \frac{3}{7}\Delta + \frac{1}{7}b - \frac{6}{7}i$$

$$\text{Therefore } i = \frac{1}{2}\Delta - \frac{3}{14}\Delta - \frac{1}{14}b + \frac{3}{7}i - \frac{1}{2}k$$

$$\frac{1}{28}i = \frac{1}{56}\Delta - \frac{1}{224}b - \frac{1}{32}k$$

[138]

One has therefore $a = \frac{1}{2}a - \frac{1}{8}\Delta - \frac{1}{8}b - \frac{1}{8}d$

$$-\frac{1}{24}\Delta - \frac{1}{24}b + \frac{1}{12}e$$ [48]

$$\frac{1}{24}\Delta - \frac{1}{48}b - \frac{1}{16}f$$

$$-\frac{1}{40}\Delta - \frac{1}{80}b + \frac{1}{20}g$$

$$\left[+\frac{1}{20}g\right]$$

$$\frac{1}{40}\Delta - \frac{1}{120}b - \frac{1}{24}h$$

$$-\frac{1}{56}\Delta - \frac{1}{168}b + \frac{1}{28}i$$

$$\frac{1}{56}\Delta - \frac{1}{224}b - \frac{1}{32}k$$

[48] This notation indicates that one can replace $-\frac{1}{8}d$ by its value, found above on page 136.

[139]

$$
\begin{array}{lll}
 & \text{A} & \text{B} \\
\text{That is to say } -a = \frac{1}{2}a & -\frac{1}{8}\Delta - \frac{1}{8}b & \\
 & -\frac{1}{24}\Delta - \frac{1}{24}b & \\
 & +\frac{1}{24}\Delta - \frac{1}{48}b & \\
 & -\frac{1}{40}\Delta - \frac{1}{80}b & \\
 & +\frac{1}{40}\Delta - \frac{1}{120}b & \\
 & -\frac{1}{56}\Delta - \frac{1}{168}b & \\
 & +\frac{1}{56}\Delta - \frac{1}{224}b - \frac{1}{32}k &
\end{array}
$$

The last quantity, here $\frac{1}{32}k$, becomes infinitely small, since k[49] does not become infinitely great, because if one has set only a single ducat more than

[49] Consult on the precise signification of k the note on page 133. One can further define $2k - \Delta$ as the *difference* of the mathematical expectations of the two players in the case where they would have divided between them a stake of nine ducats according to the rules of the game in question. On the other hand $2i$ is equal to the difference of the expectations in the case where there are eight ducats in the stake.

[140]

the other, it is rather evident that in this game he loses no more than 1 or 2 ducats, but the divisor (being here 32) becomes infinitely great, increasing each time by 4. Consequently the last quantity (here $\frac{1}{32}k$) must be counted for 0, if one supposes that one has continued the calculation to infinity. One finds therefore $-a = \frac{1}{2}a -$ the series under A $-$ the series under B.

But in the series under A all the $+$ are equal to all the $-$, with the exception of the first term $-\frac{1}{8}\Delta$. One has therefore $-a = \frac{1}{2}a - \frac{1}{8}\Delta -$ the series under B which one must continue to infinity.

But the series under B is $\frac{1}{8}$ of the series: $-\frac{1}{1}b - \frac{1}{3}b - \frac{1}{6}b - \frac{1}{10}b - \frac{1}{15}b - \frac{1}{21}b - \frac{1}{28}b$ &c. And this series is equal to $-2b$ as we will indicate it below (pages 28, 29, 30[50]), the divisors of these fractions being the triangular numbers

[50] See §6 which follows (pages 144-150) and especially the note of page 144.

[141]

beginning with the first. And $\frac{1}{8}$ of $-2b$ is $-\frac{1}{4}b$. One has therefore $-a = \frac{1}{2}a - \frac{1}{8}\Delta - \frac{1}{4}b$; $-6a = -\frac{1}{2}\Delta - b$; but $b = 3a$. see page 25;[51] $a = \frac{1}{6}\Delta$.

A, who casts first, loses therefore $\frac{1}{6}$ of a ducat.

