SUR L'INTEGRATION

D'UNE

ÉQUATION DIFFÉRENTIELLE

Á DIFFÉRENCES FINIE

QUI CONTIENT LA THÉORIE DES SUITES RÉCURRENTES

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Miscellanea Taurinensia, T. 1, 1759

1. Let the differential equation

$$dy + yXdx = Zdx,$$

be proposed where X and Z express any functions of the variable x; we know that in order to integrate this equation is suffices to make

$$y = uz,$$

that which gives

$$udz + zdu + uzXdx = Zdx,$$

where we can make two terms vanish by a convenient value of u and of z. We suppose therefore

$$zdu + usXdx = 0,$$

and dividing by z, we will have

$$du + uXdx = 0,$$

and consequently

$$\frac{du}{u} = -Xdx \quad \text{and} \quad lu = -\int Xdx,$$

namely

$$u = e^{-\int Xdx},$$

where e is the number of which the hyperbolic logarithm is 1. By this supposition the proposed will become

$$udz = Zdx,$$

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that which gives

$$dz = \frac{Zdx}{u}, \quad z = \int \frac{Zdx}{u} = \int e^{\int Xdx} Zdx,$$

and finally

$$y = uz = \frac{\int e^{\int Xdx} Zdx}{e^{\int Xdx}}.$$

2. By observing the process of this method, we will see easily that it should be able to be applied again with success to the differential equations which have the same form as the preceding, although the differences are supposed finite. Let therefore the equation be

$$dy + My = N,$$

of which the differential dy is finite, and the other quantities M and N are some functions of another variable x. We suppose in first place

$$y = uz,$$

and we will have in this case

$$dy = udz + zdu + dudz,$$

and the equation will be changed into

$$udz + zdu + dudz + Muz = N.$$

Let us put as above the two terms

$$zdu + Muz = 0,$$

and we will have

$$du + Mu = 0,$$

namely

$$\frac{du}{u} = -M;$$

in order to resolve this equation in our case where the differential du is not infinitely small, let us suppose $u = e^t$, and we will have

$$u + du = e^{t+dt} \quad \text{and} \quad du = e^t(e^{dt} - 1);$$

whence

$$\frac{du}{u} = e^{dt} - 1 = -M \quad \text{and} \quad e^{dt} = 1 - M,$$

and taking the logarithms,

$$dt = l(1 - M),$$

and next integrating,

$$t = \int l(1-M);$$

but we know that the sum of the logarithms of many numbers is equal to the logarithm of the product of all these numbers; therefore, if we express generally by $\varpi(1-M)$ the continual product of all the quantities contained in the formula $1-M$, we will have

$$t = l\varpi(1-M),$$

and consequently

$$u = e^t = \varpi(1-M).$$

By the vanishing of these two terms the equation becomes

$$udz + dudz = N,$$

whence we deduce

$$dz = \frac{N}{u+du},$$

and, by integrating,

$$z = \int \frac{N}{u+du}.$$

But having already found $u = \varpi(1-M)$, if we express by M_1, the term consecutive to M, we will have

$$u + du = \varpi(1-M_1),$$

and consequently

$$z = \int \frac{N}{\varpi(1-M_1)};$$

and, since $y = zu$,

$$y = \varpi(1-M)\int \frac{N}{\varpi(1-M_1)},$$

or else, by adding to this integration any constant A,

$$y = \varpi(1-M)\left(A + \int \frac{N}{\varpi(1-M_1)}\right).$$

3. Let at present the proposed equation be

$$y_1 = Ry + T,$$

where y_1 is the term which follows y in the series of y's; since $y_1 = y + dy$, it will be reduced to

$$dy + (1-R)y = T.$$

Let us make therefore

$$1 - R = M, \quad T = N,$$

and we will find for the value of y the following expression

$$y = \varpi R\left(A + \int \frac{T}{\varpi R_1}\right).$$

If R is a constant quantity, it is clear that ϖR and ϖR_1, become some powers of R, of which the exponent is equal to the number which denotes the place of the terms y and y_1 in the series of y's; let therefore m be this number, so that y_m is the same as y, and we will have

$$y_m = R^m\left(A + \int \frac{T}{R^{m+1}}\right).$$

If T is constant, $\int \frac{T}{R^{m+1}}$ is equal to $T\int \frac{1}{R^{m+1}}$, where the terms expressed by $\frac{1}{R^{m+1}}$ form a geometric progression, of which it will be easy to have the sum; let this sum, which begins with $\frac{1}{R}$, be equal to S, namely that

$$\frac{1}{R} + \frac{1}{R^2} + \frac{1}{R^3} + \cdots + \frac{1}{R^m} = S,$$

and we will have, by multiplying by R,

$$1 + \frac{1}{R} + \frac{1}{R^2} + \cdots + \frac{1}{R^{m-1}} = SR = S + 1 - \frac{1}{R^m};$$

