LETTRE À UN AMI SUR LES PARTIES DU JEU DE PAUME

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You indicate to me, Sir, that you have seen one of my Theses, where I advance some new Propositions, touching on the Parties[1] of the Game of Tennis; & you ask me, if these Propositions contain some reality which can be demonstrated, or if they are based only out of pure conjectures made in the air, & if they have no solidity; not being able to imagine, in that which you say, that one can measure the strengths of the players by numbers, & even less to deduce all the conclusions, that I have deduced from them. This which obliges me to put into writing all that which I have meditated on this matter, & to make it the subject of this Letter, which I have written to you in French, in order to not discourage you in its reading by the translation of the terms which are in use among the players, & which would become not very intelligible, if one put them into another Language. I will not pause to explain to you the Rules of the Game, nor the principle of the Art of conjecturing, which must serve as foundation to our research, knowing that both are perfectly known to you. But besides I enter into detail of all the particularities of my subject, without fearing reproach, that one would be able to make me entertain you too much on a trifle; because you know, that this noble Game has always been the diversion of persons of the first quality, & soon you will see, that if it is useful for the exercise of the body, it is very capable & very worthy also to fix the meditations of the mind.

I will remark to you before all things, that the reason, for which in the games of chance one can calculate exactly the advantages & disadvantages of the Players, it is because most often one knows rightly the number of cases, which are favorable or contrary to them: & I must say to you, that it is not likewise of games, which depend uniquely, or in part, on the genius, on the industry or on the skill of the players, such as are the games of tennis, of chess, & the greater part of the games of cards; being quite clear, that one would not know how to determine by causes, or *a priori*, as one speaks, by how much a man is wiser, more skillful or more able than another, without having a perfect knowledge of the nature of the soul, & of the disposition of the organs of the human body, what thousand hidden causes, which combine, render absolutely impossible. But this does not prevent, that one can know it nearly as certainly, *a posteriori*, by the observation of the event many times reiterated, by making that which can be practiced the same in the games of pure chance, when one does not know the number of cases, which can happen. We put, that there is in a sack a quantity of tickets in part white & in part black, & that I do not know the number of the ones nor of the others; what would I do to

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[1] *Translator's note*: The Partie is composed of a certain number of games, that is, it is a set.

discover it? I would draw them one after the other, (by replacing each time in the sack the ticket, that I had drawn from it, before taking the next, so that the number of tickets in the sack diminished not at all) & if I observed one hundred times that I drew a black from it, & two hundred times, that I drew a white, I would not hesitate to conclude, that the number of whites were around double of those of the black; because it is very sure, that the more I would make of these observations by drawing from it, the more I would be able to expect to approach to the true ratio, which is found between the numbers of those two sorts of tickets; being even a demonstrated thing, that one can make so much of it, that it will be in the end provable of each given probability, & consequently that it will be morally certain, that the ratio between these numbers, that one will have thus found by experience, differs from the true by as little as one will wish: which is all that one can desire. It is also in this manner, that in the games of art & of skill one can know by how much a player is stronger than the other player. I see for example two men, who play at tennis: I observe them a long time, & I note, that one of them wins 200 or 300 coups, while the other wins only one hundred of them: I judge thence, with enough certitude, that the first is two or three times a better player than the other, having so to speak two or three parts of skill, as so many cases or causes which make him to win the ball, where the other has only one of them.

I. This being understood, we put, in order to enter the matter, two equal players A & B (that is to say, by whom we have seen to win & to lose an equal number of coups) who are firstly at deuce, or thirty, or fifteen, or at goal. It is evident, that they both have an equal expectation to make the coups that are lacking to them, & thus to win the game; this is why the lot of each is estimated $\frac{1}{2}J$ or $\frac{1}{2}$ Game. We put next, that A has 30 & B 45, or (that which returns to one) that the one here has the advantage: you see, that it is just as much probable, that A will win or will lose the following coup; but if he wins it, they would become again at deuce & each will have, as I have said, $\frac{1}{2}J$; & if he loses it, he will also lose the game; it is that which it is worth to him, by the Theory that you know,

$$\frac{1 \cdot \frac{1}{2} + 1 \cdot 0}{2} = \frac{1}{4}J.$$

We put next, that A has 15 to 45; it is clear also, that it is equally possible to him, to win 30 to 45, & to have thus the preceding lot $\frac{1}{4}J$, or to lose the game (according as he wins or loses the first coup) it is this which renders now his lot

$$\frac{1 \cdot \frac{1}{4} + 1 \cdot 0}{2} = \frac{1}{8}J.$$

But if A had 15 to 30, one case would render it thirty & another 15 to 45, (of which the one there brings $\frac{1}{2}J$, & the other $\frac{1}{8}J$) this which would be worth to him then

$$\frac{1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{8}}{2} = \frac{5}{16}J.$$

One will find all the same the lots of A for the other hypotheses, as they are marked in this Table. For those of B, they are easy to supply, being always the remainders of those of A to unity.

Table I.

Points of		*Lot of*
A	B	A
45	45	$\frac{1}{2}J$
30	45	$\frac{1}{4}J$
15	45	$\frac{1}{8}J$
0	45	$\frac{1}{16}J$
30	30	$\frac{1}{2}J$
15	30	$\frac{5}{16}J$
0	30	$\frac{3}{16}J$
15	15	$\frac{1}{2}J$
0	15	$\frac{11}{32}J$
0	0	$\frac{1}{2}J$

Table II.

Games of		*Lot of*
A	B	A
3	3	$\frac{1}{2}P$
2	3	$\frac{1}{4}P$
1	3	$\frac{1}{8}P$
0	3	$\frac{1}{16}P$
2	2	$\frac{1}{2}P$
1	2	$\frac{5}{16}P$
0	2	$\frac{3}{16}P$
1	1	$\frac{1}{2}P$
0	1	$\frac{11}{32}P$
0	0	$\frac{1}{2}P$

II. Likewise if the two players are *à deux de jeux*, it is clear, that each of them can equally expect to win the Partie, by making two games in sequence; & that consequently the lot of each is $\frac{1}{2}P$ or $\frac{1}{2}$Partie. But if (the Partie is made for example with four games) A had won 2 of them, & B 3, or (this which is the same) if B had the advantage of the game, he would have as much probability as the first game rendered them *à deux de jeux*, or if he was to lose the Partie to A (according as one would win or would lose this game) this which would make him have

$$\frac{1\cdot\frac{1}{2}+1\cdot 0}{2}=\frac{1}{4}P.$$

One concludes likewise, that if A had one game, & B three, the lot of A would be $\frac{1}{8}P$. And thus of the remainder, as you see in this other Table, which contains the lots of A with respect to all the Partie. You judge, that if must be the same as the first; because this that the 4 coups of a game are in regard to this game, the 4 games are in regard of all the Partie.

III. We consider next the two players, as being *à deux de jeux*, & we give beyond it to A 30 & B 45; you see that the first coup must put them at deuce, & thus equaling their lot, if A wins the coup; & if he loses it, then B must have the advantage of the game, in which case we have found the lot of A $\frac{1}{4}P$: this is why the expectation that he has to win the Partie is now

$$\frac{1\cdot\frac{1}{2}+1\cdot\frac{1}{4}}{2}=\frac{3}{8}P.$$

We suppose next, that A has two games (or one game) & B three, & that they are at deuce, or thirty, or fifteen; it is clear, that each being able equally to win the game, this is entirely as if they had nothing in there of their games, so that the lot of A is again, as it had been found in the preceding article, $\frac{1}{4}P$ (or $\frac{1}{8}P$). But if A had 2 games to 3, & 30 to 45, he would be able equally to acquire 45, or lose the Partie with the game (according as he would win or would lose the first coup) this which would be worth to him

$$\frac{1 \cdot \frac{1}{4} + 1 \cdot 0}{2} = \frac{1}{8}P.$$

And if beyond the 2 games to 3 he had only 15 to 45, the first coup would be able equally to give to him 30 to 45 or make him lose the game & the Partie; this which then would render his lot

$$\frac{1 \cdot \frac{1}{8} + 1 \cdot 0}{2} = \frac{1}{16}P.$$

&c. It is in this manner, that I have calculated the third Table, which comprehends the lots of A for all the possible states of the two players, when beyond the entire games they have yet won some points. It is therefore general, & it contains also in the last numbers of its perpendicular ranks all of the second Table. If you take the pain to examine it, you will be able to make many reflections worthy of remark. You will see for example that 15 to 30, the players being *à deux de jeux*, are worth entirely just as much, as 30 to nothing with two games to three, or 45 to 30 with one game to two, or finally 30 to 45 with one game to one: that one game to two with 45 to 15 is worth as much or a little more for A, than if they were again at the beginning of the Partie, & that A had nothing & B 15, having only $\frac{1}{512}$ difference between the lots of these two hypotheses. &c.

Table III.

