MEMOIR

ON THE

UTILITY OF THE METHOD OF TAKING THE MEAN

AMONG

THE RESULTS OF SEVERAL OBSERVATIONS
IN WHICH ONE EXAMINES THE ADVANTAGE OF THIS METHOD
BY THE CALCULUS OF PROBABILITIES,
AND WHERE ONE SOLVES DIFFERENT PROBLEMS RELATED TO THIS MATERIAL

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Miscellanea Taurinensia, T. V, 1770–1773, pp. 167–232
Oeuvres de Lagrange, Vol. 2 (1867), pp. 173–234

When one has several observations of the same phenomenon of which the results are not entirely in accord, one is certain that these observations are all, or at least in part, not very exact, from some source that the error could originate; now one has custom to take the mean among all the results, because in this manner, the different errors themselves being apportioned equally in all the observations, the error which is able to be found in the resulting mean becomes also the mean among all the errors. Now, although everyone recognizes the utility of this practice in order to diminish, as much as it is possible, the uncertainty which is born of the imperfection of the instruments and of the inevitable errors of observations, I have thought however that it would be good to examine and to estimate by calculation the advantages that one can hope to extract from a similar method; this is the object that I have proposed to myself in this Memoir. I will begin by supposing that the errors which can slip into each observation are given, and that one knows also the number of cases which can give these errors, that is to say the facility of each error; I will suppose next that one knows only the limits between which all possible errors must be contained with the law of their facility, and I will seek through both of the hypotheses what is the probability that the error of the resulting mean be null, or equal to a given quantity, or only contained between some given limits. I will show at the same time how one can determine, *a posteriori*, the same law of facility of errors, and what is the probability that through this determination one will not be deceived by a given quantity: whence I will deduce some quite simple rules

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for the correction of instruments by some repeated verifications.

For the remainder, I will follow in all these researches the ordinary rule of calculation of probabilities, following which one estimates the probability of an event by the number of favorable cases, divided by the number of all possible cases. The difficulty consists only in the enumeration of these cases; but this enumeration demands often some quite complicated calculations, and of which one can come to the end only by some particular artifices: this is that which takes place mainly in the material that I am going to treat.

PROBLEM I

1. *One supposes that in each observation one can be deceived by one unit, as much to the greater as to the lesser, but that the number of cases which can give an exact result is to the number of cases which can give an error of one unit is as* $a : 2b$*; one demands what is the probability of having an exact result in taking the mean among the particular results of a number of observations* n*.*

Since there are a cases which give zero error, and $2b$ cases which give $+1$ and -1, that is to say b cases which give $+1$, and b cases which give -1 error, it is clear by the ordinary rules of probabilities that the probability that the error be null in each particular observation will be expressed by $\frac{a}{a+2b}$; we see therefore what will be the probability that the error be also null in taking the mean among n observations. It is easy to see that this question is reduced to this one:

Having n *dice of which each have* a *faces marked with a zero,* b *faces marked with a positive unit, and* b *faces marked with a negative unit, so that the total number of faces be* $a + 2b$*, to find the probability that there is brought zero in casting all these dice at random.*

Now one knows, by the theory of combinations, that if one raises the trinomial $a+b(x+x^{-1})$ to the power n, the coefficient of the absolute term, that is to say the one where the power of x will be zero, will denote the number of cases or chances where the sum of the points marked by all the dice will be equal to zero: therefore, naming this coefficient A, one will have, because the number of all possible combinations is $(a + 2b)^n$, one will have, I say, $\frac{A}{(a+2b)^n}$ for the sought probability.

All is reduced therefore to finding the coefficient of A; now, this is to what one can attain by several different ways.

1. If one develops the power $[a + b(x + x^{-1})]^n$ following the theorem of Newton, one will have, as one knows,

$$a^n + na^{n-1}b(x + x^{-1}) + \frac{n(n-1)}{2}a^{n-2}b^2(x + x^{-1})^2 + \ldots;$$

now, it is easy to see that the odd powers of $x + x^{-1}$ contain no term without x, and that, in the even powers, there is always a term without x, which is the one in the middle, in which the exponents of x and x^{-1} are the same. Thus, the term without x of $(x + x^{-1})^2$ will be 2, the one of $(x + x^{-1})^4$ will be $\frac{4.3}{1.2}$, the one of $(x + x^{-1})^6$ will be $\frac{6.5.4}{1.2.3}$, and thus the others; therefore one will have in general

$$\begin{aligned}A =&a^n + \frac{2}{1}\frac{n(n-1)}{2}a^{n-2}b^2 + \frac{4.3}{1.2}\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}a^{n-4}b^4 \\ &+ \frac{6.5.4}{1.2.3}\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1.2.3.4.5.6}a^{n-6}b^6 + \dots,\end{aligned}$$

that is to say

$$\begin{aligned}A =&a^n + n(n-1)a^{n-2}b^2 + \frac{n(n-1)(n-2)(n-3)}{2.2}a^{n-4}b^4 \\ &+ \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2.3.2.3}a^{n-6}b^6 + \dots,\end{aligned}$$

2. It is clear that the trinomial $a + b(x + x^{-1})$ can be factored into these two binomials $\alpha+\beta x$, $\alpha+\beta x^{-1}$, this which gives, by comparison of the terms, $\alpha^2+\beta^2 = a$ and $\alpha\beta = b$; whence one draws $\alpha \pm \beta = \sqrt{a \pm 2b}$, and from there

$$\begin{aligned}\alpha &= \frac{\sqrt{a+2b}+\sqrt{a-2b}}{2}, \\ \beta &= \frac{\sqrt{a+2b}-\sqrt{a-2b}}{2}.\end{aligned}$$

This put, one will have therefore

$$\begin{aligned}[a + b(x + x^{-1})]^n &= (\alpha+\beta x)^n\left(\alpha + \frac{\beta}{x}\right)^n \\ &= \left[\alpha^n + n\alpha^{n-1}\beta x + \frac{n(n-1)}{2}\alpha^{n-2}\beta^2x^2 + \dots\right] \\ &\times\left[\alpha^n + \frac{n\alpha^{n-1}\beta}{x} + \frac{n(n-1)}{2}\frac{\alpha^{n-2}\beta^2}{x^2} + \dots\right],\end{aligned}$$

whence it is easy to conclude that one will have

$$A = \alpha^{2n} + (n\alpha^{n-1}\beta)^2 + \left[\frac{n(n-1)\alpha^{n-2}\beta^2}{1.2}\right]^2 + \left[\frac{n(n-1)(n-2)\alpha^{n-3}\beta^3}{1.2.3}\right]^3 + \dots$$

2. COROLLARY 1. — Let $a = b$, that is to say let there be an equal number of cases which give 0, or $+1$, or -1 error; the probability of having an exact result in each particular observation will be $\frac{a}{a+2b} = \frac{1}{3}$, and that of having an exact result, in taking the middle term among the results of n observations, will be, following the first formula [in dividing the upper and the lower of the fraction $\frac{A}{(a+2b)^n}$ by a^n],

$$\frac{1 + n(n-1) + \frac{n(n-1)(n-2)(n-3)}{1.2.1.2} + \frac{n(n-1)\cdots(n-5)}{1.2.3.1.2.3} + \cdots}{3^n}.$$

Therefore, in taking successively n equal to $1,\ 2,\ 3,\ \cdots$, one will have

n	1,	2,	3,	4,	5,	$6, \cdots,$
Probability	$\frac{1}{3}$,	$\frac{1}{3}$,	$\frac{7}{27}$,	$\frac{19}{81}$,	$\frac{51}{243}$,	$\frac{141}{729}, \cdots$

One sees by this table that the probability that the error be null diminishes in measure as one take a greater number of observations, so that if one would wish to estimate the advantage that there can be in taking the mean among several observations, by the excess of the probability that the error be null in the resulting mean, over the one that the error be null also in each particular result, one will find, in the case in question here, that the advantage will always be negative, that is to say that it will change to disadvantage, which would proceed likewise by increasing the more observations there would be; whence it seems that one could conclude that, in this case, it would be more desirable to take a unique observation, than to take the mean among several observations; but there is an essential consideration to make on this matter, from which it results that it is always more advantageous in practice to multiply observations as often as one can: it is this that we will discuss more below.

3. COROLLARY II. — Let now $a = 2b$, so that the number of cases which give an exact result is equal to the number of those which can give an error of $+1$ or -1. In this case, it will be worth more to avail oneself of the second formula, because one will have $\alpha = \sqrt{b}, \beta = \sqrt{b}$, so that because $a + 2b = 4b$, one will have, on dividing the upper and lower of the fraction $\frac{A}{(a+2b)^n}$ by b^n,

$$\frac{1 + n^2 + \left[\frac{n(n-1)}{2}\right]^2 + \left[\frac{n(n-1)(n-2)}{2.3}\right]^2 + \cdots}{4^n}$$

for the probability that the error be null in taking the mean among n observations.

Therefore, making successively n equal to $1, 2, 3, \ldots$, one will have the following results

$$\begin{array}{lcccc} n & 1, & 2, & 3, & 4, \cdots, \\ \text{Probability} & \frac{1}{2}, & \frac{3}{8}, & \frac{5}{16}, & \frac{35}{128}, \cdots \end{array}$$

whence one sees that the probability diminishes in measure as n increases, as in the case of the preceding corollary.

4. COROLLARY III. — Let $b = 2a$, in such a way that the number of cases which can give an error of one unit as many by more as by less is double of the one where one would have an exact result, one will have here, for the probability that the error be null in taking the mean among several observations,

$$\frac{1 + 4n(n-1) + \frac{16n(n-1)(n-2)(n-3)}{2.2} + \frac{26n(n-1)\cdots(n-5)}{2.3.2.3} + \cdots}{5^n}.$$

Therefore, making successively n equal to $1,\ 2,\ 3, \ldots$, one will have

$$\begin{array}{lcccc} n & 1, & 2, & 3, & 4, \cdots, \\ \text{Probability} & \frac{1}{5}, & \frac{9}{25}, & \frac{1}{5}, & \frac{29}{125}, \cdots \end{array}$$

Thus, for two observations, the advantage will be of $\frac{9}{25} - \frac{1}{5} = \frac{4}{25}$, for three it will be of $\frac{1}{5} - \frac{1}{5} = 0$, for four equal to $\frac{29}{125} - \frac{1}{5} = \frac{4}{125}$, etc.; whence it appears that the greater advantage takes place in taking the mean between two observations only.

5. REMARK 1. — In order to facilitate further the solution of the preceding Problem, it is good to seek the law that follows the terms of the series which represents

the probabilities which correspond to 1, 2, $3, \ldots$ observations; now, if one takes the fraction $\frac{1}{1-z[a+b(x+x^{-1})]}$, and if one develops it into a series according to the powers of z, one will have, as one knows,

$$1+z[a+b(x+x^{-1})]+z^2[a+b(x+x^{-1})]^2+\cdots,$$

such that in this series the coefficient of z^n will the be the n^{th} power of $a+b(x+x^{-1})$; therefore, if one names A', A'', A''',..., the values of A which correspond to $n=1,2,3,\ldots$, that is to say the terms without x of the powers of $a+b(x+x^{-1})$, $[a+b(x+x^{-1})]^2$,..., it is clear that the series

$$1+A'z+A''z^2+A'''z^3+\cdots$$

will be equal to the sum of the terms without x in the fraction $\frac{1}{1-z[a+b(x+x^{-1})]}$ developed according to the powers of x and of x^{-1}; such that if one represents by

$$Z+Z'\left(x+\frac{1}{x}\right)+Z''\left(x^2+\frac{1}{x^2}\right)+\cdots$$

the series which results from the development of this fraction according to the powers of x and of $\frac{1}{x}$ (because it is easy to see that the series in question must have necessarily this form), one will have

$$Z=1+A'z+A''z^2+\cdots;$$

thus, knowing the function Z, there will be no more than to reduce it to a series according to the powers of z, in order to have the quantities A, A', A'', A''', ... For this, I reduce first the trinomial

$$1-az-bz(x+x^{-1}) \text{ to } (p-qx)(p-qx^{-1}),$$

that which gives me

$$p^2+q^2=1-az \text{ and } pq=bz;$$

next, I reduce the fraction

$$\frac{1}{(p-qx)(p-qx^{-1})} \text{ to } \alpha+\frac{\beta}{p-qx}+\frac{\beta}{p-qx^{-1}},$$

and I find

$$\alpha=\frac{1}{q^2-p^2}, \quad \beta=\frac{p}{p^2-q^2};$$

now,

$$\frac{1}{p-qx}=\frac{1}{p}+\frac{qx}{p^2}+\frac{q^2x^2}{p^3}+\cdots,$$

and, likewise,

$$\frac{1}{p-qx^{-1}}=\frac{1}{p}+\frac{q}{p^2x}+\frac{q^2}{p^3x^2}+\cdots;$$

therefore one will have

$$Z = \alpha + \frac{2\beta}{p}, \quad Z' = \frac{\beta q}{p^2}, \quad Z'' = \frac{\beta q^2}{p^3}, \quad Z''' = \frac{\beta q^3}{p^4}, \ldots;$$

therefore

$$Z = \frac{1}{q^2 - p^2} + \frac{2}{q^2 - p^2} = \frac{1}{p^2 - q^2} = \frac{1}{(p+q)(p-q)};$$

but since $p^2 + q^2 = 1 - az$, and $pq = bz$, one will have

$$p + q = \sqrt{1 - az + 2bz}, \quad p - q = \sqrt{1 - az - 2bz};$$

therefore

$$(p+q)(p-q) = \sqrt{(1-az)^2 - 4b^2z^2};$$

therefore finally

$$Z = \frac{1}{\sqrt{1 - 2az + (a^2 - 4b^2)z^2}} = 1 + A'z + A''z^2 + A'''z^3 + \cdots,$$

such that one will have, for the known functions

$$\begin{aligned}
A' &= a, \\
A'' &= \frac{3aA' + 4b^2 - a^2}{2}, \\
A''' &= \frac{5aA'' + 2(4b^2 - a^2)A'}{3}, \\
A^{\text{iv}} &= \frac{7aA''' + 3(4b^2 - a^2)A''}{4}, \\
&\vdots
\end{aligned}$$

We denote by P', P'', P''' the probabilities that the error be null in taking the mean among $1,\ 2,\ 3,\ \ldots,$ observations, and one will have

$$P' = \frac{A'}{a + 2b}, \quad P'' = \frac{A''}{(a+2b)^2}, \quad P''' = \frac{A'''}{(a+2b)^3}, \ \ldots,$$

whence

$$A' = (a+2b)P', \quad A'' = (a+2b)^2P'', \quad A''' = (a+2b)^3P''', \ldots;$$

therefore, substituting these values into the preceding formulas and making, for greater simplicity, $\frac{2b}{a} = r$, one will have

$$\begin{aligned}
P' &= \frac{1}{1+r}, \\
P'' &= \frac{3P' + r - 1}{2(1+r)}, \\
P''' &= \frac{5P'' + 2(r-1)P'}{3(1+r)}, \\
P^{\text{iv}} &= \frac{7P''' + 3(r-1)P''}{4(1+r)}, \\
P^{\text{v}} &= \frac{9P^{\text{iv}} + 4(r-1)P'''}{5(1+r)}, \\
&\vdots
\end{aligned}$$