from this equality we will deduce

$$S = \frac{R^m - 1}{R^m(R-1)},$$

consequently

$$y_m = R^m\left[A + T\frac{R^m - 1}{R^m(R-1)}\right],$$

or else

$$y_m = AR^m + T\frac{R^m - 1}{R - 1}.$$

4. In order to be convinced that this value of y satisfies entirely the conditions of the given equation

$$y_1 = Ry + T \quad \text{or else} \quad y_{m+1} = Ry_m + T,$$

we have only to multiply the found formula for y_m by R, and add to it the quantity T, and we will find the result

$$AR^{m+1} + T\frac{R^{m+1} - R}{R - 1} + T$$

which reduces to

$$AR^{m+1} + T\frac{R^{m+1} - 1}{R - 1},$$

which is the value that the general formula gives us for the term y_{m+1}.

5. After having found the method to integrate any differential equation in finite differences, comprised under the general form

$$dy + My = N,$$

we can similarly proceed to the integration of the others which depend on those. Now, Mr. d'Alembert, in the *Mémoires de l'Académie Royale de Berlin*, has shown that all the differential equations, such as

$$y + A\frac{dy}{dx} + B\frac{d^2y}{dx^2} + C\frac{d^3y}{dx^3} + \cdots = X,$$

where A, B, C, ... are arbitrary constants, and where X is any function of x, is reduced to an equation of this form:

$$z + H\frac{dz}{dx} = V,$$

where H is a constant and V a function of x, which equation is the same as we have learned to integrate in the same case of the finite differences. If therefore the process of Mr. d'Alembert can take place also when the differences are finite, we can integrate further in this circumstance every differential equation of this form:

$$y + Ady + Bd^2y + Cd^3y + \cdots = X,$$

and consequently the equation

$$y_1 + Py_2 + Qy_3 + \cdots = X,$$

which we can regard as the general formula of the recurrent series. The method of Mr. d'Alembert is found detailed in the second volume of the *Calcul intégral* of Mr. Bougainville; but, in order to spare the pain to the readers, I will try to develop it here in a few words. Let us suppose

$$\frac{dy}{dx} = p, \quad \frac{dp}{dx} = q, \quad \frac{dq}{dx} = r, \ldots,$$

and the proposed equation will be changed into

$$y + Ap + Bq + C\frac{dq}{dx} = X.$$

Let us multiply at present each of the equations which we have supposed by some indeterminate coefficients a, b, $c, \ldots$, and let us add them all to this one, we will have

$$y + (A+a)p + (B+b)q - a\frac{dy}{dx} - b\frac{dp}{dx} + C\frac{dq}{dx} = X.$$

Let it be made so that the first part of the first member of this equation becomes an exact multiple of the integral of the second, namely that

$$dy + (A+a)dp + (B+b)dq = dy + \frac{b}{a}dp - \frac{C}{a}dq,$$

and by comparing term to term there will result from it

$$A + a = \frac{b}{a}, \quad B + b = -\frac{C}{a};$$

from these two equations we deduce

$$b = -\frac{C}{a} - B = Aa + a^2 \quad \text{and} \quad a^3 + Aa^2 + Ba + C = 0,$$

of which the roots will give three values of a which will satisfy equally the requisite conditions. We suppose now

$$y + (A + a)p + (B + b)q = z,$$

the found equation will become

$$z - a\frac{dz}{dx} = X,$$

which, compared with that of No. 1, will give by integrating

$$z = -e^{\frac{x}{a}} \int \frac{X dx}{ae^{\frac{x}{a}}}.$$

Now, as the quantity a can be three different values, we name them a_1, a_2, a_3, and we express by Z_1 the value of z which contains a_1, by Z_2 the one which contains a_2, and by Z_3 the one which contains a_3; we will have therefore the following three equations:

$$\begin{aligned} y + (A + a_1)p + (B + b_1)q &= Z_1, \\ y + (A + a_2)p + (B + b_2)q &= Z_2, \\ y + (A + a_3)p + (B + b_3)q &= Z_3. \end{aligned}$$

From these three equations we will deduce the value of y, which, because of the constant quantities A, B, a_1, a_2, ..., will be reduced to this form

$$y = FZ_1 + GZ_2 + HZ_3,$$

where F, G, H are some constants of which the value depends on the others A, B, a_1, a_2, ...

