# RECHERCHES

SUR

# L'INTEGRATION DES ÉQUATIONS DIFFÉRENTIELLES

AUX DIFFÉRENCES FINIES

ET SUR

LEUR USAGE DANS LA THÉORIE DES HASARDS

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*Mémoires de l'Académie Royale des Sciences de Paris*
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I.

The first researches that one has made on the summation of arithmetic progressions and on geometric progressions contained the germ of the integral Calculus in finite differences in one and two variables; here is how: an arithmetic progression is a sequence of terms which increase equally, and it was necessary to find the sum according to this condition; it is clear that each term of the sequence is the finite difference of the sum of the preceding terms, to that same sum augmented by this term; one proposed therefore to find this sum according to the nature of its finite difference; thus by whatever manner that one is arrived there, one has truely integrated a quantity in the finite differences. The geometers who have come next have pushed further these researches; they have determined the sum of the squares and of the superior and entire powers of the natural numbers; they have arrived there first by some indirect methods: they did not perceive that that which they sought returned to finding a quantity of which the finite difference was known; but as soon as they had made this reflection, they have resolved directly, not only the cases already known, but many others more extended. In general, $\phi(x)$ representing any function whatsoever of the variable $x$, of which the finite difference

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. August 18, 2010 is supposed constant, they have proposed to find a quantity of which the finite difference is equal to that function, and this is the object of the integral Calculus in the finite differences in a single variable.

Similarly, the research of the general term of a geometric progression returns to finding the $x^{\text{th}}$ term of a sequence[1] such that each term is to the one which precedes it in constant ratio. Let $y_{x-1}$ be the $(x-1)^{\text{st}}$ term and $y_x$ be the $x^{\text{th}}$ term: the law of the sequence requires that one have $y_x = py_{x-1}$, whatever be $x$, $p$ being constant. Now it is clear that, in whatever manner that one is arrived to find $y_x$, one has veritably integrated the equation in the finite differences $y_x = py_{x-1}$. Next, one has generalized this research by proposing to find the general term of the sequences such that each of their terms is equal to many of the preceding multiplied by some constants any whatsoever; these sequences have been named for this *récurrentes*. One has arrived first to find their general term by some indirect ways, although quite ingenious; one did not perceive that this returned to integrating a linear equation in finite differences; but, when one had made this reflection, one tried to apply to these equations the methods known for the linear equations in the infinitely small differences, with the modifications that the assumption of finite differences requires, and one resolved in this manner some cases much more extended than those which were already.

Mr. Moivre is, I believe, the first who had determined the general term of the recurrent sequences; but Mr. de Lagrange is the first who is aware that this research depends on the integration of a linear equation in finite differences, and who had applied the good method of undetermined coefficients of Mr. d'Alembert (*see* Vol. I of the *Mémoires de Turin*). I myself have proposed next to deepen this interesting calculus, in a Memoir printed in Volume IV of those of Turin;[2] and next, having had occasion to reflect further there, I have made on this new researches of which I will render account shortly. I must observe here that Mr. the marquis de Condorcet has given excellent things on this matter, in his *Traité du Calcul intégral*, and in the *Mémoires de l'Académie*.

It was until then only a question of equations in ordinary finite differences and of the sequences which depend on them; but the solution of many problems on the chances has led me to a new kind of sequence which I have named *récurro-récurrentes*, and of which I believe to have given first the theory and indicated the usage in the Science of probabilities (*see* T. VI of *Savants étranges*.[3]) The equations on which these sequences depend are nearly, in the finite differences, that which the equations in the partial differences are in the infinitely small differences; that which I have given on these equations is only a trial: in deepening them, I have seen that they were quite important in the Theory of chances, and that they gave a method to treat them much more generally that one had done yet: this is that which engages me to consider them anew; but, the new researches that I have made on this object supposing those that I

[1] *Translator's note*: The word suite is used to refer to both a sequence and a series. It is rendered according to its usage.

[2] Recherches sur le calcul intégral aux differences infiniment petites, & aux différences finies. *Mélanges de philosophie et de mathématiques de la Société royale de Turin, pour les années 1766-1769 (Miscellanea Taurensia IV)*, 273-345, 1771.

[3] Mémoire sur les suites récurro-récurrentes et sur leur usages dans la théorie des hasards, *Mémoires de l'Académie Royale des Sciences de Paris (Savants étranges)* 6, 1774, p. 353-371.

have already given, I am going to begin again here all this matter.

II.

One can imagine thus the equations in finite differences; I imagine the sequence

$$y_1, y_2, y_3, y_4, y_5, \dots, y_x$$

formed following a law such as one has constantly

$$\text{(A)} \qquad X_x = M_x y_x + N_x \Delta y_x + P_x \Delta^2 y_x + \dots + S_x \Delta^n y_x;$$

the numbers 1, 2, 3, ..., $x$, placed at the base of $y$, indicating the rank which $y$ occupies in the sequence, or, that which returns to the same, the index of the series; the quantities $X_x$, $M_x$, $N_x$, ... are some functions any whatsoever of the variable $x$, of which the difference is supposed constant and equal to unity. The characteristic $\Delta$ serves to express the finite difference of the quantity before which it is placed, as in the infinitesimal Analysis the letter $d$ expresses the infinitely small difference of the quantities. This put, the preceding equation is an equation in finite differences, which can generally represent the equations of this kind, where the variable $y_x$ and its differences are under a linear form.

Although I have supposed the constant difference of $x$ equal to unity, this diminishes nothing from the generality of the preceding equation (A); because, if this difference, instead of being 1, is equal to $q$, one will make $\frac{x}{q} = x'$, and $y_x$ being a function of $x$ will become a function of $qx'$; I name $y_{x'}$ this last function. Now one has, by hypothesis,

$$\begin{aligned}\Delta y_x = y_{x+q} - y_x &= f(x+q) - f(x) \\ &= f[q(x'+a)] - f(qx') = y_{x'+1} - y_{x'} = \Delta y_{x'},\end{aligned}$$

the constant difference of $x'$ being 1. Similarly,

$$\Delta^2 y_x = y_{x+2q} - 2y_{x+q} + y_x = y_{x'+2} - 2y_{x'+1} + y_{x'} = \Delta^2 y_{x'},$$

and thus of the remaining. Equation (A) will be therefore transformed into the following

$$X_{x'} = M_{x'} y_{x'} + N_{x'} \Delta y_{x'} + \dots + S_{x'} \Delta^n y_{x'},$$

in which the difference of $x'$ is equal to unity.

One can form easily other differential equations, in which $y_x$ and its differences would enter in any manner whatsoever; but those which are contained in equation (A) are the only ones which it is truly interesting to consider.

Before researching to integrate them, I am going to recall here a principle quite useful in the analysis of the infinitely small differences, and which applies equally and with the same advantage to finite differences; here is in what it consists:

*Each function of $x$ which, containing $n$ arbitrary irreducible constants, satisfying for $y_x$ in a differential equation of order $n$, between $x$ and $y_x$, is the complete expression of $y_x$.*

By *irreducible constants*, I intend that they are such that two or many can not be reduced to one alone; it follows thence that, if a function containing $n$ irreducible arbitrary constants satisfy as $y_x$ in a differential equation of order $n-1$, this equation is surely identical; because, if it was not, the most general function of $x$ which was able to satisfy for $y_x$ would contain only $n-1$ irreducible arbitrary constants.

For the convenience of the calculus, I will suppose that the quantities noted in this manner, ${}^1H$, ${}^2H$, …, or ${}^1M$, ${}^2M$, …, express some different quantities and which can have no relation among themselves; but these here, $H_1$, $H_2$, $H_3$, …, $H_x$ or $M_1$, $M_2$, $M_3$, $ldots$, $M_x$ represent the different terms of a sequence formed according to one law any whatsoever, the numbers 1, 2, 3, …, $x$ designating the rank of the $H$ or of the $M$ in the sequence. This put, since one has

$$\begin{aligned}
&\Delta y_x = y_{x+1} - y_x,\\
&\Delta y^2 y_x = y_{x+2} - 2y_{x+1} + y_x,\\
&\Delta^3 y_x = y_{x+2} - 3y_{x+2} + 3y_{x+1} - y_x,\\
&\dots\dots\dots\dots\dots\dots ,
\end{aligned}$$

I am able to give to equation (A) this form

$$\begin{aligned}
X_x = &+y_x(M_x - N_x + P_x - \dots)\\
&+y_{x+1}(N_x - 2P_x + \dots)\\
&+\dots\dots\dots\dots\\
&+y_{x+n}S_x.
\end{aligned}$$

whence it results that each linear equation in finite differences can be generally represented by this here

$$\text{(B)} \qquad y_x = H_x y_{x-1} + {}^1H_x y_{x-2} + {}^2H_x y_{x-3} + \dots + {}^{n-1}H_x y_{x-n} + X_x;$$

the equation

$$y_x = H_x y_{x-1} + X_x$$

is of the first order, this here

$$y_x = H_x y_{x-1} + {}^1H_x y_{x-2} + X_x$$

is of the second order, and thus in sequence.

As in the series I will have need of characteristics in order to designate the finite difference of the quantities, their finite integrals, the product of all the terms of a sequence, I will serve myself for this with the following.

The characteristic $\Delta$ placed before a quantity will designate for it, as above, the finite difference: thus $\Delta H_x$ will express the finite difference of $H_x$; the characteristic $\Sigma$ placed before a quantity will designate for it the finite integral: thus $H_x$ will signify the finite integral of $H_x$; finally the characteristic $\nabla$ will designate the product of all the terms of a sequence: thus $\nabla H_x$ will represent the product $H_1H_2H_3\dots H_x$ of all the terms of the sequence $H_1, H_2, H_3, \dots, H_x$.

## III.

PROBLEM I. — *The differential equation of the first order*

$$y_x = H_x y_{x-1} + X_x$$

*being given, one proposes to integrate it.*

I make in this equation $y_x = u_x \nabla H_x$; it becomes

$$u_x \nabla H_x = H_x u_{x-1} \nabla H_{x-1} + X_x;$$

but one has

$$H_x \nabla H_{x-1} = \nabla H_x,$$

hence

$$u_x = u_{x-1} + \frac{X_x}{\nabla H_x} \qquad \text{or} \qquad \Delta u_{x-1} = \frac{X_x}{\nabla H_x};$$

and, as this equation holds whatever be $x$, one will have

$$\Delta u_x = \frac{X_{x+1}}{\nabla H_{x+1}},$$

hence, by integrating,

$$u_x = A + \sum \frac{X_{x+1}}{\nabla H_{x+1}},$$

$A$ being an arbitrary constant. One has therefore

$$y_x = \nabla H_x \left( A + \sum \frac{X_{x+1}}{\nabla H_{x+1}} \right).$$

If $H_x$ was constant and equal to $p$, one would have

$$\nabla H_x = p^x \qquad \text{and} \qquad y_x = p^x \left( A + \sum \frac{X_{x+1}}{p^{x+1}} \right).$$

## IV.

PROBLEM II. — *The differentio-differential equation*

(B) $$y_x = H_x y_{x-1} + {}^1H_x y_{x-2} + {}^2H_x y_{x-3} + \ldots + {}^{n-1}H_x y_{x-n} + X_x$$

*being given, one proposes to integrate it.*

I make

(C) $$y_x = \alpha_x y_{x-1} + T_x,$$

$\alpha_x$ and $T_x$ being two new variables, and I conclude from it the following equations:

$$\begin{aligned} y_{x-1} &= \alpha_{x-1} y_{x-2} + T_{x-1}, \\ y_{x-2} &= \alpha_{x-2} y_{x-3} + T_{x-2}, \\ y_{x-3} &= \alpha_{x-3} y_{x-4} + T_{x-3}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ y_{x-n+1} &= \alpha_{x-n+1} y_{x-n} + T_{x-n+1}; \end{aligned}$$

I multiply the first of these equations by $-{}^1\beta$, the second by $-{}^2\beta$, the third by $-{}^3\beta$, … and I add them with equation (C): this which gives me

$$\begin{aligned} y_x =&(\alpha_x + {}^1\beta)y_{x-1} + (-{}^1\beta\alpha_{x-1} + {}^2\beta)y_{x-2} \\ &+ (-{}^2\beta\alpha_{x-2} + {}^3\beta)y_{x-3} + \ldots - {}^{n-1}\beta\alpha_{x-n+1}y_{x-n} \\ &+ T_x - {}^1\beta T_{x-1} - {}^2\beta T_{x-2} - \ldots - {}^{n-1}\beta T_{x-n+1}. \end{aligned}$$

By comparing this equation with equation (B), one will have

1°

$$T_x = {}^1\beta T_{x-1} + {}^2\beta T_{x-2} + \ldots + {}^{n-1}\beta T_{x-n+1} + X_x;$$

2° The following equations:

$$\begin{aligned} {}^1\beta + \alpha_x =& H_x, \\ {}^2\beta - {}^1\beta\alpha_{x-1} =& {}^1H_x, \\ {}^3\beta - {}^2\beta\alpha_{x-2} =& {}^2H_x \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ -{}^{n-1}\beta\alpha_{x-n+1} =& {}^{n-1}H_x. \end{aligned}$$

Thence one will conclude

$$\begin{aligned} {}^1\beta =& H_x - \alpha_x, \\ {}^2\beta =& {}^1H_x + \alpha_{x-1}H_x - \alpha_x\alpha_{x-1}, \\ {}^3\beta =& {}^2H_x + \alpha_{x-2}{}^1H_x + \alpha_{x-1}\alpha_{x-2}H_x - \alpha_x\alpha_{x-1}\alpha_{x-2}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ {}^{n-1}\beta =& {}^{n-2}H_x + \alpha_{x-n+2}{}^{n-3}H_x + \alpha_{x-n+3}\alpha_{x-n+2}{}^{n-4}H_x + \ldots \\ & - \alpha_x\alpha_{x-1}\ldots\alpha_{x-n+2} = -\frac{{}^{n-1}H_x}{\alpha_{x-n+1}}, \end{aligned}$$

because of the equation

$$-{}^{n-1}\beta\alpha_{x-n+1} = {}^{n-1}H_x;$$

one will have therefore, in order to resolve the problem, the following two equations:

$$\text{(D)} \qquad \begin{cases} T_x =(H_x - \alpha_x)T_{x-1} + ({}^1H_x + \alpha_{x-1}H_x - \alpha_x\alpha_{x-1})T_{x-2} + \ldots \\ \qquad - \dfrac{{}^{n-1}H_x}{\alpha_{x-n+1}}T_{x-n+1} + X_x, \end{cases}$$

$$\text{(E)} \qquad 0 = t - \frac{H_x}{\alpha_x} - \frac{{}^1H_x}{\alpha_x\alpha_{x-1}} - \frac{{}^2H_x}{\alpha_x\alpha_{x-1}\alpha_{x-2}} - \ldots - \frac{{}^{n-1}H_x}{\alpha_x\ldots\alpha_{x-n+1}}.$$

Equations (D) and (E) are of a degree inferior to the proposed, and equation (D) is of the same form; now it is not necessary to integrate generally these equations in order to integrate equation (B) of the problem; it suffices to know for $\alpha_x$ a quantity which satisfies equation (E). I name $\delta_x$ this value; one will substitute it into equation (D), which I name (D′) after this substitution, and one will seek the complete integral of equation (D′); next, by means of the equation $y_x = \delta_x y_{x-1} + T_x$, one will conclude, by integrating by problem I,

$$y_x = \nabla\delta_x \left(A + \sum \frac{T_{x+1}}{\nabla\delta_{x+1}}\right),$$

$A$ being an arbitrary constant.

This equation is the complete integral of equation (B), because, equation (D′) being necessarily of order $n-1$, the complete expression of $T_x$ contains $n-1$ irreducible arbitrary constants; hence, $\nabla\delta_x \left(A + \sum \frac{T_{x+1}}{\nabla\delta_{x+1}}\right)$ contains $n$ arbitrary constants. These constants are moreover irreducibles, because $\nabla\delta_x \sum \frac{T_{x+1}}{\nabla\delta_{x+1}}$ contains in it $n-1$ irreducibles, and none of them is reducible with the constant $A$.

The preceding expression of $y_x$ can serve to make known the integral of equation (B) of the problem; because, since equation (D′) is linear, one can suppose that the expression of $T_x$ has this form

$$T_x = \nabla\lambda_x \left({}^1A + \sum \frac{{}^1T_{x+1}}{\nabla\lambda_{x+1}}\right),$$

${}^1T_x$ depending on the integration of a linear equation of order $n-2$; one has therefore

$$y_x = \nabla\delta_x \left[A + {}^1A \sum \frac{\nabla\lambda_{x+1}}{\nabla\delta_{x+1}} + \sum \frac{\sum \frac{{}^1T_{x+1}}{\nabla\lambda_{x+1}}}{\nabla\delta_{x+1}}\right];$$

by continuing to reason thus, one will see that the expression of $y_x$ is of this form

$$y_x = A\nabla\delta_x + {}^1A\nabla{}^1\delta_x + {}^2A\nabla{}^2\delta_x + \ldots + {}^{n-1}A\nabla{}^{n-1}\delta_x + L_x,$$

$A$, ${}^1A$, ${}^2A, \ldots$ being arbitrary.

If one supposes $X_x = 0$ in equation (B), it is easy to see, by the sequence of operations that I just indicated, that $L_x$ will be null; thus, in this case

$$y_x = A\nabla\delta_x + {}^1A\nabla{}^1\delta_x + \ldots + {}^{n-1}A\nabla{}^{n-1}\delta_x,$$

$\delta_x$ satisfying under the assumption for $\alpha_x$ in equation (E); ${}^1\delta_x$, ${}^2\delta_x$, $\ldots$ will satisfy similarly; because, since the equation $y_x = A\nabla{}^1\delta_x$, for example, satisfies equation (B) by supposing $X = 0$, one will have

$$\nabla{}^1\delta_x = H_x\nabla{}^1\delta_{x-1} + {}^1H_x\nabla{}^1\delta_{x-2} + \ldots,$$

hence

$$0 = 1 - \frac{H_x}{{}^1\delta_x} - \frac{{}^1H_x}{{}^1\delta_x\,{}^1\delta_{x-1}} - \cdots$$

## V.

I suppose, in equations (D′) and (B), $X_x = 0$; I will have the following two expressions of $y_x$:

$$y_x = \nabla\delta_x \left( A + \sum \frac{T_{x+1}}{\nabla\delta_{x+1}} \right), \tag{1}$$

$$y_x = A\nabla\delta_x + {}^1A\nabla^1\delta_x + {}^2A\nabla^2\delta_x + \ldots + {}^{n-1}A\nabla^{n-1}\delta_x. \tag{2}$$

These two expressions, different in appearance, must really coincide; I suppose therefore that the complete integral of equation (D′) is

$$T_x = {}^1AR_x + {}^2A\,{}^1R_x + \ldots + {}^{n-1}A\,{}^{n-2}R_x;$$

by substituting this value of $T_x$ into equation (1), one will have

$$y_x = \nabla\delta_x \left( A + {}^1A\frac{R_{x+1}}{\nabla\delta_{x+1}} + {}^2A\frac{{}^1R_{x+1}}{\nabla\delta_{x+1}} + \ldots + {}^{n-1}A\frac{{}^{n-2}R_{x+1}}{\nabla\delta_{x+1}} \right).$$

By comparing this last equation with equation (2), one will have

$$\nabla\delta_x \sum \frac{R_{x+1}}{\nabla\delta_{x+1}} = \nabla^1\delta_x,$$

$$\nabla\delta_x \sum \frac{{}^1R_{x+1}}{\nabla\delta_{x+1}} = \nabla^2\delta_x,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Therefore

$$R_x = \nabla\delta_x\Delta\frac{\nabla^1\delta_{x-1}}{\nabla\delta_{x-1}},$$

$${}^1R_x = \nabla\delta_x\Delta\frac{\nabla^2\delta_{x-1}}{\nabla\delta_{x-1}},$$

$${}^2R_x = \nabla\delta_x\Delta\frac{\nabla^3\delta_{x-1}}{\nabla\delta_{x-1}},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Therefore, if I know how to resolve equation (B) by supposing $X_x = 0$, I will know how to resolve equation (D′) by supposing similarly $X_x = 0$. Let therefore $u_x$, ${}^1u_x$, ${}^2u_x, \ldots$ be the particular values of $y_x$ in equation (B), so that its complete integral is

$$y_x = Au_x + {}^1A\,{}^1u_x + {}^2A\,{}^2u_x + \ldots + {}^{n-1}A\,{}^{n-1}u_x,$$

one will have

$$u_x = \nabla\delta_x, \qquad {}^1u_x = \nabla^1\delta_x, \qquad \ldots,$$

and the complete integral of equation (D′), by supposing $X_x = 0$ in it, will be

$$T_x = {}^1Au_x\Delta\frac{\overset{1}{u}_{x-1}}{u_{x-1}} + {}^2Au_x\Delta\frac{\overset{2}{u}_{x-1}}{u_{x-1}} + \ldots + {}^{n-1}Au_x\Delta\frac{\overset{n-1}{u}_{x-1}}{u_{x-1}}.$$

Presently, if I know how to integrate equation (D′) by supposing $X_x$ anything, I will be able, under the same assumption, to integrate equation (B), since one has, by that which precedes,

$$y_x = u_x\left(A + \sum\frac{T_{x+1}}{u_{x+1}}\right);$$

therefore the difficulty to integrate the equation

(B) $$y_x = H_x y_{x-1} + {}^1H_x y_{x-2} + \ldots + {}^{n-1}H_x y_{x-n} + X_x,$$

when one knows how to integrate this one

($b$) $$y_x = H_x y_{x-1} + {}^1H_x y_{x-2} + \ldots + {}^{n-1}H_x y_{x-n},$$

is reduced to integrate the equation

(D′) $$T_x = (H - \delta_x)T_{x-1} + \ldots - \frac{{}^{n-1}H_x}{\delta_{x-n+1}}T_{x-n+1} + X_x,$$

which is of degree $n-1$, and when one knows how to integrate by supposing $X_x = 0$; one will make similarly the integration of (D′) to depend on the integration of an equation of degree $n-2$, and thus in sequence; whence there results that the equation

$$y_x = H_x y_{x-1} + {}^1H_x y_{x-2} + \ldots + {}^{n-1}H_x y_{x-n} + X_x$$

is integrable in the same cases as this one

$$y_x = H_x y_{x-1} + \ldots + {}^{n-1}H_x y_{x-n}.$$

## VI.

The process which I just indicated in order to restore the integral of equation (B) to that of equation ($b$) can serve to demonstrate the liaison which these two integrals have between them; but it would be quite painful to employ it to integrate equation (B). It would be therefore very useful to have immediately the general expression of $y_x$ in equation (B), when one has that of equation ($b$).

I take for this equation

$$y_x = u_x\left(A + \sum\frac{T_{x+1}}{u_{x+1}}\right),$$

$T_x$ being supposed to be the complete expression of $T_x$ in equation (D′). Now, this equation (D′) being of the same form as equation (B), if one names $\overset{1}{u}_x$, ${}^1\overset{1}{u}_x$, ${}^2\overset{1}{u}_x, \ldots$ the particular integrals of $T_x$ in equation (D′), when one supposes $X_x = 0$ there, one will have, in the same manner and whatever be $X_x$,

$$T_x = \overset{1}{u}_x \left( {}^1A + \sum \frac{{}^1T_{x+1}}{\overset{1}{u}_{x+1}} \right),$$

${}^1T_x$ being the complete expression of ${}^1T_x$ in an equation of order $n-2$, which I name (D″) and which results from (D′) in the same manner as this one results from equation (B); one will have similarly

$${}^1T_x = \overset{2}{u}_x \left( {}^2A + \sum \frac{{}^2T_{x+1}}{\overset{2}{u}_{x+1}} \right),$$

and thus in sequence until one arrives to the equation of the first order

$${}^{n-2}T_x = S_x\, {}^{n-2}T_{x-1} + X_x,$$

of which the integral is

$${}^{n-2}T_x = \overset{n-1}{u}_x \left( {}^{n-1}A + \sum \frac{X_{x+1}}{\overset{n-1}{u}_{x+1}} \right).$$

If one substitutes presently into the expression of $y_x$ the value of $T_x$ into ${}^1T_x$, that of ${}^1T_x$ into ${}^2T_x$, etc., one will have

$$\text{(K)} \quad y_x = u_x \left\{ A + \sum \frac{\overset{1}{u}_{x+1}}{u_{x+1}} \left( {}^1A + \sum \frac{\overset{2}{u}_{x+1}}{\overset{1}{u}_{x+1}} \left[ {}^2A \ldots + \sum \frac{\overset{n-1}{u}_{x+n-1}}{\overset{n-2}{u}_{x+n-1}} \left( {}^{n-1}A + \sum \frac{X_{x+n}}{\overset{n-1}{u}_{x+n}} \right) \cdots \right] \right) \right\}.$$

It is necessary presently to determine $\overset{1}{u}_x, \overset{2}{u}_x, \ldots$; now one has, by the previous Article,

$$\overset{1}{u}_x = R_x = u_x \Delta \frac{{}^1u_{x-1}}{u_{x-1}},$$

similarly

$${}^1\overset{1}{u}_x = u_x \Delta \frac{{}^2u_{x-1}}{u_{x-1}},$$

$${}^2\overset{1}{u}_x = u_x \Delta \frac{{}^3u_{x-1}}{u_{x-1}},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots;$$

one will have likewise

$$\overset{2}{u}_x = \overset{1}{u}_x \Delta \frac{{}^{1}\overset{1}{u}_{x-1}}{\overset{1}{u}_{x-1}},$$

$$ {}^{1}\overset{2}{u}_x = \overset{1}{u}_x \Delta \frac{{}^{2}\overset{1}{u}_{x-1}}{\overset{1}{u}_{x-1}},$$

$$ {}^{2}\overset{2}{u}_x = \overset{1}{u}_x \Delta \frac{{}^{3}\overset{1}{u}_{x-1}}{\overset{1}{u}_{x-1}},$$

$$\dots\dots\dots\dots\dots\dots;$$

formula (K) will become

$$\text{(O)}\quad y_x = u_x \left\{ A + \sum \Delta \frac{{}^{1}u_x}{u_x} \left( {}^{1}A + \sum \Delta \frac{{}^{1}\overset{1}{u}_{x+1}}{\overset{1}{u}_{x+1}} \left[ {}^{2}A \dots + \sum \Delta \frac{{}^{1}\overset{n-2}{u}_{x+n-2}}{\overset{n-2}{u}_{x+n-2}} \left( {}^{n-1}A + \sum \frac{X_{x+n}}{\overset{n-1}{u}_{x+n}} \right) \cdots \right] \right) \right\};$$

if one knows only the number $n-1$ of particular integrals of $y_x$, in the equation

$$y_x = H_x y_{x-1} + {}^{1}H_x y_{x-2} + \dots + {}^{n-1}H_x y_{x-n},$$

the integration will be of difficulty no longer; I suppose that this is the integral ${}^{n-1}u_x$ which is unknown; since one knows $u_x$, ${}^{1}u_x$, …, ${}^{n-2}u_x$, one will know $\overset{1}{u}_x$, $\overset{2}{u}_x$, …until $\overset{n-1}{u}_x$ exclusively. In order to determine $\overset{n-1}{u}_x$, it is necessary to integrate the equation

$${}^{n-2}T_x = S_x\, {}^{n-2}T_{x-1} + X_x,$$

by supposing $X_x = 0$, this which will be easy by Problem I if one knows $S_x$. In order to find it, I observe that, in equation (D′), the coefficient of $T_{x-1}$ is

$$H_x - \delta_x = H_x - \frac{u_x}{u_{x-1}},$$

because of

$$\delta_x = \frac{u_x}{u_{x-1}}.$$

Similarly the one of ${}^{1}T_{x-1}$, in equation (D″), is

$$H_x - \frac{u_x}{u_{x-1}} - \frac{\overset{1}{u}_x}{\overset{1}{u}_{x-1}},$$

and thus in sequence; hence,

$$S_x = H_x - \frac{u_x}{u_{x-1}} - \frac{\overset{1}{u}_x}{\overset{1}{u}_{x-1}} - \dots - \frac{\overset{n-2}{u}_x}{\overset{n-2}{u}_{x-1}}.$$

If, instead of knowing the integral of the equation

$$y_x = H_x y_{x-1} + \ldots + {}^{n-1}H_x y_{x-n},$$

one knows a number $n$ or $n-1$ of values for $\alpha_x$, in equation (E), the preceding formulas will serve equally, because $\delta_x$, ${}^1\delta_x$, … being these values, one has

$$u_x = \nabla \delta_x, \qquad {}^1u_x = \nabla\, {}^1\delta_x, \qquad \ldots$$

VII.

Formula (O) has not at all yet the total degree of simplicity that the complete integral of $y_x$ can have, because one has seen (Art. IV) that this integral has the following form

$$y_x = Au_x + {}^1A\, {}^1u_x + \ldots + {}^{n-1}A\, {}^{n-1}u_x + L_x;$$

it is necessary therefore to restore equation (O) to this form; for this, I divide equation (O) by $u_x$, and I conclude from it, by differentiating it,

$$\Delta \frac{y_{x-1}}{u_{x-1}} = \Delta \frac{{}^1u_{x-1}}{u_{x-1}} \left\{ {}^1A + \sum \Delta \frac{{}^1\overset{1}{u}_x}{\overset{1}{u}_x} \left[ {}^2A \ldots + \sum \Delta \frac{{}^1\overset{n-2}{u}_{x+n-3}}{\overset{n-2}{u}_{x+n-3}} \left( {}^{n-1}A + \sum \frac{X_{x+n-1}}{\overset{n-1}{u}_{x+n-1}} \right) \ldots \right] \right\},$$

whence one will conclude, by dividing by $\Delta \frac{{}^1u_{x-1}}{u_{x-1}}$ and differentiating,

$$\Delta \frac{\Delta \frac{y_{x-2}}{u_{x-2}}}{\Delta \frac{{}^1u_{x-2}}{u_{x-2}}} = \Delta \frac{{}^1\overset{1}{u}_{x-1}}{\overset{1}{u}_{x-1}} [{}^2A + \ldots].$$

One will have therefore, by continuing to differentiate thus, an equation of this form

$${}^{n-1}A + \sum \frac{X_{x-1}}{\overset{n-1}{u}_{x-1}} = \gamma_x y_x + {}^1\gamma_x y_{x-1} + {}^2\gamma_x y_{x-2} + \ldots + {}^{n-1}\gamma_x y_{x-n+1},$$

$\gamma_x$, ${}^1\gamma_x, \ldots$ being some functions of $u_x$, ${}^1u_x, \ldots$ and of their finite differences. I observe now that, in order to form the values of $\overset{1}{u}_x, \overset{2}{u}_x, \overset{3}{u}_x, \ldots$, I have considered (preceding Article) the quantities $u_x$, ${}^1u_x$, ${}^2u_x, \ldots$ in this order

$$u_x,\ {}^1u_x,\ {}^2u_x, \ldots, {}^{n-1}u_x;$$

but if, instead of that, I had considered them in the following order

$${}^1u_x, u_x,\ {}^2u_x, \ldots, {}^{n-1}u_x,$$

I would arrive to the following equation

$${}^{n-1}A + \sum \frac{X_{x+1}}{\left(\overset{n-1}{u}_{x+1}\right)} = (\gamma_x) y_x + ({}^1\gamma_x) y_{x-1} + \ldots + ({}^{n-1}\gamma_x) y_{x-n+1},$$

$\left(\overset{n-1}{u}_x\right)$, $(\gamma_x)$, … being that which $\overset{n-1}{u}_x$, $\gamma_x$, … become when one changes $u_x$ into ${}^1u_x$, and ${}^1u_x$ into $u_x$. If I had supposed $X_{x+1}=0$, I would have arrived to the two equations

$$ {}^{n-1}A = \gamma_x y_x + {}^1\gamma_x y_{x-1} + \ldots + {}^{n-1}\gamma_x y_{x-n+1}, $$
$$ {}^{n-1}A = (\gamma_x)y_x + ({}^1\gamma_x)y_{x-1} + \ldots + ({}^{n-1}\gamma_x)y_{x-n+1}, $$

in which the constant ${}^{n-1}A$ is clearly the same, since I have supposed, in order to form the one and the other equation, that the complete value of $y_x$ is

$$ y_x = Au_x + {}^1A\,{}^1u_x + \ldots + {}^{n-1}A\,{}^{n-1}u_x. $$

One will have therefore, by comparing these two equations,

$$ \begin{aligned} &\gamma_x y_x + {}^1\gamma_x y_{x-1} + \ldots + {}^{n-1}\gamma_x y_{x-n+1} \\ &\quad = (\gamma_x)y_x + ({}^1\gamma_x)y_{x-1} + \ldots + ({}^{n-1}\gamma_x)y_{x-n+1}, \end{aligned} $$

an equation which must be an identity; because, if it were not, this equation being differential of order $n-1$ would have however for the complete integral

$$ y_x = Au_x + \ldots + {}^{n-1}A\,{}^{n-1}u_x, $$

an equation which contains $n$ arbitrary constants, this which would be absurd (Art. II).

One has therefore

$$ {}^{n-1}A + \sum \frac{X_{x+1}}{\left(\overset{n-1}{u}_{x+1}\right)} = {}^{n-1}A + \sum \frac{X_{x+1}}{\overset{n-1}{u}_{x+1}}, $$

hence

$$ \left(\overset{n-1}{u}_{x+1}\right) = \overset{n-1}{u}_{x+1}. $$

Thus the expression of $\overset{n-1}{u}_x$ remains always the same, whether one changes $u_x$ into ${}^1u_x$, and ${}^1u_x$ into $u_x$; one will be assured in the same manner that if in $\overset{n-1}{u}_x$ one changes $u_x$ into ${}^2u_x$, and ${}^2u_x$ into $u_x$; or ${}^1u_x$ into ${}^2u_x$, and ${}^2u_x$ into ${}^1u_x$, and generally ${}^ku_x$ into ${}^iu_x$, and ${}^iu_x$ into ${}^ku_x$, $k$ and $i$ being less than $n-1$, the expression $\overset{n-1}{u}_x$ will always remain the same, and that thus, whatever order that one gives to the quantities $u_x$, ${}^1u_x$, ${}^2u_x, \ldots$ in order to form $\overset{n-1}{u}_x$, this expression will remain always the same, provided that ${}^{n-1}u_x$ is considered as the last of these quantities.