6. REMARK II. — If one makes $r = 1$, one will have the case of Corollary II, where $a = 2b$, and one will find

$$P' = \frac{1}{2}, \quad P'' = \frac{1.3}{2.4}, \quad P''' = \frac{1.3.5}{2.4.6}, \ldots,$$

and, in general,

$$P^{(n)} = \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}$$

Thence one sees that the probability always diminishes in measure as n increases, this which we have already observed in the Corollary cited; so that in taking $n = \infty$, the probability will become infinitely small or null; indeed, by the quadrature of Wallis one has (π being the arc of 180 degrees)

$$\frac{\pi}{2} = \frac{2.2.4.4.6.6\ldots}{1.3.3.5.5.7\ldots},$$

that is to say, in taking $n = \infty$,

$$\frac{\pi}{2} = \frac{2.2.4.4.6.6\ldots 2n.2n}{1.3.3.5.5.7\ldots(2n-1)(2n-1)(2n+1)};$$

therefore, multiplying by $2n + 1$ and taking the square root, one will have

$$\sqrt{\frac{2n+1}{2}\pi} = \frac{2.4.6\ldots 2n}{1.3.5.\ldots(2n-1)};$$

therefore, when $n = \infty$, one will have

$$P^{(n)} = \frac{1}{\sqrt{n\pi}} = 0.$$

It is good to remark that, since we have found in the cited corollary for the probability P^n the expression

$$\frac{1 + n^2 + \left[\frac{n(n+1)}{2}\right]^2 + \left[\frac{n(n-1)(n-2)}{2.3}\right]^2 + \cdots}{4^n},$$

one will have, in comparing this expression with the preceding, the equation

$$1+n^2+\left[\frac{n(n+1)}{2}\right]^2+\left[\frac{n(n-1)(n-2)}{2.3}\right]^2+\cdots=\frac{1.3.5\dots(2n-1)}{1.2.3\dots n}2^n,$$

which is the much more remarkable that it does not appear easy to demonstrate *a priori*.

7. REMARK III. — By the formulas of Remark I, one will have in general

$$\begin{aligned}
P^{(n)}&=\frac{(2n-1)P^{(n-1)}+(n-1)(r-1)P^{(n-2)}}{n(r+1)},\\
P^{(n+1)}&=\frac{(2n+1)P^{(n)}+n(r-1)P^{(n-1)}}{(n+1)(r+1)},\\
P^{(n+2)}&=\frac{(2n+3)P^{(n+1)}+(n+1)(r-1)P^{(n)}}{(n+2)(r+1)},\\
&\vdots
\end{aligned}$$

where the exponents $n-2$, $n-1$, n, etc., of P do not denote the powers, but only the position of the rank. Now, if n is a very great number, it is clear that the fractions $\frac{2n-1}{n}$, $\frac{2n+1}{n+1}$, $\frac{2n+3}{n+2}$, etc., will be very nearly equal to 2, and that the fractions $\frac{n-1}{n}$, $\frac{n}{n+1}$, $\frac{n+1}{n+2}$ will be also very nearly equal to 1; so that one will have, by this hypothesis,

$$\begin{aligned}
P^{(n)}&=\frac{P^{(n-1)}+(r-1)P^{(n-2)}}{r+1},\\
P^{(n+1)}&=\frac{P^{(n)}+(r-1)P^{(n-1)}}{r+1},\\
&\vdots
\end{aligned}$$

whence one sees that the quantities $P^{(n)}$, $P^{(n+1)}$, etc., form a recurrent series of which the denominator of the generating fraction will be

$$x^2-\frac{x}{r+1}-\frac{r-1}{r+1};$$

thus, one will have in general

$$P^{(n+s)}=A\left[\frac{1+\sqrt{4r^2-3}}{2(r+1)}\right]^s+B\left[\frac{1-\sqrt{4r^2-3}}{2(1+r)}\right]^s,$$

and, in order to determine the coefficients A and B, one will suppose that the terms $P^{(n)}$ and $P^{(n+1)}$ are known, this which will give

$$P^{(n)}=A+B$$

and

$$P^{(n+1)}=A\frac{1+\sqrt{4r^2-3}}{2(r+1)}+B\frac{1-\sqrt{4r^2-3}}{2(1+r)},$$

whence

$$A = \frac{2(1+r)P^{(n+1)} - (1-\sqrt{4r^2-3})P^{(n)}}{2\sqrt{4r^2-3}},$$

$$B = \frac{(1+\sqrt{4r^2-3})P^{(n)} - 2(1+r)P^{(n+1)}}{2\sqrt{4r^2-3}},$$

whence

$$P^{(n+s)} = \left[\frac{P^{(n)}}{2} + \frac{2(1+r)P^{(n+1)} - P^{(n)}}{2\sqrt{4r^2-3}}\right]\left[\frac{1+\sqrt{4r^2-3}}{2(1+r)}\right]^s + \left[\frac{P^{(n)}}{2} - \frac{2(1+r)P^{(n+1)} - P^{(n)}}{2\sqrt{4r^2-3}}\right]\left[\frac{1-\sqrt{4r^2-3}}{2(1+r)}\right]^s$$

and this formula will be so much the more exact as one takes the number n more great.

Thus, after having calculated the terms $P^{(n)}$ and $P^{(n+1)}$, either by the formulas of No. 1, or by those of Remark I, one could find very nearly all the following terms by the preceding formula.

To the remainder, it is easy to see by this formula that the probability will be null at infinity, that is to say when $s = \infty$; indeed, it is clear that whatever be r, provided that it be a positive number, the quantities $\frac{1\pm\sqrt{4r^2-3}}{2(1+r)}$ will be always smaller than 1; because we suppose, if it is possible, $\frac{1\pm\sqrt{4r^2-3}}{2(1+r)} > 1$, one will have therefore

$$4r^2 - 2 \pm 2\sqrt{4r^2-3} > 4(1+2r+r^2),$$

namely

$$\pm\sqrt{4r^2-3} > 3 + 4r,$$

and

$$4r^2 - 3 > 16r^2 + 24r + 9,$$

namely

$$0 > 12r^2 + 24r + 12,$$

this which cannot be; therefore, by making $s = \infty$, the quantities

$$\left[\frac{1+\sqrt{4r^2-3}}{2(1+r)}\right]^s \text{ and } \left[\frac{1-\sqrt{4r^2-3}}{2(1+r)}\right]^s$$

will become null, and consequently $P^{(n+1)}$ also.

8. SCHOLIUM. — Let ρ be the result that each observation must give if it were exact: since one supposes that one can be deceived by one unit as much to the greater as to the less, one will have in each observation one of these three results: $\rho,\ \rho-1,\ \rho+1$; therefore, if one has two observations and if one takes the mean between their results, that is to say the half-sum of these results, one will have one of these five results

$$\frac{2\rho}{2},\quad \frac{2\rho-1}{2},\quad \frac{2\rho+1}{2},\quad \frac{2\rho-2}{2},\quad \frac{2\rho+2}{2},$$

namely

$$\rho, \quad \rho - \frac{1}{2}, \quad \rho + \frac{1}{2}, \quad \rho - 1, \quad \rho + 1;$$

thus, in this case, the error can be 1 or $\frac{1}{2}$, as much to the greater as to the less; one will see similarly that in taking the mean among three observations, the error could be 1, or $\frac{2}{3}$, or $\frac{1}{3}$, as much to the greater as to the less, and thus so forth. Thus, although the probability that the error be null can be smaller when one takes the mean result of several observations than when one takes the result of each observation in particular, however, if one seeks the probability that the error not surpass $\frac{1}{2}$, or $\frac{1}{3}$, ..., one will find that this probability will be greater in the first case than in the second. Indeed, in the first case, there are no other favorable cases than those where the error is absolutely null; but, in the second, the favorable cases are not only those where the error is null, but also those where the error is $\frac{1}{2}$, or $\frac{1}{3}$, ..., and it is by this consideration that it is always more advantageous to take the mean among the results of several observations than to contain oneself to the result of each observation in particular. We are going to examine the question under this point of view in the following Problem.

PROBLEM II.

9. *The same things being supposed as in the preceding Problem, to find the probability that in taking the mean among the results of n observations, the error will not surpass the fraction $\frac{m}{n}$, m being $< n$.*

In taking the mean among the results of n observations, it is clear that the error can be: either 0, or $\pm\frac{1}{n}$, or $\pm\frac{2}{n}$, or $\pm\frac{3}{n}, \ldots$, until $\pm\frac{n}{n}$, namely ± 1. Thus, the probability that the error not be greater than $\pm\frac{m}{n}$ will be the sum of the probabilities that the error will be null, or $\pm\frac{1}{n}$, or $\pm\frac{2}{n}, \ldots$, until $\pm\frac{m}{n}$. We see therefore first what it is the probability that the error will be $\pm\frac{\mu}{n}$.

In reverting this question to the dice, as we have practiced in Problem I, it is clear that it is reduced to finding the probability of bringing $+\mu$ or $-\mu$ points, with n dice of which each has a faces marked 0, b faces marked $+1$ and b faces marked -1. For this, there is only to raise the trinomial $a + b(x + x^{-1})$ to the power n, and the coefficient of x^μ will denote the number of cases where the sum of the points of all the dice will be μ, similarly that the one of $x^{-\mu}$ will denote the number of cases where the sum of the points will be $-\mu$; thus, the sum of these two coefficients divided by $(a + 2b)^n$, which is the number of all cases, will give the sought probability.

Now, one has

$$[a + b(x + x^{-1})]^n = a^n + na^{n-1}b(x + x^{-1}) + \frac{n(n-1)}{2}a^{n-2}b^2(x + x^{-1})^2 + \cdots,$$

and, moreover,

$$\begin{aligned}
(x+x^{-1})^2 &= (x^2+x^{-2})+2,\\
(x+x^{-1})^3 &= (x^3+x^{-3})+3(x+x^{-1}),\\
(x+x^{-1})^4 &= (x^4+x^{-4})+4(x^2+x^{-2})+\frac{4.3}{2},\\
(x+x^{-1})^5 &= (x^5+x^{-5})+5(x^3+x^{-3})+\frac{5.4}{2}(x+x^{-1}),\\
&\ \vdots
\end{aligned}$$

Therefore, if one supposes

$$[a+b(x+x^{-1})]^n = A+B(x+x^{-1})+C(x^2+x^{-2})+D(x^3+x^{-3})+\cdots,$$

one will have

$$\begin{aligned}
A =& a^n + \frac{2}{1}\frac{n(n-1)}{2}a^{n-2}b^2\\
&+\frac{4.3}{1.2}\frac{n(n-1)(n-2)(n-3)}{2.3.4}a^{n-4}b^4\\
&+\frac{6.5.4}{1.2.3}\frac{n(n-1)\cdots(n-5)}{2.3.4.5.6}a^{n-6}b^6+\cdots,
\end{aligned}$$

$$\begin{aligned}
B =& na^{n-1}b + \frac{3}{1}\frac{n(n-1)(n-2)}{2.3}a^{n-3}b^3\\
&+\frac{5.4}{1.2}\frac{n(n-1)(n-2)(n-3)(n-4)}{2.3.4.5}a^{n-5}b^5\\
&+\frac{7.6.5}{1.2.3}\frac{n(n-1)\cdots(n-6)}{2.3\ldots 7}a^{n-7}b^7+\cdots,
\end{aligned}$$

$$\begin{aligned}
C =& \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{4}{1}\frac{n(n-1)(n-2)(n-3)}{2.3.4}a^{n-4}b^4\\
&+\frac{6.5}{1.2}\frac{n(n-1)\cdots(n-5)}{2.3\ldots 6}a^{n-6}b^6\\
&+\frac{8.7.6}{1.2.3}\frac{n(n-1)\cdots(n-7)}{1.2\ldots 8}a^{n-8}b^8+\cdots,\\
&\vdots
\end{aligned}$$

Therefore, if one calls M the term of the series $A, B, C, \ldots,$ of which the index will be $\mu+1$, it is easy to see that one will have

$$\begin{aligned}M =&\frac{n(n-1)\cdots(n-\mu+1)}{1.2\ldots\mu}a^{n-\mu}b^{\mu}\\&+\frac{\mu+2}{1}\frac{n(n-1)\cdots(n-\mu-1)}{1.2\ldots(\mu+2)}a^{n-\mu-2}b^{\mu+2}\\&+\frac{(\mu+4)(\mu+3)}{1.2}\frac{n(n-1)\cdots(n-\mu-3)}{1.2\ldots(\mu+4)}a^{n-\mu-4}b^{\mu+4},\\&\vdots\end{aligned}$$

Now, this term M is the coefficient of the powers of x^{μ} and $x^{-\mu}$, such that one will have $\frac{2M}{(a+2b)^n}$ for the probability that the error be $\pm\frac{\mu}{n}$. Thus, the probability that the error will not surpass $\pm\frac{\mu}{n}$ will be represented by the series

$$\frac{A+2B+2C+2D+\cdots+2M}{(a+2b)^n}.$$

In order to facilitate the search for the values of $A, B, C, \ldots$, it is good to show how these quantities depend on one another; for this, one will take the equation

$$[a+b(x+x^{-1})]^n = A+B(x+x^{-1})+C(x^2+x^{-2})+D(x^3+x^{-3})+\cdots,$$

and, taking the logarithmic differential, one will have, after having divided by $\frac{dx}{x}$,

$$\frac{nb(x-x^{-1})}{a+b(x+x^{-1})} = \frac{B(x-x^{-1})+2C(x^2-x^{-2})+\cdots}{A+B(x+x^{-1})+C(x^2+x^{-2})+\cdots};$$

therefore, cross multiplying, there will happen

$$\begin{aligned}nbA(x-x^{-1})+nbB(x^2-x^{-2})+nbC(x^3-x^{-3}-x+x^{-1})\\+nbD(x^4-x^{-4}-x^2+x^{-2})+\cdots\\=&aB(x-x^{-1})+2aC(x^2-x^{-2})+3aD(x^3-x^{-3})+\cdots\\&+bB(x^2-x^{-2})+2bC(x^3-x^{-3}+x-x^{-1})\\&+3bD(x^4-x^{-4}+x^2-x^{-2})+\cdots;\end{aligned}$$

so that in comparing the terms, one will have

$$\begin{aligned}nb(A-C)&=aB+2bC,\\nb(B-D)&=2aC+b(B+3D),\\nb(C-E)&=3aD+b(2C+4E),\\&\vdots\end{aligned}$$

whence, by making for greater simplicity $\frac{a}{b} = K$, one will have

$$C = \frac{nA - KB}{n+2},$$
$$D = \frac{(n-1)B - 2KC}{n+3},$$
$$E = \frac{(n-2)C - 3KD}{n+4},$$
$$\vdots$$

Thus, by knowing the first two terms A and B, one could find successively all the others.