One has therefore $a = \frac{1}{6}\Delta$ and $b = 3$; $b = \frac{1}{2}\Delta$, but $b = \frac{\Delta - c}{2}$. See page 25.

$\frac{1}{2}\Delta = \frac{\Delta - c}{2}$; $\Delta = \Delta - c$, $c = 0$; thus, if one has set 1 against 1 the chances are equal. But $c = \frac{\Delta - b - d}{2} = 0$; $\Delta = b + d$, and $b = \frac{1}{2}\Delta$; $\frac{1}{2}\Delta = d$, hence $d = b$, that is to say: these chances are equal; namely: that which one has when one has set 0 against 1 or 1 against 2 and that one must cast.

[51] See page 133.

[142]

But $d = \frac{\Delta - c - e}{2}$, $\frac{1}{2}\Delta = \frac{\Delta - c - e}{2}$; but $c = 0$; $\frac{1}{2}\Delta = \frac{1}{2}\Delta - \frac{1}{2}e$, $e = 0$; so that if there is 2 against 2 in the stake the chances are anew equal.

But $e = \frac{\Delta - d - f}{2}$; hence $\frac{\Delta - d - f}{2} = 0$; therefore $\Delta = d + f$, but $d = \frac{1}{2}\Delta$, $\frac{1}{2}\Delta = f$. Therefore $f = d = b$.

And one can find next the other chances in the same manner. They are alternately $= 0$ and $= \frac{1}{2}\Delta$.

It would be necessary to examine if in this question one could not arrive to a certain result by a shorter path. This would be so if one could conclude that the chances are equal when one has set 1 against 1.[52] See the preceding page.[53]

[52] One can, in effect, arrive to this conclusion by a very short path. We take to this effect the notations of the note on page 133. Let therefore x_2 be the mathematical expectation of the player who must cast (that is to say the part which is due to him from the two ducats which are found in the stake) and let A be this player. One can divide the game into two periods of which the first extends to the moment where for the first time the stake is reduced to a single ducat. The second period extends from this instant to the end of the game, and it is evident that the total mathematical expectation x_2 is equal to the sum of the partial mathematical expectations concerning the two periods. Now, the expectation concerning the first period is the same as if there were only a single ducat in the stake and if one played to the depletion of the stake; it is therefore equal to x_1. As for the second period, it is certain that when it begins it will be B who must cast (since it is always his turn to cast when the number of ducats in the stake is odd.) The expectation of B will be therefore x_1, and $\Delta - x_1$ that of A. Hence, $x_2 = x_1 + \Delta - x_1 = \Delta$. The advantage of A, in the sense that Huygens attaches, is therefore $c = x_2 - \Delta = 0$; that which it was necessary to prove.

[53] The reasoning which is going to follow was written by Huygens on the preceding page of the Manuscript in a space which until then was left vacant.

[143]

The chances of 2 or of many players are equal or equivalent if the game is such that when one plays to the end with an equal success a person neither wins nor loses: And that a like excess of good chance brings as much gain or advantage to the one as it brings to another of them.

If results from it that if, in the game of which we treat here, one has set 1 against 1 (because if A casts heads and B likewise, each takes a ducat and the game is ended[54]) or 2 against 2, or 3 against 3, etc., that then the chances of players A and B are equal.[55] Now, if one has set 0 against 0 and if A must play first, we have posed for his loss:

$$-a = \begin{cases} 1 \text{ chance to } +a \text{ if he casts heads} \\ 1 \text{ chance to } -b \text{ if he casts tails and if he must therefore set } \Delta; \end{cases}$$

but

$$-b = \begin{cases} 1 \text{ chance to } -\Delta \text{ if B comes to cast heads} \\ 1 \text{ chance to } 0\text{, that is to say neither gain nor loss,} \\ \qquad \text{if B casts tails and if he must set 1 against 1.} \end{cases}$$

[54] This phrase is found in the margin of the Manuscript.