Games of A		III. or II.	II.	I.	0.	I.	0.	I.	0.	0.
Games of B		III. or II.	III.	III.	III.	II.	II.	I.	I.	0.
Points of A	B	*Lots of A:*								
45.	45	1:2	1:4	1:8	1:16	5:16	3:16	1:2	11:32	1:2
30.	45	3:8	1:8	1:16	1:32	7:32	1:8	13:32	17:64	27:64
15.	45	5:16	1:16	1:32	1:64	11:64	3:32	23:64	29:128	49:128
0.	45	9:32	1:32	1:64	1:128	19:128	5:64	42:128	53:256	93:256
45.	30	5:8	3:8	3:16	3:32	13:32	1:4	19:32	27:64	37:64
30.	30	1:2	1:4	1:8	1:16	5:16	3:16	1:2	11:32	1:2
15.	30	13:32	5:32	5:64	5:128	31:128	9:64	55:128	73:256	113:256
0.	30	11:32	3:32	3:64	3:128	25:128	7:64	49:128	63:256	103:256
45.	15	11:16	7:16	7:32	7:64	29:64	9:32	41:64	59:128	79:128
30.	15	19:32	11:32	11:64	11:128	49:128	15:64	73:128	103:256	143:256
15.	15	1:2	1:4	1:8	1:16	5:16	3:16	1:2	11:32	1:2
0.	15	27:64	11:64	11:128	11:256	65:256	19:128	113:256	151:512	231:512
45.	0	23:32	15:32	15:64	15:128	61:128	19:64	85:128	123:256	163:256
30.	0	21:32	13:32	13:64	13:128	55:128	17:64	79:128	113:256	153:256
15.	0	37:64	21:64	21:128	21:256	95:256	29:128	143:256	201:512	281:512
0.	0	1:2	1:4	1:8	1:16	5:16	3:16	1:2	11:32	1:2

IV. We try presently to discover the lots of players, when they are of unequal force: In order to shorten the calculation, let n be taken generally for the number of coups, that one has seen by the stronger A win, against whom the weaker B has won only one; so that n against 1 marks the ratio of the two players; after which we put, that they are at deuce, & that it is necessary to find their lot. If a single coup would suffice to each of them in order to win the game, the question would be already decided; since the ratio of n to 1, which is that of their forces, would also be that of their expectations for this game here; but because the laws of the game are regulated otherwise, & because they demand that one win two coups in sequence in order to win the game, the ratio that one seeks is different from that here, & a little analysis is necessary in order to find it. Knowing therefore, that after the first coup one must have the advantage, & that after the second coup the game is able to be brought back to deuce, & that being at deuce it returns the same unknown lot, that we wish to seek, we call the lot of A in this state x, & we consider that which would arrive, if one or the other won the advantage. Now if A wins it, who is n times a more able player than the other, he will have for himself n appearances to win the game, & one appearance to be returned to deuce (following he will win also or lose the other coup) this which is worth to him

$$\frac{n \cdot 1 + 1 \cdot x}{n+1} = \frac{n+x}{n+1}:$$

& if it is B who wins the advantage, there will be for A n possibilities to be returned to deuce, & one possibility to lose the game; this which makes for him

$$\frac{n \cdot x + 1 \cdot 0}{n+1} = \frac{nx}{n+1}.$$

Whence it follows, that the players being again at deuce, in which case there is for A by the same reason n times more possibilities to win the advantage, than to lose it, his lot must be

$$\frac{n \cdot \frac{n+x}{n+1} + 1 \cdot \frac{nx}{n+1}}{n+1} = \frac{nn+2nx}{nn+2n+1},$$

& because the same is called x, there will be

$$x = \frac{nn+2nx}{nn+2n+1};$$

this which gives us

$$x = \frac{nn}{nn+1},$$

& there remains for the lot of his Partie $\frac{1}{nn+1}$, so that their lots are between them in ratio of nn to 1, doubled of that of their forces n to 1. This being established, one will be able to continue in order our research for all the other hypotheses, as one has done in the preceding articles, provided that one remembers here, that at each coup it is n times more probable, that A wins this coup, than it is not probable, that he lose it: Put therefore for example, that A has 30 & B 45; there are n cases which put the game at deuce, & one case that makes A lose it; this which is worth to him

$$\frac{n \cdot \frac{nn}{nn+1} + 1 \cdot 0}{n+1} = \frac{n^3}{n^3+nn+n+1},$$

Table IV.

Points of A	B	Lots of A
45	45	$\frac{nn}{nn+1}$
30	45	$\frac{n^3}{n^3+nn+n+1}$
15	45	$\frac{n^4}{n^4+2n^3+2nn+2n+1}$
0	45	$\frac{n^5}{n^5+3n^4+4n^3+4nn+3n+1}$
45	30	$\frac{n^3+nn+n}{n^3+nn+n+1}$
30	30	$\frac{nn}{nn+1}$
15	30	$\frac{n^5+3n^4+n^3}{n^5+3n^4+4n^3+4nn+3n+1}$
0	30	$\frac{n^6+4n^5+n^4}{n^6+4n^5+7n^4+8n^3+7nn+4n+1}$
45	15	$\frac{n^4+2n^3+2nn+2n}{n^4+2n^3+2nn+2n+1}$
30	15	$\frac{n^5+3n^4+4n^3+3nn}{n^5+3n^4+4n^3+4nn+3n+1}$
15	15	$\frac{n^5+3n^4+4n^3}{n^5+3n^4+4n^3+4nn+3n+1}$
0	15	$\frac{n^7+5n^6+11n^5+5n^4}{n^7+5n^6+11n^5+15n^4+15n^3+11nn+5n+1}$
45	0	$\frac{n^5+3n^4+4n^3+4nn+3n}{n^5+3n^4+4n^3+4nn+3n+1}$
30	0	$\frac{n^6+4n^5+7n^4+8n^3+6nn}{n^6+4n^5+7n^4+8n^3+7nn+4n+1}$
15	0	$\frac{n^7+5n^6+11n^5+15n^4+10n^3}{n^7+5n^6+11n^5+15n^4+15n^3+11nn+5n+1}$
0	0	$\frac{n^7+5n^6+11n^5+15n^4}{n^7+5n^6+11n^5+15n^4+15n^3+11nn+5n+1}$

Posed that A has 15 to 45, there are n cases, which make him win 30 to 45 & again one case which makes him lose the game; this which gives birth to him the lot

$$\frac{n\cdot\frac{n^3+nn+n}{n^3+nn+n+1}+1\cdot 0}{n+1}=\frac{n^4}{n^4+2n^3+2nn+2n+1}.$$

One finds in the same manner the lot of A, when he has nothing & B 45. Where they are thirty, they have the same lot as being at deuce, because it is necessary also they win two coups in sequence, in order to make the game. One will find likewise their lot, A having 15 or 0, & B 30. Similarly one seeks the lots, A having 45, & B 30, 15 or 0; as also A having 30, & B 15 or 0. Thus one can not be ignorant of the lots, when they are fifteen, or A having 0 & B 15, or on the contrary A 15 & B 0, or finally when they are yet at goal. It is this which produces the fourth Table, where the value of the expectations of A is contained (with respect to each game) generally for all sorts of ratios, that one can imagine between the forces of the players:

V. You judge well, that if you take n for 1, there must result from it the first Table, made for the players of equal force: & if you make the value successively this letter for 2, 3, 4, &c. the Table will serve for some players, of whom one is two, three, or four times stronger than the other. If for example A is two times stronger than B, you will find his lot, being at deuce, $\frac{4}{5}J$; & having 30 to 45 you will find it $\frac{8}{15}J$; so that there will remain for the one of B, $\frac{1}{5}J$ & $\frac{7}{15}J$; & consequently the lots of the two players in these cases will be between them in ratio of 4 to 1, & of 8 to 7, & thus of all the rest, as it is represented in the fifth Table.

Table V.

Points of		*Ratios of their lots, A being stronger than B,*		
A	*B*	2*times*	3*times*	4*times*
45	45	4:1	9:1	16:1
30	45	8:7	27:13	64:21
15	45	16:29	81:79	256:169
0	45	32:103	243:397	1024:1101
45	30	14:1	39:1	84:1
30	30	4:1	9:1	16:1
15	30	88:47	513:127	1856:269
0	30	208:197	891:389	8448:2177
45	15	44:1	159:1	424:1
30	15	124:11	621:19	2096:29
15	15	112:23	297:23	2048:77
0	15	176:67	891:133	49408:3717
45	0	134:1	639:1	2124:1
30	0	392:13	1269:11	10592:33
15	0	224:19	999:25	52608:517
0	0	208:35	243:13	51968:1157