I make $\overset{n-1}{u}_{x+1} = {}^{n-1}z_{x+1}$; next, instead of considering ${}^{n-1}u_x$ as the last of the quantities $u_x$, ${}^1u_x, \ldots$ I suppose actually that ${}^{n-2}u_x$ is this last; let ${}^{n-2}z_{x+1}$ be that which becomes then ${}^{n-1}z_{x+1}$, that is to say when one changes ${}^{n-2}u_x$ into ${}^{n-1}u_x$, and ${}^{n-1}u_x$ into ${}^{n-2}u_x$. One will have, by a process similar to the preceding,

$$ {}^{n-2}A + \sum \frac{X_{x+1}}{{}^{n-2}z_{x+1}} = \underline{\gamma}_x y_x + {}^1\underline{\gamma}_x y_{x-1} + \ldots + {}^{n-1}\underline{\gamma}_x y_{x-n+1}, $$

$\underline{\gamma}$, ${}^1\underline{\gamma}_x$, … being that which $\gamma_x$, ${}^1\gamma_x$, … become when one changes ${}^{n-1}u_x$ into ${}^{n-2}u_x$ and ${}^{n-2}u_x$ into ${}^{n-1}u_x$; one will have similarly

$$ {}^{n-3}A + \sum \frac{X_{x+1}}{{}^{n-3}z_{x+1}} = \underline{\underline{\gamma}}_x y_x + {}^1\underline{\underline{\gamma}}_x y_{x-1} + \ldots + {}^{n-1}\underline{\underline{\gamma}}_x y_{x-n+1}, $$

${}^{n-3}z_{x+1}$, $\underline{\underline{\gamma}}_x$, ${}^1\underline{\underline{\gamma}}_x$ being that which ${}^{n-1}z_{x+1}$, $\gamma_x$, ${}^1\gamma_x$, … become when one changes ${}^{n-1}u_x$ into ${}^{n-3}u_x$ and ${}^{n-2}u_x$ into ${}^{n-1}u_x$. This set, by disposing in the following order all the equations that one can form thus

$$ (>) \quad \begin{cases} {}^{n-1}A + \sum \dfrac{X_{x+1}}{{}^{n-1}z_{x+1}} = \gamma_x y_x + {}^1\gamma_x y_{x-1} + {}^2\gamma_x y_{x-2} + \ldots + {}^{n-1}\gamma_x y_{x-n+1}, \\ {}^{n-2}A + \sum \dfrac{X_{x+1}}{{}^{n-2}z_{x+1}} = \underline{\gamma}_x y_x + {}^1\underline{\gamma}_x y_{x-1} + {}^2\underline{\gamma}_x y_{x-2} + \ldots + {}^{n-1}\underline{\gamma}_x y_{x-n+1}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ A + \sum \dfrac{X_{x+1}}{z_{x+1}} = \dfrac{\gamma_x}{n-1} y_x + \dfrac{{}^1\gamma_x}{n-1} y_{x-1} + \dfrac{{}^2\gamma_x}{n-1} y_{x-2} + \ldots + \dfrac{{}^{n-1}\gamma_x}{n-1} y_{x-n+1}, \end{cases} $$

and adding them altogether, after having multiplied the first by ${}^{n-1}u_x$, the second by ${}^{n-2}u_x$, etc., finally the last by $u_x$, one will have an equation of this form

$$ \begin{aligned} \lambda_x y_x + \ldots + {}^{n-1}\lambda_x y_{x-n+1} = & u_x \left( A + \sum \frac{X_{x+1}}{z_{x+1}} \right) \\ & + {}^1u_x \left( {}^1A + \sum \frac{X_{x+1}}{{}^1z_{x+1}} \right) \\ & + \ldots\ldots\ldots\ldots\ldots\ldots \\ & + {}^{n-1}u_x \left( {}^{n-1}A + \sum \frac{X_{x+1}}{{}^{n-1}z_{x+1}} \right), \end{aligned} $$

this which gives, by making $X_{x+1} = 0$,

$$ \lambda_x y_x + {}^1\lambda_x y_{x-1} + \ldots + {}^{n-1}\lambda_x y_{x-n+1} = A u_x + {}^1A\, {}^1u_x + \ldots + {}^{n-1}A\, {}^{n-1}u_x; $$

but one has in this case

$$ y_x = A u_x + {}^1A\, {}^1u_x + \ldots, $$

hence

$$ y_x = \lambda_x y_x + {}^1\lambda_x y_{x-1} + \ldots + {}^{n-1}\lambda_x y_{x-n+1}. $$

Now this equation must be an identity, because otherwise, although of order $n-1$, its integral would contain the $n$ arbitrary constants which the complete expression of $y_x$ contains; one has therefore for the complete integral of equation (B) of Problem II, whatever be $X_x$,

$$\begin{aligned} y_x = & u_x\left(A+\sum\frac{X_{x+1}}{z_{x+1}}\right)\\ & + {}^{1}u_x\left({}^{1}A+\sum\frac{X_{x+1}}{{}^{1}z_{x+1}}\right)\\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ & + {}^{n-1}u_x\left({}^{n-1}A+\sum\frac{X_{x+1}}{{}^{n-1}z_{x+1}}\right), \end{aligned}$$

Thence results this quite simple rule, in order to have the complete integral of the equation

$$y_x = H_x y_{x-1} + {}^{1}H_x y_{x-2} + \ldots + {}^{n-1}H_x y_{x-n} + X_x,$$

when one knows how to integrate this here

$$y_x = H_x y_{x-1} + {}^{1}H_x y_{x-2} + \ldots + {}^{n-1}H_x y_{x-n}.$$

Let

$$y_x = Au_x + {}^{1}A\,{}^{1}u_x + {}^{2}A\,{}^{2}u_x + \ldots + {}^{n-1}A\,{}^{n-1}u_x$$

be the integral of this last, and let one make

$$\begin{array}{lll} \overset{1}{u}_x = u_x\Delta\dfrac{{}^{1}u_{x-1}}{u_{x-1}}, & \overset{2}{u}_x = \overset{1}{u}_x\Delta\dfrac{{}^{1}\overset{1}{u}_{x-1}}{\overset{1}{u}_{x-1}}, & \overset{3}{u}_x = \overset{1}{u}_x\Delta\dfrac{{}^{1}\overset{2}{u}_{x-1}}{\overset{2}{u}_{x-1}},\\ {}^{1}\overset{1}{u}_x = u_x\Delta\dfrac{{}^{2}u_{x-1}}{u_{x-1}}, & {}^{1}\overset{2}{u}_x = \overset{1}{u}_x\Delta\dfrac{{}^{2}\overset{1}{u}_{x-1}}{\overset{1}{u}_{x-1}}, & \ldots\ldots\ldots\ldots\ldots\ldots,\\ {}^{2}\overset{1}{u}_x = u_x\Delta\dfrac{{}^{3}u_{x-1}}{u_{x-1}}, & {}^{2}\overset{2}{u}_x = \overset{1}{u}_x\Delta\dfrac{{}^{3}\overset{1}{u}_{x-1}}{\overset{1}{u}_{x-1}}, & \\ \ldots\ldots\ldots\ldots\ldots\ldots, & \ldots\ldots\ldots\ldots\ldots\ldots, & \end{array}$$

until one arrives to form $\overset{n-1}{u}_x$, let $\overset{n-1}{u}_x = {}^{n-1}z_x$. If, in the expression of ${}^{n-1}z_x$, one changes ${}^{n-1}u_x$ into ${}^{n-2}u_x$ and ${}^{n-2}u_x$ into ${}^{n-1}u_x$, one will form ${}^{n-2}z_x$; if, in the same expression of ${}^{n-1}z_x$, one changes ${}^{n-1}u_x$ into ${}^{n-3}u_x$, and reciprocally ${}^{n-3}u_x$ into ${}^{n-1}u_x$, one will form ${}^{n-3}z_x$, and thus in sequence; the complete integral of equation

(B) $$y_x = H_x y_{x-1} + {}^{1}H_x y_{x-2} + \cdots + {}^{n-1}H_x y_{x-n} + X_x$$

will be

$$(H)\quad \begin{cases} y_x = u_x\left(A + \sum \frac{X_{x+1}}{z_{x+1}}\right) \\ \quad + {}^1u_x\left({}^1A + \sum \frac{X_{x+1}}{{}^1z_{x+1}}\right) \\ \quad + \ldots\ldots\ldots\ldots\ldots\ldots \\ \quad + {}^{n-1}u_x\left({}^{n-1}A + \sum \frac{X_{x+1}}{{}^{n-1}z_{x+1}}\right). \end{cases}$$

VIII.

I take now the equations ($>$) of the preceding Article; they give

$$ {}^{n-1}A + \sum \frac{X_{x+2}}{{}^{n-1}z_{x+2}} = \gamma_{x+1}y_{x+1} + \ldots + {}^{n-1}\gamma_{x+1}y_{x-n+2}, $$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$ A + \sum \frac{X_{x+2}}{z_{x+2}} = \frac{\gamma_{x+1}}{n-1}y_{x+1} + \ldots + \frac{{}^{n-1}\gamma_{x+1}}{n-1}y_{x-n+2}; $$

if one multiplies the first by ${}^{n-1}u_x$, the second by ${}^{n-2}u_x$, ..., one will have, by adding them together, an equation of this form

$$\lambda_x y_{x+1} + {}^1\lambda_x y_{x+2} + \ldots + {}^{n-1}\lambda_x y_{x-n+2} = Au_x + {}^1A\,{}^1u_x + \ldots + {}^{n-1}A\,{}^{n-1}u_x;$$

therefore

$$\lambda_x y_{x+1} + {}^1\lambda_x y_{x+2} + \ldots + {}^{n-1}\lambda_x y_{x-n+2} = y_x,$$

an equation which must be an identity; hence,

$$\begin{aligned} y_x = {} & u_x\left(A + \sum \frac{X_{x+2}}{z_{x+2}}\right) \\ & + {}^1u_x\left({}^1A + \sum \frac{X_{x+2}}{{}^1z_{x+2}}\right) \\ & + \ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

One will find similarly

$$\begin{aligned} y_x = {} & u_x\left(A + \sum \frac{X_{x+3}}{z_{x+3}}\right) \\ & + {}^1u_x\left({}^1A + \sum \frac{X_{x+3}}{{}^1z_{x+3}}\right) \\ & + \ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

and thus in sequence until one arrives to this last equation inclusively,

$$\begin{aligned} y_x = & u_x \left( A + \sum \frac{X_{x+n}}{z_{x+n}} \right) \\ & + {}^1u_x \left( {}^1A + \sum \frac{X_{x+n}}{{}^1z_{x+n}} \right) \\ & + \ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

All these equations being the complete integral of equation (B) are identically the same; in comparing them together, one will form the following equations

$$\frac{u_x}{z_{x+1}} + \frac{{}^1u_x}{{}^1z_{x+1}} + \ldots + \frac{{}^{n-1}u_x}{{}^{n-1}z_{x+1}} = 0.$$

$$\frac{u_x}{z_{x+2}} + \frac{{}^1u_x}{{}^1z_{x+2}} + \ldots + \frac{{}^{n-1}u_x}{{}^{n-1}z_{x+2}} = 0.$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\frac{u_x}{z_{x+n-1}} + \frac{{}^1u_x}{{}^1z_{x+n-1}} + \ldots + \frac{{}^{n-1}u_x}{{}^{n-1}z_{x+n-1}} = 0.$$

## IX.

The integration of equation (B) of Problem II being reduced to the integration of this same equation when $X_x = 0$, there is no longer a question to resolve the problem but to integrate this here, but this appears very difficult in general; thus I will limit myself to the particular cases. Here is one quite expanded of it, in which the integration succeeds, and which embraces all the cases already known; it is the one in which one has

$$(\mathrm{B}') \qquad y_x = C\phi_x y_{x-1} + {}^1C\phi_x\phi_{x-1}y_{x-2} + \ldots + {}^{n-1}C\phi_x\phi_{x-1}\ldots\phi_{x-n+1}y_{x-n}.$$

If $\phi_x = 1$, one will have the equation of the recurrent sequences.

Equation (E) of Article IV becomes in this case

$$(\mathrm{E}') \qquad 0 = 1 - \frac{C\phi_x}{\alpha_x} - \frac{{}^1C\phi_x\phi_{x-1}}{\alpha_x\alpha_{x-1}} - \ldots - \frac{{}^{n-1}C\phi_x\phi_{x-1}\ldots\phi_{x-n+1}}{\alpha_x\ldots\alpha_{x-n+1}}.$$

Now (Art. IV), it suffices in order to integrate equation (B′) to know a number $n$ of values for $\alpha_x$ in equation (E′). Let therefore $\alpha_x = a\phi_x$, $a$ being constant, and equation (E′) will give

$$(h) \qquad a^n = Ca^{n-1} + {}^1Ca^{n-2} + {}^2Ca^{n-3} + \ldots + {}^{n-1}C;$$

whence one will have a number $n$ of values for $a$, and consequently for $\alpha_x$, since $\alpha_x = a\phi_x$.

Let $p$, ${}^1p$, ${}^2p$, $\ldots$, ${}^{n-1}p$ be the different values of $a$ in equation (h). One will have (Art. IV)

$$\delta_x = p\phi_x, \quad {}^1\delta_x = {}^1p\phi_x, \quad {}^2\delta_x = {}^2p\phi_x, \quad \ldots$$

Now one has (Art. V)

$$\begin{aligned} u_x &= \nabla \delta_x = \phi_1 \phi_2 \phi_3 \dots \phi_x p^x, \\ {}^1u_x &= \nabla {}^1\delta_x = \phi_1 \phi_2 \phi_3 \dots \phi_x {}^1p^x, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}$$

The complete integral of equation (B′) is therefore

$$y_x = \phi_1 \phi_2 \phi_3 \dots \phi_x (Ap^x + {}^1A\,{}^1p^x + \dots + {}^{n-1}A\,{}^{n-1}p^x).$$

One will determine the arbitrary constants $A$, ${}^1A$, ${}^2A, \dots$ by means of $n$ values of $y_x$, under as many particular assumptions for $x$. Let

$$y_1 = M, \quad y_2 = {}^1M, \quad \dots, \quad y_n = {}^{n-1}M;$$

and one will have

$$\begin{aligned} \frac{M}{\phi_1} &= Ap + {}^1A\,{}^1p + {}^2A\,{}^2p + \dots + {}^{n-1}A\,{}^{n-1}p, \\ \frac{{}^1M}{\phi_1\phi_2} &= Ap^2 + {}^1A\,{}^1p^2 + {}^2A\,{}^2p^2 + \dots + {}^{n-1}A\,{}^{n-1}p^2, \\ \frac{{}^2M}{\phi_1\phi_2\phi_3} &= Ap^3 + {}^1A\,{}^1p^3 + {}^2A\,{}^2p^3 + \dots + {}^{n-1}A\,{}^{n-1}p^3, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ \frac{{}^{n-1}M}{\phi_1\phi_2\dots\phi_n} &= Ap^n + {}^1A\,{}^1p^n + {}^2A\,{}^2p^n + \dots + {}^{n-1}A\,{}^{n-1}p^n, \end{aligned}$$

In order to resolve these equations, one can make use of the ordinary methods of elimination: but here is one of them which appears to me simpler.

I multiply the first equation by ${}^{n-1}p$, and I subtract it from the second; I multiply similarly the second by ${}^{n-1}p$, and I subtract it from the third, and thus in sequence, this which produces the following equations:

$$\begin{aligned} &\frac{{}^1M}{\phi_1\phi_2} - \frac{M}{\phi_1}\,{}^{n-1}p = Ap(p - {}^{n-1}p) + {}^1A\,{}^1p({}^1p - {}^{n-1}p) + \dots + {}^{n-2}A\,{}^{n-2}p({}^{n-2}p - {}^{n-1}p), \\ &\frac{{}^2M}{\phi_1\phi_2\phi_3} - \frac{{}^1M}{\phi_1\phi_2}\,{}^{n-1}p = Ap^2(p - {}^{n-1}p) + {}^1A\,{}^1p^2({}^1p - {}^{n-1}p) + \dots + {}^{n-2}A\,{}^{n-2}p^2({}^{n-2}p - {}^{n-1}p), \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ &\frac{{}^{n-1}M}{\phi_1\dots\phi_n} - \frac{{}^{n-2}M}{\phi_1\dots\phi_{n-1}}\,{}^{n-1}p = Ap^{n-1}(p - {}^{n-1}p) + \dots + {}^{n-2}A\,{}^{n-2}p^{n-1}({}^{n-2}p - {}^{n-1}p), \end{aligned}$$

I multiply again the first of these equations by ${}^{n-2}p$, and I subtract it from the second; I multiply similarly the second by ${}^{n-2}p$, and I subtract it from the third, this which gives

$$\begin{aligned}
&\frac{{}^{2}M}{\phi_1\phi_2\phi_3} - \frac{{}^{1}M}{\phi_1\phi_2}({}^{n-1}p - {}^{n-2}p) + \frac{M}{\phi_1}{}^{n-1}p\,{}^{n-2}p\,p \\
=&Ap(p - {}^{n-1}p)(p - {}^{n-2}p) \\
&+ {}^{1}A\,{}^{1}p({}^{1}p - {}^{n-1}p)({}^{1}p - {}^{n-2}p) \\
&+ \dots\dots\dots\dots\dots\dots \\
&+ {}^{n-3}A\,{}^{n-3}p({}^{n-3}p - {}^{n-1}p)({}^{n-3}p - {}^{n-2}p), \\
&\frac{{}^{3}M}{\phi_1\phi_2\phi_3\phi_4} - \frac{{}^{2}M}{\phi_1\phi_2\phi_3}({}^{n-1}p - {}^{n-2}p) + \frac{{}^{1}M}{\phi_1\phi_2}{}^{n-1}p\,{}^{n-3}p\,p \\
=&Ap^2(p - {}^{n-1}p)(p - {}^{n-2}p) \\
&+ \dots\dots\dots\dots\dots\dots \\
&+ {}^{n-3}A\,{}^{n-3}p^2({}^{n-3}p - {}^{n-1}p)({}^{n-3}p - {}^{n-2}p), \\
&+ \dots\dots\dots\dots\dots\dots\dots\dots\dots;
\end{aligned}$$

by operating on these last equations, as on the previous, one will have

$$\begin{aligned}
&\frac{{}^{3}M}{\phi_1\phi_2\phi_3\phi_4} - \frac{{}^{2}M}{\phi_1\phi_2\phi_3}({}^{n-1}p - {}^{n-2}p + {}^{n-3}p) \\
&\quad + \frac{{}^{1}M}{\phi_1\phi_2}[({}^{n-2}p + {}^{n-1}p)\,{}^{n-3}p + {}^{n-1}p\,{}^{n-2}p] - \frac{M}{\phi_1}{}^{n-1}p\,{}^{n-2}p\,{}^{n-3}p\,p \\
&= Ap(p - {}^{n-1}p)(p - {}^{n-2}p)(p - {}^{n-3}p) + \dots,
\end{aligned}$$

and thus in sequence.

Thence it is easy to conclude that, if one names:
$f$ the sum of the quantities ${}^{1}p,\ {}^{2}p,\ {}^{3}p,\ \dots,\ {}^{n-1}p$,
$h$ the sum of their products two by two,
$i$ the sum of their products three by three,
$q$ the sum of their products four by four, etc.,
${}^{1}f$ the sum of the quantities $p,\ {}^{2}p,\ {}^{3}p,\ \dots,\ {}^{n-1}p$,
${}^{1}h$ the sum of their products two by two,
${}^{1}i$ the sum of their products three by three, etc.,
and thus in sequence, one will have

$$\begin{aligned}
A &= \frac{{}^{n-1}M - \phi_n f\,{}^{n-2}M + \phi_n\phi_{n-1}h\,{}^{n-3}M - \phi_n\phi_{n-1}\phi_{n-2}i\,{}^{n-4}M + \dots}{\phi_1\phi_2\phi_3\dots\phi_n p(p - {}^{1}p)(p - {}^{2}p)(p - {}^{3}p)\dots}, \\
{}^{1}A &= \frac{{}^{n-1}M - \phi_n\,{}^{1}f\,{}^{n-2}M + \phi_n\phi_{n-1}\,{}^{1}h\,{}^{n-3}M - \dots}{\phi_1\phi_2\phi_3\dots\phi_n\,{}^{1}p({}^{1}p - p)({}^{1}p - {}^{2}p)({}^{1}p - {}^{3}p)\dots} \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

One can determine in a quite simple manner the quantities $f, h, i, q, {}^1f, {}^1h, {}^1i, {}^1q, \dots$; I take for this the equation

$$(h) \qquad a^n - Ca^{n-1} - {}^1C^{n-2} - \dots - {}^{n-1}C = 0;$$

I divide it by $a-p$, and the resulting equation will be

$$a^{n-1} - fa^{n-2} - ha^{n-3} - ia^{n-4} + qa^{n-5} + \dots = 0.$$

I multiply this result by $a-p$, and I will have the following equation

$$a^n - (p+f)a^{n-1} + (pf+h)a^{n-2} - (ph+i)a^{n-3} + \dots = 0;$$

I compare it with equation $(h)$, and I conclude from it

$$\begin{aligned} f &= +C - p, \\ h &= -{}^1C - pf, \\ i &= +{}^2C - ph, \\ &\dots\dots\dots\dots, \end{aligned}$$

and, consequently,

$$\begin{aligned} {}^1f &= +C - {}^1p, \\ {}^1h &= -{}^1C - {}^1p.{}^1f, \\ &\dots\dots\dots\dots, \end{aligned}$$

I have supposed until here that all the roots of equation $(h)$ are unequal, but it can happen that one or many of these roots are equal among themselves; here is in this case the method that it is necessary to follow.

I suppose that one has $p = {}^1p$; one will make ${}^1p = p + dp$, and the equation

$$y_x = \phi_1\phi_2\phi_3\dots\phi_x(Ap^x + {}^1A\,{}^1p^x + {}^2A\,{}^2p^x + \dots + {}^{n-1}A\,{}^{n-1}p^x)$$

will give, by reducing $(p+dp)^x$ into series,

$$y_x = \phi_1\phi_2\dots\phi_x\left\{p^x\left[A + {}^1A\left(1 + \frac{xdp}{p} + \frac{x(x-1)}{1.2}\frac{dp^2}{p^2} + \dots\right)\right] + {}^2A\,{}^2p^x + \dots\right\}.$$

Let

$$A + {}^1A = B \quad \text{and} \quad {}^1A\frac{dp}{p} = D,$$

$B$ and $D$ being some arbitrary and finite constants; ${}^1A$ will be therefore infinitely great of order $\frac{1}{dp}$; ${}^1A\frac{dp^2}{p^2}$, ${}^1A\frac{dp^3}{p^3}$, … will be infinitely small. Hence

$$y_x = \phi_1\phi_2\dots\phi_x[p^x(B+Dx) + {}^2A\,{}^2p^x + {}^3A\,{}^3p^x + \dots].$$

If, moreover, one has $p = {}^2p$, one will make ${}^2p = p + dp$ in this expression of $y_x$, and one will have

$$y_x = \phi_1\phi_2\dots\phi_x\left\{p^x\left[B + {}^2A + \left(D + {}^2A\frac{dp}{p}\right)x + {}^2A\frac{dp^2}{p^2}\frac{x(x-1)}{1.2} + \dots\right] + {}^3A\,{}^3p^x + \dots\right\}.$$

Let

$$ {}^2A + B = {}^1B, \qquad D + {}^2A\frac{dp}{p} = {}^1D \qquad \text{and} \qquad {}^2\mathrm{A}\frac{dp^2}{p^2} = {}^1E, $$

${}^1B$, ${}^1D$ and ${}^1E$ being some arbitrary and finite constants; one will have

$$ y_x = \phi_1\phi_2\ldots\phi_x\left\{p^x\left[{}^1B + {}^1Dx + {}^1E\frac{x(x-1)}{1.2} + \ldots\right] + {}^3A\,{}^3p^x + \ldots\right\}; $$

if moreover one had $p = {}^3p$, one would have

$$ y_x = \phi_1\phi_2\ldots\phi_x\left\{p^x\left[{}^2B + {}^2Dx + {}^2E\frac{x(x-1)}{1.2} + {}^2F\frac{x(x-1)(x-2)}{1.2.3}\right] + {}^4A\,{}^4p^x + \ldots\right\}, $$

and thus in sequence; one would determine the arbitrary constants, at least of $n$ particular values of $y_x$.

If equation ($h$) has two imaginary roots $p$ and ${}^1p$, one will make

$$ p = a + b\sqrt{-1} \qquad \text{and} \qquad {}^1p = a - b\sqrt{-1}. $$

Let

$$ \frac{a}{\sqrt{aa+bb}} = \cos q \qquad \text{and} \qquad \frac{b}{\sqrt{aa+bb}} = \sin q; $$

one will have

$$ \begin{aligned} Ap^x + {}^1A\,{}^1p^x &= (aa+bb)^{\frac{x}{2}}[A(\cos q + \sqrt{-1}\sin q)^x + {}^1A(\cos q - \sqrt{-1}\sin q)^x] \\ &= (aa+bb)^{\frac{x}{2}}[(A + {}^1A)\cos qx + (A - {}^1A)\sqrt{-1}\sin qx)^x] \end{aligned} $$

because

$$ (\cos q \pm \sqrt{-1}\sin q)^x = \cos qx \pm \sqrt{-1}\sin qx. $$

Let

$$ A + {}^1A = B \quad \text{and} \quad (A - {}^1A)\sqrt{-1} = {}^1B, $$

$B$ and ${}^1B$ being reals; one will have

$$ Ap^x + {}^1A\,{}^1p^x = (aa+bb)^{\frac{x}{2}}(B\cos qx + {}^1B\sin qx); $$

one will have therefore then

$$ y^x = \phi_1\phi_2\ldots\phi_x\left[(aa+bb)^{\frac{x}{2}}(B\cos qx + {}^1B\sin qx) + {}^2A\,{}^2p^x + \ldots\right]; $$

it will be the same process if there were a greater number of imaginaries.

If one supposes, in the preceding calculations, $\phi_x = 1$, one will have the case of the recurrent sequences. Thence results this theorem:

*If one names $Y_x$ the general term of a recurrent sequence, such that one has*

$$ Y_x = CY_{x-1} + {}^1CY_{x-2} + \ldots + {}^{n-1}CY_{x-n}, $$

*the general term of a sequence such that one has*

$$y_x = C\phi_x y_{x-1} + {}^1C\phi_x\phi_{x-1}y_{x-2} + \dots + {}^{n-1}C\phi_x \dots \phi_{x-n+1}y_{x-n},$$

*and in which the arbitrary constants which arrive by integrating are the same as in the preceding, will be*

$$y_x = \phi_1\phi_2 \dots \phi_x Y_x.$$

This is it of which it is easy to be assured besides; because, if one substitutes this value of $y_x$ into the equation

$$y_x = C\phi_x y_{x-1} + \dots,$$

one will have

$$\phi_1\phi_2 \dots \phi_x Y_x = C\phi_1\phi_2 \dots \phi_x Y_{x-1} + \dots,$$

hence

$$Y_x = CY_{x-1} + {}^1CY_{x-2} + \dots,$$

an equation which holds by assumption.

## X.

When one has, by the preceding article, the integral of the equation

$$y_x = C\phi_x y_{x-1} + {}^1C\phi_x\phi_{x-1}y_{x-2} + \dots + {}^{n-1}C\phi_x \dots \phi_{x-n+1}y_{x-n} + X_x,$$

by supposing $X_x = 0$, it is easy to conclude this same integral, $X_x$ being anything. For this, I observe that, since, $X_x$ being null, one has

$$y_x = \phi_1\phi_2 \dots \phi_x(Ap^x + {}^1A\,{}^1p^x + \dots {}^{n-1}A\,{}^{n-1}p^x),$$

one will have, by Article V,

$$\begin{aligned} u_x &= \phi_1\phi_2\phi_3 \dots \phi_x p^x, \\ {}^1u_x &= \phi_1\phi_2\phi_3 \dots \phi_x\,{}^1p^x, \\ {}^2u_x &= \phi_1\phi_2\phi_3 \dots \phi_x\,{}^2p^x, \\ &\dots\dots\dots\dots\dots\dots, \end{aligned}$$

whence one will conclude, by Article VII,

$$\begin{aligned}
\overset{1}{u}_x &= \phi_1\phi_2\dots\phi_x p^x \Delta \frac{{}^1p^{x-1}}{p^{x-1}} = \phi_1\phi_2\dots\phi_x({}^1p-p)\,{}^1p^{x-1},\\
{}^1\overset{1}{u}_x &= \phi_1\phi_2\dots\phi_x({}^2p-p){}^2p^{x-1},\\
{}^2\overset{1}{u}_x &= \phi_1\phi_2\dots\phi_x({}^3p-p){}^3p^{x-1},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,\\
\overset{2}{u}_x &= \phi_1\phi_2\dots\phi_x({}^2p-p)({}^2p-{}^1p){}^2p^{x-2},\\
{}^1\overset{2}{u}_x &= \phi_1\phi_2\dots\phi_x({}^3p-p)({}^3p-{}^1p){}^3p^{x-2},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,\\
\overset{3}{u}_x &= \phi_1\phi_2\dots\phi_x({}^3p-p)({}^3p-{}^1p)({}^3p-{}^2p){}^3p^{x-3},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

and thus in sequence, hence

$$\overset{n-1}{u}{}_{x+1} = {}^{n-1}z_{x+1} = \phi_1\phi_2\dots\phi_{x+1}({}^{n-1}p-p)({}^{n-1}p-{}^1p)({}^{n-1}p-{}^2p)\dots{}^{n-1}p^{x-n+2};$$

similarly

$$\begin{aligned}
&{}^{n-2}z_{x+1} = \phi_1\phi_2\dots\phi_{x+1}({}^{n-2}p-p)({}^{n-2}p-{}^1p)\dots{}^{n-2}p^{x-n+2},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots;
\end{aligned}$$

whence one will conclude, by substituting these values into formula (H) of article VII and making $X_x = \phi_1\phi_2\dots\phi_x\,{}^1X_x$ for brevity,

$$\begin{aligned}
y_x =& \frac{\phi_1\phi_2\dots\phi_x}{(p-{}^1p)(p-{}^2p)(p-{}^3p)\dots} p^{x+n-1}\left(G+\sum\frac{{}^1X_{x+1}}{p^{x+1}}\right)\\
&+\frac{\phi_1\phi_2\dots\phi_x}{({}^1p-p)({}^1p-{}^2p)\dots}\,{}^1p^{x+n-1}\left({}^1G+\sum\frac{{}^1X_{x+1}}{{}^1p^{x+1}}\right)\\
&+\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

If $p={}^1p$, one will make ${}^1p=p+dp$. Let $K=\frac{1}{(p-{}^2p)(p-{}^3p)\dots}$, and one will have

$$\begin{aligned}
y_x =& \phi_1\phi_2\dots\phi_x p^{x+n-1}\left\{B+Dx-\frac{K}{p}\sum\frac{{}^1X_{x+1}}{p^{x+1}}(x+1)+\left[\frac{dK}{dp}+\frac{K}{p}(x+n-1)\right]\sum\frac{{}^1X_{x+1}}{p^{x+1}}\right\}\\
&+\frac{\phi_1\phi_2\dots\phi_x}{({}^2p-p)^2({}^2p-{}^3p)\dots}\,{}^2p^{x+n-1}\left({}^2G+\sum\frac{{}^1X_{x+1}}{{}^2p^{x+1}}\right),
\end{aligned}$$

$B$ and $D$ being two arbitrary constants.

If, moreover, one has $p = {}^2p$, one will make, in this last expression of $y_x$, ${}^2p = p + dp$, and thus in sequence.

One can therefore integrate generally all the differential equations contained in the following formula

$$y_x = C\phi_x y_{x-1} + {}^1C\phi_x\phi_{x-1}y_{x-2} + \ldots + \mathrm{X}_x;$$

whence it results that, if one designates by $\theta_x$ any function whatsoever of $x$, the following equation

$$\theta_x y_x = C\theta_{x-1}\phi_x y_{x-1} + {}^1C\theta_{x-2}\phi_x\phi_{x-1}y_{x-2} + \ldots + \mathrm{X}_x$$

is generally integrable, since by making $\theta_x y_x = t_x$ this equation is of the same form as the preceding.

XI.

Here is now another kind of linear differential equations, of which the order depends on the variable $x$; let, for example,

$$\begin{aligned} y_x = &a_{x-1}y_{x-1} + b_{x-2}y_{x-2} + f_{x-3}y_{x-3} + X_x \\ &+ a_{x-4}y_{x-4} + b_{x-5}y_{x-5} + f_{x-6}y_{x-6} \\ &+ a_{x-7}y_{x-7} + b_{x-8}y_{x-8} + \ldots \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ a_3y_3 + b_2y_2 + f_1y_1. \end{aligned}$$

It is easy to bring these equations back to the form of equation (B) of problem II, because one has

$$\begin{aligned} y_{x-3} = &a_{x-4}y_{x-4} + b_{x-5}y_{x-5} + f_{x-6}y_{x-6} + X_{x-3} \\ &+ a_{x-7}y_{x-7} + b_{x-8}y_{x-8} + \ldots \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ a_3y_3 + b_2y_2 + f_1y_1. \end{aligned}$$

If one subtracts this last equation from the preceding, one will have

$$y_x = a_{x-1}y_{x-1} + b_{x-2}y_{x-2} + (f_{x-3} + 1)y_{x-3} + X_x - X_{x-3},$$

an equation contained in equation (B).

XII.

Presently here is a quite extended use of the integral Calculus in the finite differences, in order to determine directly the general expression of the quantities subject to a certain law which serves to form them, an expression that until here it seems to me that one has always sought to draw by way of induction, a method not only indirect, but which, moreover, must be often at fault.

In order to make myself better understood, I take the following example:

Let $x$ be the sine of an angle $z$ and $u$ its cosine; one has generally, as one knows,

$$\sin nz = 2u\sin(n-1)z - \sin(n-2)z,$$

whence one draws

$$\begin{aligned}
&\sin z = x,\\
&\sin 2z = x(2u),\\
&\sin 3z = x(4u^2-1),\\
&\sin 4z = x(8u^3-4u),\\
&\sin 5z = x(16u^4-12u^2+1),\\
&\dots\dots\dots\dots\dots\dots
\end{aligned}$$

It is necessary now to determine the general expression of $\sin nz$.

One can arrive by way of induction, by continuing further these expressions and seeking to discover the law of the different coefficients of the powers of $u$; but it will happen, if it is not in this example, at least in an infinity of others, that this law will be very complicated and very difficult to grasp: it matters consequently to have a general and sure method in order to find it in all the possible cases.

Let, for this, the differential equation be

$$(\nabla) \qquad y_x = \begin{cases} y_n = y_{n-1}(a_n u + b_n) \\ \quad + y_{n-2}({}^1a_n u^2 + {}^1b_n u + {}^1c_n) \\ \quad + y_{n-3}({}^2a_n u^3 + {}^2b_n u^2 + {}^2c_n u + {}^2f_n) \\ \quad + \dots\dots\dots\dots\dots\dots\dots\dots\dots \end{cases}$$

I suppose that one has

$$\begin{aligned}
&y_1 = \alpha u + \beta,\\
&y_2 = \delta u^2 + \gamma u + \Omega,\\
&y_3 = \varpi u^3 + \pi u^2 + \theta u + \sigma,\\
&\dots\dots\dots\dots\dots\dots
\end{aligned}$$

Here is how I conclude the general expression of $y_n$.

I make

$$y_n = A_n u^n + B_n u^{n-1} + C_n u^{n-2} + \dots,$$

hence,

$$\begin{aligned}
y_{n-1} &= A_{n-1}u^{n-1} + B_{n-1}u^{n-2} + C_{n-1}u^{n-3} + \dots,\\
y_{n-2} &= A_{n-2}u^{n-2} + B_{n-2}u^{n-3} + C_{n-2}u^{n-4} + \dots,
\end{aligned}$$

and thus in sequence; if one substitutes these values of $y_{n-1}, y_{n-2}, \dots$ into equation $(\nabla)$, one will have

$$\begin{aligned} y_n = &u^n(a_nA_{n-1} + {}^1a_nA_{n-2} + {}^2a_nA_{n-3} + \dots \\ &+ u^{n-1}(a_nB_{n-1} + {}^1a_nB_{n-2} + {}^2a_nB_{n-3} + \dots \\ &\qquad + b_nA_{n-1} + {}^1b_nA_{n-2} + {}^2b_nA_{n-3} + \dots) \\ &+ u^{n-2}(a_nC_{n-1} + {}^1a_nC_{n-2} + {}^2a_nC_{n-3} + \dots \\ &\qquad + b_nB_{n-1} + {}^1b_nB_{n-2} + {}^2b_nB_{n-3} + \dots \\ &\qquad + {}^1c_nA_{n-2} + {}^2c_nA_{n-3} + {}^2c_nA_{n-4} + \dots) \\ &\qquad \dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}$$

By comparing this expression of $y_n$ with the preceding, one will have the following equations

$$\begin{aligned} A_n = &a_nA_{n-1} + {}^1a_nA_{n-2} + {}^2a_nA_{n-3} + \dots, \\ B_n = &a_nB_{n-1} + {}^1a_nB_{n-2} + {}^2a_nB_{n-3} + \dots \\ &+ b_nA_{n-1} + {}^1b_nA_{n-2} + {}^2b_nA_{n-3} + \dots, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}$$

by means of which one will determine, by the preceding methods, $A_n$, $B_n$, …, and one will have thus the general expression of $y_n$.

