EXTRACTS OF THE LETTERS
REGARDING THE GAME OF TENNIS

MONTMORT & NICOLAS BERNOULLI

Extract of the Letter from Nicolas Bernoulli to Montmort
At Basel 10 November 1711
Jeux de Hazard, pp. 333-334

There is however another place which shows that this Author[1] does not understand well enough these things; it is on page 122, where he cites quite badly by the way, to that which seems to me, these words of Mr. de Fontenelle in the eloge of the my late Uncle: *It is to note that often the advantages or the forces are incommensurables, so that the two Players can never be perfectly equals,* which does not prove that the Author had intention to prove; because in order to comprehend well the sense, & that which Mr. de Fontenelle has wished to say, it is necessary to know that the late my Uncle has left, beyond the Latin Treatise *De Arte Conjectandi*, another manuscript written in French, in which he treats of the Game of Tennis in particular, & where he has resolved many questions that one can form on this Game, of which these two are the principals. 1° If one supposes the Players unequal, one demands what advantage the strongest must accord to the other in order that the game be equal or reciprocally. 2° If one supposes that one has accorded a certain advantage, & that thence their lot is made equal, one demands by how much he is stronger, or what ratio there is between their abilities; & it is in this last Problem that he has found that their forces or abilities will be incommensurables, & that one would not know how to express by any number the ratio that there is between them. I am amazed that you have not spoken of this Game in your Book. It is true that, the research on similar Problems is not easy, but it is quite curious, & does not lack usage. Here are some of them of which you could seek the solution in order to see if it will accord with that of my Uncle. 1. *Pierre & Paul play at Tennis in four linked parties, Pierre is two times more able than Paul, one demands what advantage must Pierre accord to Paul?* 2. *Pierre accords to Paul half-thirty, one demands by how much is he more able than Paul?* 3. *Pierre accords to Paul half-30, & to Jean forty-five, one demands how much can Paul accord to Jean?* 4. *Pierre & Paul play together against Jean, & their respective forces are as 1.2.3, one demands how much can this last accord to the first two?*

Extract of the Letter from Montmort to Nicolas Bernoulli
At Paris 1 March 1712
Jeux de Hazard, pp. 338, 340-344

Translated by Richard J. Pulskamp, Department of Mathematics and Computer Science, Xavier University, Cincinnati OH. August 24, 2009.

[1]Barbeyrac, Professor of Law at Lausanne, Switzerland. Bernoulli has been discussing previously his *Traité du Jeu*.

...I will limit myself in this here to make you part of the reflections of two of my friends[2] who I have left at Montmort, & that I have strongly invited, in quitting them, to examine the Problems that you proposed on Tennis, & that which you said touching on Her, I will add those that I have made rather lightly on some places of your Letter...

These Sirs have also sent to me quite long calculations on the first of your Problems on Tennis: these calculations are exact; but as there is much groping in their method, & as besides there is much lacking in them that the Problem is not resolved, I will not put them here.

For me, Sir, before undertaking the solution of it, I have believed I must demand enlightenment on that which follows.

1° When you say *Pierre is two times more able than Paul*, do you understand that Pierre has two times more facility than Paul to win each fifteen, or more exactly, that the ratios of the facility are as 2.1.

2° By this word, *parties of Tennis*, do you understand of the parties composed of six games? Do you conceive that when Pierre & Paul have each forty-five, this which is called to be at deuce, one returns necessarily into two fifteen, this which is practiced here.

3° When you say: *One demands what advantage Pierre must make to Paul*. Do you demand how many fifteen or fractions of fifteen Pierre must accord to Paul in each game? You know that the strongest gives often in order to be equal to the weakest some bisques, some entire games, to save the first or the second, to play entirely on one side, &c. all that wish to be determined. It would not be the same thing, for example, to give three games in each partie of six games, or 30 in each game of the same partie.

4° The fourth Problem that you enunciate thus: *Pierre & Paul play together against Jean, & their respective forces are as 1.2.3, one demands how much this last can accord to the first two*. This Problem, I say, seems to include no exactitude. Often two persons less strong in particular than Pierre, can play without disadvantage with him; & to the contrary two persons as strong can play with disadvantage, according as they will know or will not know to accommodate themselves together, this which is a particular talent independent of the one to play well being alone.

5° When you say in the second Problem, *Pierre accords to Paul half-thirty*. Do you intend that Paul will have 30 in the first game, & next 15, & thus in sequence 30 & 15 alternatively, or if Paul will commence by having 15, & next 30, &c. this which would be perhaps quite different.