10. COROLLARY. — Suppose, as in No. 2, $a = b$, so that one has $K = 1$, and, making successively n equal to $1, 2, 3, \ldots$, and $a = 1$, this which is permissible, one will find the following values

n	A	B	C	D	E	F	$G\ldots,$
1	1	1	0	0	0	0	$0\ldots,$
2	3	2	1	0	0	0	$0\ldots,$
3	7	6	3	1	0	0	$0\ldots,$
4	19	16	10	4	1	0	$0\ldots,$
5	51	45	30	15	5	1	$0\ldots,$
6	141	126	90	50	21	6	$1\ldots,$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Thence one will form the following table of probabilities:

VALUES of the number n of observations	PROBABILITIES THAT THE ERROR WILL NOT SURPASS THE FRACTIONS					
	$\pm\frac{0}{n}$	$\pm\frac{1}{n}$	$\pm\frac{2}{n}$	$\pm\frac{3}{n}$	$\pm\frac{4}{n}$	$\pm\frac{5}{n}$
1	$\frac{1}{3}$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
2	$\frac{3}{9}$	$\frac{7}{9}$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
3	$\frac{7}{27}$	$\frac{19}{27}$	$\frac{25}{27}$	$\ldots$	$\ldots$	$\ldots$
4	$\frac{19}{81}$	$\frac{51}{81}$	$\frac{71}{81}$	$\frac{79}{81}$	$\ldots$	$\ldots$
5	$\frac{51}{243}$	$\frac{141}{243}$	$\frac{201}{243}$	$\frac{231}{243}$	$\frac{241}{243}$	$\ldots$
6	$\frac{141}{729}$	$\frac{393}{729}$	$\frac{573}{729}$	$\frac{673}{729}$	$\frac{715}{729}$	$\frac{727}{729}$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$

One sees, by this table, that in taking the mean between two observations, the probability that the error be null will be $\frac{3}{9} = \frac{1}{3}$, and the one that the error will not surpass $\frac{1}{2}$ so much to the greater as to the lesser will be $\frac{7}{9}$; now, in each particular observation, there is a $\frac{1}{3}$ probability that the error will be 0, and since, by hypothesis, the error can be only 0 or ± 1, it is clear that the probability that the error will not surpass $\frac{1}{2}$ will be the same $\frac{1}{3}$. Thus, although the probability that the error will be null is the same, either that one take the mean result between two observations, or that one takes the particular result of a unique observation, however the probability that the error will not surpass $\frac{1}{2}$ will be greater in the first case than in the second, these two probabilities being as $\frac{7}{9} : \frac{1}{3}$, that is to say in the ratio of $7 : 3$.

Likewise, in taking the mean among three observations, one will have $\frac{7}{27}$ for the probability that the error will be null, $\frac{19}{27}$ for the probability that the error will not be greater than $\pm\frac{1}{3}$, and $\frac{25}{27}$ for the one that the error will not be greater than $\pm\frac{2}{3}$; but in each particular observation the probability that the error be null is $\frac{1}{3}$, and the one that the error not surpass $\frac{1}{3}$ or $\frac{2}{3}$, is the same $\frac{1}{3}$, because by hypothesis the error can only be null or ± 1; therefore the probability that the error be null will be in truth greater in the particular result of one unique observation than in the mean result of three observations, and this in the ratio of $9 : 7$; but in return the probability that the error will not surpass $\pm\frac{1}{3}$ will be greater in the second case than in the first by ratio of $19 : 9$, and the one that the error will not surpass $\pm\frac{2}{3}$ it will be again more, this probability being, in the second case, greater than in the first by ratio of $25 : 9$.

Here is therefore in what consists principally the advantage that there is in taking the mean among the results of several observations. In order to render the thing again more sensible, we are going to seek the probabilities that the error will not surpass the fraction $\frac{1}{2}$, in supposing successively n equal to $1,\ 2,\ 3,\ \ldots$, that is to say for a unique observation, for two, for three, ... and we will have

$$\begin{array}{ccccccc} n & 1, & 2, & 3, & 4, & 5, & 6, \cdots, \\ \text{Probabilities} & \frac{1}{3}, & \frac{7}{9}, & \frac{19}{27}, & \frac{71}{81}, & \frac{201}{243}, & \frac{673}{729}, \ldots \end{array}$$

or else, by reducing to the same denominator 729,

$$\begin{array}{ccccccc} n & 1, & 2, & 3, & 4, & 5, & 6, \cdots, \\ \text{Probabilities} & \frac{243}{729}, & \frac{567}{729}, & \frac{513}{729}, & \frac{639}{729}, & \frac{603}{729}, & \frac{673}{729}, \ldots \end{array}$$

One sees thence that the probability that the error will not surpass $\frac{1}{2}$ proceeds by increasing, in measure as one takes a greater number of observations, but with this difference that the probability is greater for two observations than for three, for four than for five, and in general for any even number than for the odd number which follows it immediately; so that, in the hypothesis in question is, it is more advantageous to take the mean only among some even number of observations.

11. REMARK. — We have seen in No. 5 that if one develops the fraction $1 - z\left[a + b\left(x + \frac{1}{x}\right)\right]$ in a series of this form

$$Z + Z'\left(x + \frac{1}{x}\right) + Z''\left(x^2 + \frac{1}{x^2}\right) + \cdots,$$

$Z,\ Z', Z'', \cdots$ being some functions of z, one will have

$$Z = \frac{1}{p^2 - q^2}, \quad Z' = \frac{\beta q}{p^2} = \frac{q}{p} Z, \quad Z'' = \frac{\beta q^2}{p^3} = \frac{q}{p} Z', \ldots,$$

p and q being such that

$$p^2 + q^2 = 1 - az \text{ and } pq = bz,$$

this which gives

$$p^2 - q^2 = \sqrt{1 - 2az + (a^2 - 4b^2)z^2},$$

and, thence,

$$\frac{q}{p} = \frac{1 - az - \sqrt{1 - 2az + (a^2 - 4b^2)z^2}}{2bz};$$

so that by making, for greater simplicity,

$$\zeta = \sqrt{1 - 2az + (a^2 - 4b^2)z^2},$$

one will have

$$Z = \frac{1}{\zeta}, \quad Z' = \frac{1 - az - \zeta}{2bz}\frac{1}{\zeta}, \quad Z'' = \left(\frac{1 - az - \zeta}{2bz}\right)^2 \frac{1}{\zeta},$$

and in general

$$Z^{(\mu)} = \left(\frac{1 - az - \zeta}{2bz}\right)^\mu \frac{1}{\zeta}.$$

Now, if one develops this quantity into a series of rational and integer powers of z, one will see easily, by that which we have said above, that the coefficient of any one power, as z^n, will denote the number of cases where the sum of the errors of n observations could be $+\mu$ or $-\mu$, so that the double of this coefficient will express the number of all cases where the mean error will be $\pm\frac{\mu}{n}$. Thence it is easy to conclude that the quantity

$$\frac{1 + 2\frac{1-az-\zeta}{2bz} + 2\left(\frac{1-az-\zeta}{2bz}\right)^2 + \cdots + 2\left(\frac{1-az-\zeta}{2bz}\right)^\mu}{\zeta},$$

being regarded as a function of z and developed according to the powers of this variable, will give a series of such nature that the coefficient of any power z^n will express correctly the number of cases where the mean error could be contained in these limits $-\frac{\mu}{n}, +\frac{\mu}{n}$; such that, this coefficient being divided by the total number of cases $(a+2b)^n$, one will have the value of the probability that the mean error will not surpass the fraction $\frac{\mu}{n}$, either to the greater or to the lesser. Now, the quantity in question being nothing other than a geometric series, it can be put under this more simple form

$$2\frac{1 - \left(\frac{1-az-\zeta}{2bz}\right)^{\mu+1}}{\zeta\left(1 - \frac{1-az-\zeta}{2bz}\right)} - \frac{1}{\zeta}.$$

Thus, the entire difficulty will consist in reducing this same quantity to an infinite series which proceeds according to the powers of z. In order to come more easily to the end, one will suppose it equal to an indeterminate y, and one will have an equation between y and z, that one could deliver by some differentiations, as much from the power $\mu + 1$ as from the irrationality of ζ; by this means, one will have a differential equation of second degree between y and z, and there will be no more than to suppose

$y = 1 + Az + Bz^2 + \cdots$, and to determine the coefficients $A, B, \ldots$ by the comparison of the terms.

Moreover, as this calculation is a little long, we will content ourselves to indicate here, in order to put on the way the ones who will wish to push this theory further.

12. SCHOLIUM. — We have supposed in the two preceding Problems that there were an equal number of cases in order to have a positive error and in order to have a negative; if this were not thus, and if the number of cases of error which would give 0, $+1$ and -1 were a, b and c, now one would solve the Problems with the same facility by considering the trinomial $a + bx + cx^{-1}$ in the place of $a + b(x + x^{-1})$, in order to have the number of cases where one would have a given mean error, and in taking next $(a+b+c)^n$ in order to have the total number of cases in the place of $(a+2b)^n$. One could likewise, without making a new calculation, adapt to this case here the formulas that we have already found; because if in the trinomial $a + bx + \frac{c}{x}$ one puts $x\sqrt{\frac{c}{b}}$ in the place of x, it will become $a + \sqrt{bc}\left(x + \frac{1}{x}\right)$; thus, there would be only to put in the trinomial $a + b\left(x + \frac{1}{x}\right)$ of the preceding Problems $\sqrt{bc}$ in the place of b, and next $x\sqrt{\frac{b}{c}}$ in the place of x. Of the rest, we are going to treat this case in a much more general manner in the following Problem.

PROBLEM III.

13. *Supposing that each observation be subject to an error of one unit to the lesser and to an error of* r *units to the greater, and that the number of cases of error which can give* 0, -1, $+r$ *are respectively* a, b, c*; one demands what is the probability that the mean error of several observations will be contained within some given limits.*

Let n be the number of observations of which one wishes to take the mean; one will form the n^{th} power of the trinomial $a + \frac{b}{x} + cx^r$, and the coefficient of any power x^μ will denote the number of cases where the sum of the errors will be μ, and consequently where the mean error will be $\frac{\mu}{n}$. We will consider therefore the quantity

$$\left(a + \frac{b}{x} + cx^r\right)^n,$$

which is reduced to

$$\frac{[b + x(a + cx^r)]^n}{x^n},$$

and one will have, as one knows,

$$[b + x(a + cx^r)]^n = b^n + nb^{n-1}x(a + cx^r) + \frac{n(n-1)}{2}b^{n-2}x^2(a + cx^r)^2 + \cdots,$$

whence it is easy to see that the coefficient of any power x^s will be

$$\begin{aligned}
&\frac{n(n-1)\cdots(n-s+1)}{2.3\ldots s}b^{n-s}a^s\\
&+\frac{n(n-1)\cdots(n-s+r)}{2.3\ldots(s-r)}\,\frac{s-2r}{1}b^{n-s+r}a^{s-r}c\\
&+\frac{n(n-1)\cdots(n-s+2r)}{2.3\ldots(s-2r)}\,\frac{(s-2r)(s-2r-1)}{1.2}b^{n-s+2r}a^{s-2r}c^2\\
&\vdots
\end{aligned}$$

this series being continued until that which one attains to some negative terms; therefore this coefficient will be the one of the power x^{s-n} in the quantity $\left(a+\frac{b}{x}+cx^r\right)^s$; therefore, if one designates in general by (μ) the coefficient of the power x^μ in this last quantity, one will have

$$\begin{aligned}
(\mu) = {} & \frac{n(n-1)\cdots(1-\mu)}{2.3\ldots(\mu+n)}b^{-\mu}a^{\mu+n}\\
&+\frac{n(n-1)\cdots(r-\mu)}{2.3\ldots(\mu+n-r-1)}b^{r-\mu}a^{\mu+n-r}c\\
&+\frac{n(n-1)\cdots(2r-\mu)}{2\times 2.3\ldots(\mu+n-2r-2)}b^{2r-\mu}a^{\mu+n-2r}c^2\\
&\vdots
\end{aligned}$$

where it will be necessary to omit the terms which would contain some negative powers of a or b.

Therefore, since for n observations the sum of all the cases is $(a+b+c)^n$, one will have for the probability that the mean error be $\frac{\mu}{n}$ the quantity $\frac{(\mu)}{(a+b+c)^n}$; and thence the probability that the mean error will be contained between these limits $-\frac{p}{n}, +\frac{q}{n}$ will be expressed by the series

$$\frac{(-p+1)+\cdots+(-1)+(0)+(1)+\cdots+(q-1)}{(a+b+c)^n}.$$

PROBLEM IV.

14. *Supposing all, as in the preceding Problem, one demands what is the mean error for which the probability is the greatest.*

We have seen that the probability that the mean error be $\frac{\mu}{n}$ is $\frac{(\mu)}{(a+b+c)^n}$, (μ) being the coefficient of the power x^μ of the trinomial $\left(a+\frac{b}{x}+cx^r\right)^{\text{n}}$; thus the question is only to know what is the term of the nth power of $a+\frac{b}{x}+cx^r$ which will have the greatest coefficient; for this it is clear that there is only to seek the greatest term of the trinomial $a+b+c$ raised to the power n; because supposing that this term be $\pi a^\alpha b^\beta c^\gamma$, α,β,γ being the exponents of a,b,c, of which the sum must be equal to n, and π the coefficient of this term, there will be only to put $\frac{b}{x}$ in the place of b, and cx^r in the place of c, and one will have

$$\pi a^\alpha b^\beta c^\gamma x^{-\beta+r\gamma}$$

for the sought term of the n th power of the trinomial $a + \frac{b}{x} + cx^r$; thus one will make $-\beta + r\gamma = \mu$, and one will have

$$\frac{r\gamma - \beta}{n}$$

for the mean error of which the probability will be greatest.