You yourself will remember however from that which I said, that these Tables serve only for each game separately; because it would be necessary yet to give one similar of it, which included the lots of the players with respect to all the Partie, when they play to many games, of which they have already won some, with some points further, if you wish; as I have made the third Table for the equal players: but because the continuation of this research by letters would be very painful, & would demand an immense calculation, I will content myself to show in a particular example, in what manner it would be necessary to take, in order to find in a few words that which one seeks. We suppose, that the Partie is made to 4 games: that A has one game & beyond this 15, B two games with 45, & that A is two times stronger than B; one wishes to know the value of the expectations that they have to win the Partie. We remark before everything that the facilities, which these players have to win each game being yet to goal, are between them by the fifth Table in ratio of 208 to 35, or else of $\frac{208}{35}$ to 1; & that consequently the one who is two times stronger than the other, will have $\frac{208}{35}$ times (this is nearly six times) more facility in order to win this game: next from which we consider, that the game, of which they have already made a Partie, being achieved, they will have either two games at deuce, or one game to three (according as one or the other will have won) in which situation there will lack to them again either two games to each, or three games to A & one game to B. Now it is quite clear, that this is then all as if there lacked only as many coups, as there lack to them of games (that is to say as if they had thirty, or 15 to 45) supposed that the facility, that the strongest has to win one entire game, was this, that he had to win a simple coup, & that we have named n. But this facility, as I just said, is expressed by $\frac{208}{35}$; if you substitute therefore this numeric fraction in the place of n in the quantities

$$\frac{nn}{n+1} \ \& \ \frac{n^4}{n^4+2n^3+2nn+2n+1},$$

which indicate, by the 4th Table, the lot of A when he is thirty or 15 to 45, you will have the lots, that fall to him when he has two games at deuce, or one game at three, which will be thus

$$\frac{43264}{44489}P \quad \& \quad \frac{1871773696}{2627030961}P.$$

And by this which one supposes, that this player has 15 to 45 of the game that one plays presently, in which state he has 16 cases to win this game, & 29 cases to lose it, by the fifth Table; it follows, that there are 16 cases which acquire to him two games at deuce, & 29 cases, which make him have one game to three, this which renders the value of his expectation to win the Partie

$$\frac{16 \cdot \frac{43264}{44489} + 26 \cdot \frac{1871773696}{2627030961}}{45} = \frac{19031314432}{23643278649}P;$$

& there remains for that of B,

$$\frac{4611964217}{23643278649}P;$$

so that these expectations are between them in ratio of 19031314432 to 4611964217, which is a little more than quadruple. But we pass beyond.

VI. If the ratio of the forces of two players is known, one can know, how much advantage one must give to the other in order to render the game equal. One has only to cast the eyes on the fifth Table, in order to see where the numbers, which indicate the ratio of their expectations, approach more. It is thus as we observe, that when A is two times stronger than B, their lots differ the least, A having nothing & B 30; so that A can give to B 30, & even with some small advantage for himself, his expectation to win the game being as much or a little greater than B. If A is three times stronger than B, & if he gives to him 45, we will see, that there is a notable advantage for B, but that he has much more advantage for himself, if he gives to him only 30. In order to render therefore the Partie equal as much as it is can be, it would be necessary that he gave to B 45, taking for himself 15. If A is four times stronger, he can give to B 45, nevertheless a small advantage for B; but if he was five times stronger than B, he would be able to give to him 45, & would still have a rather considerable advantage for himself, since their lots would be found to be as 3125 to 2491 &c.

VII. If A gives to B 15, or 30, or 45, to know to the contrary, how much is A stronger than B? In order to resolve this question it is necessary to consider, that when in order to equalize the Partie A gives to B an advantage of some points, the lot of each must be $\frac{1}{2}$; this is why one will draw from Table IV the quantities, which indicate the lot of A, when he has nothing, & B 45, or 30, or 15, & one will make them each $\frac{1}{2}$; this which furnishes us three equalities:

$$\frac{n^5}{n^5+3n^4+4n^3+4nn+3n+1}=\frac{1}{2},$$

$$\frac{n^6+4n^5+n^4}{n^6+4n^5+7n^4+8n^3+7nn+4n+1}=\frac{1}{2},$$

$$\& \qquad \frac{n^7+5n^6+11n^5+15n^4}{n^7+5n^6+11n^5+15n^4+15n^3+11nn+5n+1}=\frac{1}{2},$$

which being reduced will be

$$n^5-3n^4-4n^3-4nn-3n-1=0,$$

$$n^6+4n^5-5n^4-8n^3-7nn-4n-1=0,$$

$$n^7+5n^6+11n^5-5n^4-15n^3-11nn-5n-1=0.$$

And because the roots of these equations, which indicate the value of the unknown n, are surds, it follows that the forces of the players, which one gives to the other an advantage of some coups, are incommensurables among them. The root of the first is very near 4.216 (or around $4\frac{1}{5}$), of the second 1.946 (or $1\frac{9}{10}$), of the third 1.313 (or $1\frac{3}{10}$); this which shows, that the one who can give to the other 45, must be $4\frac{1}{5}$ times stronger: that the one who can give thirty, must be $1\frac{9}{10}$ times; & who can give fifteen, $1\frac{3}{10}$ times stronger than the other: that is to say, that the first must win 42, the second 19, & the third 13 coups, when their Opponents win 10 of them.

Now if A gives to B the advantage, which is necessary in order to render the game equal, this will be entirely the same thing, to play to one game, or to two games, or to three, or to as many as it will please you: because it is equally probable, that A win a game, or that he lose it; it is also equally possible, that he make two games in sequence, or that he lose them; when the Partie is made to two games; or else that he win or lose three games, when it is made to three games &c.

VIII. If A gives to B half-15, or half-30, or half-45,[2] namely by how much is A stronger than B? We put, that A gives to B half-45, that the Partie is played to two games; that B takes 30 in the first game, & in the other 45; next again 30, if the Partie is returned *à deux de jeux*, next 45, & thus alternately; & that every time that he takes 30, his expectation to win the game is to that of A in ratio of b to a, & every time that he takes 45, in ratio of d to c. Thus posed, we make the lot of A at the beginning of the Partie$= z$, & we consider that which would arrive, if B would win the first game: Then B would take 45, & by hypothesis A would have c possibilities to win the following game, & d possibilities to lose it. Now if A wins it, the Partie is returned *à deux de jeux*, & it is necessary that B takes back 30, entirely the same as at the beginning of the Partie: but if A loses it, he loses the Partie entirely: whence it follows, that the lot of A will be in this case

$$\frac{c\cdot z+d\cdot 0}{c+d}=\frac{cz}{c+d}.$$

[2] *Translator's note*: These represents points. 'half-fifteen: one point to be taken at the beginning of even numbered games; half-thirty: one point to be taken at the beginning of odd numbered games and 2 points with even numbered games; half-forty: 2 points to be taken at the beginning of every odd numbered game and 3 with every even numbered game.

That if on the contrary A had won the first game, B would take also 45 & A would have after this c appearances to win the game entirely & the Partie; & d appearances to return the Partie *à deux de jeux*, by losing the game: thus his lot would be then

$$\frac{c \cdot P + d \cdot z}{c+d} = \frac{cP + dz}{c+d}.$$

Finally considering the players as at the beginning of the Partie, where B takes 30, we will see that there is for A, a probabilities to win the advantage of the game, that is to say to arrive to the preceding lot $\frac{cP+dz}{c+d}$, & b probabilities to lose this advantage & to acquire thus the lot $\frac{cz}{c+d}$; this which is worth to him

$$\frac{a \cdot \frac{cP+dz}{c+d} + b \cdot \frac{cz}{c+d}}{a+b} = \frac{acP + adz + bcz}{a+b \cdot c+d}.$$

But we supposed the same lot, which A obtains at the beginning of the Partie= z; this is why there is equality between z & the said quantity

$$\frac{acP + adz + bcz}{a+b \cdot c+d},$$

which being reduced one will find

$$z = \frac{ac}{ac+bd} P.$$

And because the Partie at half-45 is supposed equal, in which before the beginning of the game the lot of each is $\frac{1}{2}P$, it is necessary to have again equality between $\frac{1}{2}P$, & the value found of z, whence there results this here $ac = bd$, which gives us the analogy $a \cdot b :: d \cdot c$.

This shows, that the Partie will be equal, when these four quantities a, b, d, c are proportionals; that is to say, when the expectation of the strongest to win the game is to the expectation of the weakest (having 30) as reciprocally the expectation of the weakest (having 45) is to that of the strongest: or else, when it is 2, 3, or 4 times greater to appear, that the weak lose the game, having 30, & that he has on the contrary as much appearance, that he win it, having 45, one can give to him half-45.

And it is to remark, that it is not important, either that B takes in the first game 30 & in the other 45; or that on the contrary he takes first 45, & then 30: because having made our calculation, for this last hypothesis, we will find

$$z = \frac{c \cdot \frac{cP+bz}{a+b} + d \cdot \frac{az}{a+b}}{c+d} = \frac{acP + bcz + adz}{a+b \cdot c+d}.$$

that is to say again

$$z = \frac{ac}{ac+bd} P,$$

as before. Consequently those are deceived, who themselves imagine, that there is advantage to take in the first game the lesser, & in the other the greater.

Now because the same reasoning always subsists, whatever ratio that the letters a, b & d, c can indicate; it follows that it will be the same likewise of the Partie, which is played to half-30, or to half-15; namely, that it will be equal every time, that the expectation of A with respect to each game surpasses that of B & is surpassed alternatively in same ratio.