I suppose that one wishes to have the general expression of $\sin nz$; it is easy to see, by that which precedes, that it will have this form

$$\sin nz = x(A_nu^{n-1} + B_nu^{n-3} + C_nu^{n-5} + D_nu^{n-7} + \dots);$$

therefore

$$\begin{aligned} \sin(n-1)z &= x(A_{n-1}u^{n-2} + B_{n-1}u^{n-4} + C_{n-1}u^{n-6} + \dots) \\ \sin(n-2)z &= x(A_{n-2}u^{n-3} + B_{n-2}u^{n-5} + C_{n-2}u^{n-7} + \dots). \end{aligned}$$

If one substitutes these values of $\sin(n-1)z$ and $\sin(n-2)z$ into the equation

$$\sin nz = 2u\sin(n-1)z - \sin(n-2)z,$$

one will have

$$\sin nz = x(2A_{n-1}u^{n-1} + 2B_{n-1}u^{n-3} + 2C_{n-1}u^{n-5} + \dots - A_{n-2}u^{n-3} - B_{n-2}u^{n-5} - \dots)$$

and, if one compares this expression with the preceding, one will have

$$(\Lambda) \qquad \begin{cases} A_n = 2A_{n-1}, \\ B_n = 2B_{n-1} - A_{n-2}, \\ C_n = 2C_{n-1} - B_{n-2}, \\ \dots\dots\dots\dots \end{cases}$$

By means of these equations one will determine $A_n, B_n, C_n, \dots$, but one must make here an observation in which it is necessary to pay attention to all the researches which depend on the integral Calculus in the finite differences; that which renders its use very delicate. This observation consists in this that the preceding equations ($\Lambda$) begin to exist not at all immediately, that is to say when $n$ has one same value in these equations. In order to demonstrate, I observe that the fundamental equation

$$\sin nz = 2u\sin(n-1)z - \sin(n-2)z,$$

by means of which I have concluded $\sin 2z, \sin 3z, \sin 4z, \ldots$, suppose known the first two sines $\sin 0z$ and $\sin 1z$; it can therefore begin to take place only when $n = 2$; hence also, equations ($\Lambda$) can begin to exist only when $n = 2$. The first of these equations begin to exist when $n = 2$, in which case one has $A_2 = 2A_1$; thus, the smallest index of $A_n$, that is to say the least value that $n$ can have in this expression, is unity; the second equation can therefore begin to take place only when $n = 3$, in which case one has $B_3 = 2B_2 - A_1$; hence, the least index of $B_n$ is 2; the third equation can therefore begin to take place only when $n = 4$, in which case one has $C_4 = 2C_3 - B_2$; hence, the smallest index of $C_n$ is 3, and thus in sequence. This put:

If one integrates the first equation, one will have

$$A_n = 2^n H,$$

$H$ being arbitrary; now, putting $n = 1, A_n = 1$, whence $H = \frac{1}{2}$, one has $A_n = 2^{n-1}$, hence $A_{n-2} = 2^{n-3}$. If one substitutes this value of $A_{n-2}$ into the second equation and if next one integrates it; one will have

$$B_n = -2^{n-3}(n+H);$$

since the differential equation in $B_n$ commences to exist when $n = 3$, the arbitrary constant $H$ must be determined by the value of $B_n$, when $n = 2$; now, $u$ not being able to have a negative exponent in the expression of $\sin nz$, it follows that $B_2 = 0$, hence $H = -2$; therefore

$$B_n = -2^{n-3}(n-2) \quad \text{and} \quad B_{n-2} = -2^{n-5}(n-4).$$

If one substitutes this value of $B_{n-2}$ into the third equation, and if next one integrates it, one will have

$$C_n = 2^{n-5}\left(\frac{n^2-7n}{2} + H\right)$$

now, putting $n = 3$, $C_n = 0$, whence $H = 6$, one has $C_n = 2^{n-5}\frac{(n-3)(n-4)}{1.2}$, and thus to infinity. Therefore

$$\sin nz = x\left[2^{n-1}u^{n-1} - \frac{n-2}{1}2^{n-3}u^{n-3} + \frac{(n-3)(n-4)}{1.2}2^{n-5}u^{n-5} \right.$$
$$\left. - \frac{(n-4)(n-5)(n-6)}{1.2.3}2^{n-7}u^{n-7} + \ldots\right].$$

Let next $z =$ angle $\sin x$; one will have, by differentiating,

$$\frac{dz}{dx} = \frac{1}{\sqrt{1-x^2}},$$

and I wish to have the general expression of $\frac{d^n z}{dx^n}$, $dx$ being supposed constant. For this, let $u = \frac{1}{\sqrt{1-x^2}}$; one will have

$$\frac{du}{dx} = \frac{x}{(1-x^2)^{\frac{3}{2}}},$$
$$\frac{d^2u}{dx^2} = \frac{2x^2+1}{(1-x^2)^{\frac{5}{2}}},$$
$$\frac{d^3u}{dx^3} = \frac{6x^3+9x}{(1-x^2)^{\frac{7}{2}}},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

It is easy to see, by considering the law of these expressions of $du$, $d^2u, \ldots$, that the general expression of $\frac{d^n u}{dx^n}$ has the following form

$$\frac{d^n u}{dx^n} = \frac{A_n x^n + B_n x^{n-2} + C_n x^{n-4} + D_n x^{n-6} + \ldots}{(1-x^2)^{n+\frac{1}{2}}};$$

by differentiating this expression, one has

$$\frac{d^{n+1}u}{dx^{n+1}} = \frac{\begin{array}{c|c|c|c} (n+1)A_n x^{n+1} + (n+3)B_n & x^{n-1} + (n+5)C_n & x^{n-3} + (n+7)D_n & x^{n-5} + \ldots \\ +nA_n & +(n-2)B_n & +(n-4)C_n & +\ldots \end{array}}{(1-x^2)^{n+\frac{3}{2}}}$$

but one has

$$\frac{d^{n+1}u}{dx^{n+1}} = \frac{A_{n+1}x^{n+1} + B_{n+1}x^{n-1} + C_{n+1}x^{n-3} + D_{n+1}x^{n-5} + \ldots}{(1-x^2)^{n+\frac{3}{2}}};$$

by comparing these two expressions of $\frac{d^{n+1}u}{dx^{n+1}}$, one will have the following equations:

$$A_{n+1} = (n+1)A_n,$$
$$B_{n+1} = (n+3)B_n + nA_n,$$
$$C_{n+1} = (n+5)C_n + (n-2)B_n,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

All these equations begin to exist immediately and when $n = 1$; this put, the first gives

$$A_n = 1.2.3\ldots n;$$

the second gives

$$B_n = 1.2.3\ldots n(n+1)(n+2)\left[H + \sum \frac{n}{(n+1)(n+2)(n+3)}\right],$$

or

$$B_n = 1.2.3\ldots n(n+1)(n+2)\left[Q + \frac{1}{2}\frac{1}{(n+1)(n+2)} - \frac{1}{n+2}\right].$$

One will determine the constant $Q$ by this condition that $B_n$ is zero when $n = 1$; one has therefore $Q = \frac{1}{2.2}$. Therefore

$$B_n = 1.2.3\ldots n\,\frac{1}{2}\,\frac{n(n-1)}{1.2}.$$

The third equation gives, by integrating and adding the appropriate constants,

$$C_n = 1.2.3\ldots n\,\frac{1.3}{2.4}\,\frac{n(n-1)(n-2)(n-3)}{1.2.3.4};$$

one will find similarly

$$D_n = 1.2.3\ldots n\,\frac{1.3.5}{2.4.6}\,\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1.2.3.4.5.6},$$

and thus in sequence. Hence

$$\begin{aligned}\frac{d^nz}{dx^n} = \frac{1.2.3\ldots(n-1)}{(1-x^2)^{n-\frac{1}{2}}}\Bigg[x^{n-1} &+ \frac{1}{2}\,\frac{(n-1)(n-2)}{1.2}\,x^{n-3} \\ &+ \frac{1.3}{2.4}\,\frac{(n-1)(n-2)(n-3)(n-4)}{1.2.3.4}x^{n-5} \\ &+ \frac{1.3.5}{2.4.6}\,\frac{(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)}{1.2.3.4.5.6}x^{n-7} \\ &+ \frac{1.3.5.7}{2.4.6.8}\,\frac{(n-1)(n-2)\ldots(n-8)}{1.2.3\ldots8}x^{n-9} \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\Bigg].\end{aligned}$$

I have supposed, in the two preceding examples, the law of the exponents known, because it was very easy to perceive; but, if it happened that it was complicated, this which must be extremely rare, one will be able to determine it by the preceding method.

## XIII.

Here is yet a remarkable usage of the integral Calculus in the finite differences in order to determine the nature of the functions according to some given conditions, this which is often useful, principally in the Calculus of partial differences.[4]

One proposes to find a function of $x$ such that by making successively $x = \phi(x)$ and $x = \psi(x)$, one has

$$(\sigma) \qquad f[\phi(x)] = H_x f[\psi(x)] + X_x,$$

[4] I had found this method at the end of 1772, on the occasion of some problems which Mr. Monge, skillful professor of Mathematics at the schools of the Genoese at Mézières, proposed to me; I did part of it for him then; at the same time, I sent it to Mr. de la Grange, and I have presented it to the Academy in the month of February 1773. Since this time, Mr. the marquis de Condorcet has had printed in the Volume of the Academy for the year 1771 a quite beautiful Memoir on this object; but the route which I have differs from his in this that he does not propose, as I do it, to restore the question to the differential equations of which the difference is constant and equal to unity. *Translator's note*: On 10 March and 17 March 1773, as reported in the Procès-Verbaux of the Paris Academy, Laplace read the paper "Recherches sur l'integration des differentielles aux différences finies et sur leur application à l'analyse des hasards."

$\phi(x)$, $\psi(x)$, $H_x$ being some given functions of $x$.

For this let

$$u_z = \psi(x) \quad \text{and} \quad u_{z+1} = \phi(x).$$

From the first of these equations, I conclude

$$x = \Gamma(u_z) \quad \text{and} \quad \phi(x) = H(u_z),$$

$\Gamma(u_z)$ and $H(u_z)$ representing some known functions of $u_z$; hence,

$$u_{z+1} = H(u_z),$$

a differential equation of which the constant difference is equal to unity, and which one can integrate in many cases.

The integral of this equation will give $u_z$ as function of $z$, and the equation $x = \Gamma(u_z)$ will give $x$ as function of $z$. Substituting this value of $x$ in $H_x$ and $X_x$, the quantities will become some functions of $z$, which I designate by $L_z$ and $Z_z$. Moreover, one has

$$f[\phi(x)] = f(u_{z+1}) \quad \text{and} \quad f[\psi(x)] = f(u_z);$$

equation ($\sigma$) will become therefore, by supposing $f(u_z) = y_z$,

$$y_{z+1} = L_z y_z + Z_z,$$

an equation integrable by Problem I.

One must observe here, consistent with a remark due to Mr. Euler, that the constants which come by integrating the finite differential equations of which the variable is $z$, and of which the constant difference is unity, can be supposed some functions any whatsoever of $\sin 2\pi z$ and $\cos 2\pi z$, $\pi$ expressing the ratio of the circumference to the diameter.

Presently, if one puts back into the expression of $y_z$ instead of $z$ its value in $x$, one will have $f[\psi(x)]$, and, if one changes $\psi(x)$ into $x$, one will have the function of $x$, which satisfies the Problem. The following examples clarify this method:

*The question is to find a function of $x$ such that by changing successively $x$ into $x^q$ and into $mx$, one has*

$$f(x^q) = f(mx) + p,$$

*$m$ and $p$ being constants.*

I make $u_z = mx$, and $u_{z+1} = x^q$; hence,

$$u_{z+1} = \left(\frac{u_z}{m}\right)^q.$$

In order to integrate this equation, I make $u_1 = a$; therefore $u_2 = \frac{a^q}{m^q}$, $u_3 = \frac{a^{q^2}}{m^{q^2+q}}, \ldots$ Let $u_z = \frac{a^{g_z}}{m^{f_z}}$; therefore

$$u_{z+1} = \frac{a^{qg_z}}{m^{qf_z+q}} = \frac{a^{g_{z+1}}}{m^{f_{z+1}}}.$$

Therefore

$$g_{z+1} = qg_z,$$

this which gives

$$g_z = Aq^z.$$

Now, putting $z = 2$, $g_z = q$, whence $A = \frac{1}{q}$, one has $g_z = q^{z-1}$. Moreover, one has $f_{z+1} = qf_z + q$. Therefore $f_z = Aq^z + \frac{q}{1-q}$. Now, putting $z = 2$, $f_z = q$; therefore $A = \frac{1}{q-1}$ and $f_z = \frac{1}{q-1}(q^z - q)$; therefore

$$u_z = \frac{a^{q^{z-1}}}{m^{\frac{1}{q-1}(q^z-q)}}.$$

This expression of $u_z$ is complete, since $a$ is arbitrary; now the equation

$$f(x^q) = f(mx) + p$$

will become

$$y_{z+1} = y_z + p.$$

Therefore

$$y_z = C + pz = f(mx).$$

It is necessary presently to have the value of $z$ in $x$; now, since one has $u_z = mx$, one will have

$$mx = \frac{a^{q^{z-1}}}{m^{\frac{1}{q-1}(q^z-q)}},$$

whence one draws[5]

$$lmx = q^z \frac{la}{q} - \frac{1}{q-1}(q^z - q)\, lm$$

or

$$q^z \left( \frac{la}{q} - \frac{lm}{q-1} \right) = l \frac{mx}{m^{\frac{q}{q-1}}};$$

let $\frac{la}{q} - \frac{lm}{q-1} = K$, and one will find

$$z = \frac{ll \frac{mx}{m^{\frac{q}{q-1}}}}{lq} - \frac{lK}{lq},$$

hence

$$y_z = A + p \frac{ll \frac{mx}{m^{\frac{q}{q-1}}}}{lq},$$

$A$ being an arbitrary constant which can be any function whatsoever of $\sin 2\pi z$ and $\cos 2\pi z$. Let $\Gamma(\sin 2\pi z, \cos 2\pi z)$ be this function; by substituting instead of $z$ its value, one will have

$$A = \Gamma \left( \sin 2\pi \frac{ll \frac{mx}{m^{\frac{q}{q-1}}}}{lq}, \cos 2\pi \frac{ll \frac{mx}{m^{\frac{q}{q-1}}}}{lq} \right).$$

[5]*Translator's note*: Laplace uses l to denote the natural logarithm. It appears as $l$ in this document.

Therefore

$$y_z = f(mx) = \Gamma\left(\sin 2\pi \frac{ll\frac{mx}{m^{\frac{q}{q-1}}}}{lq}, \cos 2\pi \frac{ll\frac{mx}{m^{\frac{q}{q-1}}}}{lq}\right) + p\frac{ll\frac{mx}{m^{\frac{q}{q-1}}}}{lq};$$

thus the function of $x$ demanded is

$$f(x) = \Gamma\left(\sin 2\pi \frac{ll\frac{x}{m^{\frac{q}{q-1}}}}{lq}, \cos 2\pi \frac{ll\frac{x}{m^{\frac{q}{q-1}}}}{lq}\right) + p\frac{ll\frac{x}{m^{\frac{q}{q-1}}}}{lq}.$$

*It is a question again to find $f(x)$ such that*

$$[f(x)]^2 = f(2x) + 2.$$

One could first think that it is impossible to satisfy this equation, at least to suppose $f(x)$ equal to a constant; this is indeed that which some able geometers have believed (*see* the second Volume of the *Mémoires de Turin*, p. 320); but one is going to see there are an infinity of other ways to satisfy it.

Let

$$u_z = x \qquad \text{and} \qquad u_{z+1} = 2x;$$

therefore

$$u_{z+1} = 2u_z \qquad \text{and} \qquad u_z = A2^z = x.$$

Moreover, one has

$$f(2x) = f(u_{z+1}), \qquad \text{which I designate by} \qquad t_{z+1},$$

and

$$f(x) = f(u_z) = t_z;$$

and one will have

$$t_{z+1} = t_z^2 - 2.$$

In order to integrate this equation, I suppose $t_1 = a + \frac{1}{a}$, therefore

$$t_2 = a^2 + \frac{1}{a^2}, \quad t_3 = a^4 + \frac{1}{a^4}, \quad \dots,$$

and generally

$$t_z = a^{2^{z-1}} + \frac{1}{a^{2^{z-1}}},$$

a complete expression of $t_x$, since $a$ is arbitrary; now one has $2^{z-1} = \frac{x}{2A}$, therefore

$$t_z = a^{\frac{x}{2A}} + a^{-\frac{x}{2A}}, \quad \text{or} \quad t_z = b^x + b^{-x},$$

$b$ being an arbitrary constant; now this constant can be supposed any function whatsoever of $\sin 2\pi z$ and $\cos 2\pi z$, and since $z = H + \frac{lx}{l2}$, $H$ being any constant whatsoever, one will have

$$b = f(\sin 2\pi\frac{lx}{l2}, \cos 2\pi\frac{lx}{l2}),$$

hence the function of $x$ demanded is

$$\left[f(\sin 2\pi\frac{lx}{l2},\cos 2\pi\frac{lx}{l2})\right]^{x}+\left[f(\sin 2\pi\frac{lx}{l2},\cos 2\pi\frac{lx}{l2})\right]^{-x}$$

*It is a question again to find* $f(x-y\sqrt{-1})$, *such that one has*

$$f(x+y\sqrt{-1})-f(x-y\sqrt{-1})=2M\sqrt{-1}.$$

By supposing $y=g+hx$, one will have

$$f[g\sqrt{-1}+x(1+h\sqrt{-1})]-f[x(1-h\sqrt{-1})-g\sqrt{-1}]=2M\sqrt{-1}.$$

Let

$$x(1+h\sqrt{-1})+g\sqrt{-1}=u_{z+1},$$
$$x(1-h\sqrt{-1})-g\sqrt{-1}=u_{z;}$$

one will have therefore

$$x=\frac{u_z+g\sqrt{-1}}{1-h\sqrt{-1}};$$

therefore

$$u_{z+1}=\frac{1+h\sqrt{-1}}{1-h\sqrt{-1}}u_z+\frac{2g\sqrt{-1}}{1-h\sqrt{-1}},$$

an equation of which the integral is

$$u_z=A\left(\frac{1+h\sqrt{-1}}{1-h\sqrt{-1}}\right)^2-\frac{g}{h}=x(1-h\sqrt{-1})-g\sqrt{-1};$$

hence,

$$zl\frac{1+h\sqrt{-1}}{1-h\sqrt{-1}}=l(g+hx)+K.$$

Now, if one names $\varpi\pi$ the angle of which the tangent is $h$, and $\pi$ the ratio of the semi-circumference to the radius, one will have

$$l\frac{1+h\sqrt{-1}}{1-h\sqrt{-1}}=2\sqrt{-1}\varpi\pi;$$

therefore

$$z=\frac{l(g+hx)}{2\sqrt{-1}\varpi\pi}+K'.$$

Now one has

$$f(u_{z+1})-f(u_z)=2M\sqrt{-1};$$

and, by representing $f(u_z)$ by $t_z$,

$$t_{z+1}=t_z+2M\sqrt{-1},$$

therefore

$$t_z = H + 2Mz\sqrt{-1};$$

substituting instead of $z$ its value, one will have

$$t_z = M\frac{l(g+hx)}{\varpi\pi} + L,$$

$L$ being an arbitrary constant, which can be any function whatsoever of $\sin 2\pi z$ and $\cos 2\pi z$, or of $\sin\frac{l(g+hx)}{\varpi\sqrt{-1}}$ and of $\cos\frac{l(g+hx)}{\varpi\sqrt{-1}}$, and consequently of $e^{\frac{l(g+hx)}{\varpi}}$; now, $e^{l(g+hx)} = g+hx$; therefore $L$ can be a function of $(g+hx)^{\frac{1}{\varpi}}$; hence

$$f(x-y\sqrt{-1}) = M\frac{l(g+hx)}{\varpi\pi} + \Gamma\left[(g+hx)\frac{1}{\varpi}\right].$$

## XIV.

*On the equations in finite differences, when one has many equations among many variables.*

I suppose that one has the following two equations among the three variables $y_x$, ${}^1y_x$ and $x$

$$(1) \qquad y_x + A_x y_{x-1} = B_x\,{}^1y_x + C_x\,{}^1y_{x-1},$$

$$(2) \qquad y_x + {}^1A_x y_{x-1} = {}^1B_x\,{}^1y_x + {}^1C_x\,{}^1y_{x-1}.$$

The simplest way to integrate them is to reduce them by elimination to two other equations, the one between $y_x$ and $x$, the other between ${}^1y_x$ and $x$; for this, I multiply the first by ${}^1C_x$, the second by $C_x$, and I subtract the one from the other; this which gives

$$({}^1C_x - C_x)y_x + ({}^1C_xA_x - C_x\,{}^1A_x)y_{x-1} = ({}^1C_xB_x - C_x\,{}^1B_x)\,{}^1y_x,$$

hence

$$(3) \qquad \begin{cases} ({}^1C_{x-1} - C_{x-1})y_{x-1} + ({}^1C_{x-1}A_{x-1} - C_{x-1}\,{}^1A_{x-1})y_{x-1} \\ \quad = ({}^1C_{x-1}B_{x-1} - C_{x-1}\,{}^1B_{x-1})\,{}^1y_{x-1}. \end{cases}$$

I multiply equation (1) by $\alpha$, equation(2) by ${}^1\alpha$, and I add them with equation (3), this which gives

$$\begin{aligned}(\alpha + {}^1\alpha)y_x &+ (\alpha A_x + {}^1\alpha\,{}^1A_x + {}^1C_{x-1} - C_{x-1})y_{x-1} + ({}^1C_{x-1}A_{x-1} - C_{x-1}\,{}^1A_{x-1})y_{x-2} \\ &= (\alpha B + {}^1\alpha\,{}^1B)\,{}^1y_x + (\alpha C_x + {}^1\alpha\,{}^1C_x + {}^1C_{x-1}B_{x-1} - C_{x-1}\,{}^1B_{x-1})\,{}^1y_{x-1};\end{aligned}$$

I make ${}^1y_x$ and ${}^1y_{x-1}$ vanish by means of the equations

$$\alpha B_x + {}^1\alpha\,{}^1B_x = 0,$$
$$\alpha C_x + {}^1\alpha\,{}^1C_x + {}^1C_{x-1}B_{x-1} - C_{x-1}\,{}^1B_{x-1} = 0,$$

and I have in this manner a differential equation between $y_x$ and $x$ alone; by an entirely similar process, one will find one of them between ${}^1y_x$ and $x$; and it would be the same thing if one has a greater number of equations and of variables.

It is easy to see that, if there was in each equation some terms such as $T_x, X_x, \dots, T_x, X_x$ being some functions any whatsoever of $x$, they would be integrable in the same cases where they are it, these terms not being there.

When one has $n-1$ equations among $n$ variables, these being able to have an infinity of different relations among them, the integration of these equations presents thus a great number of curious researches; but there is a case which merits a particular attention, in this that it is encountered sometimes and principally in the analyses of chances; it is the case in which these equations return to themselves.

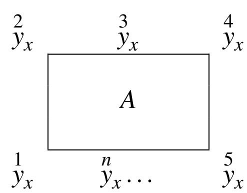
## XV.

*On the differential equations returning to themselves.*

If one has the following equations, among the $n$ variables $\overset{1}{y}_x, \overset{2}{y}_x, \overset{3}{y}_x, \dots,$

$$\overset{1}{y}_x = A\overset{2}{y}_{x-1},$$
$$\overset{2}{y}_x = A\overset{3}{y}_{x-1},$$
$$\overset{3}{y}_x = A\overset{4}{y}_{x-1},$$
$$\dots\dots\dots\dots,$$
$$\overset{n}{y}_x = A\overset{1}{y}_{x-1}.$$

These equations are those which I call *equations returning to themselves.* In general, if one disposes on the perimeter of *fig.* A the $n$ variables $\overset{1}{y}_x, \overset{2}{y}_x, \overset{3}{y}_x, \dots,$ as the figure represents them,




and if then any function whatsoever of one of these variables and of its finite differences is constantly equal to any function whatsoever of those which follow it and of their finite differences, the equation which results is that which I name an *equation returning to itself*. If, for example, each of these variables is equal to twice that which follows it, when one supposes $x$ diminishing by unity, plus three times that which follows this last, when one supposes $x$ diminishing by two units, one will have

$$\overset{1}{y}_x = 2\overset{2}{y}_{x-1} + 3\overset{3}{y}_{x-2},$$
$$\overset{2}{y}_x = 2\overset{3}{y}_{x-1} + 3\overset{4}{y}_{x-2},$$
$$\dots\dots\dots\dots\dots\dots,$$
$$\overset{n}{y}_x = 2\overset{1}{y}_{x-1} + 3\overset{2}{y}_{x-2}.$$

One sees thence that, although in the order of the variable $\overset{1}{y}_x$ is the first, one would have been able however equally to begin with any other of these variables, and the equations would have been absolutely the same, this which is the particular character of this kind of equations. This put,

XVI.

PROBLEM III. — *I suppose that one has the returning equations*

$$\begin{aligned}
&\overset{1}{y}_x + A\overset{1}{y}_{x-1} + {}^1A\overset{1}{y}_{x-2} + \ldots = B\overset{2}{y}_x + {}^1B\overset{2}{y}_{x-1} + {}^2B\overset{2}{y}_{x-2} + \ldots + X_x,\\
&\overset{2}{y}_x + A\overset{2}{y}_{x-1} + {}^1A\overset{2}{y}_{x-2} + \ldots = B\overset{3}{y}_x + {}^1B\overset{3}{y}_{x-1} + {}^2B\overset{3}{y}_{x-2} + \ldots + X_x,\\
&\\
&\overset{n}{y}_x + A\overset{n}{y}_{x-1} + {}^1A\overset{n}{y}_{x-2} + \ldots = B\overset{1}{y}_x + {}^1B\overset{1}{y}_{x-1} + {}^2B\overset{1}{y}_{x-2} + \ldots + X_x;
\end{aligned}$$

*it is necessary to determine* $\overset{1}{y}_x$, $\overset{2}{y}_x$, ...

The first equation gives

$$\begin{aligned}
&\overset{1}{y}_x + A\overset{1}{y}_{x-1} + {}^1A\overset{1}{y}_{x-2} + \ldots + A\overset{1}{y}_{x-1} + A^2\overset{1}{y}_{x-2} + \ldots + {}^1A\overset{1}{y}_{x-2}\\
&\quad = B(\overset{2}{y}_x + A\overset{2}{y}_{x-1} + {}^1A\overset{2}{y}_{x-2} + \ldots) + {}^1B(\overset{2}{y}_{x-1} + A\overset{2}{y}_{x-2} + {}^1A\overset{2}{y}_{x-3} + \ldots) + \ldots\\
&\qquad + X_x + AX_{x-1} + {}^1AX_{x-2} + \ldots
\end{aligned}$$

I substitute instead of $\overset{2}{y}_x + A\overset{2}{y}_{x-1} + \ldots$, $\overset{2}{y}_{x-1} + A\overset{2}{y}_{x-2} + \ldots$ their values which the second equation gives, this which gives me an equation among $\overset{1}{y}_x$, $\overset{1}{y}_{x-1}$, ... and $\overset{3}{y}_x$, $\overset{3}{y}_{x-1}$, ...; by operating on this here as on the first, I will have an equation among $\overset{1}{y}_x$, $\overset{1}{y}_{x-1}$, ... and $\overset{4}{y}_x$, $\overset{4}{y}_{x-1}$, ... and, by continuing to operate thus until the variable $\overset{q}{y}_x$, I will arrive to an equation of this form

$$\begin{aligned}
&\overset{1}{y}_x + b_q\overset{1}{y}_{x-1} + {}^1b_q\overset{1}{y}_{x-2} + \ldots\\
&\quad = a_q(\overset{q}{y}_x + A\overset{q}{y}_{x-1} + {}^1A\overset{q}{y}_{x-2} + \ldots) + {}^1a_q(\overset{q}{y}_{x-1} + A\overset{q}{y}_{x-2} + \ldots) + \ldots + \overset{q}{u}_x,
\end{aligned}$$

It is necessary to determine $b_q$, ${}^1b_q$, ..., $a_q$, ${}^1a_q$, ..., $\overset{q}{u}_x$.

For this I substitute into the preceding equation, instead of $\overset{q}{y}_x + A\overset{q}{y}_{x-1} + \ldots \overset{q}{y}_{x-1} + A\overset{q}{y}_{x-2} + \ldots$, their values that the $q^{\text{th}}$ of the returning equations gives, this which gives

$$\begin{aligned}
&\overset{1}{y}_x + b_q\overset{1}{y}_{x-1} + {}^1b_q\overset{1}{y}_{x-2} + \ldots\\
&= a_q(B\overset{q+1}{y}_x + {}^1B\overset{q+1}{y}_{x-1} + \ldots + X_x) + {}^1a_q(B\overset{q+1}{y}_{x-1} + {}^1B\overset{q+1}{y}_{x-2} + \ldots + X_{x-1}) + \ldots + \overset{q}{u}_x,
\end{aligned}$$

whence I conclude

$$
\begin{array}{r|r|l}
\overset{1}{y}_x + b_q & \overset{1}{y}_{x-1} + {}^1b_q & \overset{1}{y}_{x-2} + \dots = a_qB\left(\overset{q+1}{y}_x + A\overset{q+1}{y}_{x-1} + {}^1A\overset{q+1}{y}_{x-2} + \dots\right. \\
+A & +Ab_q & \qquad\quad + ({}^1a_qB + a_q{}^1B)\left(\overset{q+1}{y}_{x-1} + A\overset{q+1}{y}_{x-2} + \dots\right) \\
 & {}^1A & \qquad\quad + ({}^2a_qB + {}^1a_q{}^1B + a_q{}^2B)\left(\overset{q+1}{y}_{x-2} + \dots\right) \\
 & & \qquad\quad + \dots\dots\dots\dots\dots\dots \\
 & & \qquad\quad + X_xa_q \\
 & & \qquad\quad + X_{x-1}({}^1a_q + Aa_q) \\
 & & \qquad\quad + X_{x-2}({}^2a_q + A{}^1a_q + {}^1Aa_q) \\
 & & \qquad\quad + \dots\dots\dots\dots\dots\dots \\
 & & \qquad\quad + \overset{q}{u}_x + A\overset{q}{u}_{x-1} + {}^1A\overset{q}{u}_{x-2} + \dots;
\end{array}
$$

but one has

$$
\begin{aligned}
\overset{1}{y}_x + b_{q+1}\overset{1}{y}_{x-1} + {}^1b_{q+1}\overset{1}{y}_{x-2} + \dots = {} & a_{q+1}\left(\overset{q+1}{y}_x + A\overset{q+1}{y}_{x-1} + \dots\right) \\
& + {}^1a_{q+1}\left(\overset{q+1}{y}_{x-1} + \dots\right) \\
& + \dots\dots\dots\dots \\
& + \overset{q+1}{u}_x;
\end{aligned}
$$

whence one has, by comparing,

$$
\begin{aligned}
& b_{q+1} = b_q + A, \\
& {}^1b_{q+1} = {}^1b_q + Ab_q + {}^1A, \\
& \dots\dots\dots\dots; \\
& a_{q+1} = a_qB, \\
& {}^1a_{q+1} = {}^1a_qB + a_q{}^1B, \\
& \dots\dots\dots\dots;
\end{aligned}
$$

$$
(\Lambda) \qquad \begin{cases} \overset{q+1}{u}_x = \overset{q}{u}_x + A\overset{q}{u}_{x-1} + {}^1A\overset{q}{u}_{x-2} + \dots \\ \qquad + X_xa_q + X_{x-1}({}^1a_q + Aa_q) + X_{x-2}({}^2a_q + A{}^1a_q + {}^1Aa_q) + \dots \end{cases}
$$

By means of these equations, one will determine easily $a_q$, ${}^1a_q$, …, $b_q$, ${}^1b_q$, …; in order to determine $\overset{q}{u}_x$, I observe that one has

$$\overset{q}{u}_x = f_qX_x + {}^1f_qX_{x-1} + {}^2f_qX_{x-2} + \dots;$$

I substitute this value into equation ($\Lambda$), this which gives

$$\begin{aligned}\overset{q+1}{u}_x =&X_x(f_q+a_q)+X_{x-1}({}^1f_q+{}^1a_q+Aa_q+Af_q)\\&+X_{x-2}({}^2f_q+{}^2a_q+A\,{}^1a_q+{}^1Aa_q+{}^1Af_q+A\,{}^1f_q)\\&+\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots;\end{aligned}$$

but one has

$$\overset{q+1}{u}_x = f_{q+1}X_x + {}^1f_{q+1}X_{x-1} + {}^2f_{q+1}X_{x-2}\dots;$$

therefore

$$\begin{aligned}f_{q+1} =&f_q+a_q,\\{}^1f_{q+1} =&{}^1f_q+{}^1a_q+Af_q,\\&\dots\dots\dots\dots\dots\dots\dots\dots\end{aligned}$$

By means of these equations one will determine $f_q$, ${}^1f_q$, ..., and hence $\overset{q}{u}_x$. I suppose now $q=n$, and one will have

$$\begin{aligned}\overset{1}{y}_x+b_q\overset{1}{y}_{x-1}+\dots =&a_n(\overset{n}{y}_x+A\overset{n}{y}_{x-1}+\dots)\\&+{}^1a_n(\overset{n}{y}_{x-1}+{}^1A\overset{n}{y}_{x-2}+\dots)\\&+\dots\dots\dots\dots\dots\dots\dots\\&+\overset{n}{u}_x;\end{aligned}$$

but one has

$$\overset{n}{y}_x+A\overset{n}{y}_{x-1}+\dots = B\overset{1}{y}_x+{}^1B\overset{1}{y}_{x-1}+\dots+X_x;$$

therefore

$$\begin{aligned}\overset{1}{y}_x+b_n\overset{1}{y}_{x-1}+\dots =&a_n(B\overset{1}{y}_x+{}^1B\overset{1}{y}_{x-1}+\dots) &&+a_nX_x\\&+{}^1a_n(B\overset{1}{y}_{x-1}+\dots) &&+{}^1a_nX_{x-1}\\&+\dots\dots\dots\dots\dots &&+\dots\dots\\& &&+\overset{n}{u}_x,\end{aligned}$$

and, by ordering the different terms of this equation

$$\begin{aligned}\overset{1}{y}_x(1-a_nB)+\overset{1}{y}_{x-1}(b_n-a_n{}^1B-{}^1a_nB)+\overset{1}{y}_{x-2}({}^1b_n-a_n{}^2B+{}^1a_n{}^1B-{}^2a_nB)+\dots\\-\overset{n}{u}_x-a_nX_x-{}^1a_nX_{x-1}-\dots=0\end{aligned}$$

one will have an equation entirely similar for $\overset{2}{y}_x$, $\overset{3}{y}_x$, ...

## XVII.

PROBLEM IV. — *I suppose now that the returning equations contain three variables, and that one has*

$$
\begin{aligned}
\overset{1}{y}_x + A\overset{1}{y}_{x-1} + {}^1A\overset{1}{y}_{x-2} + \ldots = {} & B\overset{2}{y}_x + {}^1B\overset{2}{y}_{x-1} + {}^2B\overset{2}{y}_{x-2} + \ldots \\
& + C\overset{3}{y}_x + {}^1C\overset{3}{y}_{x-1} + {}^2C\overset{3}{y}_{x-2} + \ldots \\
& + X_x, \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
\overset{n}{y}_x + A\overset{n}{y}_{x-1} + {}^1A\overset{n}{y}_{x-2} + \ldots = {} & B\overset{1}{y}_x + {}^1B\overset{1}{y}_{x-1} + {}^2B\overset{1}{y}_{x-2} + \ldots \\
& + C\overset{2}{y}_x + {}^1C\overset{2}{y}_{x-1} + \ldots \\
& + X_x,
\end{aligned}
$$

*it is necessary to determine* $\overset{1}{y}_x$, $\overset{2}{y}_x, \ldots$ *By following the process of the preceding problem, one will arrive to an equation of this form*

$$
\begin{aligned}
\overset{1}{y}_x + b_q\overset{1}{y}_{x-1} + {}^1b_q\overset{1}{y}_{x-2} + \ldots = {} & a_q(\overset{q}{y}_x + A\overset{q}{y}_{x-1} + {}^1A\overset{q}{y}_{x-2} + \ldots) \\
& + {}^1a_q(\overset{q}{y}_{x-1} + A\overset{q}{y}_{x-2} + {}^1A\overset{q}{y}_{x-3} + \ldots) \\
& + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& + c_q(\overset{q+1}{y}_x + A\overset{q+1}{y}_{x-1} + {}^1A\overset{q+1}{y}_{x-2} + \ldots) \\
& + {}^1c_q(\overset{q+1}{y}_{x-1} + A\overset{q+1}{y}_{x-2} + \ldots) \\
& + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& + \overset{q}{u}_x.
\end{aligned}
$$

I substitute now into this equation, instead of

$$
\overset{q}{y}_x + A\overset{q}{y}_{x-1} + \ldots, \quad \overset{q}{y}_{x-1} + A\overset{q}{y}_{x-2} + \ldots,
$$

their values that the $q^{\text{th}}$ equation gives, this which produces the following

$$
\begin{aligned}
\overset{1}{y}_x + b_q\overset{1}{y}_{x-1} + {}^1b_q\overset{1}{y}_{x-2} + \dots = & a_q(B\overset{q+1}{y}_x + {}^1B\overset{q+1}{y}_{x-1} + {}^2B\overset{q+1}{y}_{x-2} + \dots) \\
& + {}^1a_q(B\overset{q+1}{y}_{x-1} + {}^1B\overset{q+1}{y}_{x-2} + {}^2B\overset{q+1}{y}_{x-3} + \dots) \\
& + \dots\dots\dots\dots\dots\dots \\
& + a_q(C\overset{q+2}{y}_x + {}^1C\overset{q+2}{y}_{x-1} + \dots) \\
& + {}^1a_q(C\overset{q+2}{y}_{x-1} + \dots) \\
& + \dots\dots\dots\dots \\
& + c_q(\overset{q+1}{y}_x + A\overset{q+1}{y}_{x-1} + \dots) \\
& + {}^1c_q(\overset{q+1}{y}_{x-1} + \dots) \\
& + \dots\dots\dots\dots \\
& + a_qX_x + {}^1a_qX_{x-1} + \dots \\
& + \overset{q}{u}_x;
\end{aligned}
$$

whence one will conclude easily

$$
\begin{aligned}
& \overset{1}{y}_x \begin{array}{l} +b_q \\ +A \end{array} \Bigg| \; \overset{1}{y}_{x-1} \begin{array}{l} +{}^1b_q \\ +Ab_q \\ +{}^1A \end{array} \Bigg| \; \overset{1}{y}_{x-2} + \dots \\
& = (a_qB + c_q)(\overset{q+1}{y}_x + A\overset{q+1}{y}_{x-1} + \dots) \\
& \quad + ({}^1a_qB + a_q{}^1B + {}^1c_q + Ac_q)(\overset{q+1}{y}_{x-1} + A\overset{q+1}{y}_{x-2} + \dots) \\
& \quad + ({}^2a_qB + {}^1a_q{}^1B + a_q{}^2B + {}^2c_q + A{}^1c_q + {}^1Ac_q)(\overset{q+1}{y}_{x-2} + A\overset{q+1}{y}_{x-3} + \dots) \\
& \quad + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
& \quad + a_qC(\overset{q+2}{y}_x + A\overset{q+2}{y}_{x-1} + \dots) \\
& \quad + ({}^1a_qC + a_q{}^1C)(\overset{q+2}{y}_{x-1} + A\overset{q+2}{y}_{x-2} + \dots) \\
& \quad + ({}^2a_qC + {}^1a_q{}^1C + a_q{}^2C)(\overset{q+2}{y}_{x-2} + A\overset{q+2}{y}_{x-3} + \dots) \\
& \quad + \dots\dots\dots\dots\dots\dots\dots\dots \\
& \quad + \overset{q}{u}_x + A\overset{q}{u}_{x-1} + {}^1A\overset{q}{u}_{x-2} + \dots \\
& \quad + X_xa_q + X_{x-1}({}^1a_q + Aa_q) + \dots;
\end{aligned}
$$

now one has

$$\begin{aligned}\overset{1}{y}_x + b_{q+1}\overset{1}{y}_{x-1} + \ldots = & a_{q+1}\left(\overset{q+1}{y}_x + A\overset{q+1}{y}_{x-1} + \ldots\right)\\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ & + c_{q+1}\left(\overset{q+2}{y}_x + A\overset{q+2}{y}_{x-1} + \ldots\right)\\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ & + \overset{q+1}{u}_x;\end{aligned}$$

whence one will have, by comparing,

$$\begin{aligned} b_{q+1} &= b_q + A,\\ {}^1b_{q+1} &= {}^1b_q + Ab_q + {}^1A,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots;\end{aligned}$$

thus one will determine $b_q$, ${}^1b_q, \ldots$; next

$$a_{q+1} = a_qB + c_q \quad \text{and} \quad c_{q+1} = a_qC, \quad \text{hence} \quad a_{q+1} = a_qB + a_{q-1}C;$$

whence one will have $a_q$ and $c_q$. Moreover, one will have

$$\begin{aligned} {}^1a_{q+1} &= {}^1a_qB + a_q\,{}^1B + {}^1c_q + Ac_q,\\ {}^1c_{q+1} &= {}^1a_qC + a_q\,{}^1C.\end{aligned}$$

Therefore

$${}^1a_{q+1} = {}^1a_qB + a_q\,{}^1B + {}^1c_{q-1}\,{}^1C + a_{q-1}\,{}^1C + Ac_q;$$

whence one will have ${}^1a_q$ and ${}^1c_q$, and thus of the rest; finally one will determine $\overset{q}{u}_x$, as in the preceding problem.