By you writing this, Sir, I have had the curiosity to make some tries on your four Problems. Here is the path that I have made.

You know, Sir, that naming p the number of the parties which are lacking to Pierre, q the number of parties which are lacking to Paul, a the degree of facility that Pierre has to win each point, b the degree of facility that Paul has to win each point, & supposing $p+q-1=m$, the formula which expresses the lot of Pierre is

$$a^m b^0 + m \cdot a^{m-1} b^1 + \frac{m.m-1}{1.2} a^{m-2} b^2 + \frac{m.m-1.m-2}{1.2.3} a^{m-3} b^3 + \&\text{c.}$$

& likewise that the formula which expresses the lot of Paul is

$$b^m a^0 + m \cdot b^{m-1} a^1 + \frac{m.m-1}{1.2} b^{m-2} a^2 + \frac{m.m-1.m-2}{1.2.3} b^{m-3} a^3 + \&\text{c.}$$

that it is necessary to continue the first series until the number of terms expressed by q, & the second until the number of terms expressed by p, & to divide both by $\overline{a+b}^m$.

[2]These are l'Abbé de Monsoury and Mr. Waldegrave.

I have found that if one wishes that when there is lacking more than one point to each of the two Players, one returns to deuce by necessity; these same formulas can again serve with the two restrictions which follow, 1° m must be $= p+q-2$, instead as one supposed $m = p+q-1$. 2° It is necessary to multiply the last term of the series which expresses the lot of Pierre, by $\frac{aa}{aa+bb}$; & the last term of the series which expresses the lot of Paul, by $\frac{bb}{aa+bb}$.

Next from these preliminaries, I have sought the solution of some of your Problems, or of others which have relation, here is that which I have found. 1° *Pierre plays against Paul, & he is two times stronger: there lacks to him four points, one demands how many there must lack of them to Paul, that is to say, what must be the value of q, p being $= 4$.*

Under the ordinary assumption that one not return, one has this equation

$$4m^3 - 8mm + 14m + 6 = 3^{m+1},$$

of which one can find the root by the intersection of a logarithm & of a cubic parabola, I find $m = 5 + \frac{57}{230}$, this which teaches me that Pierre must give to Paul one point, & $\frac{263}{320}$ on the 2nd point, this which I explicate in this manner. One will put three hundred twenty tokens into a pouch, of which there will be 263 whites & 57 blacks; & it will be said that if drawing a token at random one draws a white, Pierre will give two points to Paul on the partie, & that if one draws a black token, he will give to him only one of them. One can render the lots perfectly equal only by using this skill, & it is only in this manner that it is necessary to explicate the fractions of things which are not shared at all by the coups, the points, &c.

If one wishes to suppose that the Players will return when they are at deuce, that is to say every time that they will have each three points, as it is the rule in the game of Tennis.

One finds that under this assumption Pierre must give to Paul two points, & $\frac{11}{224}$ on the third point, in order that the game be equal, & it is here, it seems to me, the solution of the first of your Problems.

2° *Pierre gives to Paul two points out of four, & beyond this $\frac{11}{224}$ on the third point, one demands by how much he must be stronger than Paul in order to give to him this advantage, one finds that the ratio sought of his force to that of Paul is contained in this equality of the sixth degree,*

$$224a^6 + 830a^5b - 1142a^4bb - 1792a^3b^3 - 1568a^2b^4 - 896ab^5 - 224b^6 = 0.$$

Whence one deduces $a = 2b$.

If one supposes that *Pierre has reason to give one point to Paul, & that one demands how much he must be stronger in order to give this advantage to him, b being $= 1$*, I find $a = 1+\sqrt{2}$, that is to say that Pierre must be stronger than Paul in the ratio of 1 to $\sqrt{2}-1$; & generally that if $q = 1$, it is necessary that Pierre be stronger than Paul in the ratio of 1 to $\sqrt[p]{2}-1$.

3° I had commenced to make an attempt on one kind completely parallel to the second of your four Problems, but more simple, here is what it is.