Now, by the rules of combinations, one knows that the coefficient π of the term $\pi a^\alpha b^\beta c^\gamma$ must be

$$\frac{1.2.3\ldots n}{1.2.3\ldots\alpha\times 1.2.3\ldots\beta\times 1.2.3\ldots\gamma};$$

we denote this term by M, so that one has

$$\frac{1.2.3\ldots n\times a^\alpha b^\beta c^\gamma}{1.2.3\ldots\alpha\times 1.2.3\ldots\beta\times 1.2.3\ldots\gamma} = M,$$

and it will be necessary that in varying the exponents α, β, γ, the value of M diminishes; we make therefore α vary by one unit, so that α becomes $\alpha + 1$, and as $\alpha + \beta + \gamma = n$, it will be necessary that β or γ diminish at the same time by one unit; now, it is easy to see that if in the value of M one puts $\alpha + 1$ for α and $\beta - 1$ for β, this value will become

$$\frac{\beta}{\alpha+1}\frac{aM}{b},$$

therefore

$$\frac{\beta}{\alpha+1}\frac{aM}{b} < M,$$

and, consequently,

$$\frac{\beta}{\alpha+1}\frac{a}{b} < 1;$$

reciprocally, if one augments β by one unit, and if one diminishes α also by one unit, one will find the condition

$$\frac{\alpha}{\beta+1}\frac{b}{a} < 1;$$

thus it will be necessary that one have at the same time

$$\frac{\alpha}{\beta+1} < \frac{a}{b} \text{ and } \frac{\alpha+1}{\beta} > \frac{a}{b}.$$

Now, this is that which will take place if $\frac{\alpha}{\beta} = \frac{a}{b}$.

One will find in the same manner $\frac{\alpha}{\gamma} = \frac{a}{c}$; so that in taking an indeterminate coefficient p, one will have, in the case of the maximum,

$$\alpha = pa, \quad \beta = pb, \quad \gamma = pc;$$

but $\alpha + \beta + \gamma = n$; therefore $p = \frac{n}{a+b+c}$; therefore finally

$$\alpha = \frac{na}{a+b+c}, \quad \beta = \frac{nb}{a+b+c}, \quad \gamma = \frac{nc}{a+b+c}.$$

If the quantities $\frac{na}{a+b+c}$, $\frac{nb}{a+b+c}$, $\frac{nc}{a+b+c}$ are whole numbers, one will have exactly

$$\alpha = \frac{na}{a+b+c}, \quad \beta = \frac{nb}{a+b+c}, \quad \gamma = \frac{nc}{a+b+c},$$

as we happen to find it; but if these quantities are not some whole numbers, then it will be necessary to take for α, β, γ the whole numbers which will be the nearest. One can take however, for greater simplicity, these same quantities for the values of α, β, γ, because the error, if there is, could ever be only quite small; in this manner we will have for the mean error what has the greatest probability, the expression

$$\frac{r\gamma - \beta}{n} = \frac{rc - b}{a+b+c}.$$

15. COROLLARY. — Thence it follows that one can always regard the quantity $\frac{rc-b}{a+b+c}$ as the error of the mean result, and that also one can take the same quantity for the correction of this result.

When $r = 1$ and $c = b$, as in the hypothesis of Problem I, the correction of the mean result becomes null; it would be also it, if one would have $b = rc$; but in all the other cases it will be so much greater as rc will differ any more from b.

PROBLEM V.

16. *One supposes that each observation be subject to some errors any whatsoever given, and that one knows at the same time the number of cases where each error can take place; one demands the correction that it will be necessary to make to the mean result of several observations.*

Let $p, q, r, s, \ldots$ be the errors to which each observation is subject, and $a,\ b,\ c, d,\ \ldots$ the cases which can give these errors, namely a the number of cases which could give the error p, b the number of cases which could give the error q, and thus the others; it is clear, by that which we have demonstrated in the preceding Problems, that if one raises the polynomial

$$ax^p + bx^q + cx^r + \cdots$$

to the power n, and if one denotes by M the coefficient of the power x^μ, one will have

$$\frac{M}{(a+b+c+\cdots)^n}$$

for the probability that the error of the mean result of n observations be $\frac{\mu}{n}$. Now one knows, by the theory of combinations, that the coefficient M will be of this form

$$\frac{1.2.3.4\ldots n \times a^\alpha b^\beta c^\gamma \ldots}{1.2.3\ldots\alpha \times 1.2.3\ldots\beta \times 1.2.3\ldots\gamma \times \ldots},$$

where the exponents $\alpha, \beta, \gamma, \ldots$ must be such that

$$\alpha + \beta + \gamma + \cdots = n \text{ and } \alpha p + \beta q + \gamma r + \cdots = \mu.$$

Moreover, it is easy to demonstrate, by a similar method to that of the preceding Problem, that the coefficient M will be the greatest, when one will have

$$\alpha = \frac{na}{a+b+c+\cdots},$$
$$\beta = \frac{nb}{a+b+c+\cdots},$$
$$\gamma = \frac{nc}{a+b+c+\cdots};$$

whence it follows that the mean error, for which the probability will be the greatest, will be expressed by

$$\frac{\mu}{n} = \frac{ap+bq+cr+\cdots}{a+b+c+\cdots}.$$

Thus this quantity will represent the correction that it will be necessary to make to the mean result of several observations.

17. COROLLARY I. — If one regards the quantities $a, b, c, \ldots$ as some weights applied to an indefinite line, at some distances equal to $p, q, r, \ldots$ from a fixed point taken on this line, and if one seeks the center of gravity of these weights, the distance of this center to the fixed point will be the correction that it will be necessary to make to the mean result of several observations; this follows evidently from the formula that we have found above for the value of this correction.

18. COROLLARY II. — Therefore, if one supposes that each observation be subject to all possible errors which can be contained between the given limits, and if one knows the curve of the facility of errors in which, the abscissas being supposed to represent the errors, the ordinates represent the facilities of these errors, there will be only to seek the center of gravity of the total area of this curve, and the abscissa corresponding to this center will express the correction of the mean result. Thence one sees that if the curve in question is equal, is similar on all sides of the ordinate which passes through the origin of the abscissas, so that this ordinate is a diameter of the curve in question, then the correction will be null, the center of gravity falling necessarily on the diameter. This case takes place all the time that the errors can be equally positive and negative.

PROBLEM VI.

19. *I suppose that one has verified an arbitrary instrument, and that having repeated several times the same verification one has found different errors, of which each is found repeated a certain number of times; one demands what is the error that it will be necessary for the correction of the instrument.*

Let $p, q, r, \ldots$ be the found errors, and let $\alpha, \beta, \gamma, \ldots$ be the numbers which mark how many times each error is found repeated in making n verifications; suppose that the number of cases which can give the error p, or q, or r, ... are designated respectively by a, b, c, ...; let one raise the polynomial

$$ax^p + bx^q + cx^r + \cdots$$

to the power n, and let

$$N(ax^p)^\alpha (bx^q)^\beta (cx^r)^\gamma \cdots$$

be any term of this polynomial: the coefficient $Na^\alpha b^\beta c^\gamma \cdots$ of the power $p\alpha + q\beta + r\gamma + \cdots$ of x divided by $(a+b+c+\cdots)^n$ will denote the probability that the errors p, q, r, ... are found combined together, in a way that p is repeated α times, q β times, r γ times, and thus the others. Thus this probability will be the greatest in the combination where the value of $Na^\alpha b^\beta c^\gamma \cdots$ will be the greatest; but one has

$$N = \frac{1.2.3.4\ldots n}{1.2.3\ldots\alpha \times 1.2.3\ldots\beta \times 1.2.3\ldots\gamma \times \ldots},$$

as we have seen already in the preceding Problem; therefore, by the same Problem, the greatest value of $Na^\alpha b^\beta c^\gamma \cdots$ will take place when

$$\alpha = \frac{na}{a+b+c+\cdots},$$
$$\beta = \frac{nb}{a+b+c+\cdots},$$
$$\gamma = \frac{nc}{a+b+c+\cdots};$$

equations by which one could determine the unknowns a, b, c, ...; and one will have, by making $a+b+c+\cdots = s$,

$$a = \frac{s\alpha}{n}, \quad b = \frac{s\beta}{n}, \quad c = \frac{s\gamma}{n}, \ldots$$

Now we have demonstrated, in the Problem cited, that the correction that it is necessary to make to the mean result of an arbitrary number of observations is expressed by

$$\frac{ap+bq+cr+\cdots}{a+b+c+\cdots};$$

therefore, putting in this expression the values of a, b, c, ... that we happen to find, the correction in question will become

$$\frac{\alpha p + \beta q + \gamma r + \cdots}{n},$$

that is to say equal to the mean error among all the particular errors that the n verifications have given.

20. COROLLARY. — If one would wish to keep count also, at least in an approximate manner, of intermediate errors to which the instrument could be subject, there would be only to take on an indefinite straight line some abscissas proportional to the

found errors $p,\ q,\ r,\ \ldots$, as in No. 17; and having applied there some ordinates proportional to the quantities $a,\ b,\ c,\ \ldots$ one would pass through the extremities $p,\ q,\ r,\ \ldots$ a parabolic line; one would seek next the center of gravity of the area of all the curve, and the perpendicular dropped from this center onto the axis would cut there an abscissa which would be the correction to the instrument.

One sees thence how one can know *a posteriori* the law of facility of each of the errors to which an instrument can be subject.

21. REMARK I. — One has found above that the greatest probability takes place when

$$a = \frac{s\alpha}{n}, \quad b = \frac{s\beta}{n}, \quad c = \frac{s\gamma}{n}, \ldots,$$

so that the values of $a,\ b,\ c,\ \ldots$ are the most probable that one can suppose. If one would wish to know moreover what is the probability that these same values will not depart from the truth of any quantity $\pm\frac{rs}{n}$, there will be only to put, in the general expression of the probability (preceding Problem)

$$\frac{Na^{\alpha}b^{\beta}c^{\gamma}\cdots}{s^n}.$$

in place of $a,\ b,\ c,$ the quantities $\frac{s(\alpha+x)}{n},\ \frac{s(\beta+y)}{n},\ \frac{s(\gamma+z)}{n}, \ldots,$ and making successively $x,\ y,\ z,\ \ldots$ equal to $\pm 1,\ \pm 2,\ \pm 3,\ \ldots, \pm r$, so that meanwhile if one has always $x+y+z+\cdots = 0$, because (by hypothesis) $a+b+c+\cdots = s$ and $\alpha+\beta+\gamma+\cdots = n$, one will have as many particular probabilities, of which the sum will be the sought probability.

Let P be the probability that one has

$$a = \frac{s\alpha}{n}, \quad b = \frac{s\beta}{n}, \quad c = \frac{s\gamma}{n}, \ldots;$$

putting these values in the preceding expression, one will have

$$P = \frac{1.2.3\ldots n}{n^{\mu}} \frac{\alpha^{\alpha}}{1.2.3\ldots\alpha} \frac{\beta^{\beta}}{1.2.3\ldots\beta} \cdots$$

Let moreover Q be the probability that one has

$$a = \frac{s(\alpha+x)}{n}, \quad b = \frac{s(\beta+y)}{n}, \quad c = \frac{s(\gamma+z)}{n}, \ldots,$$

one will have the value of Q in putting these values into the same expression, and it is easy to see that one will have

$$Q = P\left(1+\frac{x}{\alpha}\right)^{\alpha}\left(1+\frac{y}{\beta}\right)^{\beta}\left(1+\frac{z}{\gamma}\right)^{\gamma}\cdots$$

Therefore, if one has in general

$$V = \left(1+\frac{x}{\alpha}\right)^{\alpha}\left(1+\frac{y}{\beta}\right)^{\beta}\left(1+\frac{z}{\gamma}\right)^{\gamma}\cdots,$$

and if $\int V$ denotes the sum of all the particular values of V, in varying x, y, z, ... from 0 to $+r$, and having care that one has always $x + y + z + \cdots = 0$, the sought probability will be equal to $P \int V$.

Since it is not easy to find the integral $\int V$, mainly when there are more than two variables, one could be content to have an approximate method; for this, there would be only to take a mean value of V and to multiply it by the number of all the particular values of V which must enter into the integral $\int V$, and the difficulty will consist only in finding this number. Now, if one designates by m the number of the quantities $\alpha, \beta, \gamma, \ldots$, it is easy to imagine that the number in question will be none other than the coefficient of u^0, that is to say the well known term of the series which represents the m power of the polynomial

$$u^{-r} + u^{-r+1} + \cdots + u^{-1} + 1 + u^1 + \cdots + u^{r-1} + u^r.$$

Let one denote this term by T, and one will have, as we will demonstrate below,

$$\begin{aligned} T =& \frac{(mr+1)(mr+2)(mr+3)\cdots(mr+m-1)}{1.2.3\ldots(m-1)} \\ &- m\frac{[(m-2)r][(m-2)r+1][(m-2)r+3]\cdots[(m-2)r+m-2]}{1.2.3\ldots(m-1)} \\ &+ \frac{m(m-1)}{2}\frac{[(m-4)r-1][(m-4)r][(m-4)r+1]\cdots[(m-4)r+m-3]}{1.2.3\ldots(m-1)} - \cdots \end{aligned}$$

by continuing this series only until when some one of the factors $mr + 1$, $(m-2)r$, $(m-4)r - 1$, ... becomes negative.

Therefore, if W is the mean value of V, one will have for the approximate value of $\int V$ the quantity VW, and the sought probability will be very nearly equal to PTW.

If, instead of taking for W the mean value of V, one takes the least, it is clear that TW will be necessarily less than the true value of $\int V$, and consequently the sought probability will be necessarily greater than PTW; thus, one could wager with advantage PTW against $1 - PTW$ that by making

$$\frac{a}{s} = \frac{\alpha}{n}, \quad \frac{b}{s} = \frac{\beta}{n}, \quad \frac{c}{s} = \frac{\gamma}{n}, \ldots,$$

one will not be deceived by a quantity greater than $\frac{r}{n}$ as much to the greater as to the lesser.