In order to make application of that which we just established, it is necessary to determine for each hypothesis the value of the letters a, b, c, d; this which is done without pain. One has only to take in the 4th Table the lots of A, when B is supposed to have 45, or 30, or 15, or 0, to nothing; which being in order

$$\frac{n^5}{n^5+3n^4+4n^3+4nn+3n+1},\qquad \frac{n^6+4n^5+n^4}{n^6+4n^5+7n^4+8n^3+7nn+4n+1}$$
$$\frac{n^7+5n^6+11n^5+5n^4}{n^7+5n^6+11n^5+15n^4+15n^3+11nn+5n+1}$$
$$\frac{n^7+5n^6+11n^5+15n^4}{n^7+5n^6+11n^5+15n^4+15n^3+11nn+5n+1}$$

those of B, as the rest to unity, will be

$$\frac{3n^4+4n^3+4nn+3n+1}{n^5+3n^4+4n^3+4nn+3n+1},$$
$$\frac{6n^4+8n^3+7nn+4n+1}{n^6+4n^5+7n^4+8n^3+7nn+4n+1},$$
$$\frac{10n^4+15n^3+11nn+5n+1}{n^7+5n^6+11n^5+15n^4+15n^3+11nn+5n+1},$$
$$\frac{15n^3+11nn+5n+1}{n^7+5n^6+11n^5+15n^4+15n^3+11nn+5n+1};$$

& consequently the expectations of A will have to those of B the ratios

$$\frac{n^5}{3n^4+4n^3+4nn+3n+1},\qquad \frac{n^6+4n^5+n^4}{6n^4+8n^3+7nn+4n+1},$$
$$\frac{n^7+5n^6+11n^5+5n^4}{10n^4+15n^3+11nn+5n+1},\qquad \frac{n^7+5n^6+11n^5+15n^4}{15n^3+11nn+5n+1}.$$

Whence it is clear, that when the Partie is played to half-45, one must make

$$\frac{a}{b}=\frac{n^6+4n^5+n^4}{6n^4+8n^3+7nn+4n+1},\quad \& \quad \frac{c}{d}=\frac{n^5}{3n^4+4n^3+4nn+3n+1}:$$

when it is played to half-30,

$$\frac{a}{b}=\frac{n^7+5n^6+11n^5+5n^4}{10n^4+15n^3+11nn+5n+1},\quad \& \quad \frac{c}{d}=\frac{n^6+4n^5+n^4}{6n^4+8n^3+7nn+4n+1}:$$

& finally when one plays it to half-15,

$$\frac{a}{b}=\frac{n^7+5n^6+11n^5+5n^4}{15n^3+11nn+5n+1},\quad \& \quad \frac{c}{d}=\frac{n^7+5n^6+11n^5+15n^4}{10n^4+15n^3+11nn+5n+1}$$

Substituting therefore these values, we will have in the place of $ac=bd$, under the first hypothesis, $\overline{n^6+4n^5+n^4}$ by $n^5=\overline{6n^4+8n^3+7nn+4n+1}$ by $\overline{3n^4+4n^3+4nn+3n+1}$: in the second, $\overline{n^7+5n^6+11n^5+5n^4}$ by $\overline{n^6+4n^5+n^4}=\overline{10n^4+15n^3+11nn+5n+1}$ by $\overline{6n^4+8n^3+7nn+4n+1}$: & in the third $\overline{n^7+5n^6+11n^5+15n^4}$ by $\overline{n^7+5n^6+11n^5+5n^4}=\overline{15n^3+11nn+5n+1}$ by $\overline{10n^4+15n^3+11nn+5n+1}$; that is to say, that the multiplication done, we will have the three equalities:

$$n^{11}+4n^{10}+n^9=18n^8+48n^7+77n^6+90n^5+77n^4+49n^3+23nn+7n+1,$$
$$n^{13}+9n^{12}+32n^{11}+54n^{10}+31n^9+5n^8$$
$$=60n^8+170n^7+256n^6+263n^5+193n^4+102n^3+38nn+9n+1,$$
$$n^{14}+10n^{13}+47n^{12}+130n^{11}+221n^{10}+220n^9+75n^8$$
$$=150n^7+335n^6+380n^5+281n^4+140n^3+47nn+10n+1;$$

which next are reduced to these here:

$$n^{11} + 4n^{10} + n^9 - 18n^8 - 48n^7 - 77n^6 - 90n^5 - 77n^4 - 49n^3 - 23nn - 7n - 1 = 0,$$

$$n^{13} + 9n^{12} + 32n^{11} + 54n^{10} + 31n^9 - 55n^8 - 170n^7 - 256n^6 - 263n^5 - 193n^4 - 102n^3 - 38nn - 9n - 1 = 0,$$

$$n^{14} + 10n^{13} + 47n^{12} + 130n^{11} + 221n^{10} + 220n^9 + 75n^8 - 150n^7 - 335n^6 - 380n^5 - 281n^4 - 140n^3 - 47nn - 10n - 1 = 0;$$

whence the unknown n indicates to us the ratio between the forces of the two players. The one who will have the leisure, will be able to seek the roots of these equations; I conjecture, that they are around $2\frac{7}{10}$, $1\frac{6}{10}$, & $1\frac{1}{10}$, so that the one who can give half-45, must win 27: who can give half-30, must win 16: & finally who can give half-15, must win 11 coups against ten coups of his Adversary.

Before finishing this article, I must yet remark, that if the advantage which one gives alternatively to player B, is such, as I have said it, that is to say that the two players make thence in each game a continual exchange of their expectations, the Partie will always be equal, not only when one plays it in one or many pairs of games, as one could imagine it, but also to whatever number of games, as one will wish to play. For posed that one plays to 3, 4 or 5 games, that A gives to B an advantage alternatively smaller & greater: to know the smallest, when the sum of the games which remain to them is an even number, & the greatest when this sum is an odd number; & that in the first case there is two times more appearance that A wins the game, & that to the other there is to the contrary two times more appearance that B wins it: one will find the lot from each to each game in order, as one sees here. (Tab. VI.) The small circles[3] indicate to you the games which remain to them to make, & it would appear, that when the number of these games is equal on both sides, the lot of each player is always $\frac{1}{2}P$.

IX. A gives to B half-30, & to C 45; how much is B able to give to C? *Resp.* Because the force of B is to that of A, as 10 to 16, by the preceding article; & that of A to that of C, as 42 to 10, by article 7, one must conclude *ex aequo perturbatè*, that the force of B is to that of C, as 42 to 16, or very nearly as 26 to 10; so that B will be able to give to C half-45, by the preceding art.

X. A gives to B half-30, & B to C half-45; what can therefore A give to C? *Resp.* the force of A being to that of B, as 16 to 10; & that of B to that of C, as 27 to 10, by article 8; if follows by the composition of the ratios, that the force of A is to that of C, as 432 to 100, that is to say that the one here can give to the one there forty-five by art. 7.

XI. A is two times stronger than B, & five times stronger than C. Therefore B is $\frac{5}{2}$ times stronger than C, & can give to him consequently nearly half-45, by art. 8.

[3] *Translator's note*: The small circles have been replaced by die face images as a close approximation.

Table VI.

Games which remain *A*	*B*	*Sum of these games*	*Lot of A*
⚁	⚁	Even	$\frac{1}{2}$
⚁	⚀	Odd	$\frac{1\cdot\frac{1}{2}+2\cdot 0}{3}=\frac{1}{6}$
⚂	⚀	E	$\frac{2\cdot\frac{1}{6}+1\cdot 0}{3}=\frac{1}{9}$
⚃	⚀	O	$\frac{1\cdot\frac{1}{9}+2\cdot 0}{3}=\frac{1}{27}$
⚄	⚀	E	$\frac{2\cdot\frac{2}{27}+1\cdot 0}{3}=\frac{2}{81}$
⚀	⚁	O	$\frac{1\cdot 1+2\cdot\frac{1}{2}}{3}=\frac{2}{3}$
⚀	⚂	E	$\frac{2\cdot 1+1\cdot\frac{2}{3}}{3}=\frac{8}{9}$
⚀	⚃	O	$\frac{1\cdot 1+2\cdot\frac{8}{9}}{3}=\frac{25}{27}$
⚀	⚄	E	$\frac{2\cdot 1+1\cdot\frac{25}{27}}{3}=\frac{79}{81}$
⚂	⚁	O	$\frac{1\cdot\frac{1}{2}+2\cdot\frac{1}{9}}{3}=\frac{13}{54}$
⚃	⚁	E	$\frac{2\cdot\frac{13}{54}+1\cdot\frac{1}{27}}{3}=\frac{14}{81}$
⚄	⚁	O	$\frac{1\cdot\frac{14}{81}+2\cdot\frac{2}{81}}{3}=\frac{2}{27}$
⚁	⚂	O	$\frac{1\cdot\frac{8}{9}+2\cdot\frac{1}{2}}{3}=\frac{17}{27}$
⚁	⚃	E	$\frac{1\cdot\frac{25}{27}+1\cdot\frac{17}{27}}{3}=\frac{67}{81}$
⚁	⚄	O	$\frac{1\cdot\frac{79}{81}+2\cdot\frac{67}{81}}{3}=\frac{71}{81}$
⚂	⚂	E	$\frac{2\cdot\frac{17}{27}+1\cdot\frac{13}{54}}{3}=\frac{1}{2}$
⚃	⚂	O	$\frac{1\cdot\frac{1}{2}+2\cdot\frac{14}{81}}{3}=\frac{137}{486}$
⚄	⚂	E	$\frac{2\cdot\frac{137}{486}+1\cdot\frac{2}{27}}{3}=\frac{155}{729}$
⚂	⚃	O	$\frac{1\cdot\frac{67}{81}+2\cdot\frac{1}{2}}{3}=\frac{148}{243}$
⚂	⚄	E	$\frac{2\cdot\frac{71}{81}+1\cdot\frac{148}{243}}{3}=\frac{574}{729}$
⚃	⚃	E	$\frac{2\cdot\frac{148}{243}+1\cdot\frac{137}{486}}{3}=\frac{1}{2}$
⚄	⚃	O	$\frac{1\cdot\frac{1}{2}+2\cdot\frac{155}{729}}{3}=\frac{1349}{4374}$
⚃	⚄	O	$\frac{1\cdot\frac{574}{729}+2\cdot\frac{1}{2}}{3}=\frac{1303}{2187}$
⚄	⚄	E	$\frac{2\cdot\frac{1303}{2187}+1\cdot\frac{1349}{4374}}{3}=\frac{1}{2}$