6. If we examine the process of these methods, it will appear clearly that if the equation had contained many more terms, for example if it had been

$$y + A\frac{dy}{dx} + B\frac{d^2y}{dx^2} + C\frac{d^3y}{dx^3} + D\frac{d^4y}{dx^4} + E\frac{d^5y}{dx^5} = X,$$

we would have found likewise

$$y = FZ_1 + GZ_2 + HZ_3 + IZ_4 + KZ_5,$$

where the quantities Z_1, Z_2, ... are some functions of X and x, such as

$$Z = -e^{\frac{x}{a}} \int \frac{X dx}{a e^x a},$$

by putting for a the roots a_1, a_2, a_3, a_4, a_5 of this equation

$$a^5 + Aa^4 + Ba^3 + Ca^2 + Da + E = 0;$$

moreover we will notice that the operations which this method requires can equally be made, either if the differences are finite, or if they are infinitely small.

7. Having therefore the equation in finite differences

$$y + Ady + Bd^2y + Cd^3y + Dd^4y + Ed^5y = X,$$

and putting

$$dy = p, \quad dp = q, \quad dq = r, \quad dr = s,$$

we will attain in the same manner to an equation such as

$$z - adz = X,$$

where

$$z = y + (A + a)p + (B + b)q + (C + c)r + (D + d)s,$$

and the quantity a will depend on this equation

$$a^5 + Aa^4 + Ba^3 + Ca^2 + Da + E = 0,$$

of which the roots have already been supposed a_1, a_2, a_3, a_4, a_5. Let us compare at present the equation

$$z - adz = X$$

with that of No. 2, namely

$$dy + My = N,$$

and we will have

$$M = -\frac{1}{a}, \qquad N = -\frac{X}{a};$$

consequently

$$1 - M = \frac{1 + a}{a},$$

that which gives next

$$z = \varpi\left(\frac{1+a}{a}\right)\left[\text{const.} + \int \frac{-\frac{X}{a}}{\varpi\left(\frac{1+a}{a}\right)}\right],$$

or else, since a is constant,

$$z_m = \left(\frac{1+a}{a}\right)^m \left[\text{const.} - \int \frac{Xa^m}{(1+a)^{m+1}}\right],$$

m expressing as above which term z in the series of z's. If we make moreover X constant, we will have, by taking the sum of the geometric progression expressed by $\int \frac{a^m}{(1+a)^{m+1}}$,

$$z_m = \left(\frac{1+a}{a}\right)^m \left[\text{const.} - X\frac{(1+a)^m - a^m}{(1+a)^m}\right].$$

Now, as a can have the values a_1, a_2, a_3, a_4, a_5, it is clear that by substituting each of them into the found formula, there will result from it as many values of z_m which will satisfy all equally. Let therefore all these values be expressed by Z_1, Z_2, Z_3, Z_4, Z_5, and since

$$z = y + (A+a)p + (B+b)q + (C+c)r + (D+d)s,$$

we will deduce, by way of the five equations

$$z = Z_1, \quad z = Z_2, \quad z = Z_3, \quad z = Z_4, \quad z = Z_5,$$

the following expression of y, namely

$$y = FZ_1 + GZ_2 + HZ_3 + IZ_4 + KZ_5.$$

8. Let next the proposed equation be

$$y_1 + Ay_2 + By_3 + Cy_4 + \cdots = X,$$

where y_1, y_2, y_3, ... express some consecutive terms of the series of y's; it is first evident that, since

$$y_2 = y_1 + dy_1, \quad y_2 = y_1 + 2y_1 + d^2y_1,$$

and thus of the others, this equation can be restored to the form of that which we just examined; but, since the calculation becomes in this fashion too long, it will be useful to resolve it directly by the same principles as we have employed to here. Moreover, in order to be able to apply more easily this equation to the recurrent series, it will be better to consider the terms $y_1, y_2, y_3, \ldots$ in a reverse order, namely as

$$y_2 + dy_2 = y_1, \qquad y_3 + dy_3 = y_2,$$

and thus of the others, so that the indices $1, 2, 3, \ldots$ denote the distance of each term to the last y_1. We suppose

$$y_2 = p_1, \quad \text{and we will have} \quad y_3 = p_2;$$

let therefore anew

$$p_2 = q_1 \quad \text{and} \quad p_3 = q_2;$$

let further

$$q_2 = r_1 \quad \text{and} \quad q_3 = r_2 = s_1,$$

and we will have

$$y_2 = p_1, \quad y_3 = q_1, \quad y_4 = r_1, \quad y_5 = s_1, \quad y_6 = s_2;$$

substituting these values into the proposed, it will become

$$y_1 + Ap_1 + Bq_1 + Cr_1 + Ds_1 + Es_2 = X.$$

If we reduce at present the preceding suppositions into equations, namely

$$p_1 - y_2 = 0, \quad q_1 - p_2 = 0, \quad r_1 - q_2 = 0, \quad s_1 - r_2 = 0,$$

and after having multiplied them by the indeterminate coefficients a, b, c, $\dots$, let us add them all to that which we just found. There will result from it the following

$$\left.\begin{array}{l} y_1 + (A+a)p_1 + (B+b)q_1 + (C+c)r_1 + (D+d)s_1 \\ \quad -ay_2 - bp_2 - cq_2 - dr_2 + Es_1 \end{array}\right\} = X.$$