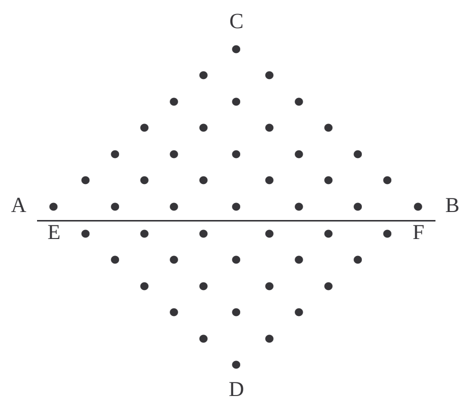
[55] This reasoning cannot be entirely satisfactory to Huygens. It seems to us that one must rather consider this part of the present Piece as a provisional annotation on which he had place perhaps to review.

[144]

Therefore $-b = -\frac{1}{2}\Delta$. Hence $-a = \begin{cases} 1 \text{ to } a \\ 1\ a - \dfrac{1}{2}\Delta. \end{cases}$ Therefore $-a = \frac{a - \frac{1}{2}\Delta}{2}$. Therefore $3a = \frac{1}{2}\Delta$. And $a = \frac{1}{6}\Delta$. As in the other solution.

§6.[56]

1. In order to find the triangle of a given number one adds this number to its square; the half of this sum is the sought triangle thus there results from the figure at the side, because if one adds to the number ACBD, that is to say to the square of the number AB, again one time the number of the line AB, one has twice the number of the triangle ACB. Hence, the half of the aforementioned sum must be equal to the triangle ACB which is the triangle of the number AB. Let therefore x be the side, then the triangle is $\frac{xx+x}{2}$.



[56] This paragraph teaches to sum the series of reciprocal values of the triangular numbers. We have not wished to suppress it since it shows the manner by which this sum has been obtained by Huygens; but one knows that one obtains it quite more easily by noting that $\frac{2}{n(n+1)} = \frac{2}{n} + \frac{2}{n+1}$ and that, consequently, the series is reduced to $1 + 1 - \frac{2}{3} + \frac{2}{3} - \frac{2}{4} + \frac{2}{4} - \frac{2}{5}$ etc.

[145]

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\frac{1}{1}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{10}$	$\frac{1}{15}$	$\frac{1}{21}$	$\frac{1}{28}$	$\frac{1}{36}$	$\frac{1}{45}$	$\frac{1}{55}$	$\frac{1}{66}$	$\frac{1}{78}$	$\frac{1}{91}$	$\frac{1}{105}$	$\frac{1}{120}$

2. This sequence of fractional numbers is such that all the numerators are equal to unity, and that the denominators are the successive triangular numbers of the numbers which one has written above, in beginning with 1 and increasing each time by 1: if of these fractions on adds two successive of them, of which the first is such that the denominator is the triangle of an even number, then the sum will be equal to the half of a fraction of the same sequence, of which the denominator is the triangle of the half of the number of which the triangle constitutes the denominator of the first of the two fractions which one has added. For example, in adding the fractions $\frac{1}{55}$ and $\frac{1}{66}$, belonging to this series, their sum will be $\frac{1}{30}$, that is to say the half of the fraction $\frac{1}{15}$, of which the denominator is the triangle of 5, that is to say of the half of 10 of which the triangle 55 is the denominator of the first of the 2 fractions; because: let the side of the triangle which constitutes the denominator of the first fraction $= x$, then its triangle, that is to say the

[146]

denominator of the first fraction, is equal to $\frac{xx+x}{2}$ and the first fraction to $\frac{2}{xx+x}$. Since the side of the denominator of the fraction which follows will be $x+1$. Therefore its triangle, which is the denominator of the second fraction, will be $\frac{xx+3x+2}{2}$. Consequently the fraction which follows will be $\frac{2}{xx+3x+2}$. In order to make the addition of this fraction to the first $\frac{2}{xx+x}$, one will note that their common denominator is $x^3+3xx+2x$ and their sum $\frac{4x+4}{x^3+3xx+2x}$, that is to say $\frac{1}{\frac{1}{4}xx+\frac{1}{2}}$. Now, by taking the side equal to $\frac{1}{2}x$, that which will represent a whole number since x is supposed even, its triangle is $\frac{\frac{1}{4}xx+\frac{1}{2}x}{2}$ or $\frac{1}{8}xx+\frac{1}{4}x$. Hence $\frac{1}{\frac{1}{8}xx+\frac{1}{4}x}$ will constitute the fraction of which the denominator is the

[147]

triangle of the half of the side x of which the triangle was the denominator of the first fraction $\frac{2}{xx+x}$. Now, in effect, $\frac{1}{\frac{1}{4}xx+\frac{1}{2}}$, that is to say the sum of the two fractions,is equal to the half of $\frac{1}{\frac{1}{8}xx+\frac{1}{4}x}$, that which it was necessary to demonstrate.