22. REMARK II. — Suppose that n is a very great number, and that consequently the numbers $\alpha, \beta, \gamma, \ldots$ of which the sum is n, are also very great; in order to find in this case the values of P and of V, one will remark:

1° That when u is a very great number, one has, very nearly,

$$\log 1 + \log\, 2 + \log\, 3 + \cdots + \log\, u = \frac{1}{2}\log\, \pi + \frac{1}{2}\log\, u - u,$$

π being the ratio of the periphery of the circle to the radius; whence it follows that one will have

$$\log \frac{1.2.3\ldots u}{u^u} = \frac{1}{2}\log\, \pi + \log\, u - u$$

and consequently
$$\frac{1.2.3\ldots u}{u^u} = \frac{\sqrt{\pi u}}{e^u};$$
therefore, because $\alpha + \beta + \gamma + \cdots = n$, one will have
$$P = \sqrt{\frac{\pi n}{\pi\alpha \times \pi\beta \times \pi\gamma \times \cdots}}.$$

2° If one takes the logarithm of V, one will have
$$\log V = \alpha \log\left(1 + \frac{x}{\alpha}\right) + \beta \log\left(1 + \frac{y}{\beta}\right) + \gamma \log\left(1 + \frac{z}{\gamma}\right) + \cdots,$$
but
$$\log\left(1 + \frac{x}{\alpha}\right) = \frac{x}{\alpha} - \frac{x^2}{2\alpha^2} + \cdots,$$
therefore, because $x + y + z + \cdots = 0$, one will have very nearly
$$\log V = -\frac{1}{2}\left(\frac{x^2}{\alpha} + \frac{y^2}{\beta} + \frac{z^2}{\gamma} + \cdots\right),$$
and thence
$$V = e^{-\frac{1}{2}\left(\frac{x^2}{\alpha} + \frac{y^2}{\beta} + \frac{z^2}{\gamma} + \cdots\right)}.$$

Now let there be
$$x = \xi\sqrt{n}, \quad y = \psi\sqrt{n}, \quad z = \zeta\sqrt{n}, \ldots,$$
and
$$\frac{\alpha}{n} = A, \quad \frac{\beta}{n} = B, \quad \frac{\gamma}{n} = C, \ldots,$$
one will have
$$\xi + \psi + \zeta + \cdots = 0 \text{ and } A + B + C + \cdots = 1;$$
therefore
$$P = \frac{1}{(\pi n)^{\frac{m-1}{2}}\sqrt{ABC\ldots}},$$
$$V = e^{-\frac{1}{2}\left(\frac{\xi^2}{A} + \frac{\psi^2}{B} + \frac{\zeta^2}{C} + \cdots\right)}$$
Now, as the increment or the difference of the quantities x, y, z, ... is 1, the difference of the variables ξ, ψ, ζ, ... will be $\frac{1}{\sqrt{n}}$, and consequently infinitely small; so that, if one calls this difference $d\theta$, one will have
$$P = \frac{d\theta^{m-1}}{\sqrt{\pi^{m-1}ABC\ldots}}.$$
Therefore
$$PV = \frac{e^{-\frac{1}{2}\left(\frac{\xi^2}{A} + \frac{\psi^2}{B} + \frac{\zeta^2}{C} + \cdots\right)}d\theta^{m-1}}{\sqrt{\pi^{m-1}ABC\ldots}}$$

Therefore, if one integrates the differential

$$\frac{d\theta^{m-1}}{e^{\frac{1}{2}\left(\frac{\xi^2}{A}+\frac{\psi^2}{B}+\frac{\zeta^2}{C}+\cdots\right)}}$$

$m-1$ times, in putting first in the place of ξ its value $-\psi-\zeta-\cdots$, and varying next successively the variables $\psi, \zeta, \ldots$, of the same differential $d\theta$, and if one completes the integral so that the values of $\xi, \psi, \zeta, \ldots,$ extend from $-\rho$ until $+\rho$ (by making $r=\rho\sqrt{n}$), one will have, by naming this integral R, the quantity

$$\frac{R}{\sqrt{\pi^{m-1}ABC\ldots}}.$$

for the probability that the values of a, b, c, … will be nearly exact to $\frac{\rho s}{\sqrt{n}}$.

Let, for example, $m=2$, so that one has found only two different errors, of which the one has been repeated α times and the other β times, in a very great number n of verifications of the instrument; in this case there will be only one integration to make, and the differential to integrate will be, in putting $-\psi$ in the place of ξ, and making $d\theta=d\psi$,

$$\frac{d\psi}{e^{\frac{1}{2}\left(\frac{1}{A}+\frac{1}{B}\right)\psi^2}},$$

which is not integrable by any of the known methods, unless one reduces to series the exponential quantity $e^{-\frac{1}{2}\left(\frac{1}{A}+\frac{1}{B}\right)\psi^2}$. In this manner one will have the differential

$$d\psi\left(1-K\psi^2+\frac{K^2\psi^4}{2}-\frac{K^3\psi^6}{2.3}+\cdots\right),$$

by making, for brevity, $K=\frac{A+B}{2AB}$; so that the integral will be

$$\psi-\frac{K\psi^3}{3}+\frac{K^2\psi^5}{2.5}-\frac{K^3\psi^7}{2.3.7}+\cdots.$$

Therefore

$$R=2\left(\rho-\frac{K\rho^3}{3}+\frac{K^2\rho^5}{2\times 5}-\frac{K^3\rho^7}{2.3\times 7}+\cdots\right);$$

therefore

$$\frac{R}{\sqrt{\pi AB}}$$

will express the probability that the values of a and b are contained between these limits

$$s\left(A\pm\frac{\rho}{\sqrt{n}}\right)\text{ and }s\left(B\pm\frac{\rho}{\sqrt{n}}\right),$$

that is to say that the facilities of the errors which are being found repeated α and β times, which are proportional to $\frac{a}{s}$ and $\frac{b}{s}$, are not separated from the quantities A and B, given by the observations, of a quantity greater than $\frac{\rho}{\sqrt{n}}$.

If one makes, for greater simplicity, $\rho = \mu\sqrt{AB}$, one will have $K = \frac{\mu^2}{2\rho^2}$, because $A + B = 1$, and the probability in question will be expressed in this manner

$$\frac{2}{\sqrt{\pi}}\left(\mu - \frac{\mu^3}{2\times 3} + \frac{\mu^5}{2.4\times 5} - \frac{\mu^7}{2.4.6\times 7} + \cdots\right).$$

Therefore, if one supposes μ =1, one will have the series

$$1 - \frac{1}{2\times 3} + \frac{1}{2.4\times 5} - \frac{1}{2.4.6\times 7} + \cdots,$$

of which the sum is very nearly 0.855624; so that the sought probability will be nearly

$$\frac{1.611248}{\sqrt{\pi}} = 0.682688.$$

Thus, one would in this case wager with advantage that, in supposing the facilities of errors respectively equal to A and B, one will not be deceived by the quantity $\mu\sqrt{\frac{AB}{n}}$, which, because n very large, is necessarily infinitely small.

It would be much more difficult to find the value of R if the variables ξ, ψ, $\zeta, \ldots$ were more than two, especially because the integration must be such, that it embraces only the values of these same variables which are contained between the limits $-\rho$ and $+\rho$; but one could, in this case, serve oneself by the approximation that we have given in the preceding number.

For this, one will remark that since we have made $r = \rho\sqrt{n}$, and that n is supposed very great, the number r must be very great also; so that one will have, very nearly,

$$T = \frac{r^{m-1}}{1.2.3\ldots(m-1)}\left[m^{m-1} - m(m-2)^{m-1} + \frac{m(m-1)}{2}(m-4)^{m-1} - \cdots\right],$$

by continuing this series until some one of the numbers $m-2$, $m-4$, ... becomes negative: therefore one will have

$$PT = \left(\frac{\rho}{\sqrt{\pi}}\right)^{m-1}\frac{m^{m-1} - m(m-2)^{m-1} + \frac{m(m-1)}{2}(m-4)^{m-1} - \cdots}{1.2.3\ldots(m-1)\sqrt{ABC\ldots}},$$

and there will be besides only to multiply this quantity by W, that is to say by the mean value, or if one wishes by the smallest value of V. Now, since one has

$$V = \frac{1}{e^{\frac{1}{2}\left(\frac{\xi^2}{A} + \frac{\psi^2}{B} + \frac{\zeta^2}{C} + \cdots\right)}},$$

it is clear that the smallest value of V will be that where the quantity $\frac{\xi^2}{A} + \frac{\psi^2}{B} + \frac{\zeta^2}{C} + \cdots$ will be the greatest; and it is easy to see that this will happen by taking $\xi = \rho$, $\psi = -\rho$, $\zeta = 0, \ldots$, because $\xi + \psi + \zeta + \cdots = 0$, and supposing that A and B are the least of all the quantities A, B, C, ...; thus, one will have

$$W = \frac{1}{e^{\frac{1}{2}(\frac{1}{A} + \frac{1}{B})\rho^2}}.$$

Therefore making, for brevity,

$$M=\frac{m^{m-1}-m(m-2)^{m-1}+\frac{m(m-1)}{2}(m-4)^{m-1}-\cdots}{1.2.3\ldots(m-1)},$$

one will have the quantity

$$M\left(\frac{\rho}{\sqrt{\pi}}\right)^{m-1}\frac{e^{-\frac{1}{2}(\frac{1}{A}+\frac{1}{B})\rho^2}}{\sqrt{ABC\ldots}},$$

which will be necessarily less than the sought probability; so that by naming H this quantity, one would always wager with advantage H against $1-H$ that in supposing the facilities of the errors equal respectively to A, B, C, ... one will not be deceived by the very small quantity $\frac{\rho}{\sqrt{n}}$.

LEMMA I.

23. *Let X be a function, rational and without divisor, of x : one demands the coefficient of the power x^μ in the series resulting from the development of the fraction $\frac{X}{(a-x)^n}$.*

One has, as one knows,

$$\begin{aligned}\frac{1}{(a-x)^n}&=\frac{1}{a^n}+\frac{nx}{a^{n+1}}+\frac{n(n+1)x^2}{2a^{n+2}}+\cdots\\&=\frac{\frac{1.2.3\ldots(n-1)}{a^n}+\frac{2.3\ldots nx}{a^{n+1}}+\frac{3.4\ldots(n+1)x^2}{a^{n+2}}+\cdots}{1.2.3\ldots(n-1)};\end{aligned}$$

therefore, if one orders the quantity X with respect to the powers of x, by beginning with the highest, in a manner that one has in general

$$X=Ax^\alpha+Bx^{\alpha-1}+Cx^{\alpha-2}+\cdots+Mx^\mu+Nx^{\mu-1}+\cdots,$$

and if one multiplies this series by that which expresses the value of $\frac{1}{(a-x)^n}$, it is easy to see that the term which will contain the power x^μ will be

$$\frac{\frac{1.2.3\ldots(n-1)}{a^n}M+\frac{2.3\ldots n}{a^{n+1}}N+\frac{3.4\ldots(n+1)}{a^{n+2}}P+\cdots}{1.2.3\ldots(n-1)}x^\mu,$$

so that the sought coefficient will be represented by the series

$$\frac{\frac{1.2.3\ldots(n-1)}{a^n}M+\frac{2.3\ldots n}{a^{n+1}}N+\frac{3.4\ldots(n+1)}{a^{n+2}}P+\cdots}{1.2.3\ldots(n-1)}.$$

We denote by X' the sum of all the terms of the value of X, where the powers of x are not higher than x^μ, so that one has

$$X'=Mx^\mu+Nx^{\mu-1}+Px^{\mu-2}+\cdots;$$

dividing by $x^{\mu+1}$, one will have

$$\frac{X'}{x^{\mu+1}} = \frac{M}{x} + \frac{N}{x^2} + \frac{P}{x^3} + \cdots ,$$

therefore, differentiating $n-1$ times and making next $x = a$, one will have

$$\pm \frac{d^{n-1}\left(\frac{X'}{x^{\mu+1}}\right)}{dx^{n-1}} = \frac{1.2.3\ldots(n-1)}{a^n} M + \frac{2.3\ldots N}{a^{n+1}} + \cdots ,$$

the upper sign being for the case where n is odd, and the lower sign for the one where n is even.

Therefore, this sought coefficient of the power x^μ will be equal to that which becomes the quantity

$$\frac{(-1)^{n-1}}{1.2.3\ldots(n-1)} \frac{d^{n-1}\left(\frac{X'}{x^{\mu+1}}\right)}{dx^{n-1}}$$

when one makes $x = a$.

24. REMARK. — If one divides the quantity X by $x^{\mu+1}$, and if one by rejecting next all the terms where there will be positive powers of x, it is clear that one will have the value of $\frac{X'}{x^{\mu+1}}$; therefore, in the place of the quantity X' one can have the same quantity X, by having care to reject the terms of which we happen to speak; in this manner one will have, for the expression of the sought coefficient of x^μ, the quantity

$$\frac{(-1)^{n-1}}{1.2.3\ldots(n-1)} \frac{d^{n-1}\left(\frac{X}{x^{\mu+1}}\right)}{dx^{n-1}},$$

by rejecting in this quantity, before or after the differentiations, all the positive powers of x, and making next $x = a$.

25. COROLLARY. — Suppose that one demands the coefficient of x^μ in the series

$$x^{-\alpha} + \cdots + x^{-2} + x^{-1} + x^0 + x^1 + x^2 + \cdots + x^\beta$$

raised to the power n.

Following the ordinary rules of the summation of the geometric progressions, one finds that the sum of this series is represented by

$$\frac{x^{-\alpha}(1 - x^{\alpha+\beta+1})}{1-x},$$

so that the n^{th} power of the same series will be equal to

$$\frac{x^{-n\alpha}(1 - x^{\alpha+\beta+1})^n}{(1-x)^n}.$$

Comparing therefore this formula with that of the preceding Lemma, one will find

$$\begin{aligned} X =& x^{-n\alpha}(1 - x^{\alpha+\beta+1})^n \\ =& x^{-n\alpha} - nx^{-(n-1)\alpha+\beta+1} + \frac{n(n+1)}{2} x^{-(n-2)\alpha+2(\beta+1)} \\ &- \frac{n(n-1)(n-2)}{2.3} x^{-(n-3)\alpha+3(\beta+1)} + \cdots ; \end{aligned}$$

therefore, dividing by $x^{\mu+1}$ and making, for brevity,

$$n\alpha + \mu = \pi, \quad \alpha + \beta = \rho,$$

one will have

$$\begin{aligned}\frac{X}{x^{\mu+1}} =& x^{-(\pi+1)} - nx^{-(\pi+1-\rho)} + \frac{n(n+1)}{2}x^{-(\pi+1-2\rho)} \\ &- \frac{n(n-1)(n-2)}{2.3}x^{-(\pi+1-3\rho)} + \cdots .\end{aligned}$$

and consequently, by differentiating $n-1$ times,

$$\begin{aligned}(-1)^{n-1}&\frac{d^{n-1}\left(\frac{X}{x^{\mu+1}}\right)}{dx^{n-1}} \\ =&(\pi+1)(\pi+2)\dots(\pi+n-1)x^{-(\pi+n)} \\ &- n(\pi+1-\rho)(\pi+2-\rho)\cdots(\pi+n-1-\rho)x^{-(\pi+n-\rho)} \\ &+ \frac{n(n-1)}{2}(\pi+1-2\rho)(\pi+2-2\rho)\cdots(\pi+n-1-2\rho)x^{-(\pi+n-2\rho)} \\ &\vdots\end{aligned}$$

One will reject therefore from this series the terms where the exponents of x will be found positive, that is to say if s is the whole number which is equal or immediately greater than $\frac{\pi+n}{\rho}$, one will continue the series only until the s th term; or else it will suffice to continue until some one of the factors $\pi+1$, $\pi+1-\rho$, ... become negative; next one will make $x=1$ and one will divide the whole by $1.2.3\dots(n-1)$; one will have thus the value of the sought coefficient, which will be consequently

$$\begin{aligned}\frac{1}{1.2.3\dots(n-1)}&[(\pi+1)(\pi+2)\dots(\pi+n-1) \\ &- n(\pi+1-\rho)(\pi+2-\rho)\cdots(\pi+n-1-\rho) \\ &+ \frac{n(n-1)}{2}(\pi+1-2\rho)(\pi+2-2\rho)\cdots(\pi+n-1-2\rho) \\ &- \frac{n(n-1)(n-2)}{2.3}(\pi+1-3\rho)(\pi+2-3\rho)\cdots(\pi+n-1-3\rho) \\ &\cdots]\end{aligned}$$

Thence one obtains the solution of the following Problem.