Pierre plays in two linked parties against Paul in the "petit palet," each of the parties is of two points. They agree that Paul will have one point in the first partie, that he will have none at all in the second, that is to say that they will play it to goal; & that in the third, if the game is not finished before, Paul will have one point. This returns to that which one calls half-fifteen in Tennis. One demands how much it is necessary that Pierre be stronger than Paul in order to give him this advantage. I have found that this Problem would depend on the resolution of this equality of the 7^{th} degree

$$a^7 + 7a^6b + 5a^5b^2 - 21a^4b^3 - 29a^3b^4 - 21a^2b^5 - 7ab^6 - b^7 = 0,$$

this which would lead me to some quite long calculations. Your second Problem demands of it yet greater, the equality being more composed; thus I pray you to say to me if you have some other secret than me in order to avoid the resolution of these equalities. My method has been every time to rest myself when I am come to the equation of it, & to leave to the curious to seek the roots of it. I will change only in order to please you in the case that you testified that you required this sacrifice here from me; I say sacrifice, because in truth it is the farthest that I myself remember to have resolved some equalities which exceed the fourth degree, & it seems to me that of all the occupations it is the least agreeable.

4° I have made further some reflections on the third of your four Problems, here is the one that I have proposed to myself, which is a little more simple, but which contains the same difficulty.

Pierre playing against Paul in two points can give one of them to him, Paul playing against Jacques can give to him one of them, one demands how much Pierre can give of them to Jacques.

I have found that he must give to him one point & $8\sqrt{2} - 11$ on the second point, this which I explicate in this way: Let be supposed $\sqrt{2} = \frac{1414}{1000}$, I say that putting 125 tokens into a pouch, of which there are 39 blacks & 86 whites; if one draws a token at random, & if it is found white, Pierre must give one point only to Paul, & if black is encountered he must give to him two of them. One sees here the example of a case where it is absolutely impossible to render the parts equal, whatever compensation that one can imagine. Here is, Sir, that which I have found quite in a hurry; if I had had more leisure to meditate on these matters, & to make long calculations, I would have perhaps better success. If I myself am deceived, give me grace in favor of my allegiance. I pass now to the other places of your Letter.

Extract of the Letter from Nicolas Bernoulli to Montmort
At Basel 2 June 1712
Jeux de Hazard, pp. 349-350

I have found a general formula for the lots of the Players, when one supposes that the degrees of facility what they have to win changes alternatively, as it arrives when a Player accords to the other either half-fifteen or half-thirty, or some other similar point, here it is: Let p be the number of parties which are lacking to Pierre, q the number of parties which are lacking to Paul, a the degree of facility that Pierre has to win the 1^{st}, 3^{rd}, 5^{th}, 7^{th}, &c. game; b the degree of facility that Paul has to win the 1^{st}, 3^{rd}, 5^{th}, &c. game; c the degree of facility that Pierre has to win the 2^{nd}, 4^{th}, 6^{th}, &c. game; d the degree of facility that Paul has to win the 2^{nd}, 4^{th}, 6^{th}, &c. game; let moreover $m + n = p + q - 1$, & $m = n$ if $m + n$ is an even number, & $m = n + 1$ if $m + n$ is an odd number; I say that the lot of

Pierre will be the sum of all the possible values of this series:

$$\begin{aligned}&b^n \times \frac{m.m-1.m-2\ldots m-s+1}{1.2.3\ldots s}a^{m-s}b^s\\&\quad + nc^{n-1}d \times \frac{m.m-1.m-2\ldots m-s+2}{1.2.3\ldots s-1}a^{m-s+1}b^{s-1}\\&\quad + \frac{n.n-1}{1.2}c^{n-2}dd \times \frac{m.m-1.m-2\ldots m-s+3}{1.2.3\ldots s-2}a^{m-s+2}b^{s-2}\\&\quad + \frac{n.n-1.n-2}{1.2.3}c^{n-3}d^3 \times \frac{m.m-1.m-2\ldots m-s+4}{1.2.3\ldots s-3}a^{m-s+3}b^{s-3} + \&c.\end{aligned}$$

the whole divided by $\overline{a+b}^m \times \overline{c+d}^n$. This is understood by taking for s successively 0, 1, 2, 3, &c. until $q-1$ inclusively. If one is *à deux de jeux*, & if it is necessary to win two games in sequence in order to win the partie, it is necessary to put $m+n-1$, instead of $m+n$; & it is necessary still to multiply this which proceeds by substituting for s the last value $q-1$ by $\frac{ac}{ac+bd}$. You see quite easily that this formula in the case of $a=c$, & $b=d$ will agree exactly with yours. I have also found throughout the same solutions as you gave in your Letter, except only this equation

