J. H. Lambert's mathematische Ergötzungen über die Glücksspiele*

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Archiv der reinen und angewandten Mathematik, published by C. F. Hindenburg, tenth Issue, Leipzig 1799, p. 209-219.

I. Lottery

1. One takes two whole decks of 52 cards. From the first, 3 are extracted and are put sight unseen on the table. The other entire deck is distributed among the players, however in a way that each one, so many as he wants to have, must buy the cards, which then are left each time to the highest bidder. If therefore now he sells all cards, and the price is put on the 3 covered cards, so one starts the second deck, that with the 3 cards removed from it, taking off after one another, whereby the card of the dealt deck corresponding to the revealed must be put aside as useless by the holder. This is continued, until the second deck becomes completely uncovered. So then finally 3 cards will be left in the hands of the gambler, which correspond to the 3 covered cards; which latter then exposed, and the price put on it is given to the owner of the corresponding cards etc.

2. During the uncoverings the players look after their cards to negotiate among each other, whereby then, because the price of the cards increases each time the fewer cards are to be still exposed, not infrequently many errors happen in that often he buys or sells the cards more dearly or more cheaply than it was fitting.

3. Since now he plays the most surely, who knows the value of the cards with every case, so the trouble to find some rules is worth it, whereby the mean value of the cards could be calculated and found.

4. The deck consists of 52 cards, 3 among them are laid down, the remaining 49 are not worth anything. Therefore a card initially risks 49 against 3.

5. Afterwards, one must know the price put on the 3 cards. This, because it normally variable, is$= a$. There now under the 52 sold cards one each has equally large claim at it, so a card is worth initially $= a : 52$.

6. Let be of the 52 cards already b cards would have been passed out, so then a card would be worth $= \frac{a}{52-b}$.

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7. The price favored on the cards laid on the table is 720 points and it is already 16 cards exposed, so the price of one card would be

$$= \frac{a}{52-b} = \frac{720}{52-16} = \frac{720}{36} = 20 \text{ Points.}$$

8. If I furthermore wanted to know my expectation for profit and would have d cards in hand, so all my cards were worth $\frac{da}{52-b}$, which compared with a, my expectation for profit shows

$$\frac{da}{52-b} : a = \frac{da}{52a-ba} = d : (52-b)$$

9. For example, let be $a = 500$ points, $b = 12$ cards, $d = 4$ cards, so the value of my cards would be $\frac{da}{52-b} = \frac{2000}{40} = 50$ points. The expectation for the profit would be then as 50 to 500, that is, as 1 to 10, or $d : (52-b) = 4 : 40 = 1$ to 10. Therefore, I would have to expect the tenth part.

Table to the lottery game, if the price is 1000 on the covered 3 cards.							
0	$19\frac{1}{4}$	13	$25\frac{2}{3}$	26	$38\frac{1}{2}$	39	77
1	$19\frac{2}{3}$	14	$26\frac{1}{3}$	27	40	40	$83\frac{1}{3}$
2	20	15	27	28	$41\frac{2}{3}$	41	91
3	$20\frac{1}{2}$	16	$27\frac{3}{4}$	29	$43\frac{1}{2}$	42	100
4	21	17	$28\frac{3}{5}$	30	45	43	111
5	$21\frac{1}{3}$	18	$29\frac{2}{5}$	31	$47\frac{2}{3}$	44	125
6	$21\frac{3}{4}$	19	$30\frac{1}{3}$	32	50	45	$142\frac{6}{7}$
7	$22\frac{1}{4}$	20	$31\frac{1}{4}$	33	$52\frac{2}{3}$	46	$166\frac{2}{3}$
8	$22\frac{2}{3}$	21	$32\frac{1}{4}$	34	$55\frac{5}{9}$	47	200
9	$23\frac{1}{4}$	22	$33\frac{1}{3}$	35	$58\frac{3}{4}$	48	250
10	$23\frac{3}{4}$	23	$34\frac{1}{2}$	36	$62\frac{1}{2}$	49	$333\frac{1}{3}$
11	$24\frac{2}{5}$	24	$35\frac{5}{7}$	37	$66\frac{2}{3}$	50	–
12	25	25	37	38	$71\frac{3}{7}$	51	–
Turn up	Price of the cards	Turn up	Price of the cards	Turn up	Price of the cards	Turn up	Price of the cards

Above calculation is for each of the 3 lying covered cards particularly. Since however, also Ambe and Tern occur, so the calculation is made different.