XII. A is $\frac{3}{2}$ times stronger than B, & B $\frac{5}{2}$ times stronger than C. Therefore A is $\frac{15}{4}$ times stronger than C, & thus will be able to give to him more than half-45, & less than 45.

XIII. Knowing the ratios between the forces of three Players A, B, C, playing one to one in each sense, one will know also the ratio of their forces, when two of these players play in company against the third. We suppose, that the absolute forces of the three players are indicated by the letters l, m, n; that A plays against the two others, & that he plays indifferently sometimes with B, sometimes with C: If he plays with B, he has l degrees of facility to win the coup, & m degrees to lose it; this which is worth to him $\frac{l}{l+m}$: & if he plays with C, he has again l degrees of appearance to win the coup, & n degrees to lose it; this which makes $\frac{l}{l+n}$. Therefore if it is equally possible, that he send the ball to B or to C, as we suppose, there is one case, which makes him have $\frac{l}{l+m}$, & one other, which makes him acquire $\frac{l}{l+n}$; this which gives to him with respect to this coup here,

$$\frac{1\cdot\frac{l}{l+m}+1\cdot\frac{l}{l+n}}{2}=\frac{l}{2l+2m}+\frac{l}{2l+2n}=\frac{2ll+lm+ln}{2ll+2lm+2ln+2mn},$$

so that there remains for the lot of the others B & C,

$$\frac{lm+ln+2mn}{2ll+2lm+2ln+2mn}.$$

Thus their forces being for example in ratio of 3, 2, 1, the lot of A is $\frac{27}{40}$, & the one of B & C $\frac{13}{40}$, that is to say that A can win 27 coups, when the others can win only 13 of them; so that he can give to them thirty with some advantage for himself, as it appears by the fifth Table. That if you make

$$\frac{2ll+lm+ln}{2ll+2lm+2ln+2mn}=\frac{lm+ln+2mn}{2ll+2lm+2ln+2mn},$$

you will have $ll=mn$; this which indicates to you, that when the absolute force of the one, who plays against the two others, is the mean proportional between the forces of those there, the Partie can be played to goal.

When we advance, as equally probable, that the player A sends the ball to B or to C, it is only an assumption, & the truth is, that the more the player is able, the more often he will send the ball to the more feeble. In order to have regard to this, suppose that every time that he plays p balls to the stronger B, he plays a greater number q of them to the more feeble C: therefore there are p cases, which make him have $\frac{l}{l+m}$, & q cases which make him obtain $\frac{l}{l+n}$; this which is worth to him

$$\begin{aligned}\frac{p\cdot\frac{l}{l+m}+q\cdot\frac{l}{l+n}}{p+q}&=\frac{pl}{\overline{p+q}\cdot\overline{l+m}}+\frac{ql}{\overline{p+q}\cdot\overline{l+n}}\\&=\frac{pll+qll+qlm+pln}{pll+qll+plm+qlm+pln+qln+pmn+qmn}:\end{aligned}$$

where if you interpret the letters l, m, n, by 3, 2, 1, as above, & beyond this p by 1, & q by 3, you will find the lot of A in regard to each coup $=\frac{57}{80}$, greater than $\frac{27}{40}$ the lot that he has, when he sends the balls indifferently to each of the others; so that he can now give to them nearly half-45. If you make

$$\frac{pll+qll+qlm+pln}{pll+qll+plm+qlm+pln+qln+pmn+qmn}=\frac{1}{2},$$

you will have $plm-pln+pmn-pll=qlm-qln-qmn+qll$; this which indicates, that the Partie at goal will be equal, when p is to q, as $lm-ln-mn+ll$ to $lm-ln+mn-ll$; & it is necessary for this effect, that mn is always greater than ll.

But one must yet here consider a thing, which counterbalances in some manner the advantage, that player A deduces from that which he plays most often to the weakest. It is that being alone against two, he is fatigued also more than each of the others, & that this fatigue seems to diminish considerably his force & his lot; because three persons of an equal force playing together, one against two, one sees well, that according to this calculation, the Partie must be equal, instead that it is more probable, that the two will win it against the third, seeing that they do not permit themselves so much, & that they defend each only the half of the Game of Tennis. In order to have therefore regard for this difference, it would be necessary to judge the absolute forces of our players by the number of coups, that they win or that they lose, not when they play each alone against A, but when they play conjointly against him: for having observed for example, that of all the coups, which are played between A & B, the number of those which A wins is to the number of those that B wins, as l to r; & that of all the coups which are played between A & C, the number of those that A wins is to the number of those that C wins, as l to s; it is clear, that the absolute forces of the three players A, B, C will be then in ratio of l, r, s; whence their lots are deduced again as above, so that one has only to substitute simply the letters r & s in the place of m & n.

XIV. Knowing the ratios of the forces of four players A, B, C, D, playing one to one in every sense, one will know the ratio of their forces, when they play two to two, A & B against C & D. We suppose that their absolute forces are expressed by k, l, m, n; it can be made, that A (likewise that B) plays with C or with D. If A plays with C, he has $\frac{k}{k+m}$; if he plays with D, he has $\frac{k}{k+n}$ possibilities to win the coup; this is what makes it to arrive to the lot

$$\frac{1 \cdot \frac{k}{k+m} + 1 \cdot \frac{k}{k+n}}{2} = \frac{2kk + km + kn}{2kk + 2km + 2kn + 2mn}.$$

By the same reason if it is B who plays, his lot is

$$\frac{1 \cdot \frac{l}{l+m} + 1 \cdot \frac{l}{l+n}}{2} = \frac{2ll + lm + ln}{2ll + 2lm + 2ln + 2mn}.$$

Now it is equally possible, that A or B play: therefore there is one case, which carries to them

$$\frac{2kk + km + kn}{2kk + 2km + 2kn + 2mn},$$

and another, which gives to them

$$\frac{2ll + lm + ln}{2ll + 2lm + 2ln + 2mn};$$

this which is worth to them

$$\frac{2kk + km + kn}{4kk + 4km + 4kn + 4mn} + \frac{2ll + lm + ln}{4ll + 4lm + 4ln + 4mn}.$$

Thus the absolute forces of the four players A, B, C, D, being as 1, 5, 2, 3, the lot of A & B with respect to each coup will be $\frac{323}{672}$, & the one of C & D $\frac{349}{672}$, if although these here can give to those there nearly half-fifteen. If in the denominators of these literal fractions you put $4kl$ instead of $4mn$, you will have

$$\begin{aligned} &\frac{2kk + km + kn}{4hk + 4km + 4kn + 4kl} + \frac{2ll + lm + ln}{4ll + 4lm + 4ln + 4kl} \\ &= \frac{2k + m + n}{4k + 4m + 4n + 4l} + \frac{2l + m + n}{4l + 4m + 4n + 4k} = \frac{2k + 2l + 2m + 2n}{4k + 4l + 4m + 4n} = \frac{1}{2}; \end{aligned}$$

this which shows, that if the forces of the players on both sides are found reciprocally proportionals, the Partie which they play to goal will be equal. Each time it is necessary here to repeat the warning of the preceding article, namely that the skilled players try always to send the balls to the weakest, to what it is necessary to have regard, if one wishes to go quite exact.