PROBLEM VII.

26. *One has several observations in each of which one supposes that one has been able to be deceived equally by any one of these quantities* $-\alpha$, ..., -2, -1, 0, 1, 2, ..., β; *one demands what is the probability that the error of the mean result of* n *observations will be* $\frac{\mu}{n}$, *or that it will be contained between these limits* $\frac{-p}{n}$ *and* $\frac{q}{n}$.

In order to find the probability that the mean result be $\frac{\mu}{n}$, it is necessary to seek the coefficient of the power x^μ of the polynomial

$$x^{-\alpha} + \cdots + x^{-2} + x^{-1} + x^0 + x^1 + x^2 + \cdots + x^\beta$$

raised to the power n, and to divide next this coefficient by the value of the same polynomial raised to the power n, which corresponds to $x = 1$, that is to say by $(\alpha + \beta + 1)^n$; this is that which follows evidently from that which we have demonstrated in the preceding Problems.

Therefore, by the preceding Corollary, one will find that the sought probability will be, by making $\pi = n\alpha + \mu$, $\rho = \alpha + \beta + 1$,

$$\begin{aligned}\frac{1}{1.2.3\ldots(n-1)\rho^n}\,[&(\pi+1)(\pi+2)\ldots(\pi+n-1)\\ &-n(\pi+1-\rho)(\pi+2-\rho)\cdots(\pi+n-1-\rho)\\ &+\frac{n(n-1)}{2}(\pi+1-2\rho)(\pi+2-2\rho)\cdots(\pi+n-1-2\rho)\\ &\cdots]\end{aligned}$$

by continuing this series until some one of the factors $\pi + 1$, $\pi + 1 - \rho$, $\ldots$ becomes negative.

Such is the general expression of the probability that the mean error of n observations be $\frac{\mu}{n}$; thus, in order to have the probability that the error be contained between the limits $\frac{-p}{n}$ and $\frac{+q}{n}$, there will be only to vary μ in the preceding quantity, and to take the sum of all the particular quantities which correspond to

$$\mu = -p,\ \ldots,\ -2,\ -1,\ 0,\ 1,\ ,2,\ \ldots q.$$

Now, since the quantity μ enters only into the value of π, there will be therefore only this variable quantity; so that the difficulty will be reduced to sum some sequence of which the general term will be of this form

$$(s+1)(s+2)(s+3)\cdots(s+k).$$

For this, let the sum of this series be represented by

$$u(s+1)(s+2)(s+3)\cdots(s+k),$$

u being an unknown function of s, and putting $s-1$ in place of s and u' in the place of u, one will have

$$u's(s+1)(s+2)\cdots(s+k-1);$$

this quantity being subtracted from the preceding, one will have the difference

$$[u(s+k) - u's](s+1)(s+2)\cdots(s+k-1);$$

but it is necessary that this difference be equal to the general term of the series of which one seeks the sum, therefore one will have the equation

$$u(s+h) - u's = s+k,$$

to which one will satisfy by making

$$u = \frac{s+k+1}{k+1};$$

so that the general sum of the series of which the general term is $(s+1)(s+2)\cdots(s+k)$ will be represented by

$$\frac{(s+1)(s+2)\cdots(s+k)(s+k+1)}{k+1},$$

and consequently the sum of all the terms contained between these two here

$$(s'+1)(s'+2)\cdots(s'+k) \text{ and } (s''+1)(s''+2)\cdots(s''+k)$$

will be equal to

$$\frac{s''(s''+1)(s''+2)\cdots(s''+k) - (s'+1)(s'+2)\cdots(s'+k+1)}{k+1}$$

Applying therefore this to the formula found above, one will have, for the probability that the mean error falls between $\frac{-p}{n}$ and $\frac{q}{n}$, the following expression, in which I have made, for brevity, $n\alpha - p = \delta$ and $n\alpha + q = \gamma$,

$$\begin{aligned}\frac{1}{1.2.3\ldots n\rho^n}\,&[\gamma(\gamma+1)\cdots(\gamma+n-1) - (\delta+1)(\delta+2)\cdots(\delta+n)\\ &-n[(\gamma-\rho)\cdots(\gamma-\rho+n-1) - (\delta-\rho+1)\cdots(\delta-\rho+n)]\\ &+\frac{n(n-1)}{2}[(\gamma-2\rho)\cdots(\gamma-2\rho+n-1) - (\delta-2\rho+1)\cdots(\delta-2\rho+n)]\\ &\cdots]\end{aligned}$$

This series must be continued until some one of the factors $\gamma-\rho$, $\gamma-2\rho$, $\ldots$ becomes negative; and as for the other factors $\delta-\rho+1$, $\delta-2\rho+1, \ldots$, if any one of among those are found negative, then it will be necessary to augment the number δ by as many units as it will be necessary to render it positive; this follows evidently from this that the series, of which the preceding is the sum, must be continued only until some one of the first factors $\pi+1-\rho$, $\pi+1-2\rho$,... becomes negative, as we have seen in the preceding number.

27. COROLLARY. — Suppose that the numbers α and β become infinite, as well as p and q, but in a way that they have between them some finite ratio; and let

$$\frac{\beta}{\alpha} = l, \quad \frac{p}{\alpha} = r, \quad \frac{q}{\alpha} = s,$$

so that one has

$$\beta = \alpha l, \quad p = \alpha r, \quad q = \alpha s,$$

l, r, s being some finite numbers; in this case one will have

$$\rho = \alpha+\beta = (1+l)\alpha,\ \delta = n\alpha - p = (n-r)\alpha,\ \gamma = n\alpha+q = (n+s)\alpha;$$

so that by substituting these values into the preceding formula, and neglecting that which one must neglect because $\alpha = \infty$, one will have this here, where $f = 1 + l$,

$$\frac{1}{1.2.3\ldots nf^n}\left[(n+s)^n - n(n+s-f)^n + \frac{n(n-1)}{2}(n+s-2f)^n - \cdots \right.$$
$$\left. - (n-r)^n + n(n-r-f)^n - \frac{n(n-1)}{2}(n-r-2f)^n + \cdots\right],$$

each of these two series before are continued only until any one of the quantities $n + s - f, n + s - 2f, \ldots$ and $n - r - f$, $n - r - 2f, \ldots$ become negative.

The case of this Corollary holds when one supposes that each observation is equally subject to all the possible errors contained between the given limits; because if one takes the greatest negative error for unity, and if one designates the greatest positive error by l, the preceding formula will denote the probability that the error of the mean result of n observations is contained between these two limits $-\frac{r}{n}$ and $+\frac{s}{n}$.

In the rest, we will give further below a method much more simple to solve these sorts of questions.

PROBLEM VIII.

28. *Supposing that the errors which one can commit in each observation are* $-\omega, \ldots, -2, -1, 0, 1, 2, \ldots, \omega$, *and that the number of cases which correspond to each of these errors is proportional respectively to* $1, 2, 3, \ldots, \alpha + 1, \ldots, 3, 2, 1$, *one demands what is the probability that the error of the mean result of* m *observations be contained between the limits* $\frac{-p}{m}$ *and* $\frac{q}{m}$.

We begin by seeking the probability that the mean error be $\frac{\mu}{m}$; this probability will be equal to the coefficient of the power x^μ of the polynomial

$$x^{-\omega} + 2x^{-\omega+1} + \cdots + \omega x^{-1} + (\omega+1)x^0 + \omega x^1 + \cdots + 2x^{\omega-1} + x^\omega$$

raised to the power m, this coefficient being next divided by the value of the same polynomial raised to the power n, which corresponds to $x = 1$.

Now one has

$$1+2x+\cdots(\omega+1)x^\omega+\cdots+2x^{2\omega-1}+x^{2\omega} = (1+x+\cdots+x^\omega)^2 = \left(\frac{1-x^{\omega+1}}{1-x}\right)^2;$$

therefore the polynomial in question will be equal to

$$x^{-\omega}\left(\frac{1-x^{\omega+1}}{1-x}\right)^2,$$

and consequently the power n of this polynomial will be represented by

$$\frac{x^{-m\omega}(1-x^{\omega+1})^{2m}}{(1-x)^{2m}}.$$

This formula being compared to that of No. 25, one will have

$$n = 2m, \quad n\alpha = m\omega, \quad \alpha + \beta = \omega + 1,$$

whence one obtains

$$n = 2m, \quad \alpha = \frac{m\omega}{n} = \frac{\omega}{2}, \text{ and } \beta = \frac{\omega}{2};$$

therefore (preceding Problem) the sought probability will be

$$\begin{aligned}\frac{1}{1.2.3\ldots 2m\rho^{2m}}\,&[(\pi+1)(\pi+2)\ldots(\pi+2m-1)\\ &-2m(\pi+1-\rho)(\pi+2-\rho)\cdots(\pi+2m-1-\rho)\\ &+\frac{2m(2m-1)}{2}(\pi+1-2\rho)(\pi+2-2\rho)\cdots(\pi+2m-1-2\rho)\\ &\cdots]\end{aligned}$$

by supposing $\pi = m\omega + \mu$ and $\rho = \omega + 1$, and continuing the series until some one of the factors $\pi + 1 - \rho$, $\pi + 1 - 2\rho$, ... becomes negative.

Thence one will find, as in the preceding Problem, that the probability that the mean error is found between the limits $\frac{-p}{n}$ and $\frac{q}{n}$ will be expressed by

$$\begin{aligned}\frac{1}{1.2.3\ldots 2m\rho^{2m}}\,&[\gamma(\gamma+1)\cdots(\gamma+2m-1)-(\delta+1)(\delta+2)\cdots(\delta+2m)\\ &-2m[(\gamma-\rho)\cdots(\gamma+2m-1-\rho)-(\delta+1-\rho)\cdots(\delta+2m-\rho)]\\ &+\frac{2m(2m-1)}{2}[(\gamma-2\rho)\cdots(\gamma+2m-1-2\rho)-(\delta+1-2\rho)\cdots(\delta+2m-2\rho)]\\ &\cdots],\end{aligned}$$

γ being $= m\omega + q$ and $\delta = m\omega - p$. In regard to the continuation of these two series, it will be necessary to follow the rules prescribed above (24).

29. COROLLARY. — Suppose now that the numbers ω, p and q become infinite, but so that one has $\frac{p}{\omega} = r$, $\frac{q}{\omega} = s$, r and s being some finite numbers, and the preceding formula will become (25)

$$\begin{aligned}&\frac{(2m+s)^{2m}-2m[2(m-1)+s]^{2m}+\frac{2m(2m-1)}{2}[2(m-2)+s]^{2m}-\cdots}{1.2.3\ldots 2m.2^{2m}},\\ &-\frac{(2m-r)^{2m}-2m[2(m-1)-r]^{2m}+\frac{2m(2m-1)}{2}[2(m-2)-r]^{2m}-\cdots}{1.2.3\ldots 2m.2^{2m}},\end{aligned}$$

these two series being continued until some one of the quantities which are raised to the power $2n$ become negative.

This formula will express therefore the probability that the mean error of n observations is contained between the limits $\frac{-r}{n}$ and $\frac{s}{n}$, in the hypothesis that each observation is subject to all the possible errors contained between these two limits -1 and $+1$, and that the facility of each error is proportional to the difference that there is between this error and the greatest possible error in the same sense; this hypothesis is more conformed to nature than that of No. 27; the curve of errors (20) would be here any isosceles triangle.

30. SCHOLIUM. — In general, one could find, by aid of the preceding Lemma, the probability that the mean error is equal to a given quantity under the hypothesis that the

errors, to which each observation is subject, form an arithmetic progression, and that the facilities of these errors form any algebraic progression, of which the differences of any order becomes null; because it is

$$Ax^{-\alpha} + Bx^{-\alpha+1} + \cdots + Px^{-1} + Qx^0 + Rx^1 + \cdots + Vx^\beta$$

the polynomial of which the exponents of x represent the errors, and the coefficients, the facilities of these errors; let one denote by ΔA, $\Delta^2 A$,... the first differences, second,..., of the series

$$A,\, B,\, C, \ldots,$$

so that

$$\Delta A = B - A, \quad \Delta^2 A = C - 2B + A, \ldots,$$

and let one denote likewise by ΔV, $\Delta^2 V$,... the differences of the series

$$V,\, X,\, Y,\, \ldots$$

supposed continued to beyond V, one will have, as one knows, for the value of the proposed polynomial, the series

$$x^{-\alpha}\left[\frac{A - Vx^{\alpha+\beta+1}}{1-x} + x\frac{\Delta A - \Delta V x^{\alpha+\beta+1}}{(1-x)^2} + \cdots\right].$$

Now, if the series

$$A, B, C, \ldots,\, V,\, X,\, \ldots$$

is such that its differences of any order m, for example, becomes null, one will have

$$\Delta^m A = 0, \quad \Delta^m V = 0,$$

and all the subsequent differences will be zero also: so that the preceding expression will become finished when likewise the proposed polynomial would contain an infinite number of terms; moreover this expression could be reduced to this form $\frac{\Xi}{(1-x)^m}$, Ξ being a rational and whole function of x; so that by raising this quantity to any power, one will have always an expression which will be in the case of that of the Lemma.

31. *One demands the coefficient of the power x^μ in the series which will result from development of the fraction*

$$\frac{X}{(a-x)^m(b-x)^n},$$

X being, as in Lemma I, a function, rational and without divisor, of x.