If we make now that each coefficient of the first part is multiplied in the same manner as its correspondent in the second, we will attain to the same equations as we have found (6), and the quantity a will be determined by the equation

$$a^5 + Aa^4 + Ba^3 + Ca^2 + Da + E = 0,$$

of which we have supposed the roots a_1, a_2, a_3, ... Therefore, if we make

$$y_1 + (A+a)p_1 + (B+b)q_1 + (C+c)r_1 + (D+d)s_1 = z_1,$$

the equation will be reduced to

$$z_1 - az_2 = X,$$

which, by an integration similar to that of No. 3, will give

$$z_m = a^m \left(\text{const.} + \int \frac{X}{a^{m+1}} \right),$$

where m will express which term z_m in the series of z's. Now, as for a, we can substitute each of the five roots a_1, a_2, $\dots$ of the equation $a^5 + Aa^4 + \dots = 0$, we will have likewise five different values of z_m which we will express as above by Z_1, Z_2, Z_3, $\dots$; therefore, because

$$z_m = y_m + (A+a)p_m + (B+b)q_m + (C+c)r_m + (D+d)s_m,$$

we will attain, by driving out the letters p_m, q_m, $\dots$, in the formula

$$y_m = FZ_1 + GZ_2 + HZ_3 + IZ_4 + KZ_5,$$

where F, G, H, $\dots$ are some constants which we must determine by the comparison of as many terms given in the series of y's.

9. If X is constant, by that which we have demonstrated (4), the sum expressed by $\int \frac{X}{a^{m+1}}$ will become equal to $X\frac{a^m-1}{a^m(a-1)}$, and naming L the constant added to this integration, we will have finally

$$Z = La^m + X\frac{a^m - 1}{a^m(a-1)},$$

whence we will deduce consequently the values Z_1, Z_2, Z_3, ..., by substituting in the place of a its values a_1, a_2, a_3, ...

10. From all this we can deduce the following general theorem; if we have the equation

$$y_m + Ay_{m-1} + By_{m-2} + Cy_{m-3} + Dy_{m-4} + Ey_{m-5} + \cdots = X,$$

where the indices of the y's denote their places, if we seek all the roots a_1, a_2, a_3, a_4, ... of the equation[1]

$$a^5 + Aa^4 + Ba^3 + Ca^2 + Da + E = 0,$$

and we will have generally

$$\begin{aligned} y_m =& Fa_1^m\left(L + \int \frac{X}{a_1^{m+1}}\right) + Ga_2^m\left(L + \int \frac{X}{a_2^{m+1}}\right) + Ha_3^m\left(L + \int \frac{X}{a_3^{m+1}}\right) \\ &+ Ia_4^m\left(L + \int \frac{X}{a_4^{m+1}}\right) + Ka_5^m\left(L + \int \frac{X}{a_5^{m+1}}\right) + \cdots, \end{aligned}$$

and, in the case where X is constant

$$\begin{aligned} y_m =& L\left(Fa_1^m + Ga_2^m + Ha_3^m + Ia_4^m + Ka_5^m + \cdots\right) \\ &+ X\left(F\frac{a_1^m-1}{a_1-1} + G\frac{a_2^m-1}{a_2-1} + H\frac{a_3^m-1}{a_3-1} + I\frac{a_4^m-1}{a_4-1} + K\frac{a_5^m-1}{a_5-1} + \cdots\right). \end{aligned}$$

If $X = 0$, we can suppress the constant L, and we will have more simply

$$y_m = Fa_1^m + Ga_2^m + Ha_3^m + Ia_4^m + Ka_5^m + \cdots,$$

a formula known for the expression of the general term of the series of y's, such that

$$y_m + Ay_{m-1} + By_{m-2} + Cy_{m-3} + Dy_{m-4} + Ey_{m-5} + \cdots = 0,$$

that which is nothing other than a recurrent series, of which the scale of relation is $-A - B - C - D - E - \cdots$

11. Here is therefore the theory of the recurrent series reduced to the differential calculus, and established in this fashion on some direct and natural principles, instead that until here it has been treated only by some entirely

[1] *Translator's note*: This equation is clearly in error. Since the difference equation is of indefinite order, the corresponding must be likewise.

indirect ways. Moreover, the researches which one has made on this material has always been limited to the case $X = 0$, and a person, who I know, has never undertaken to examine generally the other cases, where X is constant or even variable, that which can nevertheless be of the greatest importance for the resolution of many problems which lead to such equations, of which the theory of chances is principally filled, as I propose myself to show another time by applying to this kind of calculus the theory which I just explicated.