XV. If of two players A & B one can give to the other an advantage of some coups, & if he prefers to give to him this advantage in entire games than in points; one wishes to know, how many games must he give to him? For example, if A can give to B 45, & if he wishes to play to goal with him one demands, how many games can he give to him all to the reserve of one alone? In order to resolve this question, it is necessary to consider, that 1. A being able to give to B 45, the value of his force, indicated by the letter n, will be $\frac{4216}{1000}$ by the 7th art. 2. When he is at goal with B, the expectation that he has to win the game is by the 4th Table,

$$\frac{n^7 + 5n^6 + 11n^5 + 15n^4}{n^7 + 5n^6 + 11n^5 + 15n^4 + 15n^3 + 11nn + 5n + 1};$$

consequently that of B is

$$\frac{15n^3 + 11nn + 5n + 1}{n^7 + 5n^6 + 11n^5 + 15n^4 + 15n^3 + 11nn + 5n + 1};$$

& the ratio of their expectations

$$\frac{n^7 + 5n^6 + 11n^5 + 15n^4}{15n^3 + 11nn + 5n + 1}.$$

3. In order to explicate this ratio by numbers, by substituting $\frac{4216}{1000}$ in the place of n, one can be served by Logarithms, by means of which one determines it without pain at $\frac{7114529}{134167}$. We name this ratio m, & we seek successively, what is the lot of A with respect to the Partie, when there lacks to him 1, 2, 3, 4 &c. games, while to B there lacks always only one; until that we see by the progression, what must be this lot, when there lacks to him x games. Now if there lacks to him one game, likewise as to B, that is to say if the two players are at five all, it is easy to judge by this as I have demonstrated in art. 4, that the lot of A is $\frac{mm}{mm+1}$. If there lacks to him two games, it is clear that there are m cases, which would be able to put him *à deux de jeux* with B by making him win the game, & one case which makes him lose the game & the Partie; this which is worth to him

$$\frac{m \cdot \frac{mm}{mm+1} + 1 \cdot 0}{m+1} = \frac{m^3}{mm+1 \cdot m+1}.$$

If there lack to him three games, it is no less clear, that m cases will make remain to him two of them in making him win the game, & that one case will make him again lose the Partie; this which produces to him

$$\frac{m \cdot \frac{m^3}{mm+1 \cdot m+1} + 1 \cdot 0}{m+1} = \frac{m^4}{mm+1 \cdot \overline{m+1}^2}.$$

And if there lack to him four, there are m cases which would make remain to him three of them, & one case which will make lose to him the Partie; this which carries to him

$$\frac{m \cdot \frac{m^4}{mm+1 \cdot \overline{m+1}^2} + 1 \cdot 0}{m+1} = \frac{m^5}{mm+1 \cdot \overline{m+1}^3}.$$

In a word, whatever number of games there lack to him, his lot is found always expressed by a fraction, in which the exponent of m is greater, & the one of $m+1$ smaller by a unit, than the number of these games. Whence one infers, that if there lack x games to A & one game to B, this is to say if they play to x games, of which A gives $x-1$ in advance to B, the lot of A will be

$$\frac{m^{x+1}}{mm+1\cdot\overline{m+1}^{x-1}};$$

& because in this state the Partie is suppose equal, he will have

$$\frac{m^{x+1}}{mm+1\cdot\overline{m+1}^{x-1}}=\frac{1}{2},$$

that is to say $2\cdot m^{x+1}=mm+1\cdot\overline{m+1}^{x-1}$; & by taking logarithms,

$$\ln 2+\overline{x+1}\ln m=\ln\overline{mm+1}+\overline{x+1}\ln\overline{m+1}$$

or by transposition

$$x\ln\overline{m+1}-x\ln m=\ln\overline{m+1}+\ln m+\ln 2-\ln\overline{mm+1}$$

& finally by division

$$x=\frac{\ln\overline{m+1}+\ln m+\ln 2-\ln\overline{mm+1}}{\ln\overline{m+1}-\ln m}.$$

In order to achieve now the solution one has only to put $\frac{7114529}{134167}$ instead of m, & its logarithm instead of $\ln m$ &c. in consideration of which one finds that x is as much or slightly greater than 38; so that the one who can give to the other 45 could give to him up to 37 entire games of 38; if they wish to play to goal together. Whence it appears, that there is plenty of difference between giving three of 4 coups, & giving three of 4 games; seeing that we just showed, that the one who can give to the other 45, that is to say three coups of four, can give to him more advantage than three games of four. Here is the calculation of it:

m	$=\frac{7114529}{134167}$,	$m+1$	$=\frac{7248696}{134167}$,
$\ln 7114529$	$=6.8521462$,		$\ln 7248696=6.8602599$,
$\ln 134167$	$=5.1276457$,		5.1276457,
$\ln m=\ln\frac{7114529}{134167}$	$=1.7245005$	$\ln\overline{m+1}$	$=\ln\frac{7248696}{134167}=1.7326142$,
	2,		
$\ln mm=2\ln m$	$=\overline{3.4490010}$		$\ln\overline{mm+1}=3.4491555$.

$\ln\overline{m+1}$	$=1.7326142$		
$\ln m$	$=1.7245005$	81137)3089892	$(38=x)$
$\ln 2$	$=0.3010300$	243411	
$\ln\overline{m+1}+\ln m+\ln 2$	$=3.7581447$	655782	
$\ln\overline{mm+1}$	$=3.4491555$	649096	
$\ln\overline{m+1}+\ln m+\ln 2-\ln\overline{mm+1}$	$=0.3089892$	6686	
$\ln\overline{m+1}-\ln m$	$=0.0081137$		

XVI. Player A can give to B 45, one demands how many games he can give to him all to the reserve of one alone, if beyond the entire games that he gives to him, he wishes yet to give to him 15 or 30 in each game? In order to satisfy the question, you have only to put

$$\frac{n^7 + 5n^6 + 11n^5 + 15n^4}{10n^4 + 15n^3 + 11nn + 5n + 1}$$

& then

$$\frac{n^6 + 4n^5 + n^4}{6n^4 + 8n^3 + 7nn + 4n + 1}$$

(ratios of the expectations, as one finds them by the 4th Table, when B has 15 or 30 to nothing) in the place of

$$\frac{n^7 + 5n^6 + 11n^5 + 15n^4}{15n^3 + 11nn + 5n + 1}$$

(ratios of the expectations that they obtain, when they play to goal), by interpreting again n by $\frac{4216}{1000}$: this which will make you find

$$m = \frac{6798590}{450105},$$

& next

$$m = \frac{1125963}{263741};$$

whence the remainder is deduced as above, & it will provide very nearly $x = 12$, & next $x = 4$; so that A can give to B 11 games of 12, & again 15 points in each game; or else 3 games of 4 & again 30 points in each.

XVII. If A can give to B 30, & if one demands how many entire games he can give to him; it is necessary only to change the value of n, which indicates his force, into $\frac{1946}{1000}$ by the 7th art., & to make again as above, in order to find that of x. The calculation takes us, that he can give to him around four of five games, & to play to goal; or two games of three, & further 15 points in each game. If A can give to B only 15, the value of n will be counted $\frac{1313}{1000}$ by art. 7. & one will find that he would know to give to him only one game of two, if he claims to play to goal with him.

XVIII. One can form many questions on the Bisques,[4] which are of the coups given beforehand by one of the parties to the other, who profit from it when it appears good to him; & to demand for example: if in a given case it is more advantageous to take his bisque, or to not take it? if two bisques in four games are worth more than half-fifteen: or fifteen & two bisques more than half-thirty? & other similars. But as these questions would take us too far, I do not wish to undertake all: I will content myself only to pause a little on the first. We suppose, that the players play only to one game: that the force of A is to that of B in ratio of equality or of any inequality, n to 1: & that B gives bisque to A (for although this is not practical, when one knows that the players are equal: it happens often, that B does not know the forces of A, the one having concealed his game previously; or that A demands it by obstinacy, or because he has lost the preceding game that he played to goal, although one knows besides that they are equal) then we suppose, that they are at deuce & that A has not yet taken his bisque; one demands, what

[4] *Translator's note*: A bisque is the odds which one player gives to the other in allowing him to score one point once during the 'set' at any time he may elect. OED

is his expectation to win the game? & if he does better to take his bisque, or to keep it a longer time? On which I make this reasoning: If he takes his bisque, he wins the advantage, but he will no longer have bisque: consequently his lot will be by Table IV $\frac{n^3+nn+n}{n^3+nn+n+1}$; if he does not take his bisque, it can be made only that he win or lose the next coup: if he wins it, he has won the game; for having the advantage he will not lack to take after his bisque: but if he loses the coup, he will have completely again his bisque, but B will have the advantage; & since the lot of A in this encounter because of the bisque is yet unknown to me, I call it y. Having therefore by hypothesis n cases which make him win the coup, & one case which makes him lose it, the lot which he obtains when he does not take the bisque will be

$$\frac{n \cdot 1 + 1 \cdot y}{n+1} = \frac{n+y}{n+1}.$$

Now, by the privilege of the bisques, A is equally to be able to take his bisque or to not take it; that is to say, he can equally acquire

$$\frac{n^3+nn+n}{n^3+nn+n+1} \quad \text{or} \quad \frac{n+y}{n+1}:$$

this is why if the lot, which arrives to him during this indifference, is called x, he will have

$$x = \frac{n^3+nn+n}{2n^3+2nn+2n+2} + \frac{n+y}{2n+2}.$$

In order to find the lot y, it is necessary to make a similar reasoning: If A takes his bisque, he returns the game to deuce, & will no longer have the bisque; this is what gives to him by Table IV $\frac{nn}{nn+1}$. If he does not take the bisque, & if he wins the coup, he wins the lot x (because he will be at deuce, & will have yet his bisque); but if he loses it he loses altogether the game; this is that which is worth to him then

$$\frac{n \cdot x + 1 \cdot 0}{n+1} = \frac{nx}{n+1}.$$

Now A is equally at right to take his bisque or to not take it, that is to say to acquire $\frac{nn}{nn+1}$ or $\frac{nx}{n+1}$; this is why his lot during this indifference, which we call y, will be

$$\frac{nn}{2nn+2} + \frac{nx}{2n+2}.$$

Therefore putting this value of y into the equation

$$x = \frac{n^3+nn+n}{2n^3+2nn+2n+2} + \frac{n+y}{2n+2},$$

we will find

$$x = \frac{4n^4+7n^3+7nn+4n}{4n^4+7n^3+8nn+7n+4} = \frac{n+1 \cdot 4n^3+3nn+4n}{nn+1 \cdot 4nn+7n+4};$$

