Solution to Huygen's Fifth Exercise
also known as
Gambler's Ruin Problem[1]

In its most general form, Players A and B have m and n tokens respectively. On each trial, Player A wins with probability p and Player B wins with probability $q = 1 - p$. The winner of a trial takes one token from the other. The game ends when one player has obtained all the tokens of the other.

The problem posed by Pascal to Fermat may be stated as follows:

A and B each have 12 tokens. Player B obtains a token from A on each throw of three dice of 11 points, an event which can occur in 27 ways. On the other hand, B must give a token to A on each throw of 14 points, an event which can occur in 15 ways. All other outcomes are ignored. Therefore A obtains a token with probability $\frac{15}{42} = \frac{5}{14}$ and B obtains a token with probability $\frac{27}{42} = \frac{9}{14}$. The winner is the one who will be the first in possession of all the tokens.

For Player A, let $u(n)$ denote the probability of winning when in possession of n tokens. Clearly $u(0) = 0$ and $u(24) = 1$. Now, by the conditions of the problem,

$$u(n) = \frac{5u(n+1) + 9u(n-1)}{14}.$$

Such boundary value problems are easy to solve since they are linear difference equations with constant coefficients. The general solution to this equation is $u(n) = ar_1^n + br_2^n$ where r_1 and r_2 are the roots of the equation $5r^2 - 14r + 9 = 0$ and where a and b are constants to be determined by the boundary conditions.

The roots of the quadratic equation are 1 and $9/5$. Therefore the general solution is

$$u(n) = a + b\left(\frac{9}{5}\right)^n.$$

Moreover, we must have $a + b = 0$ and $a + b(9/5)^{24} = 1$ so that clearly

$$a = \frac{-5^{24}}{9^{24} - 5^{24}} = \frac{-59604644775390625}{79766383472227734472736}.$$

Consequently, we find

$$u(12) = \frac{244140625}{282673677106}.$$

Moreover, it is easy to show that the odds will be

$$\frac{u(12}{1 - u(12)} = \frac{5^{12}}{9^{12}}.$$

The solution in Letter 336 is equivalent: $150094635296999121 = 3^{12} \times 9^{12}$ and $129746337890625 = 3^{12} \times 5^{12}$.

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