Namely

$$52-b \text{ cards} \ldots 3 \text{ cards} \ldots d \text{ cards}$$

have

$$\frac{(52-b)}{1}\frac{(51-b)}{2} \text{ Ambe} \ldots 3 \text{ Ambe} \ldots d\,\frac{d-1}{2} \text{ Ambe}$$

and

$$\frac{(52-b)}{1}\frac{(51-b)}{2}\frac{(50-b)}{3} \text{ Tern} \ldots 1 \text{ Tern} \ldots d\frac{d-1}{2} \cdot \frac{d-2}{3} \text{ Tern.}$$

Therefore *one* card is worth $\frac{\frac{1}{3}\cdot a\cdot 3}{52-b} = \frac{a}{52-b}$ and d cards d times so much, since the Ambes, Terns etc. have nothing in advance.

My expectation with d cards

one to win is	$\dfrac{ad}{52-b}$
two to win is	$\dfrac{ad(d-1)}{(52-b)(51-b)}$
all *three* to win is	$\dfrac{ad(d-1)(d-2)}{(52-b)(51-b)(50-b)}$

10. If I now know, how much my cards are worth so I can sell same at a profit; however also none buy when it cannot happen for the right price.

11. If I want to sell a card in proportion more dearly, than the same was bought, so one says: how the initial price $\frac{a}{52}$ this proceeding to the present $\frac{a}{52-b}$, therefore the price of my bought cards (this is $= e$) to the price $\frac{52e}{52-b}$, for which I can sell my cards.

12. In order to know the price of the cards now, I have enclosed here § 9 a calculated table for each uncovered. Put, that on the 3 cards 1000 Pf. gold is lying; whence to find the price of all cards in proportion of other exposed points.

II. Dice

1. Each die has 6 sides 1, 2, 3, 4, 5, 6: If someone wanted now to encounter a certain number with a die on the first time, so that then 5 against 1 would be par, because only one is good among six sides, but against it 5 bad ones. His expectation is therefore $1:5$.

2. If someone wanted to meet a certain number with 2 dice, so 36 cases can occur with it, under what after here small enclosed table more of less to risk separately. For example, if someone wanted to encounter 9 dots with 2 dice, so one finds 4 cases in the small table under 9 dots, that can encounter, since however 32 can be missing. Therefore my expectation to win is $4:32 = 1:8$.

Die-Small table:

I. For 1 die

Dots	1	2	3	4	5	6	Sum
Cases	1	1	1	1	1	1	6 Cases

II. For 2 dice

Dots	2	3	4	5	6	7	8	9	10	11	12	Sum
Cases	1	2	3	4	5	6	5	4	3	2	1	36 Cases

III. For 3 dice

Dots	3	4	5	6	7	8	9	10	In
Cases	1	3	6	10	15	21	25	27	Sum
Dots	11	12	13	14	15	16	17	18	216
Cases	27	25	21	15	10	6	3	1	Cases

IV. For 4 dice

Dots	4	5	6	7	8	9	10	It
Cases	1	4	10	20	35	56	80	gives
Dots	11	12	13	14	15	16	17	in
Cases	104	125	140	146	140	125	104	all
Dots	18	19	20	21	22	23	24	1296
Cases	80	56	35	20	10	4	1	Cases

Small double-table on Encounters and Misses

1	2	1	1
3	6	4	2
9	18	13	5
27	54	40	14
81	162	121	41
243	486	364	122
729	1458	1093	365
Put	Stopped	Missed	Winning

3. It is exactly so with the other 2 small tables for 3 and 4 dice.

4. It happens occasionally, that one plays for encounters and misses with 3 dice, so then, what is over 10 dots was encountered, what is 10 and under it, was missing. One sees however from the above small table, that there are as many encounters as misses, namely 108 on both sides. Therefore my expectation to win is as 1 against 1; that is, I risk equally much as my middle player.