If one supposes presently $q = n$, one will have

$$\begin{aligned}\overset{1}{y}_x(1 - c_n) + \overset{1}{y}_{x-1}(1 - Ac_n - {}^1c_n) + \ldots = & a_n(\overset{n}{y}_x + A\overset{n}{y}_{x-1} + \ldots)\\ & + {}^1a_n(\overset{n}{y}_{x-1} + A\overset{n}{y}_{x-2} + \ldots)\\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ & + \overset{n}{u}_x.\end{aligned}$$

One will form some entirely similar equations among $\overset{n-1}{y}_x$ and $\overset{n}{y}_x$, $\overset{n-2}{y}_x$ and $\overset{n-1}{y}_x, \ldots,$ and one will have a number $n$ of returning equations in two variables, such as I have considered in the preceding problem.

The same method would succeed equally if the returning equations contained four or a greater number of variables.

XVIII. *On the integral calculus in the finite and partial differences.*

I suppose that ${}_ny_x$ represents any function whatsoever of two variables $x$ and $n$; I can in this function make $n$ vary by regarding $x$ as constant; I can make $x$ vary by regarding $n$ as constant; finally, I can vary $n$ and $x$ all together, their variations being in any ratio whatsoever; now, if there exists among ${}_ny_x$ and these different variations any equation whatsoever, it will be that which I name an *equation in the finite and partial differences.*

${}_ny_x$ represents always a function of two variables $x$ and $n$: ${}_{n-1}y_x$, ${}_{n-2}y_x$, $\dots$ signify that $n$ has diminished by one, by two, ... units in this function; ${}_ny_{x-1}$, ${}_ny_{x-2}$, $\dots$ signify that $x$ has diminished by one, by two, ... units in this function; ${}_{n-1}y_{x-2}$, $\dots$ signifies that $n$ has diminished by one unit, and $x$ by two units, and thus in sequence.

An equation in the partial differences is therefore an equation among these different quantities; such as this here:

$${}_ny_x = a.{}_ny_{x-1} + b.{}_{n-1}y_{x-1}.$$

The equations in the finite differences have been found by the consideration of the sequences (Art. II). This is similarly the consideration of certain sequences that I have named *récurro-récurrentes* (*see* volume VI of *Savants étranges*), which has led me to the finite and partial differences; here is how: I suppose that one has the sequences

$$(i)\quad \begin{cases} {}_1y_1, & {}_1y_2, & {}_1y_3, & {}_1y_4, & {}_1y_5, & \dots, & {}_1y_x, & \dots, \\ {}_2y_1, & {}_2y_2, & {}_2y_3, & {}_2y_4, & {}_2y_5, & \dots, & {}_2y_x, & \dots, \\ {}_3y_1, & {}_3y_2, & {}_3y_3, & {}_3y_4, & {}_3y_5, & \dots, & {}_3y_x, & \dots, \\ \dots, & \dots, & \dots, & \dots, & \dots, & \dots, & & \dots, \\ {}_ny_1, & {}_ny_2, & {}_ny_3, & {}_ny_4, & {}_ny_5, & \dots, & {}_ny_x, & \dots, \end{cases}$$

If any term whatsoever ${}_ny_x$ of these sequences is constantly equal to any number whatsoever of the preceding terms taken in many of these sequences, and each multiplied by a function of $x$ and of $n$, these sequences are those that I have called *récurro-récurrentes*, and the equation which expresses the law according to which they are formed is an equation in the finite and partial differences.

I will observe here that the sequences ($i$) can be considered not only in the horizontal sense, but further in the vertical sense, and, instead that in the first sense $x$ is their index, $n$ will be it in the second.

I will suppose in the following, as I have done it above in the equations in the ordinary differences, that the differences of $x$ and of $n$ are constants and equal to unity; if they are constants without being equal to unity, it will always be possible to render them such, by the introduction of new variables; I will suppose moreover (this which is yet permitted) that the smallest values that $x$ and $n$ can receive are unity; and each time that I myself will depart from this assumption, the state of the question will make it known. This put:

If one has an equation in the partial differences such that

$${}_ny_x = 2.{}_ny_{x-1} + 2.{}_{n-1}y_{x-1},$$

it begins to hold only when $x$ and $n$ are greater than unity, as in the ordinary differences the equation ${}_1y_x = a.{}_1y_{x-1}$ holds only when $x$ is greater than 1; so that ${}_1y_1$ remains arbitrary, and one determines by means of this equation only the values of ${}_1y_2$, ${}_1y_3$, ...; likewise, in the equation

$${}_ny_x = 2.{}_ny_{x-1} + 2.{}_{n-1}y_{x-1},$$

${}_1y_x$ and ${}_ny_1$ are arbitrary; thus the general expression of ${}_ny_x$ contains an arbitrary function.

In general, the number of arbitrary functions that the integral of an equation in the partial differences contains will be determined by the degree of the difference of that of the two quantities $x$ and $n$ which varies the least; thus, in the equation

$${}_ny_x = {}_ny_{x-1} + 3.{}_{n-1}y_{x-1},$$

the number of arbitrary functions which the integral contains is 1, because, $n$ being here that of the two variables of which the difference is the least, it varies only by one unit; indeed, it is clear that, if one knows ${}_1y_x$, one can determine ${}_2y_x$, ${}_3y_x$, ${}_4y_x$, ... by means of the equation

$${}_ny_x = {}_ny_{x-1} + 3.{}_{n-1}y_{x-1};$$

there is therefore then only ${}_1y_x$ arbitrary.

## XIX.

PROBLEM V. — *The equation in the finite and partial differences*

$${}_ny_x = {}_nH_x \cdot {}_{n-1}y_{x-1} + {}^1_nH_x \cdot {}_{n-2}y_{x-2} + {}^2_nH_x \cdot {}_{n-1}y_{x-1} + \ldots + {}_nP_x$$

*being given, one proposes to integrate it.*

Since, in each term of this equation, the variable $n$ decreases according to the same law as the variable $x$, I can suppose $x = n + K$, $K$ being any constant whatsoever; ${}_ny_x$, ${}_nH_x$, ${}^1_nH_x$, ... become then functions of $x$ and of $K$; I represent in this case ${}_ny_x$ by $u_x$; ${}_nH_x$, ${}^1_nH_x$, ... by $L_x$, ${}^1L_x$, ..., finally ${}_nP_x$ by $X_x$; the proposed equation becomes therefore

$$u_x = L_xu_{x-1} + {}^1L_xu_{x-2} + {}^2L_xu_{x-1} + \ldots + X_x,$$

an equation in the ordinary differences, and of which the integral has this form by the preceding Articles, by restoring instead of $K$ its value $x-n$,

$$u_x = C.{}_nz_x + {}^1C.{}^1_nz_x + {}^2C.{}^2_nz_x + \ldots + {}_nR_x;$$

$C$, ${}^1C$, ${}^2C$, ... are some arbitrary constants, which can be functions of $K$ or of $x-n$; one will have therefore

$${}_ny_x = {}_nz_x.\phi(x-n) + {}^1_nz_x.{}^1\phi(x-n) + {}^2_nz_x.{}^2\phi(x-n) + \ldots + {}_nR_x;$$

one will determine the arbitrary functions $\phi(x-n)$, ${}^1\phi(x-n)$, ... by means of the values of ${}_n y_x$, in as many particular assumptions for $x$ as there are of these arbitrary functions.

The proposed equation in the partial differences is therefore generally integrable, that which comes from this that in each term $n$ and $x$ vary in the same manner; but, if one excepts this case and some others quite rare, it is impossible to have an integral entirely rid of any sign of integration. In order to show it by a quite simple example, I suppose that one has to integrate the equation

$${}_n y_x = {}_n y_{x-1} + {}_{n-1} y_{x-1};$$

by supposing ${}_1 y_x = \phi(x)$, one will have

$${}_2 y_x - {}_2 y_{x-1} = \phi(x-1) \quad \text{or} \quad \Delta.{}_2 y_x = \phi(x),$$

hence ${}_2 y_x = \Sigma\phi(x)$; one will find similarly

$${}_3 y_x = \Sigma^2\phi(x), \qquad {}_4 y_x = \Sigma^3\phi(x),$$

and generally

$${}_n y_x = \Sigma^{n-1}\phi(x);$$

such is therefore the complete value of ${}_n y_x$ by taking care to add to each integration an arbitrary constant.

One can simplify this value and reduce it to some quantities affected with the simple sign of integration, in the following manner.

It is necessary to reduce the double integral $\Sigma^2\phi(x)$ to some simple integrals; I make for this

$$\Sigma^2\phi(x) = z_x\Sigma\phi(x) - \Sigma t_x\phi(x);$$

by differentiating, there comes

$$\Sigma\phi(x) = (z_x+\Delta z_x)[\phi(x)+\Sigma\phi(x)] - z_x\Sigma\phi(x) - t_x\phi(x)$$

or

$$\Sigma\phi(x) = (z_x+\Delta z_x - t_x)\phi(x) + \Delta z_x\Sigma\phi(x).$$

Therefore $\Delta z_x = 1$ and $t_x = z_x + \Delta z_x$; I can therefore suppose $z_x = x$ and $t_x = x+1$, this which gives

$$\Sigma^2\phi(x) = x\Sigma\phi(x) - \Sigma(x+1)\phi(x);$$

one will reduce, by a similar process, $\Sigma^3\phi(x)$ to some quantities affected by a single sign of integration; but it will be impossible to rid it of it entirely.

Here is now a method to integrate equations in the partial differences, in which the inconvenience of the quantities affected by many signs of integration is not at all to fear.

## XX.

PROBLEM VI. — *The equation in the finite and partial differences*

$$(h)\qquad \begin{cases} {}_n y_x = +A_n \cdot {}_n y_{x-1} + {}^1A_n \cdot {}_n y_{x-2} + {}^2A_n \cdot {}_n y_{x-3} + \dots + N_n \\ \qquad +B_n \cdot {}_{n-1} y_x + {}^1B_n \cdot {}_{n-1} y_{x-1} + {}^2B_n \cdot {}_{n-1} y_{x-2} + \dots \end{cases}$$

*being given, one proposes to integrate it.*

For this I seek to restore the integration to that of an equation in the ordinary differences. I suppose therefore that one has ${}_1 y_x = \phi(x)$; equation ($h$) will give the following

$$(1)\qquad {}_2 y_x = A_2 \cdot {}_2 y_{x-1} + {}^1A_2 \cdot {}_2 y_{x-2} + \dots + N_2 + B_2 \phi(x) + {}^1B_2 \phi(x-1) + \dots,$$

next

$${}_3 y_x = A_3 \cdot {}_3 y_{x-1} + {}^1A_3 \cdot {}_3 y_{x-2} + \dots + N_3 + B_3 \cdot {}_2 y_x + {}^1B_3 \cdot {}_2 y_{x-1} + \dots;$$

whence it is easy to conclude

$$\begin{aligned} {}_3 y_x - A_3 \cdot {}_3 y_{x-1} - {}^1A_3 \cdot {}_3 y_{x-2} - \dots - A_2 ({}_3 y_{x-1} - A_3 \cdot {}_3 y_{x-2} - \dots) - {}^1A_1 ({}_2 y_{x-2} - \dots) \\ = B_3 ({}_2 y_x - A_2 \cdot {}_2 y_{x-1} - \dots) + {}^1B_3 ({}_2 y_{x-1} - A_2 \cdot {}_2 y_{x-2} - \dots) + \dots \\ + N_2 (1 - A_2 - {}^1A_2 - \dots). \end{aligned}$$

If one substitutes, instead of

$$\begin{aligned} &{}_2 y_x - A_2 \cdot {}_2 y_{x-1} - \dots, \\ &{}_2 y_{x-1} - A_2 \cdot {}_2 y_{x-2} - \dots, \\ &\dots\dots\dots\dots\dots\dots\dots\dots, \end{aligned}$$

their values drawn from equation (1), one will have an equation of this form:

$${}_3 y_x - a_3 \cdot {}_3 y_{x-1} - {}^1a_3 \cdot {}_3 y_{x-2} - \dots = {}_3 u_x.$$

This equation is in the ordinary differences; in order to integrate it by the preceding Articles, it is necessary to know ${}_3 u_x$ and the roots of the equation

$$1 = \frac{a_3}{f} + \frac{{}^1a_3}{f^2} + \frac{{}^2a_1}{f^3} + \dots;$$

now this equation is the same as this here

$$0 = 1 - \frac{A_3}{f} + \frac{{}^1A_3}{f^2} - \dots - \frac{A_2}{f}\left(1 - \frac{A_3}{f} - \dots\right) - \frac{{}^1A_2}{f^1}\left(1 - \frac{A_3}{f} - \dots\right)$$

and, hence, it is equal to the following

$$0 = \left(1 - \frac{A_2}{f} - \frac{{}^1A_2}{f^2} - \frac{{}^2A_2}{f^3} - \dots\right)\left(1 - \frac{A_3}{f} - \frac{{}^1A_3}{f^2} - \dots\right).$$

By following the same process for ${}_4y_x$, ${}_5y_x$ and generally for ${}_ny_x$, one will transform equation (h) of the Problem in the following

$$(2) \qquad {}_ny_x = a_n \cdot {}_ny_{x-1} + {}^1a_n \cdot {}_ny_{x-2} + \ldots + {}_nu_x,$$

that it will be easy to integrate it when one will know ${}_nu_x$ and the roots of the equation

$$1 - \frac{a_n}{f} + \frac{{}^1a_n}{f^2} + \frac{{}^2a_n}{f^3} - \ldots;$$

one will see easily that this equation is the same as this here

$$0 = \left(1 - \frac{A_2}{f} - \frac{{}^1A_2}{f^2} - \frac{{}^2A_2}{f^3} - \ldots\right)\left(1 - \frac{A_3}{f} - \frac{{}^1A_3}{f^2} - \ldots\right)\ldots\left(1 - \frac{A_n}{f} - \frac{{}^1A_n}{f^2} - \ldots\right),$$

whence it is easy to conclude $a_n$, ${}^1a_n$, ....

In order to determine presently the value of ${}_nu_x$, I observe that, from equation

$$(2) \qquad {}_ny_x = a_n \cdot {}_ny_{x-1} + {}^1a_n \cdot {}_ny_{x-2} + \ldots + {}_nu_x,$$

one draws

$$\begin{aligned} B_n \cdot {}_{n-1}y_x &= B_n . a_{n-1} \cdot {}_{n-1}y_{x-1} + B_n . {}^1a_{n-1} \cdot {}_{n-1}y_{x-2} + \ldots + B_n \cdot {}_{n-1}u_x, \\ {}^1B_n \cdot {}_{n-1}y_{x-1} &= {}^1B_n . a_{n-1} \cdot {}_{n-1}y_{x-2} + {}^1B_n . {}^1a_{n-1} \cdot {}_{n-1}y_{x-3} + \ldots + {}^1B_n \cdot {}_{n-1}u_{x-1}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

If one adds all these equations member by member, one will have

$$\begin{aligned} &B_n \cdot {}_{n-1}y_x + {}^1B_n \cdot {}_{n-1}y_{x-1} + \ldots \\ &= a_{n-1}(B_n \cdot {}_{n-1}y_{x-1} + {}^1B_n \cdot {}_{n-1}y_{x-2} + \ldots) \\ &\quad + {}^1a_{n-1}(B_n \cdot {}_{n-1}y_{x-2} + \ldots) \\ &\quad + \ldots\ldots\ldots\ldots\ldots\ldots \\ &\quad + B_n \cdot {}_{n-1}u_x + {}^1B_n \cdot {}_{n-1}u_{x-1} + \ldots \end{aligned}$$

Now, if one substitutes, instead of

$$\begin{aligned} &B_n \cdot {}_{n-1}y_x + {}^1B_n \cdot {}_{n-1}y_{x-1} + \ldots, \\ &B_n \cdot {}_{n-1}y_{x-1} + {}^1B_n \cdot {}_{n-1}y_{x-1} + \ldots, \end{aligned}$$

their values given by the equation of the problem, one will have

$$\begin{aligned} {}_ny_x - A_n \cdot {}_ny_{x-1} - \ldots - N_n &= a_{n-1}({}_ny_{x-1} - A_n \cdot {}_ny_{x-2} - \ldots - N_n) \\ &\quad + {}^1a_{n-1}({}_ny_{x-2} - \ldots - N_n) \\ &\quad + \ldots\ldots\ldots\ldots\ldots\ldots \\ &\quad + B_n \cdot {}_{n-1}u_x + {}^1B_n \cdot {}_{n-1}u_{x-1} + \ldots \end{aligned}$$

By ordering the different terms of this equation, one will have

$$\begin{aligned}{}_n y_x ={} & {}_n y_{x-1}(a_{n-1}+A_n)\\ & + {}_n y_{x-2}({}^1a_{n-1}-a_{n-1}.A_n+{}^1A_n)\\ & + {}_n y_{x-3}({}^2a_{n-1}-{}^1a_{n-1}.A_n+a_{n-1}.{}^1A_n+{}^2A_n)\\ & + \dots\dots\dots\dots\\ & + B_n\cdot{}_{n-1}u_x+{}^1B_n\cdot{}_{n-1}u_{x-1}+\dots\\ & + N_n(1-a_{n-1}-{}^1a_{n-1}-\dots).\end{aligned}$$

If one compares now term by term this last equation with equation (2), one will have the following:

$$\begin{aligned} a_n &= a_{n-1}+A_n,\\ {}^1a_n &= {}^1a_{n-1}-a_{n-1}.A_n+{}^1A_n,\\ {}^2a_n &= {}^2a_{n-1}-{}^1a_{n-1}.A_n-a_{n-1}.{}^1A_n+{}^2A_n,\\ & \dots\dots\dots\dots\end{aligned}$$

One could, by integrating these equations, determine $a_n$, ${}^1a_n$, …, if it was not much more simple to conclude them by the preceding method.

Finally one will have

$$(3)\quad {}_nu_x = N_n(1-a_{n-1}-{}^1a_{n-1}-{}^2a_{n-1}-\dots)+B_n\cdot{}_{n-1}u_x+{}^1B_n\cdot{}_{n-1}u_{x-1}+\dots$$

In order to integrate this last equation, I observe that, since ${}_1y_x=\phi(x)$, one will have ${}_1u_x=\phi(x)$; whence I conclude

$${}_2u_x = N_2(1-a_1-\dots)+B_2\phi(x)+{}^1B_2\phi(x-1)+\dots;$$

one would have in the same manner ${}_3u_x$, ${}_4u_x$, …, and one sees that by preceding thus one will have generally

$$(4)\qquad {}_nu_x = b_n\phi(x)+{}^1b_n\phi(x-1)+{}^2b_n\phi(x-2)+\dots+C_n;$$

therefore

$$\begin{aligned}{}_{n-1}u_x &= b_{n-1}\phi(x)+{}^1b_{n-1}\phi(x-1)+\dots+C_{n-1},\\ {}_{n-1}u_{x-1} &= b_{n-1}\phi(x-1)+{}^1b_{n-1}\phi(x-2)+\dots+C_{n-1},\\ & \dots\dots\dots\dots\end{aligned}$$

If one substitutes these values into equation (3), one will have

$$\begin{aligned}{}_nu_x ={} & N_n(1-a_{n-1}-{}^1a_{n-1}-\dots)+C_{n-1}(B_n+{}^1B_n+\dots)\\ & +b_{n-1}B_n\phi(x)+\phi(x-1)({}^1b_{n-1}B_n+b_{n-1}\,{}^1B_n+\dots;\end{aligned}$$

whence, by comparing with equation (4), one will have

$$
\begin{aligned}
b_n &= B_n . b_{n-1}, \\
{}^1b_n &= B_n . {}^1b_{n-1} + {}^1B_n . b_{n-1}, \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
C_n &= C_{n-1}(B_n + {}^1B_n + \dots) + N_n(1 - a_{n-1} - {}^1a_{n-1} - \dots).
\end{aligned}
$$

By integrating these different equations and adding the appropriate constants, one will have the values of $b_n$, ${}^1b_n$, ..., $C_n$, and hence that of ${}_nu_x$. The constants must be such, that by supposing $n = 1$ one has ${}_nu_x = \phi(x)$; so that one must have $C_1 = 0$, $b_1 = 1$, ${}^1b_1 = 0$, ${}^2b_1 = 0$, ....

By integrating equation (2) to which the equation of the problem is reduced, this operation introduces in the expression of ${}_ny_x$ some arbitrary constants, which can be functions of $n$; but these functions are not arbitrary, since the integral of equation (h) can contain no other arbitrary function than $\phi(x)$; one will determine them in this manner.

If one names $p_n$, ${}^1p_n$, ${}^2p_n$, ... the roots of the equation

$$1 = \frac{a_n}{f} + \frac{{}^1a_n}{f^2} + \frac{{}^2a_n}{f^3} + \dots;$$

one will have, by Article X,

$${}_ny_x = C_n . p_n^x + {}^1C_n . {}^1p_n^x + {}^2C_n . {}^2p_n^x + \dots + {}_nL_x.$$

If one substitutes this expression of ${}_ny_x$ into equation (h), one will draw from it, by comparing the terms homologous with respect to $x$, as many differential equations as there are functions $C_n$, ${}^1C_n$, ..., and, by integrating these equations, one will determine these functions.

Instead of making ${}_1y_x = \phi(x)$, one can imagine a differential equation any whatsoever between ${}_1y_x$ and $x$; I suppose that this equation is that of a recurrent sequence, so that one has

$${}_1y_x = F . {}_1y_{x-1} + {}^1F . {}_1y_{x-2} + \dots + L,$$

$F$, ${}^1F$, ... and $L$ being constants; by following the method of the problem, one will arrive to the following equation

$$\text{(5)} \qquad {}_ny_x = a_n . {}_ny_{x-1} + {}^1a_n . {}_ny_{x-2} + {}^2a_n . {}_ny_{x-3} + \dots + u_n,$$

and one will find that the equation

$$1 = \frac{a_n}{f} + \frac{{}^1a_n}{f^2} + \frac{{}^2a_n}{f^3} + \dots$$

is the same as this here:

$$0 = \left(1 - \frac{F}{f} - \frac{{}^1F}{f^2} - \dots\right)\left(1 - \frac{A_2}{f} - \frac{{}^1A_2}{f^2} - \dots\right)\dots\left(1 - \frac{A_n}{f} - \frac{{}^1A_n}{f^2} - \dots\right)$$

One will have next

$$u_n = u_{n-1}(B_n + {}^1B_n + \ldots) + N_n(1 - a_{n-1} - {}^1a_{n-1} - \ldots),$$

whence it will be easy to conclude the value of ${}_ny_x$.

The case in which the equation between ${}_1y_x$ and $x$ is that of a recurrent sequence is the one which is encountered most frequently in the application of this theory.

One can observe here that the quantities $B_n$, ${}^1B_n, \ldots$ enter not at all into the formation of $a_n$, ${}^1a_n, \ldots$, but simply in that of $u_n$; whence it follows that, when this quantity is null (this which must happen very often), equation (5) will remain the same thing as the quantities $B_n$, ${}^1B_n, \ldots$ are; thence there results that, in this case, these quantities influence in the solution of the problem only on the determination of the arbitrary constants which come from the integration of equation (5).

## XXI.

In order to clarify the preceding theory with some examples, I suppose that one has the two equations

$${}_1y_x = 2.{}_1y_{x-1},$$
$${}_ny_x = 2.{}_ny_{x-1} + 2.{}_{n-1}y_{x-1}.$$

If in the first equation one makes ${}_1y_1 = 1$, one will form in its way the following sequence 1, 2, 4, 8, 16, ... The second equation gives

$${}_2y_x = 2.{}_2y_{x-1} + 2.{}_1y_{x-1},$$

and, if one supposes ${}_2y_1 = 0$, one will have ${}_2y_2 = 2$, ${}_2y_3 = 8, \ldots$; one will form in this manner the sequence 0, 2, 8, 24, ... By continuing thus and supposing always ${}_3y_1 = 0$, ${}_4y_1 = 0$, ${}_5y_1 = 0, \ldots$ one will form the *récurro-récurrentes* sequences:

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ... | $x$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | ... | . |
| 2 | 0 | 2 | 8 | 24 | 64 | 160 | 384 | 896 | ... | . |
| 3 | 0 | 0 | 4 | 24 | 96 | 320 | 960 | 2688 | ... | . |
| 4 | 0 | 0 | 0 | 8 | 64 | 320 | 1280 | 4480 | ... | . |
| 5 | 0 | 0 | 0 | 0 | 16 | 160 | 960 | 4880 | ... | . |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $n$ | . | . | . | . | . | . | . | . | . | . |

It is necessary presently to determine the general term of these sequences or, this which reverts to the same, the expression of ${}_ny_x$.

For this, I observe that one has, by the preceding Article,

$${}_ny_x = a_n \cdot {}_ny_{x-1} + {}^1a_n \cdot {}_ny_{x-2} + \ldots + u_n,$$

next the equation

$$1 = \frac{a_n}{f} + \frac{{}^1a_n}{f^2} + \ldots$$

is in this case this here

$$0 = \left(1 - \frac{2}{f}\right)^n,$$

of which all the roots are equal to 2; one has, moreover, $u_n = 2u_{n-1}$; therefore $u_n = H.2^n$. Now, putting $n = 1$, one has $u_n = 0$, therefore $H = 0$; one will have thus, by Article IX,

$$\begin{aligned} {}_n y_x = 2^{x-1} \bigg[ & C_n \frac{(x-1)(x-2)\dots(x-n+1)}{1.2.3\dots(n-1)} \\ & + D_n \frac{(x-1)(x-2)\dots(x-n+2)}{1.2.3\dots(n-2)} \\ & + E_n \frac{(x-1)\dots(x-n+3)}{1.2.3\dots(n-3)} + \cdots \bigg]. \end{aligned}$$

In order to determine the arbitrary constants $C_n, D_n, \dots$, one will substitute this value of ${}_n y_x$ into the equation

$${}_n y_x = 2.{}_n y_{x-1} + 2.{}_{n-1} y_{x-1},$$

by observing that

$$\frac{(x-1)(x-2)\dots(x-n+1)}{1.2.3\dots(n-1)} = \frac{(x-2)(x-3)\dots(x-n)}{1.2.3\dots(n-1)} + \frac{(x-2)\dots(x-n+1)}{1.2.3\dots(n-2)},$$
$$\frac{(x-1)(x-2)\dots(x-n+2)}{1.2.3\dots(n-2)} = \frac{(x-2)\dots(x-n+1)}{1.2.3\dots(n-2)} + \frac{(x-2)\dots(x-n+2)}{1.2.3\dots(n-3)},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$

and one will have

$$\begin{aligned} & C_n \frac{(x-2)(x-3)\dots(x-n)}{1.2.3\dots(n-1)} + (C_n + D_n) \frac{(x-2)\dots(x-n+1)}{1.2.3\dots(n-2)} \\ & \qquad + (D_n + E_n) \frac{(x-2)\dots(x-n+2)}{1.2.3\dots(n-3)} + \dots \\ & = C_n \frac{(x-2)\dots(x-n)}{1.2.3\dots(n-1)} + (D_n + C_{n-1}) \frac{(x-2)\dots(x-n+1)}{1.2.3\dots(n-2)} \\ & \qquad + (E_n + D_{n-1}) \frac{(x-3)\dots(x-n+3)}{1.2.3\dots(n-3)} + \dots \end{aligned}$$

By comparing term by term, one will have:

1° $C_n = C_{n-1}$; therefore $C_n = A$. Now, putting $n = 1$, the quantity

$$\frac{1(x-1)(x-2)\dots(x-n+1)}{1.2.3\dots(n-1)}$$

is reduced to its first factor 1, and the quantities following

$$\frac{(x-1)(x-2)\dots(x-n+2)}{1.2.3\dots(n-2)}, \quad \dots$$

become nulls; therefore ${}_1y_x = A.2^{x-1}$. Now one has ${}_1y_1 = 1$; therefore $A = 1 = C_n$.

2° $D_n = D_{n-1}$, hence $D_n = A$ and ${}_2y_x = 2^{x-1}\left(\frac{x-1}{1} + A\right)$. Now putting $x = 1$, one has ${}_2y_1 = 0$ by the formation of the previous sequences; therefore $A = 0$ and $D_n = 0$.

One will find similarly $E_n = 0$, $F_n = 0, \ldots$; therefore

$$ {}_ny_x = 2^{x-1}\frac{(x-1)(x-2)\ldots(x-n+1)}{1.2.3\ldots(n-1)}. $$

Let, for example, $x = 8$ and $n = 5$; one will have

$$ {}_5y_8 = 2^7\frac{7.6.5.4}{1.2.3.4} = 4480. $$

I take further for example the two equations

$$ {}_1y_x = 2.{}_1y_{x-1}, $$
$$ {}_ny_x = (n+1).{}_ny_{x-1} + {}_{n-1}y_{x-1}. $$

If one supposes

$$ {}_1y_1 = 1, \quad {}_2y_1 = 0, \quad {}_3y_1 = 0, \quad {}_4y_1 = 0, \quad \ldots, $$

one will form the following sequences:

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | … | $x$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 4 | 8 | 16 | 32 | 64 | . | … | . |
| 2 | 0 | 1 | 5 | 19 | 65 | 211 | 665 | . | … | . |
| 3 | 0 | 0 | 1 | 9 | 55 | 285 | 1351 | . | … | . |
| 4 | 0 | 0 | 0 | 1 | 14 | 125 | 910 | . | … | . |
| 5 | 0 | 0 | 0 | 0 | 1 | 20 | 245 | . | … | . |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $n$ | . | . | . | . | . | . | . | . | . | . |

In order to find now the general term of these sequences, or the expression of ${}_ny_x$, I observe that one has, by the previous Article,

$$ {}_ny_x = a_n \cdot {}_ny_{x-1} + {}^1a_n \cdot {}_ny_{x-2} + {}^2a_n \cdot {}_ny_{x-3} + \ldots + u_n, $$

and that the equation

$$ 1 = \frac{a_n}{f} + \frac{{}^1a_n}{f^2} + \frac{{}^2a_n}{f^3} + \ldots $$

is the same as this here

$$ 0 = \left(1 - \frac{2}{f}\right)\left(1 - \frac{3}{f}\right)\left(1 - \frac{4}{f}\right)\cdots\left(1 - \frac{n+1}{f}\right); $$

finally, that one has

$$ u_n = 2u_{n-1}; $$

whence, by integrating,

$$u_n = H2^n.$$

Now, putting $n = 1$, one has $u_1 = 0$; therefore

$$H = 0 \quad \text{and} \quad u_n = 0.$$

By integrating, one will have therefore

$$_n y_x = C_n 2^{x-1} + {}^1C_n 3^{x-1} + {}^2C_n 4^{x-1} + \ldots + {}^{n-1}C_n (n+1)^{x-1},$$

an equation in which it is necessary presently to determine the arbitrary constants $C_n$, ${}^1C_n$, ... For this, I substitute this value of $_n y_x$ into the equation

$$_n y_x = (n+1).{}_n y_{x-1} + {}_{n-1}y_{x-1},$$

this which gives

$$\begin{aligned} &C_n 2^{x-1} + {}^1C_n 3^{x-1} + \ldots \\ &\quad = (n+1)C_n 2^{x-2} + (n+1).{}^1C_n 3^{x-2} + \ldots + C_{n-1} 2^{x-2} + {}^1C_{n-1} 3^{x-2} + \ldots; \end{aligned}$$

whence, by comparing term by term, I will have

$$\begin{aligned} 2.\ C_n &= (n+1).C_n + C_{n-1}, \\ 3.{}^1C_n &= (n+1).{}^1C_n + {}^1C_{n-1}, \\ &\ldots\ldots \end{aligned}$$

It is clear that the first equation begins to hold only when $n = 2$; the second, when $n = 3$; the third, when $n = 4$, ... By integrating the first, one will have

$$C_n = \frac{C_1}{(1-2)(1-3)(1-4)\ldots(1-n)}.$$

Now, since one has $_1 y_x = 2^{x-1}$, one will have $C_1 = 1$; therefore

$$C_n = \pm \frac{1}{1.2.3\ldots(n-1)},$$

the $+$ sign holding if $n$ is odd, and the $-$ sign if it is even.