One knows that the fraction $\frac{1}{(a-x)^m(b-x)^n}$ can be decomposed into different fractions such as these here

$$\frac{A}{(a-x)^m} + \frac{A'}{(a-x)^{m-1}} + \frac{A''}{(a-x)^{m-2}} + \cdots + \frac{A^{(m-1)}}{a-x}$$
$$+\frac{B}{(b-x)^n} + \frac{B'}{(b-x)^{n-1}} + \frac{B''}{(b-x)^{n-2}} + \cdots + \frac{B^{(n-1)}}{b-x},$$

the coefficients $A,\ A',\ A'',\ldots$ being equal to these which become the quantities

$$+\frac{1}{(b-x)^n},\quad -\frac{d\frac{1}{(b-x)^n}}{dx},\quad +\frac{d^2\frac{1}{(b-x)^n}}{2dx^2},\ldots,$$

when $x=a$, and the coefficients $B,\ B',\ B'',\ \ldots$ being equal to these which become the quantities

$$+\frac{1}{(a-x)^n},\quad -\frac{d\frac{1}{(a-x)^n}}{dx},\quad +\frac{d^2\frac{1}{(a-x)^n}}{2dx^2},\ldots,$$

when $x=b$. Therefore the proposed fraction will be changed into these two sequences of fractions

$$\frac{AX}{(a-x)^m}+\frac{A'X}{(a-x)^{m-1}}+\cdots+\frac{A^{(m-1)}X}{a-x}$$
$$+\frac{BX}{(b-x)^n}+\frac{B'X}{(b-x)^{n-1}}+\cdots+\frac{B^{(n-1)}X}{b-x},$$

But, by Lemma I, the coefficient of the power x^μ in the resulting series of a fraction such as $\frac{X}{(a-x)^{m-s}}$ can be expressed by

$$\frac{(-1)^{m-s-1}}{1.2.3\ldots(m-s-1)}\frac{d^{m-s-1}\frac{X}{x^{\mu+1}}}{dx^{m-s-1}}$$

by making, after the differentiations, $x=a$.

Therefore, in general, the fraction $\frac{AX}{(a-x)^{m-s}}$ will give for the coefficient of x^μ the quantity

$$\frac{(-1)^s}{1.2.3\ldots s}\frac{d^s\frac{1}{(b-x)^n}}{dx^s}\times\frac{(-1)^{m-s-1}}{1.2.3\ldots(m-s-1)}\frac{d^{m-s-1}\frac{X}{x^{\mu+1}}}{dx^{m-s-1}},$$

where it is necessary to make $x=a$. Therefore, since

$$d^{m-1}yz=yd^{m-1}z+(m-1)dyd^{m-2}z+\frac{(m-1)(m-2)}{2}d^2yd^{m-3}z+\cdots+sd^{m-1}y,$$

it is easy to see that the fractions

$$\frac{AX}{(a-x)^m}+\frac{A'X}{(a-x)^{m-1}}+\cdots+\frac{A^{(m-1)}X}{a-x},$$

taken altogether, will give for the coefficient of x^μ the quantity

$$\frac{(-1)^{m-1}}{1.2.3\ldots(m-1)}\frac{d^{m-1}\frac{X}{x^{\mu+1}(a-x)^n}}{dx^{m-1}},$$

x being made equal to a.

Likewise the fractions

$$\frac{BX}{(b-x)^n}+\frac{B'X}{(b-x)^{n-1}}+\cdots+\frac{B^{(n-1)}X}{b-x}$$

will give for the coefficient of x^μ the quantity

$$\frac{(-1)^{n-1}}{1.2.3\dots(n-1)}\frac{d^{n-1}\frac{X}{x^{\mu+1}(a-x)^m}}{dx^{n-1}},$$

x being made equal to b.

Therefore, in reuniting these two quantities, one will have for the coefficient of x^μ in the resulting series of the fraction $\frac{X}{(a-x)^m(b-x)^n}$, the expression

$$\begin{aligned} &\frac{(-1)^{m-1}}{1.2.3\dots(m-1)}\frac{d^{m-1}\frac{X}{x^{\mu+1}(b-x)^n}}{dx^{m-1}} \qquad (x=a)\\ +&\frac{(-1)^{n-1}}{1.2.3\dots(n-1)}\frac{d^{n-1}\frac{X}{x^{\mu+1}(a-x)^m}}{dx^{n-1}} \qquad (x=b)\end{aligned}$$

by having care to reject in the value of $\frac{X}{x^{\mu+1}}$ all positive powers of x.

32. COROLLARY. — It is easy to conclude from there that if one would develop into series the fraction

$$\frac{X}{(a-x)^m(b-x)^n(c-x)^p\cdots},$$

one would have for the coefficient of x^μ the following expression

$$\begin{aligned} &\frac{(-1)^{m-1}}{1.2.3\dots(m-1)}\frac{d^{m-1}\left[\frac{X}{x^{\mu+1}(b-x)^n(c-x)^p\cdots}\right]}{dx^{m-1}} \qquad (x=a)\\ +&\frac{(-1)^{n-1}}{1.2.3\dots(n-1)}\frac{d^{n-1}\left[\frac{X}{x^{\mu+1}(a-x)^m(c-x)^p\cdots}\right]}{dx^{n-1}} \qquad (x=b)\\ +&\frac{(-1)^{p-1}}{1.2.3\dots(p-1)}\frac{d^{p-1}\left[\frac{X}{x^{\mu+1}(a-x)^m(b-x)^n\cdots}\right]}{dx^{p-1}} \qquad (x=c)\\ &\vdots \qquad\qquad\qquad\qquad \vdots\end{aligned}$$

by taking in the value of $\frac{X}{x^{\mu+1}}$ only the negative powers of x, and rejecting all the positives.

33. REMARK. — By means of the preceding Lemma, one could therefore determine easily the probability that the mean error, resulting from as many observations as one will wish, be null or equal to a given quantity, when the polynomial (30)

$$Ax^{-\alpha}+Bx^{-\alpha+1}+\cdots+Px^{-1}+Qx^0+Rx^1+\cdots+Vx^\beta$$

forms any recurrent series; because then the sum of this series could be expressed, as one knows, by a rational fraction such as

$$\frac{\Xi}{(a-x)^\lambda(b-x)^\pi(c-x)^\rho\cdots},$$

Ξ being a function, rational and without divisor, of x; so that by raising this quantity to any power, one will have always an expression which could correspond to those of the Lemma above.

In the rest, the hypothesis most conforming to nature is that where one supposes each observation be subject to all the errors contained between some given limits, so that the number of all the possible errors are infinite, as in Nos. 27 and 29; now, in order to find in this case the probability that the mean error of any number of observations is also contained between some given limits, it is not necessary to consider first a finite number of errors and to suppose next that this number become infinite, as we have practiced in the numbers cited; but one can attain it directly by a much more simple and more general method, which is based on the following Lemma.

LEMMA III.

34. *If y denotes any function of x, such that $\frac{d^m y}{dx^m}$ is a constant quantity, one will have*

$$\int y a^x dx = a^x \left[\frac{y}{\log a} - \frac{dy}{dx(\log a)^2} + \frac{d^2 y}{dx^2(\log a)^3} - \cdots \pm \frac{d^m y}{dx^m(\log a)^{m+1}}\right] + \text{const.};$$

this is that which is easy to verify by differentiation.

35. COROLLARY I. — If one makes $y = x^m$, m being a positive integer, one will have therefore

$$\begin{aligned}\int x^m a^x dx =& a^x \left[\frac{x^m}{\log a} - \frac{m x^{m-1}}{(\log a)^2} + \frac{m(m-1)x^{m-1}}{(\log a)^3} - \cdots \right. \\ & \left. \pm \frac{m(m-1)(m-2)\dots 2.1}{(\log a)^{m+1}}\right] + \text{ const.}\end{aligned}$$

Let one take the integral $\int x^m a^x dx$ so that it is null when $x = 0$, and one will have

$$\int x^m a^x dx = a^x \left[\frac{x^m}{\log a} - \frac{m x^{m-1}}{(\log a)^2} + \cdots + (-1)^m \frac{1.2.3\dots m}{(\log a)^{m+1}}\right].$$

Now if one supposes that a is a fraction smaller than unity, so that $\frac{1}{a}$ is a number greater than unity, and if one makes $x = \infty$, it is easy to see that $\left(\frac{1}{a}\right)^x$ will be an infinite quantity of an order infinitely greater than x^m and than any finite power of x; therefore $\frac{x^m}{\left(\frac{1}{a}\right)^x}$ or else $a^x x^m$ will be null, and, by all the more reason also, all the other quantities $a^x x^{m-1}$, $a^x x^{m-2}$, … will be null, so that one will have in these cases

$$\int x^m a^x dx = \frac{1.2.3\dots m}{(-\log a)^{m-1}}.$$

Whence I conclude that the quantity $\frac{1}{(-\log a)^m}$ is equal to the integral of $x^{m-1} a^x dx$ taken from $x = 0$ to $x = \infty$, and divided by $1.2.3\dots(m-1)$, provided that a be a positive number less than unity.

If a were a positive number greater than unity, there would be only to put $\frac{1}{a}$ in the place of x in the preceding formula, and one would conclude from it that the quantity

$\frac{1}{(\log a)^m}$ would be equal to the integral of $\frac{x^{m-1}dx}{a^x}$ taken likewise from $x=0$ to $x=\infty$, and divided by $1.2.3\dots(m-1)$; one sees thence how one can reduce any powers of $\frac{1}{\log a}$ in some infinite series which proceeds following the powers of a.

36. COROLLARY II. — Therefore, if one has any function, rational and without divisor, of a such that

$$A = Pa^p + Qa^{p-1} + Ra^{p-2} + \cdots,$$

and if one demands the coefficient of the power a^{p-x} in the function $\frac{A}{(\log a)^m}$, there will be only to put, in the place of $\frac{1}{(\log a)^m}$, the sum of the values of $\frac{x^{m-1}dx}{a^x}$ from $x=0$ to $x=\infty$, divided by $1.2.3\dots(m-1)$ (preceding Corollary), and collecting all the terms where a will be found raised to a given power, one will have for the coefficient of this power the series

$$\frac{Px^{m-1} + Q(x-1)^{m-1} + R(x-2)^{m-2} + \cdots}{1.2.3\dots(m-1)}dx,$$

which must be continued only to some one of the terms $x-1$, $x-2$, $\dots$ become negative; and as this coefficient no longer depends on the value of a, it is clear that the formula that we happen to find will always take place, either when a be greater or less than unity.

If, instead of the function $\frac{A}{(\log a)^m}$, one has this here

$$\frac{A}{(\log a - x)^m},$$

as $\log a - \alpha = \log\frac{a}{e^\alpha}$, it would be necessary to substitute in the place of $\frac{1}{(\log a - x)^m}$ the sum of the values of $\frac{e^{\alpha x}x^{m-1}dx}{a^s}$ from $x=0$ to $x=\infty$, divided by $1.2.3\dots(m-1)$, and one would have for the coefficient of a^{p-x} the series

$$\frac{Pe^{\alpha x}x^{m-1} + Qe^{\alpha(x)}(x-1)^{m-1} + Re^{\alpha(x)}(x-2)^{m-2} + \cdots}{1.2.3\dots(m-1)}dx.$$

Finally, if one had the function

$$\frac{A}{(\log a - \alpha)^m(\log a - \beta)^n\cdots},$$

one would decompose first, by the known methods, the fraction

$$\frac{1}{(\log a - \alpha)^m(\log a - \beta)^n\cdots}$$

into these here

$$\begin{aligned}&\frac{F}{(\log a - x)^m} + \frac{F'}{(\log a - x)^{m-1}} + \cdots + \frac{F^{(m-1)}}{\log a - x}\\ +&\frac{G}{(\log a - \beta)^m} + \frac{G'}{(\log a - \beta)^{m-1}} + \cdots + \frac{G^{(n-1)}}{\log a - \beta}\\ &\dots;\end{aligned}$$

next, multiplying each of these fractions by A, one would have as many functions of a, in which one could find the coefficient of the power a^x by the formula above.

37. REMARK. — By means of the preceding Lemma, one can find the integral

$$\int ya^x dx,$$

when $y = Xe^{-\alpha x}$, X being a function, rational and without divisor, of x, such that its differential of an arbitrary order is constant; because for this there will be only to put, in the formula of the Lemma, X in place of y and $ae^{-\alpha}$ in place of a; for which one will have

$$\int \frac{Xa^x}{e^{x\alpha}} dx = \frac{a^x}{e^{x\alpha}} \left[\frac{X}{\log a - \alpha} - \frac{dX}{dx(\log a - \alpha)^2} + \frac{d^2X}{dx^2(\log a - \alpha)^3} - \cdots \right] + \text{const.}$$

And one will find likewise the integral of $ya^x dx$, when y will be composed of different functions of the same kind $\frac{X}{e^{x\alpha}}$.

Whence it follows that one could also find the integral of $ya^x dx$ when y will be of this form: $X \cos \alpha x$ or $X \sin \alpha x$, or composed of several functions of a similar form; because there will be only to put in the place of the sines and cosines the imaginary exponential expressions which are their equivalents, and, the calculation achieved, one will put back in the place of these expressions the sines or cosines which correspond to it.

These are the only cases where the formula $ya^x dx$ is integrable, at least by the methods known till now; in all the other cases the integration can be executed only by approximation.

PROBLEM X.

38. *One supposes that each observation is subject to all the possible errors contained between these two limits, p and $-q$, and that the facility of each error x, that is to say the number of cases where it can take place divided by the total number of cases, is represented by any function of x designated by y; one demands the probability that the mean error of n observations be contained between the limits r and $-s$.*

One will begin first by seeking the probability that the mean error be z, and this probability being represented by a function of z, there will be only to take the integral from $z = r$ to $z = s$; this will be the sought probability.

Now, in order to have the probability that the mean error of n observations be z, it will be necessary to consider the polynomial which is represented by the integral of $ya^x dx$, by supposing this integral taken in such a way that it is extended from $x = p$ to $x = -q$; one will raise this polynomial to the power n, and one will find the coefficient of power z of a, by the rules given in the Corollaries of the preceding Lemma; this coefficient, which will be a function of z, will express the probability that the mean error be z, as it is easy to see after that which has been demonstrated above.

39. EXAMPLE I. — We suppose first that y is a constant quantity K, so that all the errors are equally probable, and the integral of $ya^x dx$ will be $\frac{Ka^x}{\log a}$, so that in taking this integral from $x = p$ to $x = -q$, one will have its complete value $\frac{K(a^p - a^{-q})}{\log a}$; let

one raise therefore this quantity to the power n, and one will have a quantity of the form $\frac{A}{(\log a)^n}$, where (making $p+q=t$)

$$A = K^n\left[a^{pn} - na^{pn-t} + \frac{n(n-1)}{2}a^{pn-2t} - \cdots\right].$$

Therefore, by Corollary II of the Lemma (36), the coefficient of the power a^{pn-x} will be

$$\frac{K^n}{1.2.3\ldots(n-1)}\left[x^{n-1} - n(x-t)^{n-1} + \frac{n(n-1)}{2}(x-2t)^{n-1} \right.$$
$$\left. - \frac{n(n-1)(n-2)}{2.3}(x-3t)^{n-1} + \cdots\right]dx,$$

in it taking care to continue the series only until one arrives to some terms $x - mt$ which are negatives. Making therefore $pn - x = z$, that is to say $x = pn - z$ one will have the probability that the mean error of n observations be z. One will integrate now the preceding formula in y making x vary, and one will take the integral so that it is null when $x = pn - r$, and complete when $x = pn + s$; one will have in this way the quantity

$$\frac{K^n}{1.2.3\ldots n}\left[(pn+s)^n - n(pn+s-t)^n + \frac{n(n-1)}{2}(pn+s-2t)^n - \cdots \right.$$
$$\left. - (pn-r)^n + n(pn-r-t)^n - \frac{n(n-1)}{2}(pn-r-2t)^n + \cdots\right],$$

which will express the probability that the mean error of n observations is contained between the limits r and $-s$; in the rest this formula returns to the same as that of No. 27.