5. If one however wants to win this way certainly, so one must place initially little; get it lost, one must double or quadruple, which then replaces the previous loss with a win again; as soon as one won however, one must again as initially, place little.

6. If someone with 2 dice wants to meet two with the same dots, so 6 cases come to him, namely: $1, 1$; $2, 2$; $3, 3$; $4, 4$; $5, 5$; $6, 6$. Since now among 36 cases only 6 are good, however 30 bad ones, his expectation is as $6 : 30$ or $1 : 5$.

7. If someone plays with 3 dice, however in a way, that to win 2 of those have the

same dots, so of 216 cases only 96 are good. Namely

111	331	551	121	313	515	*111*	144	155
112	332	552	131	323	525	211	244	255
113	333	553	141	343	535	311	344	355
114	334	554	151	353	545	411	444	455
115	335	555	161	363	565	511	544	555
116	336	556	*111*	333	555	611	644	655
221	441	661	212	414	616	122	133	166
222	442	662	232	424	626	222	233	266
223	443	663	242	434	636	322	333	366
224	444	664	252	454	646	422	433	466
225	445	665	262	464	656	522	533	566
226	446	666	222	444	666	622	633	666

8. If someone wants to play with 3 dice in a way, that he wins, if all 3 dots are the equal, so are for him only 6 cases, namely 111, 222, 333, 444, 555, 666, but against him 210. His expectation therefore is as 6 : 210 or 1 : 35.

9. If someone therefore wants to play with 2 dice, that one of it hits a certain number of dots, for example, 6, so he has 11 cases to win and 25 to lose. His expectation therefore is 11 : 25.

III. On Münz or Ummünz

Croix ou pile

1. It happens on occasion, in fact also among children, that one tosses up different coins, and, after they fall on the table, one sees, if more *Münz*, that is heads, or more *Ummünz*, that is what stands on the other side, for example, coats of arms etc. lies above? The two players guess with each other, in such a way, that each one draws the one, which he guessed. Posed, I would have tossed up 12 times, and the other wants to have this, what is tails; it posed furthermore 7 tails would have fallen from it, so he would receive *seven*, but I however receive only the remaining *five* from it, and consequently, since each one had put in 6 previously, I lose one of those.

2. There is also with this game different cases. For example, if one tossed up 10 times, so it would be much more likely, that heads would fall 6 times as 1 time, because 1 time only can fall on the same 10 ways, however 6 times on the same 210 ways. If I therefore wanted to bet, that I should throw in the first toss a head once only, so my expectation would be much more inferior, than if I should meet 6 in the first time.

3. So that however, we may determine these cases, so is heads introduced through Δ, tails through 0. If now one tosses up *only one* Thaler, so only 2 cases are possible, namely one Δ and 0.

If one tosses up *two* Thalers, so 4 cases are possible, namely

$$\Delta\Delta,\ \Delta 0,\ 0\Delta,\ 00.$$

Amongst them $\Delta\Delta$ falls once, $\Delta 0$ twice and 00 once.

With 3 Thaler 8 cases are possible, namely

$$\left| \begin{array}{c|c|c|c} \Delta\Delta\Delta & \Delta 0 \Delta & 0 \Delta\Delta & 00\Delta \\ \Delta\Delta\, 0 & \Delta 0\, 0 & 0\Delta\, 0 & 00\, 0 \end{array} \right|$$

amongst them $\Delta\Delta\Delta$ falls once, $\Delta\Delta$ 3 times, Δ 3 times, 000 once.