One will have similarly

$${}^1C_n = \frac{{}^1C_2}{(2-3)(2-4)\ldots(2-n)}.$$

Now, putting $n = 2$, one has

$$_2 y_x = C_2 2^{x-1} + {}^1C_2 3^{x-1} = {}^1C_2 3^{x-1} - 2^{x-1}.$$

Therefore, since $_2 y_1 = 0$, one will have ${}^1C_2 = 1$; hence

$${}^1C_n = \mp \frac{1}{1.2.3\ldots(n-2)},$$

the $+$ sign having place if $n$ is even, and the $-$ sign if it is odd. One will find, by a similar calculation,

$$^{2}C_{n} = \pm\frac{1}{1.2}\,\frac{1}{1.2.3\ldots(n-3)},$$
$$^{3}C_{n} = \mp\frac{1}{1.2.3}\,\frac{1}{1.2.3\ldots(n-4)},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Therefore

$$_{n}y_{x} = \pm\frac{1}{1.2.3\ldots(n-1)}\left[2^{x-1} - \frac{n-1}{1}3^{x-1} + \frac{(n-1)(n-2)}{1.2}4^{x-1} - \frac{(n-1)(n-2)(n-3)}{1.2.3}5^{x-1} + \ldots \pm (n+1)^{x-1}\right],$$

the $+$ sign having place if $n$ is odd, and the $-$ sign if it is even. Let $n = 4$ and $x = 7$; one will have

$$_{4}y_{7} = -\frac{1}{1.2.3}(2^{6} - 3.3^{6} + 4.4^{6} - 5^{6}) = 910.$$

## XXII.

PROBLEM VII. — *The differential equation*

$$_{n}y_{x} + A_{n\cdot n}y_{x-1} + {}^{1}A_{n\cdot n}y_{x-2} + \ldots + N_{n} = B_{n\cdot n-1}y_{x} + {}^{1}B_{n\cdot n-1}y_{x-1} + \ldots + C_{n\cdot n-2}y_{x} + {}^{1}C_{n\cdot n-2}y_{x-1} + \ldots$$

*being given, one proposes to integrate it.*

In following the analysis of the preceding Problem, I make $_{1}y_{x} = \phi(x)$ and $_{2}y_{x} = {}^{1}\phi(x)$; the proposed equation will give therefore

$$_{3}y_{x} + A_{3\cdot 3}y_{x-1} + {}^{1}A_{3\cdot 3}y_{x-2} + \ldots + N_{3} = B_{3}.{}^{1}\phi(x) + {}^{1}B_{3}.{}^{1}\phi(x-1) + \ldots + C_{3}.\phi(x) + {}^{1}C_{3}.\phi(x-1) + \ldots$$

and

$$_{4}y_{x} + A_{4\cdot 4}y_{x-1} + {}^{1}A_{4\cdot 4}y_{x-2} + \ldots + N_{4}$$
$$= B_{4\cdot 3}y_{x} + {}^{1}B_{4\cdot 3}y_{x-1} + \ldots$$
$$+ C_{4}.{}^{1}\phi(x) + {}^{1}C_{4}.{}^{1}\phi(x-1) + \ldots$$

whence one will draw

$$\begin{aligned}
&{}_4y_x + A_4 \cdot {}_4y_{x-1} + {}^1A_4 \cdot {}_4y_{x-2} + \ldots + N_4 \\
&\quad + A_4({}_4y_{x-1} + A_4 \cdot {}_4y_{x-2} + \ldots) \\
&\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\qquad = B_4({}_3y_x + A_3 \cdot {}_3y_{x-1} + \ldots) \\
&\quad + {}^1B_4({}_3y_{x-1} + A_3 \cdot {}_3y_{x-2} + \ldots) \\
&\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\quad + C_4 . {}^1\phi(x) + {}^1C_3 . {}^1\phi(x-1) + \ldots \\
&\quad + A_3 . C_4 . {}^1\phi(x-1) + \ldots
\end{aligned}$$

Now, if one substitutes into this equation, instead of

$$\begin{gathered}
{}_3y_x + A_3 \cdot {}_3y_{x-1} + \ldots, \\
{}_3y_{x-1} + A_3 \cdot {}_3y_{x-2} + \ldots,
\end{gathered}$$

their values, one will have an equation of this form

$${}_4y_x = a_4 \cdot {}_4y_{x-1} + {}^1a_4 \cdot {}_4y_{x-2} + {}^2a_4 \cdot {}_4y_{x-3} + \ldots + {}_4u_x$$

This equation will be integrated by that which precedes, as soon as one will know ${}_4u_x$ and the roots of the equation

$$1 = \frac{a_4}{f} + \frac{{}^1a_4}{f^2} + \frac{{}^2a_4}{f^3} + \ldots$$

Now it is easy to see that this equation is the same as this one here

$$0 = \left(1 + \frac{A_3}{f} + \frac{{}^1A_3}{f^2} + \ldots\right)\left(1 + \frac{A_4}{f} + \frac{{}^1A_4}{f^2} + \ldots\right).$$

By following the same process for ${}_5y_x$, ${}_6y_x$, …, and generally for ${}_ny_x$, one will arrive to an equation of this form

$$\text{(A)} \qquad {}_ny_x = a_n \cdot {}_ny_{x-1} + {}^1a_n \cdot {}_ny_{x-2} + {}^2a_n \cdot {}_ny_{x-3} + \ldots + {}_nu_x,$$

an equation which will be easily integrable when one will know ${}_nu_x$ and the roots of the equation

$$1 = \frac{a_n}{f} + \frac{{}^1a_n}{f^2} + \frac{{}^2a_n}{f^3} + \ldots$$

Now one will find easily that this equation is the same as this one

$$0 = \left(1 + \frac{A_3}{f} + \frac{{}^1A_3}{f^2} + \ldots\right)\left(1 + \frac{A_4}{f} + \frac{{}^1A_4}{f^2} + \ldots\right)\ldots\left(1 + \frac{A_n}{f} + \frac{{}^1A_n}{f^2} + \ldots\right),$$

whence it is easy to conclude $a_n$, ${}^1a_n, \ldots$

In order to determine presently ${}_nu_x$, I observe that the equation of the Problem gives the following:

$$\begin{aligned}
&{}_ny_x - a_n \cdot {}_ny_{x-1} - {}^1a_n \cdot {}_ny_{x-2} - \ldots + N_n(1 - a_n - {}^1a_n - \ldots)\\
&\quad + A_n({}_ny_{x-1} - {}^1a_n \cdot {}_ny_{x-2} - \ldots)\\
&\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\qquad = B_n({}_{n-1}y_x - a_n \cdot {}_{n-1}y_{x-1} - \ldots)\\
&\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\quad + C_n({}_{n-2}y_x - a_n \cdot {}_{n-2}y_{x-1} - \ldots)\\
&\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

Now one has, by equation (A),

$$\begin{aligned}
{}_ny_x - a_n \cdot {}_ny_{x-1} - \ldots &= {}_nu_x,\\
{}_ny_{x-1} - a_n \cdot {}_ny_{x-2} - \ldots &= {}_nu_{x-1},\\
\ldots\ldots\ldots\ldots\ldots&\ldots\ldots\ldots;
\end{aligned}$$

moreover,

$$\begin{aligned}
&{}_{n-1}y_x - a_n \cdot {}_{n-1}y_{x-1} - {}^1a_n \cdot {}_{n-1}y_{x-2} - \ldots\\
&= {}_{n-1}y_x - a_{n-1} \cdot {}_{n-1}y_{x-1} - \ldots + A_n({}_{n-1}y_{x-1} - a_{n-1} \cdot {}_{n-1}y_{x-2} - \ldots) + \ldots\\
&= {}_{n-1}u_x + A_n \cdot {}_{n-1}u_{x-1} + \ldots;
\end{aligned}$$

similarly,

$$\begin{aligned}
&{}_{n-2}y_x - a_{n-1} \cdot {}_{n-2}y_{x-1} - \ldots\\
&= {}_{n-2}y_x - a_{n-2} \cdot {}_{n-2}y_{x-1} - \ldots + A_{n-1}({}_{n-2}y_{x-1} - \ldots) + \ldots\\
&= {}_{n-2}u_x + A_{n-1} \cdot {}_{n-2}u_{x-1} + \ldots
\end{aligned}$$

and

$$\begin{aligned}
&{}_{n-2}y_x - a_n \cdot {}_{n-2}y_{x-1} - \ldots\\
&= {}_{n-2}y_x - a_{n-1} \cdot {}_{n-2}y_{x-1} - \ldots + A_n({}_{n-2}y_{x-1} - \ldots) + \ldots\\
&= {}_{n-2}u_x + A_{n-1} \cdot {}_{n-2}u_{x-1} + \ldots + A_n({}_{n-2}u_{x-1} + A_{n-1} \cdot {}_{n-2}u_{x-2} + \ldots) + \ldots;
\end{aligned}$$

therefore

$$\text{(V)}\quad \left\{\begin{aligned}
&{}_nu_x + A_n \cdot {}_nu_{x-1} + {}^1A_n \cdot {}_nu_{x-2} + \ldots + N_n(1 - a_n - {}^1a_n - \ldots)\\
&= B_n({}_{n-1}u_x + A_n \cdot {}_{n-1}u_{x-1} + \ldots) + {}^1B_n({}_{n-1}u_{x-1} + \ldots) + \ldots\\
&\quad + C_n[{}_{n-2}u_x + A_{n-1} \cdot {}_{n-2}u_{x-1} + \ldots + A_n({}_{n-2}u_{x-1} + A_{n-1} \cdot {}_{n-2}u_{x-2} \ldots)\\
&\qquad\qquad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots]\\
&\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right.$$

In order to integrate this equation, one will observe that the value of ${}_nu_x$ must have this form

$$\begin{aligned}{}_nu_x =&b_n\phi(x)+{}^1b_n\phi(x-1)+{}^2b_n\phi(x-2)+\dots\\&+c_n{}^1\phi(x)+{}^1c_n{}^1\phi(x-1)+{}^2c_n{}^1\phi(x-2)+\dots+g_n.\end{aligned}$$

There is no longer now a question but to determine $b_n$, ${}^1b_n, \dots, c_n$, ${}^1c_n, \dots, g_n$. For this, one will substitute this value of ${}_nu_x$ into equation (V), this which gives

$$\begin{aligned}&b_n\phi(x)+\phi(x-1)({}^1b_n+A_n.b_n)+\dots\\&\quad+c_n{}^1\phi(x)+{}^1\phi(x-1)({}^1c_n+A_n.c_n)+\dots\\&\quad=\phi(x)(B_nb_{n-1}+C_nb_{n-2})\\&\quad+\phi(x-1)[B_n{}^1b_{n-1}+B_nA_nb_{n-1}+{}^1B_nb_{n-1}\\&\qquad+C_n{}^1b_{n-2}+C_nA_{n-1}b_{n-2}+C_nA_nb_{n-2}+{}^1C_nb_{n-2}]\\&\quad+\dots\dots\dots\dots\dots\dots\\&\quad+{}^1\phi(x)(B_nc_{n-1}+C_nc_{n-2})\\&\quad+{}^1\phi(x-1)[B_n{}^1c_{n-1}+B_nA_nc_{n-1}+{}^1B_nc_{n-1}\\&\qquad+C_n{}^1c_{n-2}+C_nA_{n-1}c_{n-2}+C_nA_nc_{n-2}+{}^1C_nc_{n-2}]\\&\quad+\dots\dots\dots\dots\dots\dots\end{aligned}$$

whence one will have

$$\begin{aligned}b_n =&B_nb_{n-1}+C_nb_{n-2}\\{}^1b_n =&B_n{}^1b_{n-1}+C_n^1b_{n-2}+b_{n-1}(B_nA_n+{}^1B_n+C_nA_{n-1})+b_{n-2}(C_nA_n+{}^1C_n),\\&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\c_n =&B_nc_{n-1}+C_nc_{n-2},\\&\dots\dots\dots\dots;\end{aligned}$$

by integrating, one will have the values of $b_n$, ${}^1b_n, \dots, c_n$, ${}^1c_n, \dots$

These equations ascend to the second differences, their integral must contain two arbitrary constants. Now, by supposing $n=1$,

$${}_ny_x=\phi(x).$$

One must therefore have then

$$\begin{aligned}&b_n=1, &&{}^1b_n=0, &&{}^2b_n=0, &&\dots,\\&c_n=0, &&{}^1c_n=0, &&{}^2c_n=0, &&\dots,\end{aligned}$$

Moreover, by supposing $n=2$,

$${}_ny_x={}^1\phi(x).$$

Therefore then

$$b_n = 0, \qquad {}^1b_n = 0, \qquad {}^2b_n = 0, \qquad \ldots,$$
$$c_n = 1, \qquad {}^1c_n = 0, \qquad {}^2c_n = 0, \qquad \ldots,$$

By means of these conditions, it will be easy to determine the arbitrary constants. Knowing thus the expression of ${}_nu_x$, there is no longer a question but to integrate equation (A), and the arbitrary constants that the integration introduces, which can be functions of *n*, will be determined by the method that I have given (Art. XX).

If, instead of the two equations

$$\begin{aligned} {}_1y_x &= \phi(x) \\ {}_2y_x &= {}^1\phi(x). \end{aligned}$$

one had the two following

$$\begin{aligned} &{}_1y_x + E.{}_1y_{x-1} + {}^1E.{}_1y_{x-2} + \ldots + K = 0, \\ &{}_2y_x + H.{}_2y_{x-1} + {}^1H.{}_2y_{x-2} + \ldots + L = F.{}_1y_x + {}^1F.{}_1y_{x-1} + \ldots, \end{aligned}$$

one will arrive, by the preceding method, to an equation of this form

$${}_ny_x = a_n \cdot {}_ny_{x-1} + {}^1a_n \cdot {}_ny_{x-2} + \ldots + {}_nu_x,$$

and one will find that the equation

$$1 = \frac{a_n}{f} + \frac{{}^1a_n}{f^2} + \ldots$$

is the same as this one here:

$$\begin{aligned} 0 = &\left(1 - \frac{E}{f} + \frac{{}^1E}{f^2} + \ldots\right)\left(1 - \frac{H}{f} + \frac{{}^1H}{f^2} + \ldots\right) \\ &\times \left(1 - \frac{A_3}{f} + \ldots\right) \ldots \left(1 - \frac{A_n}{f} + \ldots\right). \end{aligned}$$

In order to determine $u_n$, one must observe that in this case equation (V) becomes

$$\begin{aligned} u_n&(1 + A_n + {}^1A_n + \ldots) + N_n(1 - a_n - {}^1a_n - \ldots) \\ &= u_{n-1}(1 + A_n + \ldots)(B_n + {}^1B_n + \ldots) \\ &\quad + u_{n-1}(1 + A_{n-1} + \ldots)(1 + A_n + \ldots)(C_n + \ldots); \end{aligned}$$

now

$$1 - a_n - {}^1a_n - \ldots = (1 - a_{n-1} - {}^1a_{n-1} - \ldots)(1 + A_n + {}^1A_n + \ldots);$$

therefore

$$\begin{aligned} u_n =& N_n(a_{n-1} + {}^1a_{n-1} + \ldots - 1) \\ &+ u_{n-1}(B_n + {}^1B_n + \ldots) + u_{n-2}(1 + A_{n-1} + \ldots)(C_n + {}^1C_n + \ldots). \end{aligned}$$

This equation being differential of the second order contains two arbitrary constants; they will be determined by means of the values of $u_1$ and $u_2$. Now one has

$$u_1 = -L,$$
$$u_2 = -L(1 + E + {}^1E + \dots) - K(F + {}^1F + \dots).$$

XXIII.

Although, in the last two problems, the equations in the partial differences considered with respect to the variable $n$ do not pass the second order, one sees however that the method will succeed generally, whatever be the degree of the difference of the variables. This method supposes in truth that ${}_1y_x$ or ${}_1y_x$ and ${}_2y_x$, ... according to the degree of the difference of $n$, are given as functions of $x$, or by some linear equations between $x$ and these quantities; now it can happen that this is not. I suppose, for example, that one has the following equations:

$$\begin{aligned}
&{}_1y_x = {}_2y_{x-1},\\
&{}_2y_x = {}_1y_{x-1} + {}_3y_{x-1},\\
&\dots\dots\dots\dots\dots\dots\dots,\\
&{}_ny_x = {}_{n-1}y_{x-1} + {}_{n+1}y_{x-1},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots,\\
&{}_my_x = {}_{m-1}y_{x-1}.
\end{aligned}$$

The equation

$${}_ny_x = {}_{n-1}y_{x-1} + {}_{n+1}y_{x-1}$$

is in the partial differences; but it differs from the preceding equations:

1° In this that ${}_1y_x$ and ${}_2y_x$ are not at all given as functions of $x$, or by two differential equations;

2° In this that it ceases to hold when $n = m$.

As this kind of equations are encountered sometimes, and principally in the analysis of hazards, I am going to give here the manner to integrate them.

I observe for this that, if one was able to reduce the equation

$${}_ny_x = {}_{n-1}y_{x-1} + {}_{n+1}y_{x-1},$$

which is of the third order with respect to $n$, to another of the second order, the problem would be resolved; I suppose indeed that the equation of the second order is

$${}_ny_x = a_n \cdot {}_ny_{x-1} + {}^1a_n \cdot {}_ny_{x-2} + \dots + u_n + b_n \cdot {}_{n+1}y_x + {}^1b_n \cdot {}_{n+1}y_{x-1} + \dots$$

In the case $n = m - 1$, one will have

$${}_{m-1}y_x = a_{m-1} \cdot {}_{m-1}y_{x-1} + {}^1a_{m-1} \cdot {}_{m-1}y_{x-2} + \dots + u_{m-1} + b_{m-1} \cdot {}_my_x + \dots,$$

whence, eliminating ${}_{m-1}y_x$ by means of the equation ${}_my_x = {}_{m-1}y_{x-1}$, one will have an equation in the ordinary differences between $x$ and ${}_my_x$.

All difficulty consists therefore to lower the equation from the third order, with respect to $n$,

$$ {}_{n}y_{x} = {}_{n-1}y_{x-1} + {}_{n+1}y_{x-1} $$

to one of the second order; this is the object of the following problem.

PROBLEM VIII. — *The equation in the partial differences of the second order, with respect to n,*

$$ (\gamma) \qquad \begin{cases} {}_{n}y_{x} = A_{n} \cdot {}_{n}y_{x-1} + {}^{1}A_{n} \cdot {}_{n}y_{x-2} + \ldots + N_{n} \\ \qquad + B_{n} \cdot {}_{n+1}y_{x} + {}^{1}B_{n} \cdot {}_{n+1}y_{x-1} + {}^{2}B_{n} \cdot {}_{n+1}y_{x-2} + \ldots \\ \qquad + C_{n} \cdot {}_{n+1}y_{x} + {}^{1}C_{n} \cdot {}_{n+1}y_{x-1} + {}^{2}C_{n} \cdot {}_{n+1}y_{x-2} + \ldots \end{cases} $$

*being given, it is necessary to lower it to another of the first order with respect to n.*

It is necessary for this that, under a particular assumption for $n$, this equation is reduced to one of the first order. I suppose therefore that, by making $n = 1$, one has this here

$$ (\eta) \qquad {}_{1}y_{x} = F \cdot {}_{1}y_{x-1} + {}^{1}F \cdot {}_{1}y_{x-2} + \ldots + L + H \cdot {}_{2}y_{x} + {}^{1}H \cdot {}_{2}y_{x-1} + \ldots $$

It is easy to see, this put, that equation $(\gamma)$ can always be transformed into the following $(\theta)$, of the second order with respect to $n$,

$$ (\theta) \qquad \begin{cases} {}_{n}y_{x} = a_{n} \cdot {}_{n}y_{x-1} + {}^{1}a_{n} \cdot {}_{n}y_{x-2} + {}^{2}a_{n} \cdot {}_{n}y_{x-1} + \ldots + u_{n} \\ \qquad + b_{n} \cdot {}_{n+1}y_{x} + {}^{1}b_{n} \cdot {}_{n+1}y_{x-1} + {}^{2}b_{n} \cdot {}_{n+1}y_{x-2} + \ldots, \end{cases} $$

from which one will determine the coefficients $a_{n}$, ${}^{1}a_{n}$, ..., $b_{n}$, ${}^{1}b_{n}$, ... in this manner: the equation $(\theta)$ gives this here

$$ \begin{aligned} C_{n} \cdot {}_{n+1}y_{x} &= C_{n}(a_{n-1} \cdot {}_{n-1}y_{x-1} + {}^{1}a_{n-1} \cdot {}_{n-1}y_{x-2} + {}^{2}a_{n} \cdot {}_{n-1}y_{x-3} + \ldots + u_{n-1} \\ &\quad + b_{n-1} \cdot {}_{n}y_{x} + {}^{1}b_{n-1} \cdot {}_{n}y_{x-1} + {}^{2}b_{n-1} \cdot {}_{n}y_{x-2} + \ldots), \\ {}^{1}C_{n} \cdot {}_{n-1}y_{x-1} &= {}^{1}C_{n}(a_{n-1} \cdot {}_{n-1}y_{x-2} + {}^{1}a_{n-1} \cdot {}_{n-1}y_{x-3} + \ldots + u_{n-1} \\ &\quad + b_{n-1} \cdot {}_{n}y_{x-1} + {}^{1}b_{n-1} \cdot {}_{n}y_{x-2} + \ldots) \\ &\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} $$

If one adds these different equations member by member, and if one substitutes in their sum, instead of

$$ C_{n} \cdot {}_{n-1}y_{x} + {}^{1}C_{n} \cdot {}_{n-1}y_{x-1} + \ldots, $$
$$ C_{n} \cdot {}_{n-1}y_{x-1} + {}^{1}C_{n} \cdot {}_{n-1}y_{x-2} + \ldots, $$

their values which furnish equation ($\gamma$), one will have, after having ordered,

$$
\begin{aligned}
{}_ny_x = \frac{1}{1-b_{n-1}C_n}[&{}_ny_{x-1}(a_{n-1}+A_n+b_{n-1}\,{}^1C_n+{}^1b_{n-1}C_n)\\
&+{}_ny_{x-2}({}^1a_{n-1}-a_{n-1}A_n+{}^1A_n\\
&\quad+b_{n-1}\,{}^2C_n+{}^1b_{n-1}\,{}^1C_n+{}^2b_{n-1}C_n)\\
&+{}_ny_{x-3}({}^2a_{n-1}-{}^1a_{n-1}A_n-a_{n-1}\,{}^1A_n+{}^2A_n\\
&\quad+b_{n-1}\,{}^3C_n+{}^1b_{n-1}\,{}^2C_n+{}^2b_{n-1}\,{}^1C_n+{}^3b_{n-1}C_n)\\
&+\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&+{}_{n+1}y_xB_n\\
&+{}_{n+1}y_{x-1}({}^1B_n-a_{n-1}B_n)\\
&+{}_{n+1}y_{x-2}({}^2B_n-a_{n-1}\,{}^1B_n-{}^1a_{n-1}B_n)\\
&+\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&+u_{n-1}(C_n+{}^1C_n+{}^2C_n+\dots)\\
&\quad+N_n(1-a_{n-1}-{}^1a_{n-1}-{}^2a_{n-1}-\dots)].
\end{aligned}
$$

By comparing this equation with equation ($\theta$), one will have

1 °
$$b_n = \frac{B_n}{1-C_nb_{n-1}}.$$

In order to integrate this equation, I make $b_n = \frac{z_{n-1}}{z_n}$; this which gives

$$0 = z_{n-1}+C_nz_{n-2}+B_nz_n,$$

a linear equation in the ordinary differences.

2 °
$${}^1b_n = \frac{{}^1B_n-a_{n-1}B_n}{1-C_nb_{n-1}},$$

3 °
$$a_n = \frac{A_n+a_{n-1}+{}^1b_{n-1}C_n+b_{n-1}\,{}^1C_n}{1-C_nb_{n-1}}.$$

From the first of these equations, one will have

$${}^1b_{n-1} = \frac{{}^1B_{n-1}-a_{n-2}B_{n-1}}{1-C_{n-1}b_{n-2}};$$

substituting this value of $b_{n-1}$ into the second, one will have

$$a_n = \frac{A_n+a_{n-1}+C_n\frac{{}^1B_{n-1}-a_{n-2}B_{n-1}}{1-C_{n-1}b_{n-2}}+b_{n-1}\,{}^1C_n}{1-C_nb_{n-1}},$$

whence one will have $a_n$, hence ${}^1b_n$, and thus the rest.

Finally, one will determine $u_n$ by this equation

$$u_n = u_{n-1}\frac{C_n + {}^1C_n + \ldots + N_n(1 - a_{n-1} - {}^1a_{n-1} - \ldots)}{1 - C_n b_{n-1}}.$$

Equation ($\gamma$) of the second order with respect to $n$ will be lowered to another ($\theta$) of the first order; and one sees that the preceding method will succeed generally, whatever be the order of the proposed.

## XXIV.

*On the equations in finite and partial differences in four variables.*

Until now I have considered the equations in the partial differences among three variables ${}_ny_x$, $n$ and $x$; I am going presently to say a word on those which contain a greater number of them.

I suppose that ${}_{m,n}y_x$ represents a function of three variables $x$, $m$ and $n$, of which I regard the differences as constants and equal to unity; I am able, in this function, to make $m$, $n$ and $x$ vary separately, or two of these quantities at once, or all three together in any relation whatsoever; now, if there exists an equation among these different variations, it will be that which I name an *equation in the partial differences in four variables.* This put,

PROBLEM IX. — *I suppose that one has the equation in the partial differences in four variables*

$$(\Omega)\qquad \begin{cases} {}_{m,n}y_x + {}_mA_n \cdot {}_{m,n}y_{x-1} + {}^1_mA_n \cdot {}_{m,n}y_{x-2} + \ldots + {}_mN_n \\ \qquad + {}_mB_n \cdot {}_{m,n-1}y_x + {}^1_mB_n \cdot {}_{m,n-1}y_{x-1} + {}^2_mB_n \cdot {}_{m,n-1}y_{x-2} + \cdots \\ \qquad = {}_mC_n \cdot {}_{m-1,n}y_x + {}^1_mC_n \cdot {}_{m-1,n}y_{x-1} + {}^2_mC_n \cdot {}_{m-1,n}y_{x-2} + \ldots; \end{cases}$$

*one proposes to determine* ${}_{m,n}y_x$.

I suppose that, in the case of $n = 1$, one has, or one can have the following equation

$${}_{m,1}y_x + D_m \cdot {}_{m,1}y_{x-1} + {}^1D_m \cdot {}_{m,1}y_{x-2} + \ldots + L_n = 0,$$

and that, in the case of $m = 1$, one has, or one can have this here

$${}_{1,n}y_x + E_n \cdot {}_{1,n}y_{x-1} + {}^1E_n \cdot {}_{1,n}y_{x-2} + \ldots + {}^1H_n = 0;$$

one will be able, in this case, to transform equation ($\Omega$) into the following

$$(\varpi)\qquad {}_{m,n}y_x = {}_ma_n \cdot {}_{m,n}y_{x-1} + {}^1_ma_n \cdot {}_{m,n}y_{x-2} + {}^2_ma_n \cdot {}_{m,n}y_{x-3} + \ldots + {}_mu_n,$$

from which one will determine the coefficients in this manner.

This equation gives

$${}_mC_n \cdot {}_{m-1,n}y_x = {}_mC_n({}_{m-1}a_n \cdot {}_{m-1,n}y_{x-1} + {}^1_{m-1}a_n \cdot {}_{m-1,n}y_{x-2} + \ldots + {}_{m-1}u_n),$$
$${}^1_mC_n \cdot {}_{m-1,n}y_{x-1} = {}^1_mC_n({}_{m-1}a_n \cdot {}_{m-1,n}y_{x-2} + {}^1_{m-1}a_n \cdot {}_{m-1,n}y_{x-3} + \ldots + {}_{m-1}u_n),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

If one adds all these equations member by member, and if one eliminates the quantities

$$
\begin{gathered}
{}_{m}C_{n}\cdot {}_{m-1,n}y_{x}+{}^{1}_{m}C_{n}\cdot {}_{m-1,n}y_{x-1}+\dots \\
{}_{m}C_{n}\cdot {}_{m-1,n}y_{x-1}+{}^{1}_{m}C_{n}\cdot {}_{m-1,n}y_{x-2}+\dots
\end{gathered}
$$

by means of equation ($\Omega$), one will have

$$
(\sigma)\quad \left\{
\begin{aligned}
{}_{m,n}y_{x} ={}& {}_{m,n}y_{x-1}({}_{m-1}a_{n}-{}_{m}A_{n})\\
&+{}_{m,n}y_{x-2}({}^{1}_{m-1}a_{n}+{}_{m-1}a_{n}\cdot {}_{m}A_{n}-{}^{1}_{m}A_{n})\\
&+{}_{m,n}y_{x-3}({}^{2}_{m-1}a_{n}+{}^{1}_{m-1}a_{n}\cdot {}_{m}A_{n}+{}_{m-1}a_{n}\cdot {}_{m}A_{n}-{}^{2}_{m}A_{n})\\
&+\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&-{}_{m,n-1}y_{x}\cdot {}_{m}B_{n}\\
&+{}_{m,n-1}y_{x-1}(-{}^{1}_{m}B_{n}+{}_{m-1}a_{n}\cdot {}_{m}B_{n})\\
&+{}_{m,n-1}y_{x-2}(-{}^{2}_{m}B_{n}+{}^{1}_{m-1}a_{n}\cdot {}_{m}B_{n}+{}_{m-1}a_{n}\cdot {}^{1}_{m}B_{n})\\
&+\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&+{}_{m-1}u_{n}({}_{m}C_{n}+{}^{1}_{m}C_{n}+\dots)\\
&-{}_{m}N_{n}(1-{}_{m-1}a_{n}-{}^{1}_{m-1}a_{n}-\dots).
\end{aligned}\right.
$$

This equation is in the partial differences among three variables by considering $m$ as a constant, and it is contained in that of Problem IV of Article XX. Now, since equation ($\sigma$) can be transformed into equation ($\varpi$), one will have, by Article XX, the following equations:

$$
\begin{aligned}
{}_{m}a_{n} &={}_{m}a_{n-1}+{}_{m-1}a_{n}-{}_{m}A_{n},\\
{}^{1}_{m}a_{n} &={}^{1}_{m}a_{n-1}+{}^{1}_{m-1}a_{n}+{}_{m-1}a_{n}\cdot {}_{m}A_{n}-{}^{1}_{m}A_{n}-{}_{m}a_{n-1}({}_{m-1}a_{n}-{}_{m}A_{n}),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
{}_{m}u_{n} &={}_{m}u_{n-1}(-{}_{m}B_{n}-{}^{1}_{m}B_{n}+{}_{m-1}a_{n}\cdot {}_{m}B_{n}+\dots)\\
&\quad+{}_{m-1}u_{n}({}_{m}C_{n}+{}^{1}_{m}C_{n}+\dots)(1-{}_{m}a_{n-1}-{}^{1}_{m}a_{n-1}-\dots)\\
&\quad-{}_{m}N_{n}(1-{}_{m-1}a_{n}-{}^{1}_{m-1}a_{n}-\dots)(1-{}_{m}a_{n-1}-{}^{1}_{m}a_{n-1}-\dots).
\end{aligned}
$$

These equations are in the partial differences in three variables; in order to integrate them, I observe that they are all contained in this here:

$$
(b)\qquad {}_{n}y_{x}={}_{n}R_{x}\cdot {}_{n}y_{x-1}+{}_{n}T_{x}\cdot {}_{n-1}y_{x}+{}_{n}M_{x}.
$$

I suppose therefore that, in the case of $n=1$, one has ${}_{1}y_{x}=\phi(x)$. This put, one will be able always to transform equation (b) into the following

$$
(l)\qquad {}_{n}y_{x}={}_{n}b_{x}\cdot {}_{n}y_{x-1}+{}^{1}_{n}b_{x}\cdot {}_{n}y_{x-2}+{}^{2}_{n}b_{x}\cdot {}_{n}y_{x-3}+{}_{n}z_{x},
$$

whence one will have this here:

$$
{}_{n-1}y_{x}\cdot {}_{n}T_{x}={}_{n}T_{x}({}_{n-1}b_{x}\cdot {}_{n}y_{x-1}+{}^{1}_{n-1}b_{x}\cdot {}_{n}y_{x-2}+\dots)+{}_{n}T_{x}\cdot {}_{n-1}z_{x}]
$$

If one substitutes, instead of ${}_nT_x \cdot {}_{n-1}y_x$, ${}_nT_x \cdot {}_{n-1}y_{x-1}$, …, their values drawn from equation (b), one will have

$$
\begin{aligned}
{}_ny_x = {}_nR_x \cdot {}_ny_{x-1} &+ {}_nM_x \\
&+ {}_{n-1}b_x \left({}_ny_{x-1} - {}_nR_{x-1} \cdot {}_ny_{x-2} - {}_nM_{x-1}\right) \frac{{}_nT_x}{{}_nT_{x-1}} \\
&+ {}_{n-1}^{\;\;1}b_x \left({}_ny_{x-2} - {}_nR_{x-2} \cdot {}_ny_{x-3} - {}_nM_{x-2}\right) \frac{{}_nT_x}{{}_nT_{x-2}} \\
&+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&+ {}_nT_x \cdot {}_{n-1}z_x
\end{aligned}
$$

whence one will draw, by comparing this equation with equation (l),

$$
\begin{aligned}
&{}_nb_x = {}_{n-1}b_x \frac{{}_nT_x}{{}_nT_{x-1}} + {}_nR_x, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
&{}_nz_x = {}_{n-1}z_x \cdot {}_nT_x + {}_nM_x - {}_{n-1}b_x \cdot {}_nM_{x-1} \frac{{}_nT_x}{{}_nT_{x-1}} - \ldots,
\end{aligned}
$$

equations which are integrated easily by Problem I by regarding $n$ as the only variable.

One could make some analogous researches on the partial differences in five, six, etc. variables, and one sees that the preceding method will succeed generally, whatever be the number of these variables.

## XXV.

*Application of the preceding researches to the analysis of chances.*

The present state of the system of Nature is evidently a sequel of that which was in the preceding moment, and, if we imagine an intelligence who, for a given instant, embraces all the relationships of the beings of this universe, she could determine for any time taken in the past or in the future the respective position, the movements, and generally the attachments of all these beings.

Physical astronomy, this of all our attainments which gives the greatest credit to the human spirit, offers us an idea, although imperfect, of that which could be a similar intelligence. The simplicity of the law which moves the celestial bodies, the relationships of their masses and of their distances, permits the analysis to follow, up to a certain point, their movements; and, in order to determine the state of the system of these great bodies in the past or future centuries, it suffices to the geometer that observation gives to him their position and their velocity for any instant: man owes then this advantage to the power of the instrument which he employs, and to the small number of relationships which he embraces in his calculations; but the ignorance of the different causes which compete in the production of events, and their complication, joining to the imperfection of the analysis, prevents pronouncing with the same certitude on the great number of phenomena; there are for him therefore some uncertain things, these are more or less probable. In the impossibility to know them, he has sought to compensate himself by determining their different degrees of possibility, so that we owe to the feebleness of the human mind one of the most delicate and most ingenious theories of Mathematics, known as the science of chances or of probabilities.

Before going further, it is important to fix the sense of these words *chance* and *probability*. We regard a thing as the effect of chance, when it offers to our eyes no regularity, or which announces no design, and when we are ignorant moreover of the causes which have produced it. Chance has therefore no reality in itself; it is only a proper term to designate our ignorance of the manner in which the different parts of a phenomenon are coordinated among themselves and with the rest of Nature.

The notion of probability depends upon our ignorance. If we are assured that, of two events which cannot exist together, one or the other must necessarily happen, and if we see no reason in order that one would happen rather than the other, the existence and the nonexistence of each of them is equally probable. Similarly, if of three events which are mutually exclusive, one must necessarily happen, and if we see no reason in order that one would happen rather than the other, their existence is equally probable, but the nonexistence of each of them is more probable than its existence, and this in the ratio of 2 to 1, because on three equally probable cases there are two which are favorable to it, and one alone which is contrary to it.

The number of possible cases remaining the same, the probability of an event increases with the number of favorable cases; on the contrary, the number of favorable cases remaining the same, it diminishes in measure as the number of possible cases increases; so that it is in direct proportion to the number of favorable cases and in inverse to the number of all the possible cases.

The probability of the existence of an event is thus only the ratio of the number of favorable cases to that of all the possible cases, when we see moreover no reason in order that one of these cases would happen rather than the other. It can be consequently represented by a fraction of which the numerator is the number of favorable cases, and the denominator that of all the possible cases.

Similarly, the probability of the nonexistence of an event is the ratio of the number of the cases which are contrary to it to that of all the possible cases, and must be consequently expressed by a fraction of which the numerator is the number of contrary cases, and the denominator that of all the possible cases.

It follows thence that the probability of the existence of an event added to the probability of its nonexistence makes a sum equal to unity which represents consequently entire certitude, because it is clear that an event must necessarily either rightly happen or fail.

Moreover, a thing happens certainly when all the possible cases are favorable to it, and the fraction which expresses its probability is then unity itself. Certitude can therefore be represented by the unit, and probability by a fraction of certitude; it can approach more and more to unity, and even differ from it less than any given quantity; but it can never become greater. The theory of chances has for object to determine these fractions, and one sees thence that it is the most happy supplement that one can imagine to the uncertainty of our knowledge.

Certitude and probability, such as we just defined them, are evidently comparable between them and can be subjected to a rigorous calculus; it is not therefore some different states of the human mind when it sees only all the possible cases favoring an event, or when, in this number, it realizes many of them which are contrary to it.

These two states are absolutely incomparable, and one cannot say of the first that it is the double, or triple of the second, because truth is indivisible. There happens here the same thing as in all the physico-mathematical sciences; we measure the intensity of light, the different degrees of heat of bodies, their forces, their resistances, etc. In all these researches, the physical causes of our sensations, and not the sensations themselves, are the object of Analysis.

The problem of events serves to determine the expectation or the fear of the persons interested in their existence, and it is under this point of view that the science of chances is one of the most useful of the civil life. This word *expectation* has different meanings: it ordinarily expresses the state of the human mind when there must happen to it any good under certain assumptions which are only probable. In the theory of chances, expectation is the product of the expected sum by the probability to obtain it. In order to distinguish the two meanings of this term, I will call the first *moral expectation*, and the second, *mathematical expectation*.

We imagine $n$ persons who have an equal probability to obtain the sum $a$, and that this sum must certainly belong to one among them; the total probability being 1, or equal to certitude, it is clear that the probability of each of these persons is $\frac{1}{n}$, and consequently their mathematical expectation $\frac{a}{n}$. This is thus the sum which ought to return to them, if they wished, without incurring the risks of the events, sharing the entire sum $a$.