40. EXAMPLE II. — One supposes that the quantity y be $K(p^2-x^2)$, and that the two limits of the errors are p and $-p$, it will be necessary to integrate the differential $Ka^x(p^2-x^2)dx$, and to take the integral so that it is extended from $x=-p$ to $x=p$. Now, since the second differential of p^2-x^2 is constant, one will have by the Lemma this integral

$$Ka^x\left[\frac{p^2-x^2}{\log a} + \frac{2x}{(\log a)^2} - \frac{2}{(\log a)^3}\right],$$

which being completed, as one has just said, will give

$$\frac{2Kp(a^p+a^{-p})}{(\log a)^2} - \frac{2K(a^p-a^{-p})}{(\log a)^3};$$

one will raise therefore this quantity to the power n, and one will have

$$\frac{(2Kp)^n(a^p+a^{-p})^n}{(\log a)^{2n}} - n\frac{(2K)^n p^{n-1}(a^p+a^{-p})^{n-1}(a^p-a^{-p})}{(\log a)^{2n+4}}$$
$$+\frac{n(n-1)}{2}\frac{(2K)^n p^{n-2}(a^p+a^{-p})^{n-2}(a^p-a^{-p})^2}{(\log a)^{2n+2}} + \cdots;$$

one will develop the powers of $a^p + a^{-p}$ and of $a^p - a^{-p}$, and one will seek next by the rules of No. 36 the coefficient of the power a^z. In order to facilitate these operations we will suppose

$$\begin{aligned}
&(a^p + a^{-p})^n = a^{pn} + Pa^{pn-2p} + Qa^{pn-4p} + \cdots \\
&(a^p + a^{-p})^{n-1}(a^p - a^{-p}) = a^{pn} + P'a^{pn-2p} + Q'a^{pn-4p} + \cdots \\
&(a^p + a^{-p})^{n-2}(a^p - a^{-p})^2 = a^{pn} + P''a^{pn-2p} + Q''a^{pn-4p} + \cdots \\
&\vdots
\end{aligned}$$

and one will find, for the coefficient of the power $np - x$, the series

$$\begin{aligned}
&\frac{(2Kp)^n}{1.2.3\ldots(2n-1)}[x^{2n-1} + P(x-2p)^{2n-1} + Q(x-4p)^{2n-1} + \cdots]dx \\
&- \frac{(2K)^n p^{n-1}}{1.2.3\ldots 2n}[x^{2n} + P'(x-2p)^{2n} + Q'(x-4p)^{2n} + \cdots]dx \\
&+ \frac{n(n-1)}{2}\frac{(2K)^n p^{n-2}}{1.2.3\ldots(2n+1)}[x^{2n+1} + P''(x-2p)^{2n+1} + Q''(x-4p)^{2n+1} + \cdots]dx \\
&\vdots
\end{aligned}$$

One will have therefore $z = np - x$, that is to say $x = np - z$, and one will integrate in such a way that the integral is null when $z = r$ and complete when $z = -s$, that is to say null when $x = np - r$ and complete when $x = np + s$; one will have the quantity,

$$\begin{aligned}
&\frac{(2Kp)^n}{1.2.3\ldots 2n}\ \big[(np+s)^{2n} + P[(n-2)p+s]^{2n} + Q[(n-4)p+s]^{2n} + \cdots \\
&\qquad\qquad -(np-r)^{2n} - P[(n-2)p-r]^{2n} - Q[(n-4)p-r]^{2n} - \cdots\big] \\
&- n\frac{(2K)^n p^{n-1}}{1.2.3\ldots(2n+1)}\ \big[(np+s)^{2n+1} + P'[(n-2)p+s]^{2n+1} + Q'[(n-4)p+s]^{2n+1} + \cdots \\
&\qquad\qquad -(np-r)^{2n+1} - P'[(n-2)p-r]^{2n+1} - Q'[(n-4)p-r]^{2n+1} - \cdots\big] \\
&+ \frac{n(n-1)}{2}\frac{(2K)^n p^{n-2}}{1.2.3\ldots(2n+2)}\ \big[(np+s)^{2n+2} + P''[(n-2)p+s]^{2n+2} + Q''[(n-4)p+s]^{2n+2} + \cdots \\
&\qquad\qquad -(np-r)^{2n+2} - P''[(n-2)p-r]^{2n+2} - Q''[(n-4)p-r]^{2n+2} - \cdots\big] \\
&\vdots
\end{aligned}$$

which will express the probability that the mean error of n observations is contained between the limits r and $-s$; in the rest it will be necessary always to remember that the preceding series must be continued only until some one of the quantities which are raised to the powers $2n$, $2n+1, \ldots$ become negatives.

41. REMARK. — The hypothesis of the last example appears the simplest and most natural that one can imagine; it is true that the one of Problem VIII appears again simpler, since one supposes that the facility of the errors x and $-x$ are represented by $p - x$, p being the greatest value possible of x, that is to say the limit of the errors, as many positives as negatives; but this hypothesis has the inconvenience that the law of

continuity is not observed in passing from positive errors to the negatives; this is why, if one would wish to apply the method of the preceding Problem, it will be necessary, by making $y = K(p-x)$, to take first the integral $\int ya^x dx$ from $x = 0$ to $x = p$, which will be

$$K\left[\frac{a^p-1}{(\log a)^2} - \frac{p}{\log a}\right];$$

next, by making x negative and preserving the same value of y, it will be necessary to take the same integral $\int ya^{-x}dx$ from $x = 0$ to $x = p$, which will be (making in it only to put $\frac{1}{a}$ in the place of a in the preceding expression)

$$K\left[\frac{a^{-p}-1}{(\log a)^2} + \frac{p}{\log a}\right],$$

and the sum of these two particular integrals will be the complete integral of $\int ya^x dx$ from $x = p$ to $x = -p$ under the hypothesis in question; one will have therefore the quantity

$$K\left[\frac{a^p + a^{-p} - 2}{(\log a)^2}\right] \text{ or else } K\left(\frac{a^{\frac{p}{2}} + a^{-\frac{p}{2}}}{\log a}\right)^2,$$

which it will be necessary to raise to the power n, and on which one could next operate, as in Example I; one could likewise, without making a new calculation, apply here the formulas of that Example by putting $2n$ in the place of n, $\frac{p}{2}$ in the place of p and of q, and consequently p in the place of $t = p+q$; in this manner one will have immediately the expression of the probability that the mean error of n observations is contained between the limits r and $-s$, which will be

$$\frac{K^{2n}}{1.2.3\dots 2n}\left[(np+s)^{2n} - 2n[(n-1)p+s]^{2n} + \frac{2n(2n-1)}{2}[(n-2)p+s]^{2n} - \cdots \right.$$
$$\left. -(np-r)^{2n} + 2n[(n-1)p-r]^{2n} - \frac{2n(n-1)}{2}[(n-2)p-r]^{2n} - \cdots\right]$$

this which agrees with the formula of No. 29.

PROBLEM XI.

42. *Supposing that each observation is subject to all the possible errors contained between the limits p and $-p$ (p being the arc of 90 degrees), and that the facility of each error x is proportional to $\cos x$, one demands the probability that the mean error of n observations will be contained between the limits r and $-s$.*

One will have therefore here $y = K\cos x$, and the first question is to integrate the differential $Ka^x \cos x\, dx$, of which the integral (by taking $\frac{e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}}{2}$ in the place of $\cos x$) will be found by No. 37,

$$\frac{Ka^x}{2}\left(\frac{e^{x\sqrt{-1}}}{\log a + \sqrt{-1}} + \frac{e^{-x\sqrt{-1}}}{\log a - \sqrt{-1}}\right);$$

that is to say, by passing over from the imaginary exponential to the sines and cosines,

$$Ka^x \frac{\log a . \cos x + \sin x}{(\log a)^2 + 1};$$

this integral must now be taken so that it extends from $x = -p$, in which case $\cos x = 0$ and $\sin x = 1$ to $x = p$, where $\cos x = 0$ and $\sin x = 1$; thus one will have for the complete integral

$$\frac{K(a^p + a^{-p})}{(\log a)^2 + 1}.$$

Let one raise therefore this quantity to the power n, and making, for brevity,

$$A = K^n \left[a^{pn} + na^{pn-2p} + \frac{n(n-1)}{2} a^{pn-4p} + \cdots \right],$$

one will have the quantity

$$\frac{A}{[(\log a)^2 + 1]^n},$$

in which the question is now to seek the coefficient of the power a^x.

For this it will be necessary (36) to factor the fraction

$$\frac{1}{[(\log a)^2 + 1]^n}, \text{ that is to say } \frac{1}{(\log a + \sqrt{-1})^n (\log a - \sqrt{-1})^n},$$

into these simple fractions

$$\begin{aligned} &\frac{F}{(\log a + \sqrt{-1})^n} + \frac{F'}{(\log a + \sqrt{-1})^{n-1}} + \frac{F''}{(\log a + \sqrt{-1})^{n-2}} + \cdots \\ +&\frac{G}{(\log a - \sqrt{-1})^n} + \frac{G'}{(\log a - \sqrt{-1})^{n-1}} + \frac{G''}{(\log a - \sqrt{-1})^{n-2}} + \cdots, \end{aligned}$$

and one will have by the known method (31)

$$\begin{aligned} F &= \frac{1}{(-2\sqrt{-1})^n}, & F' &= -\frac{n}{(-2\sqrt{-1})^{n+1}}, & F'' &= \frac{n(n+1)}{2(-2\sqrt{-1})^{n+2}} + \cdots, \\ G &= \frac{1}{(2\sqrt{-1})^n}, & G' &= -\frac{n}{(2\sqrt{-1})^{n+1}}, & G'' &= \frac{n(n+1)}{2(2\sqrt{-1})^{n+2}} + \cdots; \end{aligned}$$

multiplying next by A each of these fractions, one will find, by the method of No. 36, that the coefficient of the power a^{pn-x} will be expressed in this way:

$$
\begin{aligned}
&\frac{K^n dx}{1.2.3\ldots(n-1)}\Big[(Fe^{-x\sqrt{-1}}+Ge^{x\sqrt{-1}})x^{n-1}\\
&\qquad +n[Fe^{-(x-2p)\sqrt{-1}}+Ge^{(x-2p)\sqrt{-1}}](x-2p)^{n-1}\\
&\qquad +\frac{n(n-1)}{2}[Fe^{-(x-4p)\sqrt{-1}}+Ge^{(x-4p)\sqrt{-1}}](x-4p)^{n-1}+\cdots\Big]\\
&+\frac{K^n dx}{1.2.3\ldots(n-2)}\Big[(F'e^{-x\sqrt{-1}}+G'e^{x\sqrt{-1}})x^{n-2}\\
&\qquad +n[F'e^{-(x-2p)\sqrt{-1}}+G'e^{(x-2p)\sqrt{-1}}](x-2p)^{n-2}\\
&\qquad +\frac{n(n-1)}{2}[F'e^{-(x-4p)\sqrt{-1}}+G'e^{(x-4p)\sqrt{-1}}](x-4p)^{n-2}+\cdots\Big]\\
&+\frac{K^n dx}{1.2.3\ldots(n-3)}\Big[(F''e^{-x\sqrt{-1}}+G''e^{x\sqrt{-1}})x^{n-3}\\
&\qquad +n[F''e^{-(x-2p)\sqrt{-1}}+G''e^{(x-2p)\sqrt{-1}}](x-2p)^{n-3}\\
&\qquad +\frac{n(n-1)}{2}[F''e^{-(x-4p)\sqrt{-1}}+G''e^{(x-4p)\sqrt{-1}}](x-4p)^{n-3}+\cdots\Big]\\
&\ \vdots
\end{aligned}
$$

Now one has $e^{\pm x\sqrt{-1}}=\cos x\pm\sqrt{-1}\sin x$, and thus of the others; therefore, substituting these values, and making, for brevity,

$$
\begin{gathered}
G+F=f,\qquad G-F=\tfrac{g}{\sqrt{-1}},\\
G'+F'=f',\qquad G'-F'=\tfrac{g'}{\sqrt{-1}},\\
G''+F''=f'',\qquad G''-F''=\tfrac{g''}{\sqrt{-1}},\\
\vdots
\end{gathered}
$$

where the quantities f, g, f', g', … will be necessarily real, the preceding formula will become

$$\begin{aligned}
&\frac{K^n dx}{1.2.3\ldots(n-1)}\ \Big[(f\cos x + g\sin x)x^{n-1} \\
&\qquad + n[f\cos(x-2p) + g\sin(x-2p)](x-2p)^{n-1} \\
&\qquad + \frac{n(n-1)}{2}\,[f\cos(x-4p) + g\sin(x-4p)](x-4p)^{n-1} + \cdots\Big] \\
&+\frac{K^n dx}{1.2.3\ldots(n-2)}\ \Big[(f'\cos x + g'\sin x)x^{n-2} \\
&\qquad + n[f'\cos(x-2p) + g'\sin(x-2p)](x-2p)^{n-2} \\
&\qquad + \frac{n(n-1)}{2}\,[f'\cos(x-4p) + g'\sin(x-4p)](x-4p)^{n-2} + \cdots\Big] \\
&+\frac{K^n dx}{1.2.3\ldots(n-3)}\ \Big[(f''\cos x + g''\sin x)x^{n-3} \\
&\qquad + n[f''\cos(x-2p) + g''\sin(x-2p)](x-2p)^{n-3} \\
&\qquad + \frac{n(n-1)}{2}\,[f''\cos(x-4p) + g''\sin(x-4p)](x-4p)^{n-3} + \cdots\Big] \\
&\ \vdots
\end{aligned}$$

where it will be necessary to continue the different series until the quantities x, $x-2p$, $x-4p$, ... or their exponents become negatives; this quantity will express therefore the probability that the mean error of n observations be $pn-x$; consequently there will be further only to integrate in such a way that the integral be null when $x = pn-r$ and complete when $x = pn+s$ in order to have the sought expression of the probability that the mean error be contained between the given limits r and $-s$; but as this integration is easy by the known methods, we will not enter into that in greater detail; and we will terminate likewise here our researches, by which one must see that there remains no more difficulty in the solution of the questions that one can propose on this subject.