3. If now one forms likewise a sequence of fractions which are proportional to those of the preceding triangular sequence, it is certain that, since the sum of two successive fractions of the triangular sequence is equal to the half of the fraction indicated above, the sum of the 2 fractions proportional to the 2 said fractions will be equal to the half of the fraction proportional to the aforesaid fraction.

1	2	3	4	5	For example, since $\frac{1}{10}+\frac{1}{15}=$ the half of $\frac{1}{3}$, that is to say
$\frac{1}{1}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{10}$	$\frac{1}{15}$	equal to $\frac{1}{6}$, one will have also in the inferior proportional sequence
$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{1}{40}$	$\frac{1}{60}$	$\frac{1}{40}+\frac{1}{60}=$ the half of $\frac{1}{12}$, that is to say equal to $\frac{1}{24}$.

[148]

4. In the sequence of fractions of which the denominators are the successive triangular numbers and the numerators always 1, the sum of the total series prolonged to infinity is equal to 2.

	1	2		3	4		5		6		7	8		9		10		11		12		13		14		15
first sequence	$\frac{1}{1}$	$\frac{1}{3}$		$\frac{1}{6}$	$\frac{1}{10}$		$\frac{1}{15}$		$\frac{1}{21}$		$\frac{1}{28}$	$\frac{1}{36}$		$\frac{1}{45}$		$\frac{1}{55}$		$\frac{1}{66}$		$\frac{1}{78}$		$\frac{1}{91}$		$\frac{1}{105}$		$\frac{1}{120}$
second sequence			$\frac{1}{2}$			$\frac{1}{6}$				$\frac{1}{12}$			$\frac{1}{20}$				$\frac{1}{30}$				$\frac{1}{42}$				$\frac{1}{56}$	
third sequence								$\frac{1}{4}$							$\frac{1}{12}$								$\frac{1}{24}$			
4th sequence																			$\frac{1}{8}$							

We suppose that the first sequence continues to infinity and that the second sequence is composed of the sums of the fractions of the first sequence, taken 2 by 2, in a manner that the smallest denominator of each pair is the triangle of an even number. Then the numbers of the second sequence are, according to the second proposition,[57] the halves

[57] See on page 145 the line numbered 2.

[149]

of the numbers of the first sequence, taken in the same order. If now the 3rd sequence is composed anew of the sums of the numbers of the 2nd sequence, taken 2 by 2, by leaving again from the first side of these numbers, the numbers of the 3rd sequence will be, according to the third proposition, [58] each the half of the number of the 2nd sequence.

And likewise, if the 4th sequence is composed of the sums of each pair of 2 numbers of the 3rd sequence, leaving from the first side, the numbers of this 4th sequence will be each the half of those of the 3rd sequence. And in the same manner one will treat all the inferior sequences to infinity.

Now, it follows that the first number of the second sequence, that is to say $\frac{1}{2}$, is the sum of the 2 numbers of the first sequence which follow after the first number.

And that the first number of the 3rd sequence is the sum of 4 numbers following from the first sequence.

And that the first number of the 4th sequence is the sum of 8 numbers following from the first sequence.

And likewise that the first number of the 5th sequence will be the sum of 16 numbers following from the first sequence, etc.

[58] See page 146, the line numbered 3.

[150]

Since therefore the first numbers of the sequences constitute the sum of all the numbers of the first sequence, and since one has demonstrated that these first numbers of the sequences are each the half of the one which precedes, and since, consequently, all together to infinity are equal to 2 times the first number $\frac{1}{1}$, there results from it that all the numbers of the first sequence together are also $= 2$ times $\frac{1}{1}$, that is to say $= 2$. That which it was necessary to demonstrate.