If one of these persons $p$ had a probability double of that of the others, his mathematical expectation and, consequently, the sum which ought to return to him in the sharing would be similarly two times greater; because, if one imagines $n+1$ persons who have an equal probability on the sum $a$, their probability to obtain it will be $\frac{1}{n+1}$, and their mathematical expectation $\frac{a}{n+1}$. Now one can suppose that one among them cedes his claims and his expectation to $p$; this one will acquire consequently a double probability and a double expectation expressed by $\frac{2a}{n+1}$; and in the sharing he must have a sum $\frac{2a}{n+1}$ double of that of the other persons.

We see thence that the mathematical expectation is nothing other than the partial sum which must be returned when one does not wish to incur the risks of the event, by supposing that the apportionment of the entire sum is made proportionally to the probability to obtain it; it is in fact the only equitable manner to apportion it when we set aside all strange circumstances, because with an equal degree of probability one has an equal right to the expected sum.

Moral expectation depends, in this way as the mathematical expectation, on the expected sum and on the probability to obtain it; but is not always proportional to the product of these two quantities; it is ruled by a thousand variable circumstances, that it is nearly always impossible to define, and even more to subject to Analysis; these circumstances, it is true, serve only to increase or to diminish the advantage that procures the expected sum, and so we can regard the moral expectation itself as the product of this advantage by the probability to obtain it; but we must distinguish, in the expected good, its relative value to its absolute value; this here is absolutely independent of the need and of the other reasons which make it wished for, instead of which the first increases with these different motives.

Now we cannot give any determinate rule to appreciate this relative value; there is however a most ingenious point that Mr. Daniel Bernoulli proposes in the Volume of Petersburg for the year 1730. The relative value of a very small sum is, according to this illustrious geometer, proportional to its absolute value divided by the total wealth of the interested person.

This rule is however not general, but it must serve in a great number of circumstances, and it is all that one can desire in this matter.

Most of that which was written on chances has seemed to confuse expectation and moral probability with expectation and mathematical probability, or to regulate at least one by the other; they have wanted thus to give to their theories an extent to which they are not susceptible, this has rendered them obscure and little fit to satisfy the mind accustomed to the rigorous clarity of Geometry. Mr. d'Alembert has proposed against them some very fine objections, which have awakened the attention of the geometers; he has made felt the absurdity which it would have lead them, in a great number of circumstances, after the results of the Calculus of Probabilities, and, consequently, the necessity to establish in these matters a distinction between the mathematical and the moral; this part of the sciences owes to him therefore the advantage to be supported hereafter on some clear principles and to be tightened in its true bounds.

Let one permit me here the following digression on the difficulties of which the analysis of chances has seemed susceptible: the probability of uncertain things and the expectation which is found linked to their existence are, as I have said, the two objects of this Analysis; the distinction established previously between moral expectation and mathematical expectation responds, it seems to me, to all the objections that one could make against the second of these two objects; we examine consequently those which have relationship to the first.

In the research of the probability of events, one starts from this principle, namely that the probability is the number of favorable cases divided by those of all the possible cases, this is evident; there therefore can be difficulty only as much as one would assume an equal possibility to two unequally possible cases; now we cannot be prevented from agreeing that the applications that have been made hitherto of the Calculus of the Probabilities to the objects of civil life are subject to this difficulty. I suppose, for example, that in the game of *heads* and of *tails* the piece that one casts into the air has greater inclination to fall back on one side than the other, but that the two players are unaware of which side has the greatest inclination; it is clear that there are equal odds for *heads* as for *tails*; one can therefore assume on the first toss, as one does ordinarily, that *heads* and *tails* are equally probable; but this assumption is no longer permitted if, for example, one of the players wagers that on two tosses he will bring about *heads*; because then one must take into consideration the possible inequality of *heads* and of *tails*, since, just as one is unaware on what side is found the greatest, however this inequality encourages always the one who wagers that *heads* will not occur in two tosses, in such a way that its probability is greater than if *heads* and *tails* were equally possible; the cause of the error into which one falls comes from this that one assumes equally possible these four cases: 1° *heads* on the first toss, *heads* on the second, that which I designate in this manner (*heads, heads*); 2° (*heads, tails*); 3° (*tails, heads*); 4° (*tails, tails*), that which is not; because these two here (*heads, heads*), (*tails, tails*), are more probable than the two others; in fact, I suppose that $\frac{1+\varpi}{2}$ represents the probability of a side which has the greatest inclination, and $\frac{1-\varpi}{2}$ that of the other side; this put, the probability of (*heads, heads*) will be $\frac{1+2\varpi+\varpi^2}{4}$ if *heads* were the most probable, and $\frac{1-2\varpi+\varpi^2}{4}$ if it were the least probable; but, as there is no more reason to suppose it the one rather than the other, it is necessary to add together these two probabilities and by taking the mean, which gives $\frac{1+\varpi^2}{4}$ for the probability of (*heads, heads*), and hence likewise for that of (*tails, tails*); one will find similarly the probability of (*heads, tails*), or of (*tails, heads*), equal to $\frac{1-\varpi^2}{4}$; one sees therefore that these four cases are not equally possible, and that the inequality of the probabilities of *heads* and of *tails*, provided that one is unaware of what side has the greatest, favors the player who wagers that on two tosses *heads* will not occur.

This which I just said of the game of *heads* and of *tails* is able to be applied to the game of dice, and generally to all the games in which the different events are susceptible to one physical inequality; but, having developed besides this remark with enough extension (*see* in Volume VI of the *Savants étranges* a Memoir *Sur la probabilité des causes par les événements*), I will observe only that, even if one is unaware which are the most probable of these events, however there occurs this of the remarkable, namely, that one can, in nearly all cases, determine to which of the players this inequality is advantageous.

The Theory of chances supposes again that if *heads* and *tails* are equally possible, it will be likewise for all the combinations of them (*heads, heads, heads,* etc.), (*tails, heads, tails,* etc.), etc. Many philosophers have thought that this assumption is incorrect, and that the combinations in which an event occurs many times in sequence are less possible than the others; but it would be necessary to assume for this that the past events have some influence on those which must occur, which is not admissible. I admit, the ordinary march of nature is to intermingle the events, but this comes, it seems to me, from this that the combinations where they are mixed are much more numerous. Here is, however, a specious difficulty, to which it is good to respond. If *heads* happened, for example, twenty times in sequence, one could be quite tempted to believe that this is not the effect of chance, while if *heads* and *tails* were intermingled in any manner, one would not seek the cause. Now, why this difference between these two cases, if it is only because the one is physically less possible than the other? To this, I respond generally that, there where we perceive the symmetry, we believe always to recognize the effect of a cause acting with order, and we reason by this consistently with probabilities, because, a symmetric effect must be necessarily the effect of chance or the one of a regular cause, the second of these assumptions is more probable than the first. Let $\frac{1}{m}$ be the probability of its existence in the case where it would be due to chance, and $\frac{1}{n}$ this probability if it started from a regular cause; the probability of the existence of this cause will be (*see* Volume VI of *Savants étranges*)

$$\frac{\frac{1}{n}}{\frac{1}{m}+\frac{1}{n}} = \frac{1}{1+\frac{n}{m}};$$

whence one sees that the more $m$ will be great with respect to $n$, the more also the probability that the symmetric event is the effect of a regular cause will increase. This is not because the symmetric event is less possible than the others, but because there is greater odds that it is due to a cause acting with order than to pure chance, that we seek this cause. A quite simple example will clarify this remark. I suppose that one finds on a table some printed characters arranged in this order, INFINITÉSIMAL; the reason which leads us to believe that this arrangement is not the effect of chance can come only from this that, physically speaking, it is less possible than the others, because, if the word *infinitésimal* were not used in any language, this arrangement would be neither greater, nor less possible, and yet we would suspect then no particular cause. But, as this word is in use among us, it is incomparably more probable that a person will have thus arranged the preceding characters, than it is only that this disposition is due to chance. I return now to my object.

The uncertainty of human knowledge carries either on the events, or on the cause of the events. If we are assured, for example, that an urn contains only some black and white tickets in a given ratio, and that we ask the probability that by taking at random one of these tickets it will be white, the event is uncertain, but the cause on which depends the probability of its existence, that is to say the ratio of the white tickets to the black, is known.

In the following problem: *An urn being supposed to contain a given number of black and white tickets, if one draws from it a white ticket, to determine the probability that the proportion of the white tickets to the black in the urn is that of $p$ to $q$;* the event is known and the cause unknown.

We can restore to these two classes of problems all those which depend on the Theory of chances. There exists, in truth, a very great number in which the cause and the event seem equally unknown; such is the one: *An urn being supposed able equally to contain all the numbers of white and black tickets from 2 to n inclusively, to determine the probability that by drawing at random two of these tickets, they will be white.* The ratio of the white tickets to the black, the total number of tickets and the event which must result from it are unknown; but one must regard here as cause of the event the equal possibility of all the numbers from 2 to $n$, and the indifference of the tickets to be white or black; thus this problem is of the genre of those in which, the cause being known, the event is unknown.

My design being not to give here a complete treatise on the Theory of chances, I will be content to apply the preceding researches to the solution of many problems related to this Theory; I will limit myself even here to those in which, the cause being known, the question is to determine the events, having considered in one other Memoir the case where one proposes to reascend again from the events to the causes (*see* Volume VI of *Savants étrangers*).

## XXVI.

PROBLEM X. — *If in a pile of $x$ pieces one takes a number at random, it is necessary to determine the probability that this number be even or odd.*

I suppose that we can take indifferently, or one alone, or many, or all these pieces at one time.

This put, let $y_x$ be the sum of the cases in which the number can be even, and ${}^1y_x$ that of the cases in which it can be odd; it is clear that, if we increase the number $x$ of pieces by one unit, the sum of the even cases, represented thus by $y_{x+1}$ will be equal:

1° to the preceding number of even cases; 2° to the preceding number of odd cases, since each of these cases, combined with the new piece, give an even case. We will have therefore

(1) $$y_{x+1} = y_x + {}^1y_x;$$

next the number of odd cases, represented by ${}^1y_{x+1}$ will be equal: 1° to the preceding number ${}^1y_x$ of odd cases; 2° to the preceding number of even cases; 3° to the unit, since the new piece can be taken alone. We will have consequently

(2) $${}^1y_{x+1} = {}^1y_x + y_x + 1.$$

In order to integrate these two equations, I observe that the equation (1) gives

$$\Delta y_x = {}^1y_x \quad \text{hence,} \quad \Delta^2 y_x = \Delta.{}^1y_x.$$

Now equation (2) gives

$$\Delta.{}^1y_x = y_x + 1, \quad \text{therefore} \quad \Delta^2 y_x = y_x + 1;$$

whence it is easy to conclude

$$y_{x+1} = 2y_x + 1,$$

By integrating this equation by Problem I, we will have

$$y_x = A2^x - 1,$$

$A$ being an arbitrary constant; in order to determine it, I observe that, $x$ being 1, we have

$$y_x = 0, \quad \text{therefore} \quad A = \frac{1}{2}, \quad \text{hence} \quad y_x = 2^{x-1} - 1.$$

Now, since we have ${}^1y_x = \Delta y_x$, we will have ${}^1y_x = 2^{x-1}$. The sum of all the possible cases is clearly

$$y_x + {}^1y_x = 2^x - 1.$$

If therefore we call $z_x$ the probability that the number of pieces is even, and ${}_1z_x$ that it is odd, we will have

$$z_x = \frac{2^{x-1} - 1}{2^x - 1} \quad \text{and} \quad {}_1z_x = \frac{2^{x-1}}{2^x - 1};$$

whence there results that there is always more advantage to wager for the odd numbers than for the evens.

I suppose that one is assured that the number $x$ cannot exceed the number $n$, but that this number and all the lesser are equally possible, we will have the sum of all the odd cases $= 2^x + C$. Now, $x$ being 1, we must have $2^x + C = 1$; therefore $C = -1$. We will find similarly the sum of all the even cases $= 2^x - x + C$; now, $x$ being 1, we have

$2^x - x + C = 0$. Therefore $C = -1$; hence, the sum of the odd cases is $2^n - 1$, and the sum of the even cases is $2^n - n - 1$; thus, the probability for the odds is

$$\frac{2^n - 1}{2^{n+1} - n - 2},$$

and the probability for the evens

$$\frac{2^n - n - 1}{2^{n+1} - n - 2}.$$

XXVII.

PROBLEM XI. — *Let a be a sum which Paul constitutes to an annuity, in a way that the interest is $\frac{1}{m}$ of that which is due to him: I suppose that, for some arbitrary reasons, one keeps each year the fraction $\frac{1}{n}$ of this interest, so that Paul, at the end of the first year, for example, must collect only the quantity $\frac{a}{m} - \frac{a}{mn}$, this put, if one pays to him every year the sum $\frac{a}{m}$, and, consequently, more than is due to him, and let the surplus be used to amortize the capital, one asks what this capital will become in the year x.*

Let $y_x$ be this capital in the year $x$; it is clear that, at the end of the year $x$, there will be due to Paul only $y_x\left(\frac{1}{m} - \frac{1}{mn}\right)$. Therefore, since one pays the sum $\frac{a}{m}$, the capital will be diminished by the quantity $\frac{a}{m} - y_x \frac{n-1}{mn}$; hence, we will have

$$y_{x+1} = y_x - \frac{a}{m} + y_x \frac{n-1}{mn}$$

and, integrating as in Problem I,

$$y_x = \frac{na}{n-1} + A\left(1 + \frac{n-1}{mn}\right)^{x-1};$$

now, setting $x = 1$, $y_x = a$; thus,

$$A = -\frac{a}{n-1};$$

hence,

$$y_x = \frac{a}{n-1}\left[n - \left(1 + \frac{n-1}{mn}\right)^{x-1}\right].$$

If we ask the year $x$ at which this capital will be zero, we will have

$$\left(1 + \frac{n-1}{mn}\right)^{x-1} = n;$$

therefore

$$x = 1 + \frac{\ln n}{\ln\left(1 + \frac{n-1}{mn}\right)}.$$

I suppose that the interest be 5 for 100, and that one collects $\frac{1}{10}$ on this interest, we will have

$$m = 20 \quad \text{and} \quad n = 10;$$

hence,

$$x = 53.3.$$

One can resolve in the same manner the following problem:

*A person owes the sum a, and wishes to release himself at the end of h years, so that she owes nothing in the year $h+1$, the interest being always $\frac{1}{m}$ of the quantity due; the question is to find what must she give for this each year.*

Let $p$ be this quantity, and $y_x$ that which she owes in year $x$, we will have, by the preceding method,

$$y_{x+1} = y_x \left(1 + \frac{1}{m}\right) - p,$$

whence I conclude by integrating $y_x = mp + A\left(1 + \frac{1}{m}\right)^{x-1}$. Now, putting $x = 1$, $y_x = a$; thus

$$a = mp + A;$$

hence,

$$y_x = mp + (a - mp)\left(1 + \frac{1}{m}\right)^{x-1};$$

but, by making $x = h+1$, we have

$$y_x = 0,$$

by assumption; therefore

$$p = \frac{a\left(1 + \frac{1}{m}\right)^h}{m\left[\left(1 + \frac{1}{m}\right)^h - 1\right]}.$$

XXVIII.

PROBLEM XII. — *I imagine a solid composed of a number n of perfectly equal faces, and which I designate by the numbers* $1, 2, 3, \ldots, n$*; I wish to have the probability that, in a number x of casts, I will bring about these n faces in sequence in the order* $1, 2, 3, 4, \ldots, n$.

I call $y_x$ this probability, and $u_x$ the number of favorable cases: the number of all the possible cases is $n^x$; because, if we call $t_x$ this number at the cast $x$, it will be $t_{x-1}$ at the cast $x-1$. Now, the number of cases at the cast $x-1$ must be combined with all the faces of the solid, in order to form all the possible cases at the cast $x$; we have therefore

$$t_x = nt_{x-1},$$

this which gives

$$t_x = An^x.$$

Now, setting $x = 1$, $t_x = n$; thus

$$A = 1 \quad \text{and} \quad t_x = n^x.$$

We will have therefore

$$\frac{u_x}{n^x} = y_x.$$

Now $u_x$ is evidently equal to the number of favorable cases at the cast $x - 1$ multiplied by the number of faces of the solid, plus to the number of cases in which the combination $1, 2, 3, \ldots, n$ can happen precisely at the cast $x$; moreover, all the cases in which this combination does not happen at the cast $x - n$ each gives a case in which it will happen precisely at the cast $x$. The number of these cases is $n^{x-n} - u_{x-n}$; we will have therefore

$$u_x = nu_{x-1} + n^{x-n} - u_{x-n}; \quad \text{hence,} \quad y_x = y_{x-1} - \frac{y_{x-n}}{n^n} + \frac{1}{n^n},$$

an equation which we will integrate easily by the preceding methods.

Let $n = 2$: we will have

$$y_x = y_{x-1} - \frac{y_{x-2}}{4} + \frac{1}{4};$$

whence I conclude, by integrating,

$$y_x = 1 + \frac{Ax + B}{2^{x-1}};$$

now, setting $x = 1$, $y_x = 0$, and setting $x = 2$, $y_x = \frac{1}{4}$; thus, $A = -\frac{1}{2}$, and $B = -\frac{1}{2}$; hence, $y_x = 1 - \frac{x+1}{2^x}$.

XXIX.

PROBLEM XIII. — *I suppose a number n of players* $(1), (2), (3), \ldots, (n)$ *play in this way:* $(1)$ *plays with* $(2)$*, and if he wins he wins the game; if he neither loses nor wins, he continues to play with* $(2)$*, until one of the two wins. But if* $(1)$ *loses,* $(2)$ *plays with* $(3)$*; if he wins it, he wins the game; if he neither loses nor wins, he continues to play with* $(3)$*; but if he loses,* $(3)$ *plays with* $(4)$*, and thus in sequence until one of the players has defeated the one who follows him; that is to say* $(1)$ *must be winner over* $(2)$*, or* $(2)$ *over* $(3)$*, or* $(3)$ *over* $(4)$*, …, or* $(n-1)$ *over* $(n)$*, or* $(n)$ *over* $(1)$*. Moreover, the probability of anyone of the players to win over the other equals* $\frac{1}{3}$*, and that of neither winning nor losing equals* $\frac{1}{3}$*. This put, it is necessary to determine the probability that one of these players will win the game at trial x.*

Let $\overset{n}{u}_x$ be the probability that at trial $x$, $(n)$ will be the winner over $(n-1)$: we will have

$$\overset{n}{u}_x = \frac{1}{3}\overset{n}{u}_{x-1} + \frac{1}{3}\overset{n-1}{u}_{x-1}$$

Let now $\overset{1}{z}_x$ be the probability that $(n)$, at trial $x$, will win the game, $\overset{2}{z}_x$ the probability that it will be $(n-1)$, and thus in sequence: we will have $\overset{1}{z}_x = \frac{1}{3}\overset{n}{u}_{x-1}$. Hence,

$$\overset{1}{z}_x - \frac{1}{3}\overset{1}{z}_{x-1} = \frac{1}{3}\overset{2}{z}_{x-1}.$$

We will have likewise

$$\overset{2}{z}_x - \frac{1}{3}\overset{2}{z}_{x-1} = \frac{1}{3}\overset{3}{z}_{x-1},$$
$$\overset{3}{z}_x - \frac{1}{3}\overset{3}{z}_{x-1} = \frac{1}{3}\overset{4}{z}_{x-1},$$
$$\vdots$$

such that these equations are reentrant. This put, by following the method set forth previously for this type of equations, we will have

$$\overset{1}{z}_x - \frac{2}{3}\overset{1}{z}_{x-1} + \frac{1}{3^2}\overset{1}{z}_{x-2} = \frac{1}{3}(\overset{2}{z}_{x-1} - \frac{1}{3}\overset{2}{z}_{x-1}) = \frac{1}{3^2}\overset{4}{z}_{x-3};$$

hence,

$$\overset{1}{z}_x - \frac{3}{3}\overset{1}{z}_{x-1} + \frac{3}{3^2}\overset{1}{z}_{x-2} - \frac{1}{3^3}\overset{1}{z}_{x-3} = \frac{1}{3^3}(\overset{3}{z}_{x-2} - \frac{1}{3}\overset{3}{z}_{x-2}) = \frac{1}{3^3}\overset{4}{z}_{x-3};$$

whence, by continuing to operate so, we will have

$$\overset{1}{z}_x - \frac{n}{3}\overset{1}{z}_{x-1} + \frac{n(n-1)}{1.2}\frac{1}{3^2}\overset{1}{z}_{x-2} - \frac{n(n-1)(n-2)}{1.2.3}\frac{1}{3^3}\overset{1}{z}_{x-3} + \cdots = \frac{1}{3^n}\overset{1}{z}_{x-n};$$

we will have similarly

$$\overset{2}{z}_x - \frac{n}{3}\overset{2}{z}_{x-1} + \frac{n(n-1)}{1.2}\frac{1}{3^2}\overset{2}{z}_{x-2} - \frac{n(n-1)(n-2)}{1.2.3}\frac{1}{3^3}\overset{2}{z}_{x-3} + \cdots = \frac{1}{3^n}\overset{2}{z}_{x-n},$$

and thus in sequence for the other variables $\overset{3}{z}_x, \overset{4}{z}_x, \ldots$

In order to integrate these different equations, it is necessary to solve this here $(f-\frac{1}{3})^n = \frac{1}{3^n}$; or, by making $f - \frac{1}{3} = q$, $q^n - \frac{1}{3^n} = 0$, this which is easy to do, by the beautiful theorem of Cotes. There remains in this way no more difficulty than the determination of the arbitrary constants which come from the integration. For this, it is necessary to have the probability of winning of each player for a number $n$ of trials. Now, for that which regards player $(1)$, his probability of winning on the first trial is $\frac{1}{3}$; on the second trial it is $\frac{1}{3^2}$; on the third trial it is $\frac{1}{3^3}$, ..., so that we have

$$\begin{array}{cccccc} 1, & 2, & 3, & 4, & \ldots, & n, \\ \frac{1}{3}, & \frac{1}{3^2}, & \frac{1}{3^3}, & \frac{1}{3^4}, & \ldots, & \frac{1}{3^n}, \end{array}$$

by setting under each trial the probability of player $(1)$ winning at this trial; we will form likewise for player $(2)$ the sequence

$$\begin{array}{cccccc} 2, & 3, & 4, & 5, & \ldots, & n+1, \\ \frac{1}{3^2}, & \frac{2}{3^3}, & \frac{3}{3^4}, & \frac{4}{3^5}, & \ldots, & \frac{n}{3^{n+1}}, \end{array}$$

and for player $(3)$ this one:

$$\begin{array}{cccccc} 3, & 4, & 5, & 6, & \ldots, & n+2, \\ \frac{1}{3^3}, & \frac{3}{3^4}, & \frac{6}{3^5}, & \frac{10}{3^6}, & \ldots, & \frac{\frac{n(n+1)}{1.2}}{3^{n+2}}, \end{array}$$

and thus in sequence for the other players.

## XXX.

PROBLEM XIV. — *Two players* A *and* B*, of whom the respective skills are in ratio of p to q, play together in a way that, out of a number x of trials, there lacks n of them to player* A*, and consequently $x-n$ to player* B*, in order to win; the question is to determine the respective probabilities of these two players.*

Let ${}_{n}y_{x}$ be the probability of B winning; it is clear that on the following trial it will be, either ${}_{n-1}y_{x-1}$, if B loses, or ${}_{n}y_{x-1}$, if he wins. Now, the probability that he will win is $\frac{q}{p+q}$, and that he will lose, $\frac{p}{p+q}$. We have therefore

$$(g) \qquad {}_{n}y_{x} = \frac{q}{p+q}\,{}_{n}y_{x-1} + \frac{p}{p+q}\,{}_{n-1}y_{x-1}.$$

This equation is in partial differences. In order to integrate I observe that, when $n=1$, we have ${}_{1}y_{x} = \frac{q}{p+q}\,{}_{1}y_{x-1}$, since in this case ${}_{n-1}y_{x} = 0$; we will have therefore by Problem VI, article XX,

$${}_{n}y_{x} = a_{n}\cdot{}_{n}y_{x-1} + {}^{1}a_{n}\cdot{}_{n}y_{x-2} + {}^{1}a_{n}\cdot{}_{n}y_{x-3} + \cdots + u_n,$$

and we will find that the equation

$$0 = 1 - \frac{a_n}{f} - \frac{{}^{2}a_n}{f} - \cdots$$

is the same as this one:

$$0 = \left(f - \frac{q}{p+q}\right)^n.$$

We will have besides $u_n = \frac{p}{p+q}u_{n-1}$, therefore $u_n = H\left(\frac{p}{p+q}\right)^n$. Now, setting $n=1$, $u_n = 0$; thus $H=0$, and $u_n = 0$. The expression of ${}_{n}y_{x}$ will be therefore (art. IX)

$$\begin{aligned}{}_{n}y_{x} = \frac{q^{x-1}}{(p+q)^{x-1}}\Bigg[&C_n + D_n(x-1) + E_n\frac{(x-1)(x-2)}{1.2} + \cdots \\ &+ L_n\frac{(x-1)(x-2)\cdots(x-n+1)}{1.2.3\cdots(n-1)}\Bigg].\end{aligned}$$

In order to determine the arbitrary constants $C_n, D_n, E_n, \ldots$, which can be functions of $n$, I observe that, if one makes $x=n$, we will have ${}_{n}y_{n} = 1$; because it is clear that A loses necessarily, when out of $n$ trials there lacks $n$ of them to him; if one makes $x=n-1$, we will have similarly ${}_{n}y_{n-1} = 1$; because equation $(g)$ gives

$${}_{n}y_{n} = \frac{q}{p+q}\,{}_{n}y_{n-1} + \frac{p}{p+q}\,{}_{n-1}y_{n-1}$$

or

$$1 = \frac{q}{p+q}\,{}_{n}y_{n-1} + \frac{p}{p+q},$$

hence ${}_{n}y_{n-1} = 1$; similarly, if one makes $x = n-2$, we will have ${}_{n}y_{n-2} = 1$, and so in sequence. If therefore one makes in the expression of ${}_{n}y_{x}$, $x = 1$, we will have ${}_{n}y_{1} = 1$; hence, $C_n = 1$. If one makes $x = 2$, we will have

$$1 = (C_n + D_n)\frac{q}{p+q};$$

hence, $D_n = \frac{p}{q}$. If one makes $x = 3$, we will have

$$1 = (C_n + 2D_n + E_n)\frac{q^2}{(p+q)^2} = (1 + 2\frac{p}{q} + E_n)\frac{q^2}{(p+q)^2},$$

therefore $E_n = \frac{p^2}{q^2}$, and thus in sequence; whence it is easy to conclude

$$ {}_{n}y_{x} = \frac{1}{(\frac{p}{q}+1)^{x-1}}\left[1 + \frac{p}{q}(x-1) + \frac{p^2}{q^2}\frac{(x-1)(x-2)}{1.2} + \frac{p^3}{q^3}\frac{(x-1)(x-2)(x-3)}{1.2.3} + \cdots \right.$$
$$\left. + \frac{p^{n-1}}{q^{n-1}}\frac{(x-1)(x-2)\cdots(x-n+1)}{1.2.3\cdots(n-1)}\right].$$

XXXI.

PROBLEM XV. — *Three players* A, B, C*, of whom the respective abilities are represented by the letters* $p, q, r$*, play together in a manner that, out of a number* $x$ *of trials, there lacks* $m$ *to* A*,* $n$ *to* B *and* $x-m-n$ *to* C*; one proposes to determine the respective probability of these three players for winning.*

Let ${}_{m,n}y_{x}$ be the probability of C winning; it is clear that after a new trial it will be, either ${}_{m-1,n}y_{x-1}$, or ${}_{m,n-1}y_{x-1}$, or ${}_{m,n}y_{x-1}$; now, the probability that it will be ${}_{m-1,n}y_{x-1}$ is $\frac{p}{p+q+r}$; the probability that it will be ${}_{m,n-1}y_{x-1}$ is $\frac{q}{p+q+r}$; and the probability that it will be ${}_{m,n}y_{x-1}$ is $\frac{r}{p+q+r}$. We will have therefore

$$(o) \qquad {}_{m,n}y_{x} = \frac{p}{p+q+r}\,{}_{m-1,n}y_{x-1} + \frac{q}{p+q+r}\,{}_{m,n-1}y_{x-1} + \frac{r}{p+q+r}\,{}_{m,n}y_{x-1}.t$$

This equation is in partial differences in four variables, and is integrated by Problem IX; but, for this, it is necessary that one have two particular equations for the case of $m = 1$ and of $n = 1$; in order to find them, I observe that, if one makes $m = 1$, we will have

$$(p) \qquad {}_{1,n}y_{x} = \frac{r}{p+q+r}\,{}_{1,n}y_{x-1} + \frac{q}{p+q+r}\,{}_{1,n-1}y_{x-1},$$

because, when $m = 1$, we have ${}_{m-1,n}y_{x-1} = 0$.

Equation $(p)$ is in partial differences in two variables; in order to integrate it, I observe that, if one supposes $n = 1$, we have

$$ {}_{1,1}y_{x} = \frac{r}{p+q+r}\,{}_{1,1}y_{x-1};$$

from this equation and from equation ($p$), we will conclude easily, by Problem VI,

$$(q)\qquad \begin{cases} {}_{1,n}y_x = n\dfrac{r}{p+q+r}\,{}_{1,n}y_{x-1} - \dfrac{n(n-1)}{1.2}\dfrac{r^2}{(p+q+r)^2}\,{}_{1,n}y_{x-2} \\ +\dfrac{n(n-1)(n-2)}{1.2.3}\dfrac{r^3}{(p+q+r)^3}\,{}_{1,n}y_{x-3} - \cdots \end{cases}$$

We will have similarly

$$(q')\qquad \begin{cases} {}_{m,1}y_x = m\dfrac{r}{p+q+r}\,{}_{m,1}y_{x-1} - \dfrac{m(m-1)}{1.2}\dfrac{r^2}{(p+q+r)^2}\,{}_{m,1}y_{x-2} \\ +\dfrac{m(m-1)(m-2)}{1.2.3}\dfrac{r^3}{(p+q+r)^3}\,{}_{m,1}y_{x-3} - \cdots \end{cases}$$

By means of these equations and of equation ($o$), we will determine, by Problem IX, the general expression of ${}_{m,n}y_x$; thus the problem proposed has no other difficulty than the length of the calculation.

The general method of Problem IX leads to one final very elevated equation; but, by means of particular considerations, I have arrived at the solution of the preceding problem by a much simpler method, that I have developed. I have for brevity $p+q+r=1$, and equation (o) gives

$$(o')\qquad {}_{2,n}y_x = p\cdot{}_{1,n}y_{x-1} + q\cdot{}_{2,n-1}y_{x-1} + r\cdot{}_{2,n}y_{x-1},$$

and if one makes $m=2$, equation ($q'$) gives

$${}_{2,1}y_x = 2r\cdot{}_{2,1}y_{x-1} - r^2\cdot{}_{2,1}y_{x-2}.$$

Let

$$(s)\qquad {}_{2,n}y_x = a_n\cdot{}_{2,n}y_{x-1} + {}^1a_n\cdot{}_{2,n}y_{x-2} + \cdots + {}_nX_x;$$

therefore

$$q\cdot{}_{2,n-1}y_{x-1} = a_{n-1}q\cdot{}_{2,n-1}y_{x-2} + {}^1a_{n-1}q\cdot{}_{2,n-1}y_{x-3} + \cdots + q\cdot{}_{n-1}X_{x-1}.$$

Substituting into this equation, in place of ${}_{2,n-1}y_{x-2}$, ${}_{2,n-1}y_{x-3}$, …, their values deduced from equation ($o'$), we will have

$$\begin{aligned} {}_{2,n}y_x =& (r+a_{n-1})\cdot{}_{2,n}y_{x-1} + ({}^1a_n - a_{n-1}r)\,{}_{2,n}y_{x-2} \\ &+ p\cdot{}_{1,n}y_{x-1} - a_{n-1}p\cdot{}_{1,n}y_{x-2} - \cdots + q\cdot{}_{n-1}X_{x-1}, \end{aligned}$$

whence, by comparing with equation ($s$), we will have:

1° $a_n = a_{n-1} + r$, hence, $a_n = (n+1)r + C$; now, setting $n=1$, $a_n = 2r$; thus, $C=0$.

2° ${}^1a_n = {}^1a_{n-1} - a_{n-1}r$, hence, ${}^1a_n = -\frac{n(n+1)}{1.2}r^2 + C$; now, putting $n=1$, ${}^1a_n = -r^2$; thus, $C=0$.

3° ${}^{2}a_{n} = {}^{2}a_{n-1} + \frac{n(n-1)}{1.2}r^3$; therefore, ${}^{2}a_{n} = \frac{(n-1)n(n+1)}{1.2.3}r^3 + C$; now, setting $n = 1$, ${}^{2}a_{n} = 0$; therefore, $C = 0$, and thus the rest. Hence,

$$\begin{aligned} &p({}_{1,n}y_{x-1} - a_{n-1} \cdot {}_{1,n}y_{x-2} - \cdots) \\ &\quad = p\left[{}_{1,n}y_{x-1} - nr.{}_{1,n}y_{x-2} + \frac{n(n-1)}{1.2}r^2 \cdot {}_{1,n}y_{x-3} - \cdots\right] = 0, \end{aligned}$$

by virtue of equation ($q$).

4° ${}_{n}X_{x} = q.{}_{n-1}X_{x-1}$. Now, we have ${}_{1}X_{x} = 0$; therefore, ${}_{2}X_{x} = 0$, and generally ${}_{n}X_{x} = 0$. We have therefore

$${}_{2,n}y_{x} = (n+1)r.{}_{2,n}y_{x-1} - \frac{n(n+1)}{1.2}r^2 \cdot {}_{2,n}y_{x-2} + \frac{(n-1)n(n+1)}{1.2.3}{}_{2,n}y_{x-3} - \cdots$$

We will have, by an entirely similar process,

$${}_{3,n}y_{x} = (n+2)r.{}_{3,n}y_{x-1} - \frac{(n+2)(n+1)}{1.2}r^2 \cdot {}_{3,n}y_{x-2} + \cdots$$

and generally

$${}_{m,n}y_{x} = (m+n-1)r.{}_{m,n}y_{x-1} - \frac{(m+n-1)(m+n-2)}{1.2}r^2 \cdot {}_{m,n}y_{x-2} + \cdots,$$

an equation of which the integral is

$$\begin{aligned} {}_{m,n}y_{x} = r^{x-2}\bigg[&{}_{m}N_{n}\frac{(x-2)(x-3)\cdots(x-m-n+1)}{1.2.3\ldots(m+n-2)} + {}_{m}M_{n}\frac{(x-2)\cdots(x-m-n+2)}{1.2.3\ldots(m+n-3)} \\ &+ {}_{m}L_{n}\frac{(x-2)\cdots(x-m-n+3)}{1.2.3\ldots(m+n-4)} + {}_{m}K_{n}\frac{(x-2)\cdots(x-m-n+4)}{1.2.3\ldots(m+n-5)} \\ &+ {}_{m}I_{n}\frac{(x-2)\cdots(x-m-n+5)}{1.2.3\ldots(m+n-6)} + \cdots + {}_{m}C_{n}\bigg]. \end{aligned}$$

The difficulty consists presently in determining the arbitrary constants ${}_{m}N_{n}$, ${}_{m}M_{n}, \ldots$, which are able to be functions of $m$ and of $n$.