$$a^7 + 7a^6b + a^5bb - 15a^4b^3 - 29a^3b^4 - 21aab^5 - 7ab^6 - b^7 = 0,$$

instead of which I have found this here

$$a^7 + 7a^6b + 13a^5bb - 21a^4b^3 - 35a^3b^4 - 21aab^5 - 7ab^6 - b^7 = 0,$$

in order to determine by how much Pierre must be stronger than Paul, so that by playing in two parties, of which each is of two points, he is able to give to him one point in the first game, & nothing in the second; & if the partie is not ended before, further a point in the third game; but I have come up no more than you of another secret in order to avoid the resolution of these equalities than the approximations, & I would take also for a sacrifice if it was necessary for me to seek the roots of these sorts of equations, it is a work that I leave quite gladly to the curious.

Extract of the Letter from Montmort to Nicolas Bernoulli
At Montmort 8 June 1712
Jeux de Hazard, pp. 352-353

In reading your Letter & my response I have noticed in it a fault of which I have believed I must caution you. Instead of this equality

$$a^7 + 7a^6b + a^5bb - 15a^4b^3 - 29a^3b^4 - 21aab^5 - 7ab^6 - b^7 = 0,$$

it is necessary

$$a^7 + 7a^6b + 13a^5bb - 21a^4b^3 - 35a^3b^4 - 21aab^5 - 7ab^6 - b^7 = 0,$$

of which the root[3] is nearly 1.77. I have again made some tries on some of the Problems similar to those which you proposed to me, but always uselessly. I fall into some equalities which appear to me to demand immense calculations, of which the difficulty belongs to the algebra, & which demands no invention. I suppose therefore, Sir, that I am not in the good way with respect to these Problems, & I pray you to put me there: here is one of them very simple in appearance that I had proposed to myself:

Pierre plays for three points, Paul for two, & Jacques for one, their lots being equal, one demands what must be the ratio of their forces.

---

[3] The root is approximately 1.746.

By naming their forces respectively $a,\ b,\ c$, one has, conforming to the Problem of page 175, the lot[4] of Pierre

$$= a^3c + 4a^3b + a^4,$$

the one of Paul

$$b^4 + 4ab^3 + 4abbc + 6aabb + bbcc + 2bc^3,$$

the one of Jacques

$$c^4 + 2cb^3 + 4bc^3 + 5bbcc + 4ac^3 + 8abbc + 12abcc + 6aacc + 12aabc + 3a^3c,$$

the whole divided by $\overline{a+b+c}^4$.

I see well that by comparing these equalities, of which each $= \frac{1}{3}$, I will come in the end to determine the value of the three unknowns; but it will not be at all without resolving an equality so composite that would demand perhaps 30 hours of calculation, without counting the risk to be deceived.[5] I will await therefore on all these questions your help & your illuminations. In awaiting, & in order to fill the paper, I am going to make you part of two rather curious remarks, it seems to me, that I have made (it is a long time) on the occasion of this Problem.

Extract of the Letter from Nicolas Bernoulli to Montmort
At London 11 October 1712
Jeux de Hazard, p. 371

I am quite comfortable, Sir, that you have noticed with me in your Letter of March 1st that it is necessary to write

$$a^7 + 7a^6b + 13a^5bb - 21a^4b^3 - 35a^3b^4 - 21aab^5 - 7ab^6 - b^7 = 0,$$

instead of the equality

$$a^7 + 7a^6b + a^5bb - 15a^4b^3 - 29a^3b^4 - 21aab^5 - 7ab^6 - b^7 = 0,$$

The way that you follow in order to resolve the Problems parallel to the one for which you have found the preceding equation, it appears to me good, you arrive always to some solutions which I find myself, & it seems to me that you are wrong to believe that one must expect a better method; the resolution of the algebraic equalities is inevitable in these sorts of Problems; & when these resolutions are too difficult, it is necessary to be content with approximations. I have found the same lots for the three Players Pierre, Paul & Jacques, to whom there are lacking respectively 3, 2, 1 points: you have forgotten in the quantity

$$b^4 + 4ab^3 + 2b^3c + 4abbc + 6aabb + bbcc,$$

which expresses the lot of Paul, the third term $2b^3c$.

[4] See page 242 of the 2nd edition. Here Montmort makes a Remark on the Problem of Points.

[5] The approximate solution is $a = 0.5803289$, $b = 0.3508267$, $c = 0.12307937$. Or, in the ratio of $4.72 : 2.85 : 1$.