& then substituting reciprocally this here we will have

$$y\left(\frac{nn}{2nn+2} + \frac{nx}{2n+2}\right) = \frac{nn \cdot 4nn+5n+4}{nn+1 \cdot 4nn+7n+4}.$$

Thus the game being at deuce, there are presented three quantities,

$$\frac{n^3+nn+n}{n^3+nn+n+1}\left(\frac{n^3+nn+n}{nn+1 \cdot n+1}\right), \quad \frac{n+y}{n+1}, \quad \frac{n+1 \cdot 4n^3+3nn+4n}{nn+1 \cdot 4nn+7n+4},$$

which indicate the lot of A under three different hypotheses: the one, when he takes his bisque: the other, when he does not take it: & the third (which must be mean between the two others), when he is again under the indifference of taking or not taking it. And because the first after the reduction to one same denominator is found greater than the third, it follows that all the more reason it will be greater than the second, & that consequently A does better to take his bisque, than to guard it for another time. If one examines these three other quantities

$$\frac{nn}{nn+1}, \quad \frac{nx}{n+1}, \quad \frac{nn \cdot 4nn + 5n + 4}{nn + 1 \cdot 4nn + 7n + 4},$$

that we have found by the same operation, & which indicate the lot of A under the said hypotheses, when B has the advantage, or (this which is as much) when he has 45 to 30, one can remark, that the first is also greater than the two others; so that in this state A does better again to take his bisque.

You will find finally with these reasonings the lots of the player A, for all the other constitutions of the game, when B has 45 to 15, or 45 to nothing, or 30 to 15 &c. & even with less pain, if you go in order; because you will encounter no more in your operation but some lots already found & known. I content myself to give them to you for some equal players in the three columns marked I. II. III. of the seventh Table: the first considers player A, as taking his bisque; the third, as not taking it; & that of the middle, as not yet being determined if he will take it or not: & one notes through all, that the fractions of the first column are a little greater than those of the others; whence one can generally conclude that it is always more advantageous for A to take first his bisque, than to keep it longer.

Table VII.

NB. A & B are some equal players:						*A has a bisque to take.*					
Points of		*Lots of A*			*Col. of*	*Points of*		*Lots of A*			*Col. of*
A	*B*	I.	II.	III.	*chase*	*A*	*B*	I.	II.	III.	*chase*
45	45	$\frac{3}{4}$	$\frac{11}{15}$	$\frac{43}{60}$	$\frac{19}{15}$	30	15	$\frac{7}{8}$	$\frac{209}{240}$	$\frac{13}{15}$	$\frac{17}{15}$
30	45	$\frac{1}{2}$	$\frac{13}{30}$	$\frac{11}{30}$	$\frac{15}{7}$	15	15	$\frac{11}{16}$	$\frac{219}{320}$	$\frac{109}{160}$	$\frac{47}{44}$
15	45	$\frac{1}{4}$	$\frac{7}{30}$	$\frac{13}{60}$	$\frac{15}{11}$	0	15	$\frac{1}{2}$	$\frac{319}{640}$	$\frac{159}{320}$	$\frac{61}{59}$
0	45	$\frac{1}{8}$	$\frac{29}{240}$	$\frac{7}{60}$	$\frac{15}{13}$						
30	30	$\frac{3}{4}$	$\frac{11}{15}$	$\frac{43}{60}$	$\frac{19}{15}$	30	0	$\frac{15}{16}$	$\frac{899}{960}$	$\frac{449}{480}$	$\frac{16}{15}$
15	30	$\frac{1}{2}$	$\frac{59}{120}$	$\frac{29}{60}$	$\frac{8}{7}$	15	0	$\frac{13}{16}$	$\frac{779}{960}$	$\frac{389}{480}$	$\frac{123}{119}$
0	30	$\frac{5}{16}$	$\frac{99}{320}$	$\frac{49}{160}$	$\frac{46}{43}$	0	0	$\frac{21}{32}$	$\frac{1007}{1536}$	$\frac{503}{768}$	$\frac{303}{298}$

XIX. The calculation of the preceding article supposes player A in a perfect indifference in regard to the bisque, which gives to him always an equal penchant to take it or to not take it: however it is necessary to remark, that although he is equally able to take it at each coup, it is not always equally probable that he take it; having places, where he can make better worth of it than of others; if this is perhaps when one plays without making the chase, in which case I see no reason, why it would be necessary to postpone the bisque on a single coup; but making the chases, he has some encounters, where one can employ it so usefully, that it serves nearly of thirty; for having a difficult chase to win for A, it is as much as a loss for him; taking therefore his bisque, he prevents not only his Adversary to win 15, but he wins them himself, this which is worth to him 30. As therefore the determination of the lot of the players, which demands the consideration of the bisques, depends on the particular constitution of the game, of the diversity of the chases, & even of the caprice of the players, who observe not at all the rules, it is difficult to form from them some quite certain conjectures. Here is however the way, in which I myself would wish to take, if it would be necessary yet to have regard for the chases: Posed that the players are thirty or at deuce, & that there is a more difficult chase to win by the one than by the other (the number of times, that one has seen to win one similar by player A, being to the number of times, that one has seen to win by B, by reason of the any inequality of m to 1) although the two players are equal besides; I consider, that if player A wins the chase without taking his bisque, he wins the game, because he will not lack to take it after: & if he loses the chase, B will have the advantage, but A will retain his bisque, which is worth to him, by the II$^{\text{nd}}$ column of the VII$^{\text{th}}$ Table, $\frac{13}{30}$. Because therefore by hypothesis this player has m degrees of facility to win the chase against one degree to lose it, the lot, which he possessed when he did not take his bisque, will be

$$\frac{m \cdot 1 + 1 \cdot \frac{13}{30}}{m+1} = \frac{30m+13}{30m+30}.$$

But if on the contrary he takes this bisque, the chase is dead, & his lot is found, by column I of the said Table, $\frac{3}{4}$. I have therefore only to seek, which of the two fractions, either of $\frac{30m+13}{30m+30}$ or of $\frac{3}{4}$, surpasses the other; by making on them the same operations, as if there were equality between them until only m remains only on one side: in consideration of which I find, that player A does better as much to keep, as to take his bisque, according as m is greater or smaller than $\frac{19}{15}$; & that he must be indifferent to it to take it or to keep it, if m is exactly $\frac{19}{15}$. Put anew, that A has 30 to 45, or that B has the advantage, & that he has the same chase; it is clear, that if A wins without taking his bisque, he will be at deuce, consequently by column II of the VII$^{\text{th}}$ Table he will have $\frac{11}{15}$; but that if he loses the chase, he will lose the game. Having therefore m cases to win it & one case to lose it, he will have (when he does not take the bisque)

$$\frac{m \cdot \frac{11}{15} + 1 \cdot 0}{m+1} = \frac{11m}{15m+15}.$$

If A wishes on the contrary to take the bisque, the game is set at deuce, & the chase being dead the lot of each will be $\frac{1}{2}$. Making therefore comparison between $\frac{11m}{15m+15}$ and $\frac{1}{2}$, we find, that it is worth more for A to keep or to take the bisque, according as m is greater or smaller than $\frac{15}{7}$, & that one is worth as much as the other, if $m = \frac{15}{7}$. From this which I just showed, we can again conclude that the facility, that player A has to win a chase, being expressed by a number contained between $\frac{19}{15}$ & $\frac{15}{7}$, it will be better to keep his bisque, if the game is at deuce; but that if B has the advantage, it would be better to keep it. Finally it is in this manner, that I have determined all the other numbers of the column of the chases of the VII$^{\text{th}}$ Table, which can indicate to us, when player A must take or keep his bisque: because if he has more facility to win some chase, which is not carried by these numbers, he does better to keep the bisque; if he has less, he does better to take it; & if he has all so much exactly, he can make without prejudice that which he wishes.