For this, I assume first $m = 1$, and we will have

$$(\sigma) \quad {}_{1,n}y_{x} = r^{x-2}\left[{}_{1}C_{n} + {}_{1}D_{n}(x-2) + {}_{1}E_{n}\frac{(x-2)(x-3)}{1.2} + \cdots + {}_{1}N_{n}\frac{(x-2)\cdots(x-n)}{1.2.3\ldots(n-1)}\right]$$

Now we have ${}_{1,n}y_{n+1} = 1$, as it is clear, since then no trials lack to player $C$; I take next the equation

$${}_{1,n}y_{x} = r.{}_{1,n}y_{x-1} + q.{}_{1,n-1}y_{x-1}.$$

If one makes $x = n+1$, we have

$${}_{1,n}y_{n+1} = 1 = r.{}_{1,n}y_{n} + q,$$

thus

$${}_{1,n}y_{n} = \frac{1-q}{r};$$

next
$$ {}_{1,n}y_n = \frac{1-q}{r} = r.\,{}_{1,n}y_{n-1} + q\frac{1-q}{r}, $$
thus
$$ {}_{1,n}y_{n-1} = \left(\frac{1-q}{r}\right)^2. $$
We will find similarly
$$ {}_{1,n}y_{n-2} = \left(\frac{1-q}{r}\right)^3, $$
and thus in sequence. This put, if one makes $x = 2$, equation ($\sigma$) will give $\left(\frac{1-q}{r}\right)^{n-1} = {}_1C_n$; if one makes $x = 3$, we will have
$$ \left(\frac{1-q}{r}\right)^{n-2} = r\left[\left(\frac{1-q}{r}\right)^{n-1} + {}_1D_n\right], $$
therefore
$$ {}_1D_n = \left(\frac{1-q}{r}\right)^{n-2}\frac{q}{r}. $$
By making $x = 4$, we will have
$$ {}_1E_n = \left(\frac{1-q}{r}\right)^{n-3}\frac{q^2}{r^2}, $$
and thus in sequence; hence
$$ \begin{aligned} {}_{1,n}y_x = r^{x-2}\Bigg[&\frac{q^{n-1}}{r^{n-1}}\frac{(x-2)\cdots(x-n)}{1.2.3\ldots(n-1)} + \frac{q^{n-2}}{r^{n-2}}\frac{1-q}{r}\frac{(x-2)\cdots(x-n+1)}{1.2.3\ldots(n-2)} \\ &+\frac{q^{n-3}}{r^{n-3}}\left(\frac{1-q}{r}\right)^2\frac{(x-2)\cdots(x-n+2)}{1.2.3\ldots(n-3)} + \cdots + \left(\frac{1-q}{r}\right)^{n-1}\Bigg] \end{aligned} $$
We will have, likewise,
$$ {}_{m,1}y_x = r^{x-2}\left[\frac{p^{m-1}}{r^{m-1}}\frac{(x-2)\cdots(x-m)}{1.2.3\ldots(m-1)} + \frac{p^{m-2}}{r^{m-2}}\frac{1-p}{r}\frac{(x-2)\cdots(x-m+1)}{1.2.3\ldots(m-2)} + \cdots\right]. $$
If one substitutes now into equation ($o$), in place of ${}_{m,n}y_x$, its value found above,

we will have the following equation

$$
\begin{aligned}
&{}_mN_n \frac{(x-3)(x-4)\cdots(x-m-n)}{1.2.3\ldots(m+n-2)} + ({}_mM_n + {}_mN_n)\frac{(x-3)\cdots(x-m-n+1)}{1.2.3\ldots(m+n-3)} \\
&\quad + ({}_mL_n + {}_mM_n)\frac{(x-3)\cdots(x-m-n+2)}{1.2.3\ldots(m+n-4)} + \cdots \\
&\quad = + \frac{p}{r}\,{}_{m-1}N_n \frac{(x-3)\cdots(x-m-n+1)}{1.2.3\ldots(m+n-3)} \\
&\quad + \frac{p}{r}\,{}_{m-1}M_n \frac{(x-3)\cdots(x-m-n+2)}{1.2.3\ldots(m+n-4)} + \cdots \\
&\quad + \frac{q}{r}\,{}_mN_{n-1} \frac{(x-3)\cdots(x-m-n+1)}{1.2.3\ldots(m+n-3)} \\
&\quad + \frac{q}{r}\,{}_mM_{n-1} \frac{(x-3)\cdots(x-m-n+2)}{1.2.3\ldots(m+n-4)} + \cdots \\
&\quad + {}_mN_n \frac{(x-3)\cdots(x-m-n)}{1.2.3\ldots(m+n-2)} \\
&\quad + {}_mM_n \frac{(x-3)\cdots(x-m-n+1)}{1.2.3\ldots(m+n-3)} + \cdots,
\end{aligned}
$$

whence we will form the following equations:

$$
\begin{aligned}
{}_mN_n &= \frac{p}{r}\,{}_{m-1}N_n + \frac{q}{r}\,{}_mN_{n-1}, \\
{}_mM_n &= \frac{p}{r}\,{}_{m-1}M_n + \frac{q}{r}\,{}_mM_{n-1}, \\
{}_mL_n &= \frac{p}{r}\,{}_{m-1}L_n + \frac{q}{r}\,{}_mL_{n-1}, \\
&\vdots
\end{aligned}
$$

Now we have

$$ {}_1N_n = \frac{q^{n-1}}{r^{n-1}}; $$

therefore

$$ {}_2N_n = \frac{p}{r}\frac{q^{n-1}}{r^{n-1}} + \frac{q}{r}\,{}_2N_{n-1}, $$

hence

$$ {}_2N_n = \frac{q^{n-1}}{r^{n-1}}\frac{p}{r}(n+C); $$

now, putting $n=1$, ${}_2N_1 = \frac{p}{r}$; therefore

$$ C = 0. $$

Next

$$ {}_3N_n = \frac{p^2}{r^2}\frac{q^{n-1}}{r^{n-1}}n + \frac{q}{r}\,{}_3N_{n-1}; $$

therefore

$$ {}_3N_n = \frac{q^{n-1}}{r^{n-1}} \left[ \frac{p^2}{r^2} \frac{n(n+1)}{1.2} + C \right]; $$

now, putting $n = 1$, ${}_3N_1 = \frac{p^2}{r^2}$; therefore

$$ C = 0, $$

and generally

$$ {}_mN_n = \frac{p^{m-1}q^{n-1}}{r^{m+n-2}} \frac{n(n+1)\cdots(n+m-2)}{1.2.3\cdots(m-1)}. $$

We have next

$$ {}_1M_n = \frac{1-q}{r} \frac{q^{n-2}}{r^{n-2}}; $$

therefore

$$ {}_2M_n = \frac{q}{r} {}_2M_{n-1} + \frac{p}{r} \frac{1-q}{r} \frac{q^{n-2}}{r^{n-2}}; $$

hence,

$$ {}_2M_n = \frac{q^{n-2}}{r^{n-2}} \frac{p}{r} \frac{1-q}{r} (n-1) + C \frac{q^{n-1}}{r^{n-1}}; $$

now, putting $n = 1$, ${}_2M_n = \frac{1-p}{r}$; therefore

$$ C = \frac{1-p}{r} $$

and

$$ {}_2M_n = \frac{q^{n-2}}{r^{n-2}} \frac{p}{r} \frac{1-q}{r} (n-1) + \frac{q^{n-1}}{r^{n-1}} \frac{1-p}{r}. $$

We will have similarly

$$ {}_3M_n = \frac{q^{n-2}}{r^{n-2}} \frac{p^2}{r^2} \frac{1-q}{r} \frac{(n-1)n}{1.2} + \frac{q^{n-1}}{r^{n-1}} \frac{p}{r} \left( \frac{1-p}{r} n + C \right); $$

now, putting $n = 1$, ${}_3M_n = \frac{p}{r} \left( \frac{1-p}{r} \right)$; therefore

$$ C = 0. $$

By continuing to operate so, we will find generally

$$ \begin{aligned} {}_mM_n = {} & \frac{p^{m-1}q^{n-2}}{r^{m+n-3}} \frac{1-q}{r} \frac{(n-1)n\cdots(n+m-3)}{1.2.3\cdots(m-1)} \\ & + \frac{q^{n-1}p^{m-2}}{r^{m+n-3}} \frac{1-p}{r} \frac{n(n+1)\cdots(n+m-3)}{1.2.3\cdots(m-2)}. \end{aligned} $$

I will observe here, relative to these expressions for ${}_mN_n$ and for ${}_mM_n$, that

$$ \frac{n(n+1)\cdots(n+m-2)}{1.2.3\cdots(m-1)} = \frac{m(m+1)\cdots(m+n-2)}{1.2.3\cdots(n-1)} $$

and that

$$\frac{n(n+1)\cdots(n+m-3)}{1.2.3\cdots(m-2)} = \frac{(m-1)m\cdots(m+n-3)}{1.2.3\cdots(n-1)};$$

whence there results that the quantities ${}_mN_n$ and ${}_mM_n$ remain the same when one changes $p$ to $q$, $m$ to $n$, and reciprocally; this which must be moreover by the nature of the problem. We must say as much of the other quantities ${}_mL_n$, ${}_mK_n$, ....

Presently

$${}_mL_n = \frac{p}{r}\,{}_{m-1}L_n + \frac{q}{r}\,{}_mL_{n-1};$$

now, ${}_1L_n = \frac{q^{n-3}}{r^{n-3}}\frac{p}{r}\left(\frac{1-q}{r}\right)^2$; therefore we will have, by integrating,

$${}_2L_n = \frac{q^{n-3}}{r^{n-3}}\frac{p}{r}\left(\frac{1-q}{r}\right)^2(n-2) + C\frac{q^{n-2}}{r^{n-2}};$$

now, putting $n=2$, $m=2$ and $x=4$, in the expression found above for ${}_{m,n}y_x$, we have

$${}_{2,2}y_4 = r^2({}_2L_2 + 2.{}_2M_2 + {}_2N_2);$$

therefore, since ${}_{2,2}y_4 = 1$,

$${}_2L_2 = \frac{1}{r^2} - \frac{2p}{r^2}(1-q) - \frac{2q}{r^2}(1-p) - \frac{2pq}{r^2};$$

moreover, C equals ${}_2L_2$ in the expression for ${}_2L_n$.

We will find similarly

$$\begin{aligned}{}_3L_n = {} & \frac{q^{n-3}}{r^{n-3}}\frac{p^2}{r^2}\left(\frac{1-q}{r}\right)^2\frac{(n-2)(n-1)}{1.2}\\ & + \frac{q^{n-2}}{r^{n-2}}\frac{p}{r}\,{}_2L_2(n-1)\\ & + C\frac{r^{n-1}}{q^{n-1}},\end{aligned}$$

$C$ being an arbitrary constant; now, putting $n=1$, ${}_3L_n = \left(\frac{1-p}{r}\right)^2$; therefore

$$C = \left(\frac{1-p}{r}\right)^2;$$

hence,

$$\begin{aligned}{}_3L_n = {} & \frac{q^{n-3}}{r^{n-3}}\frac{p^2}{r^2}\left(\frac{1-q}{r}\right)^2\frac{(n-2)(n-1)}{1.2}\\ & + \frac{q^{n-2}}{r^{n-2}}\frac{p}{r}\,{}_2L_2(n-1)\\ & + \left(\frac{1-p}{r}\right)^2\frac{q^{n-1}}{p^{n-1}},\end{aligned}$$

and generally we will have

$$
\begin{aligned}
{}_mL_n = {} & \frac{q^{n-3}p^{m-1}}{r^{m+n-4}}\left(\frac{1-q}{r}\right)^2\frac{(n-2)(n-1)\cdots(n+m-4)}{1.2.3\cdots(m-1)} \\
& + \frac{q^{n-2}p^{m-2}}{r^{m+n-4}}\,{}_2L_2\,\frac{(n-1)\cdots(m+n-4)}{1.2.3\cdots(m-2)} \\
& + \frac{q^{n-1}p^{m-3}}{r^{m+n-4}}\left(\frac{1-p}{r}\right)^2\frac{n\cdots(n+m-4)}{1.2.3\cdots(m-3)}.
\end{aligned}
$$

We have next

$$ {}_2K_n = \frac{q^{n-4}}{r^{n-4}}\frac{p}{r}\left(\frac{1-q}{r}\right)^3 + \frac{q}{r}\,{}_2K_{n-1}; $$

hence,

$$ {}_2K_n = \frac{q^{n-4}}{r^{n-4}}\frac{p}{r}\left(\frac{1-q}{r}\right)^3(n-3) + C\frac{q^{n-3}}{r^{n-3}}; $$

now, putting $n = 3$, we have

$$ C = {}_2K_3. $$

Likewise,

$$ {}_3K_n = \frac{q^{n-4}}{r^{n-4}}\frac{p^2}{r^2}\left(\frac{1-q}{r}\right)^3\frac{(n-3)(n-2)}{1.2} + \frac{q^{n-3}}{r^{n-3}}\frac{p}{r}\,{}_2K_3(n-2) + \frac{q^{n-2}}{r^{n-2}}\,{}_3K_2, $$

and generally we will have

$$
\begin{aligned}
{}_mK_n = {} & \frac{q^{n-4}p^{m-1}}{r^{m+n-5}}\left(\frac{1-q}{r}\right)^3\frac{(n-3)\cdots(m+n-5)}{1.2.3\cdots(m-1)} \\
& + \frac{q^{n-3}p^{m-2}}{r^{m+n-5}}\,{}_2K_3\,\frac{(n-2)\cdots(n+m-5)}{1.2.3\cdots(m-2)} \\
& + \frac{q^{n-2}p^{m-3}}{r^{m+n-5}}\,{}_3K_2\,\frac{(n-1)\cdots(n+m-5)}{1.2.3\cdots(m-3)} \\
& + \frac{q^{n-1}p^{m-4}}{r^{m+n-5}}\left(\frac{1-p}{r}\right)^3\frac{n\cdots(n+m-5)}{1.2.3\cdots(m-4)}.
\end{aligned}
$$

We will determine ${}_2K_3$ and ${}_3K_2$ by means of the following equations:

$$
\begin{aligned}
r^3({}_2K_3 + 3\,{}_2L_3 + 3\,{}_2M_3 + {}_2N_3) = 1, \\
r^3({}_3K_2 + 3\,{}_3L_2 + 3\,{}_3M_2 + {}_3N_2) = 1.
\end{aligned}
$$

The law of the other coefficients ${}_mI_n$, ${}_mH_n, \ldots$ is clear, and it is easy, consequently, to determine them. As for the coefficient ${}_mC_n$, we will determine it by this equation

$$ 1 = r^{m+n-2}\left[{}_mC_n + (m+n-2)\,{}_mD_n + \frac{(m+n-2)(m+n-3)}{1.2}\,{}_mE_n + \cdots\right]. $$

Thus we have therefore a general expression for ${}_{m,n}y_x$ and, consequently, the probability of player C winning; by the same method, and by means of analogous formulas, we would have that of the two other players A and B; in such a way that we have a solution of the Problem of points in the case of three players; a Problem which had not yet been solved, as I know, although the geometers who have occupied themselves in the analysis of chances seemed to desire the solution. (See *Mr. Montmort*, in his work *Sur l'analyse des jeux de hasard*, second edition, page 247.)

I assume in the expression ${}_{m,n}y_x$, $m = 2, n = 3$ and $x = 9$, that is to say that the number of trials which fall to player C is 4: I assume, moreover, $p = q = r = \frac{1}{3}$. This put, we will have

$$ {}_{2,3}y_x = \frac{x-3}{3^{x-2}}\left(\frac{xx+2}{2}\right), $$

and, by supposing $x = 9$, we will have the probability of C, for winning, equal to ${}_{2,3}y_9 = \frac{83}{729}$; in order to have the probability of B, I observe that it is equal to ${}_{2,4}y_9$; now we have

$$ {}_{2,4}y_x = \frac{1}{3^{x-2}}\left[4\frac{(x-2)(x-3)(x-4)(x-5)}{1.2.3.4} + 8\frac{(x-2)(x-3)(x-4)}{1.2.3} \right. $$
$$ \left. +7\frac{(x-2)(x-3)}{1.2} + 5(x-2) - 17\right] $$

If we suppose $x = 9$, we will have

$$ {}_{2,4}y_9 = \frac{195}{729}; $$

the probability of A equals $1 - \frac{83}{729} - \frac{195}{729} = \frac{451}{729}$.

The preceding method could take place again, if, instead of three players, one supposed a greater number.

One can solve the preceding Problem by the method of combinations in an extremely simple manner that is here:

The same things being assumed as in the preceding Problem; let, moreover, $i$ be the number of trials which lacks to player $C$, so that we have $x = m+n+i$; it is evident that the game must end at the latest in $x-2$ trials; therefore the number of all the possible cases, multiplied each by their particular probability, is $(p+q+r)^{m+n+i-2}$. In order to have the number of all the cases in which the player A wins, it is necessary to develop the trinomial $(p+q+r)^{m+n+i-2}$ and to admit only the terms in which $p$ has an exponent equal or superior to $m$; let therefore $Hp^{m+\mu}q^{\nu}r^{n+i-2-\mu-\nu}$ be one of the terms; if the exponents of $q$ and of $r$ are one less than $n$, and the other less than $i$, it is necessary to admit this term in whole; but, if the exponent of $q$, for example, is equal or greater than $n$, it is necessary to reject from this term all the combinations in which $q$ happens $n$ times before $p$ happens $m$ times. Let therefore $\nu = n+\lambda$; I observe, this put, that these combinations are: 1° those in which, $p$ having happened $m-1$ times, $q$ has happened precisely $n$ times; 2° those in which, $p$ having happened $m-2$ times, $q$ has happened precisely $n+1$ times; 3° those in which, $p$ having happened $m-3$ times, $q$ has happened precisely $n+2$ times, etc., and thus in sequence until the combination in which, $p$ having happened $m-\lambda-1$ times, $q$ has happened $n+\lambda$ times, if however $\lambda$ does not exceed $m-1$; because, otherwise, it would be necessary to stop at the combination in which $p$ does not happen at all; presently, the number of cases in which, out of $m+n-1$ trials, $p$ will happen $m-1$, and $q$, $n$ times, is, as one knows,

$$\frac{\Delta(m+n-1)}{\Delta(n)\Delta(m-1)};$$

but, as in the term $Hp^{m+\mu}q^{n+\lambda}r^{i-2-\mu-\lambda}$, $p$ happens $m+\mu$ times, and $q$, $n+\lambda$ times, it is necessary to multiply $\frac{\Delta(m+n-1)}{\Delta(n)\Delta(m-1)}$ by the number of combinations in which, $p$ happening $\mu+1$ times, $q$ happens $\lambda$ times; now the number of these combinations is

$$\frac{\Delta(\mu+\lambda+1)}{\Delta(\mu+1)\Delta(\lambda)};$$

therefore we will have

$$\frac{\Delta(m+n-1)\Delta(\mu+\lambda+1)}{\Delta(n)\Delta(\lambda)\Delta(m-1)\Delta(\mu+1)}$$

for the number of combinations in which $q$ has happened $n$ times, when $p$ has yet happened only $m-1$ times; we will find similarly

$$\frac{\Delta(m+n-1)\Delta(\mu+\lambda+1)}{\Delta(n+1)\Delta(\lambda-1)\Delta(m-2)\Delta(\mu+2)}$$

for the number of cases in which $q$ has happened $n+1$ times, when $p$ has not yet happened $m-2$ times, and thus in sequence. Let therefore

$$Q_{\mu+\lambda} = \left[1+\frac{\lambda(m-1)}{(n-1)(\mu+2)}+\frac{\lambda(\lambda-1)(m-1)(m-2)}{(n+1)(n+2)(\mu+2)(\mu+3)}+\cdots\right] \times\frac{\Delta(m+n-1)\Delta(\mu+\lambda+1)}{\Delta(n)\Delta(m-1)\Delta(\mu+1)\Delta(\lambda)}p^{m+\mu}q^{n+\lambda}r^{i-2-\mu-\lambda};$$

let us designate as $(Q_{\mu+\lambda})$ the sum of all the terms which one can form, by giving to $\mu$ and to $\lambda$, in $Q_{\mu+\lambda}$, all the possible values in whole and positive numbers from zero, in a manner however that $\mu+\lambda$ never exceed $i-2$; let us express next by $(R_{\mu+\lambda})$ that which $(Q_{\mu+\lambda})$ becomes, when we change $q$ to $r$, $n$ to $i$, and reciprocally; this put, the probability of A, for winning, will be

$$\frac{1}{(p+q+r)^{m+n+i-2}} = \left[p^{m+n+i-2}+\frac{m+n+i-2}{1}p^{m+n+i-3}(q+r)+\cdots +\frac{(m+n+i-2)\cdots(m+i-1)}{1.2.3\cdots(n-2)}p^{m}(q+r)^{n+i-2}-(Q_{\mu+\lambda})-(R_{\mu+\lambda})\right].$$

The same method has equal place, whatever be the number of players.

## XXXII.

PROBLEM XVI. — *I suppose the tickets* A1, A2, B1 *and* B2*, contained in an urn, and that two players* A *and* B *play on this condition that* A *choosing the tickets* A1 *and* A2*, and* B *the two others, if one draws each time one alone of these tickets at random, the one of the two players will win, who first will have attained the number $i$, the tickets* A1 *and* B1 *counting for* 1*, and the tickets* A2 *and* B2 *counting for* 2*. This put, if there lacks $n$ units to the player* A*, and $x-n$ units to player* B*, one asks the respective probabilities of the two players* A *and* B *to win.*

Let ${}_n y_x$ be the probability of B winning; if one draws from the urn the ticket A1, it will become ${}_{n-1} y_{x-1}$; if one draws the ticket A2, it will become ${}_{n-2} y_{x-2}$; if the ticket B1 comes out, it will be ${}_n y_{x-1}$; if it is the ticket B2, it will be ${}_n y_{x-2}$; we will have therefore

$$(1) \qquad {}_n y_x = \frac{1}{4}\,{}_n y_{x-1} + \frac{1}{4}\,{}_n y_{x-2} + \frac{1}{4}\,{}_{n-1} y_{x-1} + \frac{1}{4}\,{}_{n-2} y_{x-2}.$$

This equation is integrated as in Problem VII; but, for this, it is necessary to have two particular equations in the two particular suppositions for $n$. Now, if one supposes $n = 0$, we have ${}_0 y_x = 0$, and if one supposes $n = 1$, ${}_1 y_x = \frac{1}{2}\,{}_1 y_{x-1}$, because I suppose that then the two players exclude the tickets A2 and B2. We have therefore, by Problem VII,

$${}_n y_x = a_n \cdot {}_n y_{x-1} + {}^1 a_n \cdot {}_n y_{x-2} + {}^2 a_n \cdot {}_n y_{x-3} + \cdots,$$

and the equation

$$1 = \frac{a_n}{f} + \frac{{}^1 a_n}{f^2} + \frac{{}^2 a_n}{f^3} + \cdots$$

is the same as this

$$0 = \left(1 - \frac{1}{2f}\right)\left(1 - \frac{1}{4f} - \frac{1}{4ff}\right)^{n-1};$$

we will have thus

$$\begin{aligned} {}_n y_x = \frac{A_n}{2^x} + p^x \Bigg[ & N_n \frac{x(x-1)\cdots(x-n+3)}{1.2.3\ldots(n-2)} + M_n \frac{x(x-1)\cdots(x-n+4)}{1.2.3\ldots(n-3)} \\ & + L_n \frac{x(x-1)\cdots(x-n+5)}{1.2.3\ldots(n-4)} + K_n \frac{x(x-1)\cdots(x-n+6)}{1.2.3\ldots(n-5)} \\ & + \cdots + C_n \\ + {}^1 p^x \Bigg[ & {}^1 N_n \frac{x(x-1)\cdots(x-n+3)}{1.2.3\ldots(n-2)} + \cdots \Bigg], \end{aligned}$$

$p$ and ${}^1 p$ being the two roots of the equation

$$f^2 - \frac{1}{4} f = \frac{1}{4},$$

that is $p$ being $\frac{1+\sqrt{17}}{8}$, and ${}^1 p$ being $\frac{1-\sqrt{17}}{8}$.

It is necessary now to determine the arbitrary constants $A_n, N_n, \ldots$. Now, if one substitutes into equation (1), in place of ${}_n y_x$, ${}_n y_{x-1}$, ${}_{n-1} y_{x-1}$, $\ldots$ their values drawn from the expression of ${}_{n}y_{x}$, we will have

$$\begin{aligned}
&\frac{A_n}{2^x}+p^x\left[N_n\frac{(x-2)\cdots(x-n+1)}{1.2.3\ldots(n-2)}+(2N_n+M_n)\frac{(x-2)\cdots(x-n+2)}{1.2.3\ldots(n-3)}\right.\\
&\qquad+(N_n+2M_n+L_n)\frac{(x-2)\cdots(x-n+3)}{1.2.3\ldots(n-4)}\\
&\qquad+(M_n+2L_n+K_n)\frac{(x-2)\ldots(x-n+4)}{1.2.3\ldots(n-5)}\\
&\qquad\qquad\left.+\cdots C_n\right]\\
&+{}^1p^x\left[{}^1N_n\frac{(x-2)\cdots(x-n+1)}{1.2.3\ldots(n-2)}+\cdots\right]
\end{aligned}$$

$$\begin{aligned}
=\frac{1}{4}p^x&\left\{N_n\left(\frac{1}{p}+\frac{1}{p^2}\right)\frac{(x-2)\cdots(x-n+1)}{1.2.3\ldots(n-2)}\right.\\
&+\left[\frac{N_n}{p}+M_n\left(\frac{1}{p}+\frac{1}{p^2}\right)+\frac{N_{n-1}}{p}\right]\frac{(x-2)\cdots(x-n+2)}{1.2.3\ldots(n-3)}\\
&+\left[\frac{M_n}{p}+L_n\left(\frac{1}{p}+\frac{1}{p^2}\right)+\frac{M_{n-1}}{p}+\frac{N_{n-1}}{p}+\frac{N_{n-2}}{p^2}\right]\frac{(x-2)\cdots(x-n+3)}{1.2.3\ldots(n-4)}\\
&\quad\left.+\left[\frac{L_n}{p}+K_n\left(\frac{1}{p}+\frac{1}{p^2}\right)+\frac{L_{n-1}}{p}+\frac{M_{n-1}}{p}+\frac{M_{n-2}}{p^2}\right]\frac{(x-2)\cdots(x-n+4)}{1.2.3\ldots(n-5)}+\cdots\right\}\\
&+\frac{1}{4}{}^1p^x\left\{{}^1N_n\left(\frac{1}{{}^1p}+\frac{1}{{}^1p^2}\right)\frac{(x-2)\cdots(x-n+1)}{1.2.3\ldots(n-2)}+\cdots\right\}\\
&+\frac{1}{4}\frac{A_n}{2^{x-1}}+\frac{1}{4}\frac{A_n}{2^{x-2}}+\frac{1}{4}\frac{A_{n-1}}{2^{x-1}}+\frac{1}{4}\frac{A_{n-2}}{2^{x-2}}.
\end{aligned}$$

Whence, by considering that

$$1=\frac{1}{4p}+\frac{1}{4pp},$$

we will form the following equations:

$$\begin{aligned}
0&=\frac{1}{2}A_n+\frac{1}{2}A_{n-1}+A_{n-2},\\
2N_n&=\frac{1}{4}\frac{N_n}{p}+\frac{1}{4}\frac{N_{n-1}}{p},\\
2M_n+N_n&=\frac{1}{4}\frac{M_n}{p}+\frac{1}{4}\frac{M_{n-1}}{p}+\frac{1}{4}\frac{N_{n-2}}{p^2}+\frac{1}{4}\frac{N_{n-1}}{p},\\
2L_n+M_n&=\frac{1}{4}\frac{L_n}{p}+\frac{1}{4}\frac{L_{n-1}}{p}+\frac{1}{4}\frac{M_{n-2}}{p^2}+\frac{1}{4}\frac{M_{n-1}}{p},\\
&\vdots
\end{aligned}$$

We will have some similar equations for ${}^1N_n$, ${}^1M_n,\ldots$ We will determine the quantities $C_n$ and ${}^1C_n$, by considering that, when $n=x$, ${}_{n}y_{x}=1$, and that, when $x=2n$,

${}_{n}y_{x} = \frac{1}{2}$; whence we obtain the equations

$$1 = \frac{A_n}{2^n} + p^n \left[ C_n + nD_n + \cdots + \frac{n(n-1)\cdots 3}{1.2.3\ldots(n-2)} N_n \right] + {}^1p^n \left[ {}^1C_n + n\,{}^1D_n + \cdots \right]$$

and

$$\frac{1}{2} = \frac{A_n}{2^{2n}} + p^{2n} \left[ C_n + 2nD_n + \cdots + N_n \frac{2n\cdots(n+3)}{1.2.3\ldots(n-2)} \right] + {}^1p^{2n} \left[ {}^1C_n + 2n\,{}^1D_n + \cdots + {}^1N_n \frac{2n\cdots(n+3)}{1.2\ldots(n-2)} \right].$$

It is necessary now to integrate the preceding equations. Now, if one makes $-\frac{1}{2\sqrt{2}} = \cos q$ and $\frac{\sqrt{7}}{2\sqrt{2}} = \sin q$, which gives very nearly $q = 110\,^\circ\,42'$, we will find (article IX)

$$A_n = 2^{\frac{n}{2}} (\alpha \cos nq + \beta \sin nq),$$

$\alpha$ and $\beta$ being two arbitrary constants. Now, if one makes $n = 0$, we have

$$A_0 = 0 = \alpha;$$

and if one makes $n = 1$, we have

$$A_n = \frac{1}{2},$$

because ${}_{1}y_{x} = \frac{1}{2^{x-1}}$; therefore

$$\beta\sqrt{2}\sin q = \frac{1}{2} \quad \text{and} \quad \beta = \frac{1}{2\sqrt{2}\sin q};$$

hence

$$A_n = 2^{\frac{n-2}{2}} \frac{\sin nq}{\sin q}.$$

The equation

$$2N_n = \frac{1}{4}\frac{N_n}{p} + \frac{1}{4}\frac{N_{n-1}}{p}$$

gives

$$N_n = \frac{Q}{(8p-1)^{n-2}}.$$

This value of $N_n$ commences to take place only when $n = 2$; therefore

$$Q = N_2 \quad \text{and} \quad N_n = \frac{N_2}{(8p-1)^{n-2}};$$

similarly

$${}^1N_n = \frac{{}^1N_2}{(8\,{}^1p-1)^{n-2}}.$$

We will determine $N_2$ and ${}^1N_2$ by these equations

$$1 = \frac{A_2}{2^2} + p^2 \cdot N_2 + {}^1p^2 \cdot {}^1N_2$$
$$\frac{1}{2} = \frac{A_2}{2^4} + p^4 \cdot N_2 + {}^1p^4 \cdot {}^1N_2$$

We will determine in the same manner the other coefficients $M_n, L_n, K_n, \ldots$.

## XXXIII.

PROBLEM XVII. — *Two players* A *and* B *play to this condition, that at each trial, the one who loses will give an écu to the other; I suppose that the skill of* A *be to that of* B*, as p is to q, and that both have a number m of écus; we ask what is the probability that the game will end before, or at the number x of trials.*

I suppose first $p = q$. Let

${}_0y_x$ be the number of cases according to which, at trial $x$, the gain of the two players is null;

${}_1y_x$ be the number of cases according to which the gain of one or the other is 1;

${}_2y_x$ be the number of cases following which the gain is 2, and thus in sequence. This put, we will form the following equations:

$$(\psi)\quad \begin{cases} {}_0y_x = {}_1y_{x-1}, \\ {}_1y_x = 2 \cdot {}_0y_{x-1} + {}_2y_{x-1}, \\ {}_2y_x = {}_1y_{x-1} + {}_3y_{x-1}, \\ {}_3y_x = {}_2y_{x-1} + {}_4y_{x-1}, \\ \vdots \\ (\sigma) \qquad {}_ny_x = {}_{n-1}y_{x-1} + {}_{n+1}y_{x-1}, \\ \vdots \\ {}_{m-1}y_x = {}_{m-2}y_{x-1} \end{cases}$$

In order to show by what process one obtains these equations, I observe that, at each trial, there can happen two different cases, namely, that A wins, or that it is B; now it is clear that the gain cannot be zero at the trial $x$, without having been 1 at the trial $x-1$, and each case in which it is 1 at trial $x-1$ gives a case in which it is null at trial $x$; whence I deduce the equation

$${}_0y_x = {}_1y_{x-1}.$$

Next all the cases in which the gain is null at trial $x-1$ each give two cases in which there is 1 at trial $x$; whence we will have

$${}_1y_x = 2 \cdot {}_0y_{x-1} + {}_2y_{x-1}.$$

It is likewise in the other equations. Finally, we will obtain the last by considering that one must exclude the term ${}_{m}y_{x-1}$, because this term cannot take place, as long as the game is supposed not finite.

The number of all possible cases is $2^x$; because, by naming $h_x$ this number, as there can happen at the following trial two different cases, namely, that A beats B or that B beats A, the number $h_x$, being able to be combined with these two cases, gives consequently $2h_x$ for the number of all possible cases at trial $x+1$; we have therefore

$$h_{x+1} = 2h_x;$$

whence, by integrating,

$$h_x = A2^x,$$

$A$ being an arbitrary constant. Now, putting $x = 1$, $h_x = 2$; therefore

$$A = 1 \quad \text{and} \quad h_x = 2^x.$$

Let presently $u_x$ be the probability that the game will end precisely at the number $x$ of trials: we will have

$$u_x = \frac{{}_{m}y_{x}}{2^x};$$

but we have clearly

$${}_{m}y_{x} = {}_{m-1}y_{x-1};$$

therefore

$$u_x = \frac{{}_{m-1}y_{x-1}}{2^x}.$$

Let $z_x$ be the probability that the game will end before or at the number $x$ of trials, we will have

$$z_x = z_{x-1} + u_x;$$

therefore

$$\Delta z_{x-1} = \frac{{}_{m-1}y_{x-1}}{2^x} \quad \text{or} \quad 2^{x+1}\Delta z_x = {}_{m-1}y_{x}.$$

There is therefore no more but to determine the value of ${}_{m-1}y_{x}$, which can be made by means of the preceding equations ($\psi$). For this, I observe that these equations are able to correspond to Problem VIII by means of a simple preparation; now this preparation consists to form, by means of the first two, an equation among three variables, which we will make by substituting into the second, in place of ${}_{0}y_{x-1}$, its value ${}_{1}y_{x-2}$ deduced from the first, and we will have

$${}_{1}y_{x} = 2 \cdot {}_{1}y_{x-2} + {}_{2}y_{x-1}.$$

Let now

$$(\Omega) \quad {}_{n}y_{x} = a_n \cdot {}_{n}y_{x-2} + {}^{1}a_n \cdot {}_{n}y_{x-4} + \cdots + u_n + b_n \cdot {}_{n+1}y_{x-1} + {}^{1}b_n \cdot {}_{n+1}y_{x-3} + \cdots,$$

It is not necessary to take account, in this equation, of the terms ${}_{n}y_{x-1}$, ${}_{n}y_{x-3}$, $\ldots$, ${}_{n+1}y_{x-2}$, ${}_{n+1}y_{x-4}$, $\ldots$, because these terms are null as soon as ${}_{n}y_{x}$ has any value, seeing that, if the gain is even or odd at trial $x$, it is necessarily odd or even at the trials $x-1, x-3, \ldots$. This put, the equation ($\Omega$) gives

$$
\begin{aligned}
{}_{n-1}y_{x-1} = a_{n-1}\cdot {}_{n-1}y_{x-3} + {}^1a_{n-1}\cdot {}_{n-1}y_{x-5} + \cdots + u_{n-1} \\
+ b_{n-1}\cdot {}_{n}y_{x-2} + {}^1b_{n-1}\cdot {}_{n}y_{x-4} + \cdots
\end{aligned}
$$

If one substitutes into this equation, in place of ${}_{n-1}y_{x-1}$, ${}_{n-1}y_{x-3}, \ldots$, their values that equation ($\sigma$) gives, we will have, after having ordered,

$$
\begin{aligned}
{}_{n}y_{x} = (a_{n-1}+b_{n-1})\,{}_{n}y_{x-2} + ({}^1a_{n-1} + {}^1b_{n-1})\,{}_{n}y_{x-4} + ({}^2a_{n-1} + {}^2b_{n-1})\,{}_{n}y_{x-6} + \cdots \\
+ {}_{n+1}y_{x+1} - a_{n-1}\cdot {}_{n+1}y_{x-3} - {}^1a_{n-1}\cdot {}_{n+1}y_{x-5} - \cdots + u_{n-1}.
\end{aligned}
$$

By comparing this equation with equation ($\Omega$), we will have

$$
\begin{aligned}
b_n &= 1,\\
a_n &= a_{n-1} + b_{n-1},\\
{}^1b_n &= -a_{n-1},\\
{}^1a_n &= {}^1a_{n-1} + {}^1b_{n-1},\\
{}^2b_n &= -{}^1a_{n-1},\\
{}^2a_n &= {}^2a_{n-1} + {}^2b_{n-1},\\
&\vdots\\
u_n &= u_{n-1}.
\end{aligned}
$$

In order to integrate these equations, it is necessary to make the following considerations:

The first equation begins to take place when $n = 1$.

The second begins to exist only when $n = 2$; thus, the arbitrary constant which comes by integrating must be determined by means of the value of $a_n$ when $n = 1$.

The third equation begins to exist when $n = 2$.

The fourth begins to exist only when $n = 3$; and the arbitrary constant which comes by integrating must be determined by means of the value of ${}^1a_n$, when $n = 2$; and thus for the rest.