With 4 Thaler 16 cases are possible, namely

$$\left| \begin{array}{c|c|c|c} \Delta\Delta\Delta\Delta & \Delta 0 \Delta\Delta & 0\Delta\Delta\Delta & 00\Delta\Delta \\ \Delta\Delta\Delta\, 0 & \Delta 0\Delta\, 0 & 0\Delta\Delta\, 0 & 00\Delta\, 0 \\ \Delta\Delta\, 0\, \Delta & \Delta 0\, 0\, \Delta & 0\Delta\, 0\, \Delta & 00\, 0\, \Delta \\ \Delta\Delta\, 0\ 0 & \Delta 0\, 0\ 0 & 0\Delta\, 0\ 0 & 00\, 0\ 0 \end{array} \right|$$

under these are $1{\cdot}\Delta\Delta\Delta\Delta$, $4{\cdot}\Delta\Delta\Delta$, $6{\cdot}\Delta\Delta$, $4{\cdot}\Delta$, $1{\cdot}0000$.

4. The possible cases are therefore

With Thalers	1	2	3	4	5	etc.
Heads 0	1	1	1	1	1	etc.
1	1	2	3	4	5	etc.
2		1	3	6	10	etc.
3			1	4	10	etc.
4				1	5	etc.
5					1	etc.
In all	2	4	8	16	32	etc.

5. We see from this table, that the number of all possible cases with 1, 2, 3, 4, 5, 6 etc, x Thalers, is the 1st, 2nd, 3rd, 4th, 5th, 6th etc., xth power of the number or root 2. Consequently, if there are x Thalers, so 2^x cases are possible.

6. We see furthermore from this, that, if one adds 2 numbers of a column standing under each other, the sum is a number of the following column. E.g. in the 3rd column are 3 and 3 under each other; the sum 6 is located in the 4th columne beside the lower 3.

7. It follows from this, that these numbers are found, if one raises $1 + 1$ to the power. E.g.

$$\begin{array}{lll} 2 = 1+1 & & \text{Cases for 1 Thaler} \\ \quad\ \ \underline{\ \ 1+1} & & \\ 4 = 1+2+1 & & \text{Cases for 2 Thaler} \\ \quad\ \ \underline{\ \ 1+2+1} & & \\ 8 = 1+3+3+1 & & \text{Cases for 3 Thaler} \\ \quad\ \ \underline{\ \ 1+3+3+1} & & \\ 16 = 1+4+6+4+1 & & \text{Cases for 4 Thaler} \\ \quad \text{etc.} & & \end{array}$$

8. This leads to the long well-known figurate numbers, according to the following small table, that one can easily continue and widen, so far as one wants.

0		1	1	1	1	1	1	1	1	1	1	etc.
1	$\frac{x}{1} =$	1	2	3	4	5	6	7	8	9	10	
2	$\frac{x}{1} \cdot \frac{x+1}{2} =$		1	3	6	10	15	21	28	36	45	
3	$\frac{x}{1} \cdot \frac{x+1}{2} \cdot \frac{x+2}{3} =$			1	4	10	20	35	56	84	120	
4	$\frac{x}{1} \cdot \frac{x+1}{2} \cdot \frac{x+2}{3} \cdot \frac{x+3}{4} =$				1	5	15	35	70	126	252	
5	$\frac{x}{1} \cdot \frac{x+1}{2} \cdot \frac{x+2}{3} \cdot \frac{x+3}{4} \cdot \frac{x+4}{5} =$					1	6	21	56	126	252	
6	$\cdots$						1	7	28	84	210	
7	$\cdots$							1	8	36	120	
8	$\cdots$								1	9	45	
9	$\cdots$									1	10	
10	$\cdots$										1	
	possible cases	2	4	8	16	32	64	128	256	512	1024	

9. The use of this table can be together with (§ 1) also. It is the question, how much I can bet if I should throw with 10 Thalers 6 heads on the first time? From the table I find that 1024 cases are possible with 10 Thalers, and that 6 heads can fall on the same kinds 210. My expectation is therefore $\frac{210}{1024} = \frac{105}{512}$, that is, somewhat more than $\frac{1}{5}$. Therefore I would have 1 to win, since the other has 4 to win. It therefore is necessary that I bet 1 Thaler against 4.