XX. There remains yet to me to speak of the *services*, & of the advantage that there is to give them. You know, that the first coup of each ball, which one gives out of home, is called *service*. The one who gives it seems to have some advantage over the one who receives it, for two reasons: the one, because the coup of service is a sure coup, which the ball is given by the hand; instead that the coups which are played next the ball in the air are subjects to be missed: the other, because when the one who serves lacks some ball, it is a *chase*, instead that when the other lacks it, he always loses fifteen (at least if the ball enters into the game; because for the *chases toward the game*, I do not wish to speak, for fear of being extended too much, & it suffices to me to indicate to you the route in large, that it is necessary to take in this research.) We put that there are two players A & B, that A gives the service, that against one coup that he has lacked, one has observed that he has made p good coups; & that against one coup that B has lacked, one has seen him make q of them good: we put again that in the times that it is to A to play, his expectation to win the ball is y, but that this expectation becomes z, when the other B must play; & we consider firstly what will be of these expectations, if one would play without making the chases, that is to say if the ball that one lacks were always lost for the one who ought to play it. Now because we come to establish it is easy to see, that if A must play, there is one case, which will make him lose the ball, & p cases which making him successful in his coup will put B into the necessity to play, will change thus the lot y of player A into the one of z. If it is on the contrary B who plays in his turn, there is one case which will make the ball winning to A (by making it losing to B) & q cases which will return player A into the necessity to play, & restore to him the lot y. Therefore we will have on one side

$$y = \frac{1 \cdot 0 + p \cdot z}{1+p} = \frac{pz}{1+p};$$

on the other

$$z = \frac{1 \cdot 1 + q \cdot y}{1+q} = \frac{1+qy}{1+q},$$

that is to say, putting in place of y its found value

$$\frac{pz}{1+p}, \quad z = \frac{1+p+pqz}{1+p \quad 1+q} = \frac{1+p+pqz}{1+p+q+pq};$$

whence one deduces

$$z = \frac{1+p}{1+p+q}.$$

Now because player A would not know how to miss his coup of service, it follows that it would not be necessary to count this coup, & to imagine when he plays it, as if it were to B to play: therefore the expectation which he has to win the ball will be counted then $\frac{1+p}{1+p+q}$ consequently that of B $\frac{q}{1+p+q}$, & the ratio of these expectations $1+p$ to q. Whence it would appear, that if the two players are equals, & if each can strike for example ten good coups against one which is worth nothing, the letters p & q being worth each 10, the advantage of the one who gives the service on the one who receives it is as of 11 to 10; but that this advantage increases in measure as the players are more feeble, & that it diminishes until it is extinguished entirely, in measure as they are found more able.

XXI. We join now the consideration of the chases, but without constraining ourselves by their inequality, by we imagining, as if there were all under the cord; that is to say, as if all the balls which pass the cord, were able to win them. You know that when there is a chase, the players make an exchange of their places, & passing each on the other side of the game, the one who has given the services, is obliged to take them after. Let these four letters v, x, y, z, indicate therefore the expectation of A in four different states: namely, the first two v & x, before he has chase; the others y & z after the chase, when the players have passed: the first v & the third y, when it is to A to play; & the second x & 4^{th} z, when the other B must play. This posed, & the reasoning of the preceding article understood, you understand also without pain the reason of the four equalities following, unless there is need to extend any further this discourse:

$$v = \frac{1 \cdot y + p \cdot x}{1+p} = \frac{y+px}{1+p}, \qquad x = \frac{1 \cdot 1 + q \cdot v}{1+q} = \frac{1+qv}{1+q},$$
$$y = \frac{1 \cdot 0 + p \cdot z}{1+p} = \frac{pz}{1+p}, \qquad z = \frac{1 \cdot 1 + q \cdot y}{1+q} = \frac{1+qy}{1+q}.$$

You chase from the equality x the letter v, & from the equality y the letter z, you will have

$$x = \frac{1+p+qy+pqx}{1+p \cdot 1+q},$$

that is to say

$$x = \frac{1+p+qy}{1+p+q}, \quad \& \quad y = \frac{p+pqy}{1+p \cdot 1+q},$$

that is to say

$$y = \frac{p}{1+p+q}.$$

Chase again y from the new equality x, you will find finally

$$x = \frac{1+2p+q+pp+2pq}{\overline{1+p+q}^2},$$

& its remainder from unity

$$1 - x = \frac{q+qq}{\overline{1+p+q}^2}.$$

Whence it is necessary to conclude, that the expectation of A, in the time that B must receive from him the coup of service, is to that of B in ratio of $1+2p+q+pp+2qp$ to $q+qq$; where you can remark, that p & q being equals, the more one increases their value, the more this ratio approaches the triple, so that of the two players, who play equally & perfectly well, the one who serves has around three times more expectation to win the ball, than the other: but you remember, that it is under the assumption, that one makes no point of distinction between the chases, & that one does not admit those, that one calls *de vers le jeu*; because otherwise this double regard would diminish his advantage much.

XXII. I must not end my Letter, Sir, without having prevented certain false reasonings, which could come to the mind on this matter, for fear that they dazzle by their magnificence to mislead, & to cast doubt on the solidity of the principles established above. In the seventh article one demands, how many times must player A be stronger than B, in order that he is able to give 45 to him? Someone would have been able to reason the above thus: If B played against a third player C of similar force as he, & if they were 45 to 0, their lots would be by Table I in ratio of 15 to 1, that is to say that B would be able to win the game 15 times, when C would do it only one time. Now A giving 45 to B the Partie is supposed equal, that is to say such, that when B wins 15 times the game A can also 15 times. Therefore A & C playing together to goal, A can win it 15 times, whence C can win it only one time; & consequently A must be 15 times stronger than C, or (this which is as much) than B, who is of a same force: instead that by our analysis we have found, that he must be only $4\frac{1}{5}$ times stronger than he. To which I respond, that when this reasoning would be as evident as it is not, he deduces badly from the conclusion this result which is false: *Consequently* A must be &c., A, who can give 45 to B, can win 15 games against one, if he plays to goal with him, I concede it, because he can well wager $\frac{7114529}{134167}$ that is to say more than 50, by the 15$^{\text{th}}$ art. but it does not follow from it thence, that he is 15 times stronger, being able to make that he wins 15 games, or even 50 games, if you wish, against one, unless he has won more than 4 or 5 times more coups; because all the coups, that B wins during each game that he uses, are not counted for nothing, which carrying together would make perhaps the fourth Partie of the coups of A. We remark therefore, that it is worth more to measure the forces of the players by the number of coups that each wins, than by the one of the games or of the Parties which they make, when they play to goal.

In the thirteenth article one has researched, by how much must A be counted stronger, if he would play against two others B & C, posed that their absolute forces make in ratio of $3 \cdot 2 \cdot 1$? It would be well of people, who in order to respond to this question would be served by the analogy drawn from the mixture of the things: If there were for example three sorts of wine, of which the prices are in ratio of $3 \cdot 2 \cdot 1$, it is certain, that having mixed the two smaller together in equal quantity, the price of the mixed will be $1\frac{1}{2}$, & consequently the price of the best to the one of the other, as 3 to $1\frac{1}{2}$, or as 2 to 1. All the same, I say, they would be able to think, that the two players B & C who play in company against the third A, passing only for one player, their game being mixed more or less, & that thus the force of A must also be double of that of the two others taken together. Of others they would reason perhaps as this: Since by hypothesis A wins three coups, where B wins only two of them, & since he wins again three of them, where C makes only one of them; it follows that he must win six coups, when the two others together make only $2 + 1 = 3$ of them; & that consequently his force must again surpass by double that of the others, as we have concluded by the first discourse: Now this is contrary to the calculation of the 13$^{\text{th}}$ article, which we have made to find the lot of A more than the double of the one of the others. I am able to respond in a few words to these two reasonings: For the first, you know, that the analogies prove nothing; & for the other, its paralogism avoided, in this that one must reasonably suppose, that A plays as many times or as often to the weakest C as to B, & that following this reasoning it is made all the contrary of it; because A would play to B five coups, of which he would win three; & to C he would play only four coups, of which he would win again three; instead that our calculation replenishes perfectly this condition; for you put that A plays 20 coups to B, he must win 12 of them; if he plays therefore as many of them to C, he must win 15 of them; this which makes in all 27, & B & C win the other 13: but if he plays three times as much, that is to say 60 coups, to C, he must win 45 of them, which joined to the 12, that he wins on B, makes 57, & there remains for B & C the other 23; this which is immediately conformed to that which the calculation carries in the 13^{th} article.

I end, Sir, with this reflection: it is that it is extremely easy to be mistaken in all these understandings, if one does not pay always serious attention: because the reasonings, which one makes commonly in the world, are not better, than those that I just reported, but often much worse: one sees everyday, that the most scholarly reason on pure analogies; where if they imagine to see clear in the things, they take for very evident that which is not, & therefore there are only those, to whom the usage of Mathematics has illuminated the mind, who would be capable to discover the fraud of it.

I am &c.