This put, if one integrates the second equation, we will have

$$a_n = n + C,$$

$C$ being an arbitrary constant; now, putting $n = 1$, we have

$$a_n = 2, \quad \text{thus} \quad C = 1;$$

hence

$${}^1b_n = -a_{n-1} = -n.$$

One must observe that this equation begins to exist only when $n = 2$; now, $n$ being 1, we have

$${}^1b_1 = 0, \quad {}^2b_1 = 0, \quad \ldots,$$

moreover, by making $n=2$, we have

$$^{2}b_2 = -\,^{1}a_1 = 0;$$

likewise,

$$^{3}b_2 = 0, \quad ^{4}b_2 = 0, \quad \ldots, \quad ^{1}a_2 = \,^{1}a_1 + \,^{1}b_1 = 0;$$

similarly,

$$^{2}a_2 = 0, \quad ^{3}a_2 = 0, \quad \ldots,$$

If one integrates the fourth equation, we will have

$$^{1}a_n = -\frac{(n+1)(n-2)}{1.2} + C;$$

in order to determine the constant $C$, one avails oneself of the value of $^{1}a_2$; we have

$$^{1}a_2 = 0, \quad \text{therefore} \quad C = 0;$$

hence

$$^{2}b_n = \frac{n(n-3)}{1.2};$$

this expression of $^{2}b_n$ is able to begin to take place, by the remarks preceding, only when $n=3$; moreover, by making $n=3$, we have

$$^{3}b_3 = -\,^{2}a_2 = 0;$$

similarly,

$$^{4}b_3 = 0, \quad ^{5}b_3 = 0, \quad \ldots, \quad ^{2}a_3 = \,^{2}a_2 + \,^{2}b_2 = 0;$$

similarly,

$$^{3}a_3 = 0, \quad ^{4}a_3 = 0, \quad \ldots$$

The sixth equation gives, by integrating,

$$^{2}a_n = \frac{(n+1)(n-3)(n-4)}{1.2.3} + C.$$

In order to determine $C$, I observe that $^{2}a_3$ equals 0; therefore, $C=0$. Hence

$$^{2}b_n = -\frac{n(n-4)(n-5)}{1.2.3},$$

an expression which is able to begin to exist only when $n=4$, and thus in sequence.

Finally, $u_n = u_{n-1}$; therefore, $u_n = C$. Now, putting $n=1$, $u_n=0$; therefore, $C=0$. Thus we will have

$$\begin{aligned} {}_{n}y_{x} =& (n+1)\,{}_{n}y_{x-2} - \frac{(n+1)(n-2)}{1.2}\,{}_{n}y_{x-4} \\ &+ \frac{(n+1)(n-3)(n-4)}{1.2.3}\,{}_{n}y_{x-6} - \cdots \\ &+ {}_{n+1}y_{x-1} - n\cdot{}_{n+1}y_{x-3} + \frac{n(n-3)}{1.2}\,{}_{n+1}y_{x-5} - \cdots \end{aligned}$$

If one supposes now $n = m-1$, then it is not necessary to take account of the terms ${}_{n+1}y_{x-1}$, ${}_{n+1}y_{x-3}$, ... because these terms are excluded from the equations ($\psi$); we will have therefore

$$ {}_{m-1}y_x = m \cdot {}_{m-1}y_{x-2} - \frac{m(m-3)}{1.2}\,{}_{m-1}y_{x-4} + \frac{m(m-4)(m-5)}{1.2.3}\,{}_{m-1}y_{x-6} - \cdots $$

If one substitutes presently into this equation, in place of ${}_{m-1}y_x$, its value $2^{x+1}\Delta z_x$, we will have, after having integrated,

$$ z_x = m\frac{1}{2^2}z_{x-2} - \frac{m(m-3)}{1.2}\frac{1}{2^4}z_{x-4} + \frac{m(m-4)(m-5)}{1.2.3}\frac{1}{2^6}z_{x-6} + \cdots + C. $$

I suppose now the skills of two players unequal in the ratio of $p$ to $q$; let $p+q=1$. This put, if one asks for the probability of the following combination

$$ \begin{array}{cccccccccc} 1, & 2, & 3, & 4, & 5, & 6, & 7, & \ldots, & x, \\ p, & q, & q, & p, & p, & p, & q, & \ldots, & q, \end{array} $$

which signifies A wins on the first trial, B on the second and on the third, A on the fourth, fifth, and sixth, etc. It is clear that, in order to have this probability, one must multiply all these quantities by one another; naming therefore $r$ the number of times that $p$ is found repeated in this combination, $x-r$ will express how many times $q$ is found repeated; the probability of this combination will be consequently $p^r q^{x-r}$.

If one makes $x-r = r+s$, and if in some place one stops the combination, the number of times that one of the quantities $p$ and $q$ is found more often repeated than the other is always less than $m$, this combination will be one of those in which B will gain $s$ écus to player $A$; now, one is able to make a corresponding combination in which A will gain $s$ écus to B, and the probability of this combination will be $q^r p^{r+s}$, the ratio of this probability to the preceding is that of $p^s$ to $q^s$; whence there results that generally the number of cases according to which A gains $s$ écus to B, each multiplied by their particular probability, is to the number of cases according to which B gains $s$ écus to player A, multiplied by their probability, as $p^s : q^s$.

This put, let ${}_0y_x$ be the number of cases according to which at trial $x$ the gain of the two players is null, each multiplied by their probability. Let ${}_1y_x$, ${}_2y_x$, ... be the number of cases according to which the gain of player A is $1, 2, \ldots$ écus, each multiplied by their particular probability, and if ${}_1\overset{1}{y}_x$, ${}_2\overset{1}{y}_x$, ... express the analogous quantities for player B; it is easy, now by some considerations entirely similar to those according to which I have formed the equations ($\psi$), to obtain the following:

$$ (\psi') \qquad \left\{ \begin{array}{ll} & {}_0y_x = q \cdot {}_1y_{x-1} + p \cdot {}_1\overset{1}{y}_{x-1}, \\ & {}_1y_x = p \cdot {}_0y_{x-1} + q \cdot {}_2y_{x-1}, \\ & {}_2y_x = p \cdot {}_1y_{x-1} + q \cdot {}_3y_{x-1}, \\ & \vdots \\ (\sigma') & {}_ny_x = p \cdot {}_{n-1}y_{x-1} + q \cdot {}_{n+1}y_{x-1}, \\ & \vdots \\ & {}_{m-1}y_x = p \cdot {}_{m-2}y_{x-1} \end{array} \right. $$

Now we have, by the preceding remarks,

$$p \cdot {}_1\overset{1}{y}_{x-1} = q \cdot {}_1y_{x-1}.$$

The first equation becomes therefore

$${}_0y_x = 2q \cdot {}_1y_{x-1},$$

hence

$${}_0y_{x-1} = 2q \cdot {}_1y_{x-2};$$

substituting this value of ${}_0y_{x-1}$ into the second, we will have

$${}_1y_x = 2qp \cdot {}_1y_{x-1} + q \cdot {}_2y_{x-1};$$

it is easy to see that the equations $(\psi')$ correspond in this way to Problem VIII. Let there be therefore

$${}_ny_x = a_n \cdot {}_ny_{x-2} + {}^1a_n \cdot {}_ny_{x-4} + \cdots + u_n + b_n \cdot {}_{n+1}y_{x-1} + {}^1b_n \cdot {}_{n+1}y_{x-3} + \cdots,$$

and we will find, by operating exactly as I have done above, when $p$ and $q$ were equal,

$$\begin{aligned} {}_ny_x = (n+1)pq \cdot {}_ny_{x-2} - \frac{(n+1)(n-2)}{1.2}p^2q^2 \cdot {}_ny_{x-4} + \cdots \\ + q \cdot {}_{n+1}y_{x-1} - npq^2 \cdot {}_{n+1}y_{x-3} + \cdots \end{aligned}$$

Therefore, if one supposes $n = m-1$, we will have

$$(\varpi) \qquad {}_{m-1}y_x = mpq \cdot {}_{m-1}y_{x-2} - \frac{m(m-3)}{1.2}p^2q^2 \cdot {}_{m-1}y_{x-4} + \cdots;$$

by rejecting the terms ${}_my_{x-1}$, ${}_my_{x-3}$, ... which can have no place, according to the supposition that the game does not end before the trial $x$. Let now $u_x$ be the probability that the game will end precisely at trial $x$, it is clear that we will have

$$u_x = {}_my_x + {}_m\overset{1}{y}_x;$$

now we have ${}_my_x : {}_m\overset{1}{y}_x :: p^m : q^m$; therefore

$$u_x = \left(1 + \frac{q^m}{p^m}\right) {}_my_x;$$

moreover,

$${}_my_x = p \cdot {}_{m-1}y_{x-1};$$

hence,

$$u_x = p\left(1 + \frac{q^m}{p^m}\right) {}_{m-1}y_{x-1}.$$

Let $z_x$ be the probability that the game will end before or at trial $x$, we will have

$$\Delta z_x = u_{x+1} = p\left(1+\frac{q^m}{p^m}\right) {}_{m-1}y_x;$$

by substituting therefore, in place of ${}_{m-1}y_x$ this value in equation ($\varpi$), we will have, after having integrated,

$$(\varpi) \qquad \begin{cases} z_x = mpqz_{x-2} - \dfrac{m(m-3)}{1.2}p^2q^2z_{x-4} \\ \qquad\qquad + \dfrac{m(m-3)(m-5)}{1.2.3}p^3q^3z_{x-6} - \cdots + C. \end{cases}$$

In order to determine the arbitrary constant $C$, I observe that, as long as $x$ is less than $m$, $z_x$ equals 0, and that $x$ being equal to $m$, $z_x$ equals $p^m+q^m$; therefore,

$$C = p^m + q^m.$$

Let $1-t_x = z_x$; $t_x$ will express consequently the probability that the game will not end before or at trial $x$, and we will have

$$\begin{aligned} t_x &= mpqt_{x-2} - \frac{m(m-3)}{1.2}p^2q^2t_{x-4} + \cdots \\ &\qquad - p^m - q^m + \left[1 - mpq + \frac{m(m-3)}{1.2}p^2q^2 - \cdots\right]. \end{aligned}$$

Now it is remarkable that we have, whatever be $m$, and by supposing $p+q=1$,

$$0 = 1 - p^m - q^m - mpq + \frac{m(m-3)}{1.2}p^2q^2 - \cdots,$$

or, generally, by supposing any $p$ and $q$,

$$(p+q)^m = mpq(p+q)^{m-2} - \frac{m(m-3)}{1.2}p^2q^2(p+q)^{m-4} + \cdots + p^m + q^m;$$

it is this of which would be able to be convinced by induction, by giving to $m$ different numerical values, but here is a general demonstration of it. We have

$$\begin{aligned} p+q &= p+q, \\ (p+q)^2 &= 2pq(p+q)^0 + p^2 + q^2, \\ (p+q)^3 &= 3pq(p+q) + p^3 + q^3, \\ &\vdots \end{aligned}$$

Let therefore, in general,

$$(\tau) \qquad (p+q)^m = A_m(p+q)^{m-2} + {}^1A_m(p+q)^{m-4} + \cdots + p^m + q^m,$$

and we will have

$$(p+q)^{m+1} = A_m(p+q)^{m-1} + {}^1A_m(p+q)^{m-3} + \cdots \\ + p^{m+1} + q^{m+1} + pq(p^{m-1}+q^{m-1}).$$

Now we have

$$p^{m-1}+q^{m-1} = (p+q)^{m-1} - A_{m-1}(p+q)^{m-3} - \cdots;$$

therefore

$$(p+q)^{m+1} = (A_m+pq)(p+q)^{m-1} \\ + ({}^1A_m - A_{m-1}pq)(p+q)^{m-3} + \cdots + p^{m+1} + q^{m+1}.$$

We have moreover

$$(p+q)^{m+1} = A_{m+1}(p+q)^{m-1} + {}^1A_{m+1}(p+q)^{m-3} + \cdots + p^{m+1} + q^{m+1};$$

whence, by comparing, we will have

$$\begin{aligned} A_{m+1} &= A_m + pq, \\ {}^1A_{m+1} &= {}^1A_m - A_{m-1}pq, \\ {}^2A_{m+1} &= {}^2A_m - {}^1A_{m-1}pq, \\ &\vdots \end{aligned}$$

All these equations are not able to exist at once; the first begins to take place only when $m=1$; the second, when $m=2$; the third, when $m=3$; etc. Moreover, as they assume necessarily known the expressions of $p+q$ and $(p+q)^2$, in order to determine next, in their way, $(p+q)^3$, $(p+q)^4, \ldots$, there results that the law represented by these equations begins to take place when $m+1=3$; thus, the first equation begins to exist when $m=2$; the second, when $m=3$; the third, when $m=4$, etc.

This put, by integrating the first, we have

$$A_m = mpq + C.$$

Now, putting $m=2$, we have

$$A_2 = 2pq;$$

therefore, $C=0$.

Next, the second gives

$${}^1A_m = -\frac{m(m-3)}{1.2}p^2q^2 + C;$$

now, putting $m=3$, ${}^1A_3 = 0$, because $(p+q)$ is not able to have negative exponent in the formula $(\tau)$; therefore $C=0$, and thus for the rest. Therefore

$$(p+q)^m = mpq(p+q)^{m-2} - \frac{m(m-3)}{1.2}p^2q^2(p+q)^{m-4} + \cdots + p^m + q^m;$$

thus we will have

$$(\delta) \qquad t_{x} = mpqt_{x-2} - \frac{m(m-3)}{1.2} p^2q^2 t_{x-4} + \cdots$$

In order to integrate this equation, I begin by observing that it is differential of order $\frac{m}{2}$ or $\frac{m-1}{2}$, according as $m$ is even or odd. Moreover, it is easy to see, by inspection of the equations ($\psi'$), that it begins to exist when $x = m$. Thus, the arbitrary constants which come by the integration must be determined by the values of $t_x$, when one makes $x = 0, x = 2, x = 4, \ldots, x = m-2$ or $x = 1, x = 3, x = 5, \ldots, x = m-2$, according as $m$ is even or odd. Now, all these values are equal to unity, because it is certain that the game cannot end before $m$ trials.

Presently, if one supposes $x'$ equal to $\frac{x}{2}$ or $\frac{x-1}{2}$, according as $m$ is even or odd, we will have

$$t_{x'} = mpqt_{x'-1} - \frac{m(m-3)}{1.2} p^2q^2 t_{x'-2} + \cdots$$

The integral of this equation depends on the resolution of this algebraic equation

$$f^{\frac{m}{2}} = mpqf^{\frac{m}{2}-1} - \frac{m(m-3)}{1.2} p^2q^2 f^{\frac{m}{2}-2} + \cdots,$$

if $m$ is even, or of this

$$f^{\frac{me-1}{2}} = mpqf^{\frac{m-1}{2}-1} - \frac{m(m-3)}{1.2} p^2q^2 f^{\frac{m-1}{2}-2} + \cdots,$$

if $m$ is odd.

Now, if one makes $\cos\phi = y$, we have, as one knows,

$$\cos m\phi = 2^{m-1}y^m - m2^{m-3}y^{m-2} + \frac{m(m-3)}{1.2} 2^{m-5}y^{m-4} - \cdots$$

Let $\cos m\phi = 0$, and we will have

$$0 = y^m - m\frac{1}{4}y^{m-2} + \frac{m(m-3)}{1.2}\frac{1}{4^2}y^{m-4} - \cdots$$

when $m$ is even, or

$$0 = y^{m-1} - m\frac{1}{4}y^{m-3} + \frac{m(m-3)}{1.2}\frac{1}{4^2}y^{m-5} - \cdots$$

when $m$ is odd.

The different values of $y$ in this equation are the cosines of the different arcs, which, multiplied by $m$, have their cosines equal to zero; now the arcs which have their cosines null are $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots, \pi$ expressing the semi-circumference of which the radius is unity. The different values of $y$ are, consequently, plus and minus the cosines of the arcs $\frac{\pi}{2m}, \frac{3\pi}{2m}, \frac{5\pi}{2m}, \ldots$ to $\frac{(m-1)\pi}{2m}$ or $\frac{(m-2)\pi}{2m}$ inclusively, according as $m$ is even or odd; the cosines of the following arcs being the same, with the difference of signs excepted, the one of $\frac{\pi}{2}$ being null; let therefore $l, l_1, l_2, \ldots$ be these different cosines, the values of $y$

will be therefore $\pm l, \pm l_1, \ldots$. Now it is easy to see that $f = 4y^2pq$, hence, the different values of $f$ will be $4l^2pq$, $4l_1^2pq, \ldots$, whence we will have

$$t_x = A(2l\sqrt{pq})^x + A_1(2l_1\sqrt{pq})^x + \cdots,$$

$A, A_1, \ldots$ being some arbitrary constants which will be determined by the method of article IX.

## XXXIV.

PROBLEM XVIII. — *I have supposed, in the preceding problem, that the two players* A *and* B *had an equal number m écus; I suppose actually that player* A *has i écus, and player* B*, m écus; the rest subsisting, as above, we ask the probability that the game will end before, or at the number x of trials.*

It is easy to see that we will have first the equations ($\psi'$) of the preceding Problem. Moreover, we will have the following:

$$(\psi'')\quad \begin{cases} {}_1\overset{1}{y}_x = q\cdot {}_0y_{x-1} + p\cdot {}_2\overset{1}{y}_{x-1}, \\ {}_2\overset{1}{y}_x = q\cdot {}_1\overset{1}{y}_{x-1} + p\cdot {}_3\overset{1}{y}_{x-1}, \\ {}_3\overset{1}{y}_x = q\cdot {}_2\overset{1}{y}_{x-1} + p\cdot {}_4\overset{1}{y}_{x-1}, \\ \quad\vdots \\ {}_n\overset{1}{y}_x = q\cdot {}_{n-1}\overset{1}{y}_{x-1} + p\cdot {}_{n+1}\overset{1}{y}_{x-1}, \\ \quad\vdots \\ {}_{i-1}\overset{1}{y}_x = q\cdot {}_{i-2}\overset{1}{y}_{x-1}. \end{cases}$$

Let

$$\begin{array}{llll} {}_{i-1}\overset{1}{y}_x = {}_1\lambda_x & {}_{i-2}\overset{1}{y}_x = {}_2\lambda_x & {}_{i-3}\overset{1}{y}_x = {}_3\lambda_x & \ldots, \\ {}_0y_x = {}_i\lambda_x & {}_1y_x = {}_{i+1}\lambda_x & {}_2y_x = {}_{i+2}\lambda_x & \ldots, \end{array}$$

and we will have, by reuniting the equations ($\psi'$) and ($\psi''$),

$$\begin{aligned} {}_1\lambda_x &= q\cdot {}_2\lambda_{x-1}, \\ {}_2\lambda_x &= q\cdot {}_3\lambda_{x-1} + p\cdot {}_1\lambda_{x-1}, \\ &\vdots \\ {}_{i+m-1}\lambda_x &= p\cdot {}_{i+m-2}\lambda_{x-1}. \end{aligned}$$

Let

$$(\Omega'')\quad \begin{cases} {}_n\lambda_{x-1} = a_n\cdot {}_n\lambda_{x-2} + {}^1a_n\cdot {}_n\lambda_{x-4} + {}^2a_n\cdot {}_n\lambda_{x-6} + \cdots + u_n \\ \qquad\qquad + b_n\cdot {}_{n+1}\lambda_{x-1} + {}^1b_n\cdot {}_{n+1}\lambda_{x-3} + {}^2b_n\cdot {}_{n-1}\lambda_{x-5} + \cdots, \end{cases}$$

and we will have

$$p\cdot {}_{n-1}\lambda_{x-1} = a_{n-1}p\cdot {}_{n-1}\lambda_{x-3} + {}^{1}a_{n-1}p\cdot {}_{n-1}\lambda_{x-5} + {}^{2}a_{n-1}p\cdot {}_{n-1}\lambda_{x-7} + \cdots + u_{n-1}p$$
$$+ b_{n-1}p\cdot {}_{n}\lambda_{x-2} + {}^{1}b_{n-1}p\cdot {}_{n}\lambda_{x-4} + \cdots$$

Now we have

$${}_{n}\lambda_{x} = q\cdot {}_{n+1}\lambda_{x-1} + p\cdot {}_{n-1}\lambda_{x-1};$$

therefore

$${}_{n}\lambda_{x} = (a_{n-1}+b_{n-1}p)\,{}_{n}\lambda_{x-2} + ({}^{1}a_{n-1}+{}^{1}b_{n-1}p)\,{}_{n}\lambda_{x-4} + ({}^{2}a_{n-1}+{}^{2}b_{n-1}p)\,{}_{n}\lambda_{x-6} + \cdots + u_{n-1}p$$
$$+ q\cdot {}_{n+1}\lambda_{x-1} - a_{n-1}q\cdot {}_{n+1}\lambda_{x-3} - {}^{1}a_{n-1}q\cdot {}_{n+1}\lambda_{x-5} - \cdots,$$

whence we will have, by comparing with equation ($\Omega''$),

$$\begin{aligned}
b_n &= q,\\
a_n &= a_{n-1}+b_{n-1}p,\\
{}^{1}b_n &= -a_{n-1}q,\\
{}^{1}a_n &= {}^{1}a_{n-1}+{}^{1}b_{n-1}p,\\
{}^{2}b_n &= -{}^{1}a_{n-1}q,\\
{}^{2}a_n &= {}^{2}a_{n-1}+{}^{2}b_{n-1}p,\\
&\vdots\\
u_n &= u_{n-1}p.
\end{aligned}$$

One must observe that the first of these equations begins to exist when $n=1$; the second and the third, when $n=2$; the fourth and the fifth, when $n=3$; etc.

This put, if one integrates the second, we will have

$$a_n = (n-1)pq + C;$$

now, putting $n=1$, $a_n=0$; thus $C=0$, hence

$${}^{1}b_n = -a_{n-1}q = -(n-2)pq^2.$$

If we integrate the fourth, we will have

$${}^{1}a_n = -\frac{(n-2)(n-3)}{1.2}p^2q^2 + C;$$

in order to determine the constant $C$, I observe that, when $n=2$, we have

$${}^{1}a_2 = {}^{1}a_1 + {}^{1}b_1 p = 0;$$

therefore $C=0$, hence,

$${}^{2}b_2 = \frac{(n-3)(n-4)}{1.2}p^2q^3.$$

If we integrate the sixth equation, we will have

$$^2a_n = \frac{(n-3)(n-4)(n-5)}{1.2.3}p^3q^3 + C;$$

now we have

$$^2a_3 = {}^2a_2 + {}^1b_2 \quad \text{and} \quad {}^2a_2 = {}^2a_1 + {}^1b_1 = 0;$$

therefore $^2a_3 = 0$, hence $C = 0$, and thus the rest.

Finally, we have $u_n = u_{n-1}p$, therefore $u_n = Cp^n$; now, putting $n = 1$, $u_n = 0$; therefore $C + 0$ and $u_n = 0$; therefore

$$\begin{aligned} {}_n\lambda_x =& (n-1)pq \cdot {}_n\lambda_{x-2} - \frac{(n-2)(n-3)}{1.2}p^2q^2 \cdot {}_n\lambda_{x-4} \\ &+ \frac{(n-3)(n-4)(n-5)}{1.2.3}p^3q^3 \cdot {}_n\lambda_{x-6} - \cdots \\ &+ q \cdot {}_{n+1}\lambda_{x-1} - (n-2)pq^2 \cdot {}_{n+1}\lambda_{x-3} + \frac{(n-3)(n-4)}{1.2}p^2q^3 \cdot {}_{n+1}\lambda_{x-5} \\ &- \cdots \end{aligned}$$

If we make $n = i + m - i$, we will have

$${}_{i+m-1}\lambda_x = {}_{m-1}y_x \quad \text{and} \quad {}_{i+m}\lambda_x = 0;$$

therefore

$$(\mathfrak{u}) \quad \begin{cases} {}_{m-1}y_x = (i+m-2)pq \cdot {}_{m-1}y_{x-2} - \dfrac{(i+m-3)(i+m-4)}{1.2}p^2q^2 \cdot {}_{m-1}y_{x-4} \\ \qquad + \dfrac{(i+m-4)(i+m-5)(i+m-6)}{1.2.3}p^2q^2 \cdot {}_{m-1}y_{x-6} - \cdots \end{cases}$$

If therefore we name $z_x$ the probability that A will win before or at trial $x$, we will have, by a process similar to that of the preceding Problem,

$$(\pi) \qquad z_x = (m+i-2)pqz_{x-2} - \frac{(m+i-3)(m+i-4)}{1.2}p^2q^2z_{x-4} + \cdots + C.$$

Similarly, if we name $\overset{1}{z}_x$ the probability of player B winning before, or at trial $x$, we will have

$$(\pi') \qquad \overset{1}{z}_x = (m+i-2)pq\overset{1}{z}_{x-2} - \frac{(m+i-3)(m+i-4)}{1.2}p^2q^2\overset{1}{z}_{x-4} + \cdots + {}^1C.$$

In order to determine the arbitrary constants which enter into the expressions of $z_x$ and $\overset{1}{z}_x$, I observe that they are to the number of $\frac{m+i}{2}$ if $m+i$ is even, or $\frac{m+i+1}{2}$ if it is odd; now here is in what manner we will have them.

I suppose $m$ and $i$ odd; the equation $(\mathfrak{u})$ will begin visibly to take place only when $x - i - m + 2$ will equal 0, this gives $x = i + m - 2$. The equation $(\pi)$ will begin to exist therefore only when $x$ will equal $i + m + 1$; it is necessary, consequently, to have all the values of $z_x$, from $z_1$ to $z_{i+m+1}$, in order to determine the arbitrary constants of equation $(\pi)$.

If $m$ and $i$ are some even numbers, the equation $(\mathfrak{u})$ will begin to take place only when $x-i-m+2$ will equal 1; this gives $x=i+m-1$. The equation $(\pi)$ begins therefore to take place only when $x$ equals $i+m+2$; it is necessary, consequently, to have the values of $z_x$, from $z_2$ to $z_{i+m+2}$.

If, $m$ being even, $i$ is odd, equation $(\mathfrak{u})$ will begin to take place only when $x-i-m+1$ will equal 1, this gives $x=i+m$. The equation $(\pi)$ has therefore a place only when $x$ equals $i+m+3$; thus it is necessary to have the values of $z_x$, from $z_2$ to $z_{i+m+3}$.

Finally, if, $m$ being odd, $i$ is even, equation $(\mathfrak{u})$ will begin to take place only when $x-i-m+1$ will equal 0, this gives $x=i+m-1$. Equation $(\pi)$ begins therefore to exist only when $x$ equals $i+m+2$. It is necessary consequently to have the values of $z_x$, from $z_1$ to $z_{i+m+2}$.

This put, the number of all the possible cases to trial $m$, each multiplied by their particular probability, will be

$$p^m + mp^{m-1}q + \frac{m(m-1)}{1.2}p^{m-2}q^2 + \cdots + q^m.$$

The number of cases which make A win at trial $m$ equals $p^m$. In order to have the number of cases which make him win precisely at trial $m+2$, it is clear that it is necessary to subtract $p^m$ from the preceding quantity, and to multiply the rest by $p^2+2pq+q^2$, this gives

$$(\chi)\qquad \begin{cases} mp^{m+1}q + \dfrac{m(m-1)}{1.2}p^m q^2 + \dfrac{m(m-1)(m-2)}{1.2.3}p^{m-1}q^3 + \cdots \\ \qquad + 2mp^m q^2 + \dfrac{2m(m-1)}{1.2}p^{m-1}q^3 + \cdots + mp^{m-1}q^3 + \cdots \end{cases}$$

Now, the number of cases which make him win precisely at trial $m+2$ is clearly $mp^{m+1}q$; we have therefore

$$z_{m+2} = p^m(1+mpq).$$

In order to have the number of cases which make A win at trial $m+4$, it is necessary to subtract from the preceding quantity $(\chi)$, $mp^{m+1}q$, to multiply the rest by $p^2+2pq+q^2$, and we will have $\frac{m(m+3)}{1.2}p^{m+2}q^2$ for the number of these cases; thus,

$$z_{m+4} = p^m\left[1 + mpq + \frac{m(m+3)}{1.2}p^2q^2\right].$$

We will find, likewise,

$$z_{m+6} = p^m\left[1 + mpq + \frac{m(m+3)}{1.2}p^2q^2 + \frac{m(m+4)(m+5)}{1.2.3}p^3q^3\right],$$

and thus in sequence; the law of these values of $z_x$ holds to $z_{m+i-2}$; if we have need of further values of $z_x$, one could obtain them easily by this process.

In order to integrate now the equation ($\pi$), it is necessary to have the roots of the equation

$$f^{\frac{m+i-1}{2}} = (m-i-2)pqf^{\frac{m+i-3}{2}} - \frac{(m+i-3)(m+i-4)}{1.2}p^2q^2f^{\frac{m+i-5}{2}} + \cdots,$$

if $m+i$ is odd, or

$$f^{\frac{m+i}{2}-1} = (m-i-2)pqf^{\frac{m+i}{2}-2} - \cdots$$

if $m+i$ is even; now we will find these roots by considering that we have

$$\sin(m+i)z = x\left[2^{m+i-1}u^{m+i-1} - (m+i-2)2^{m+i-3}u^{m+i-3} + \cdots\right],$$

$x$ being the sine and $u$ the cosine of angle $z$; now, putting

$$\sin(m+i)z = 0,$$

we will have

$$u^{m+i-1} = (m+i-2)\frac{1}{4}u^{m+i-3} - \cdots$$

Let $u = \frac{\sqrt{f}}{2\sqrt{pq}}$, and we will have

$$f^{\frac{m+i-1}{2}} = (m+i-2)pqf^{\frac{m+i-3}{2}} - \cdots$$

if $m+i$ is odd, or

$$f^{\frac{m+i}{2}-1} = (m-i-2)pqf^{\frac{m+i}{2}-2} - \cdots$$

if $m+i$ is even; the different values of $u$ are the cosines of the angles $z$, such that $\sin(m+i)z$ equals 0, this gives

$$z = \frac{\pi}{m+i}, \quad z = \frac{2\pi}{m+i}, \quad z = \frac{3\pi}{m+i}, \quad \cdots$$

Let $l$, $l_1$, $l_2$, ... be the cosines of these angles to $\frac{m+i}{2}$ if $m+i$ is even, or $\frac{m+i-1}{2}$ if it is odd; the different values of $f$ will be $4l^2pq$, $4l_1^2pq$, .... These values one time determined, it is easy to find those of $z_x$ and $\overset{1}{z}_x$.

## XXXV.

PROBLEM XIX. — *I suppose two players* A *and* B*, with an equal number m of écus, playing to this condition, that the one who loses will give an écu to the other; let the probability of* A *winning a trial be p; let that of* B *be q; but let it be able to happen that any of them not win, and let the probability of this be r. This put, we ask the probability that the game will end before or at the number x of trials.*

Let ${}_0y_x$ be the number of cases according to which, at the trial $x$, the gain of the two players is null, multiplied by their probabilities; ${}_1y_x$, ${}_2y_x$, ${}_3y_x$, ... the number of cases according to which the gain of player A is 1, 2, 3, ... at trial $x$, multiplied by their probability, and let $\overset{1}{{}_1y}_x$, $\overset{1}{{}_2y}_x$, $\overset{1}{{}_3y}_x$, ... express the same things for player B. This put, we will form the following equations:

$$(-)\qquad \begin{cases} {}_0y_x = r\cdot{}_0y_{x-1} + q\cdot{}_1y_{x-1} + p\cdot\overset{1}{{}_1y}_{x-1}, \\ {}_1y_x = r\cdot{}_1y_{x-1} + q\cdot{}_2y_{x-1} + p\cdot{}_0y_{x-1}, \\ {}_2y_x = r\cdot{}_2y_{x-1} + q\cdot{}_3y_{x-1} + p\cdot{}_1y_{x-1}, \\ \vdots \\ {}_ny_x = r\cdot{}_ny_{x-1} + q\cdot{}_{n+1}y_{x-1} + p\cdot{}_{n-1}y_{x-1}, \\ \vdots \\ {}_{m-1}y_x = r\cdot{}_{m-1}y_{x-1} + p\cdot{}_{m-2}y_{x-1} \end{cases}$$

Now we have

$$p\cdot\overset{1}{{}_1y}_{x-1} = q\cdot{}_1y_{x-1};$$

the first equation will become therefore

$${}_0y_x = r\cdot{}_0y_{x-1} + 2q\cdot{}_1y_{x-1};$$

and, if one combines it with the second, we will have

$${}_1y_x = 2r\cdot{}_1y_{x-1} + (2pq - r^2)\,{}_1y_{x-2} + q\cdot{}_2y_{x-1} - qr\cdot{}_2y_{x-2}.$$

Let now

$${}_ny_x = a_n\cdot{}_ny_{x-1} + {}^1a_n\cdot{}_ny_{x-2} + \cdots + u_n + b_n\cdot{}_{n+1}y_{x-1} + {}^1b_n\cdot{}_{n+1}y_{x-3} + \cdots;$$

therefore

$$\begin{aligned} p\cdot{}_{n-1}y_{x-1} =& a_{n-1}p\cdot{}_{n-1}y_{x-2} + {}^1a_{n-1}p\cdot{}_{n-1}y_{x-2} + \cdots + pu_{n-1} \\ & + b_{n-1}p\cdot{}_ny_{x-2} + {}^1b_{n-1}p\cdot{}_ny_{x-3} + \cdots \end{aligned}$$

Substituting in place of $p\cdot{}_{n-1}y_{x-1}$, $p\cdot{}_{n-1}y_{x-2}$, ... their values that equation $(-)$ gives, we will have

$$\begin{aligned} {}_ny_x =& (a_{n-1} + r)\cdot{}_ny_{x-1} + ({}^1a_{n-1} - a_{n-1}r + pb_{n-1})\,{}_ny_{x-2} \\ & + ({}^2a_{n-1} - {}^1a_{n-1}r + p\cdot{}^1b_{n-1})\,{}_ny_{x-3} + \cdots \\ & + q\cdot{}_{n+1}y_{x-1} - a_{n-1}q\cdot{}_{n+1}y_{x-3} - {}^1a_{n-1}q\cdot{}_{n+1}y_{x-5} - \cdots + pu_{n-1}; \end{aligned}$$

whence, by comparing, we will have

$$\begin{aligned} a_n &= a_{n-1} + r, \\ b_n &= q, \\ {}^1a_n &= {}^1a_{n-1} - a_{n-1}r + pb_{n-1}, \\ {}^1b_n &= -a_{n-1}q, \\ {}^2a_n &= {}^2a_{n-1} - {}^1a_{n-1}r + p\cdot{}^1b_{n-1}, \\ &\vdots \end{aligned}$$

The first of these equations begins to exist when $n$ equals 2; the second, when $n$ equals 1; the third, when $n$ equals 2; etc. We will have therefore, by integrating and adding the appropriate constants,

$$\begin{aligned}
&a_n = r(n+1),\\
&b_n = q,\\
&{}^1a_n = -r^2\cdot\frac{n(n+1)}{1.2} + pq(n+1),\\
&{}^1b_n = -a_{n-1}q = -qrn.
\end{aligned}$$

This last equation being true, when $n$ equals 1, it follows that the fifth equation begins to exist when $n$ equals 2; this gives

$${}^2a_n = r^3\frac{(n+1)n(n-1)}{1.2.3} - pqr(n+1)(n-1).$$

Therefore

$${}^2b_n = qr^2\frac{n(n-1)}{1.2},$$

an equation which begins to exist when $n$ equals 1, because ${}^2b_1$ equals 0. Therefore, the sixth equation begins to exist when $n$ equals 2, and we will have

$$\begin{aligned}
{}^3a_n =& -r^4\frac{(n+1)n(n-1)(n-2)}{1.2.3.4}\\
&+pqr^2(n+1)(n-1)(n-2) - p^2q^2\frac{(n+1)(n-2)}{1.2} + C.
\end{aligned}$$

Now, putting $n=2$, we have

$${}^3a_2 = {}^3a_1 - {}^2a_1r + p\cdot{}^2b_1 = 0,$$

therefore $C=0$, and thus in sequence; finally, $u_n = 0$. We will have therefore, by making $n = m-1$ and rejecting the terms ${}_my_{x-1}$, ${}_my_{x-2}$, $\cdots$

$$\begin{aligned}
{}_{m-1}y_x =& mr\cdot{}_{m-1}y_{x-1} - \left[r^2\frac{m(m-1)}{1.2} - pqm\right]{}_{m-1}y_{x-2}\\
&+\left[r^3\frac{m(m-1)(m-2)}{1.2.3} - pqrm(m-2)\right]{}_{m-1}y_{x-3}\\
&-\left[r^4\frac{m(m-1)(m-2)(m-3)}{1.2.3.4} - pqr^2\frac{m(m-2)(m-3)}{1.2} + p^2q^2\frac{m(m-3)}{1.2}\right]{}_{m-1}y_{x-4}\\
&+\cdots
\end{aligned}$$

If one supposes $r=0$, we will have

$${}_{m-1}y_x = mpq\cdot{}_{m-1}y_{x-2} - \frac{m(m-3)}{1.2}p^2q^2\cdot{}_{m-1}y_{x-4} + \cdots,$$

the same equation as I have found above for that case.

If we name $z_x$ the probability of A winning before or at trial $x$, we will have

$$z_x = mrz_{x-1} - \left[r^2\frac{m(m-1)}{1.2} - pqm\right] z_{x-2} + \cdots + C,$$

$C$ being an arbitrary constant.

Similarly, if we name $\overset{1}{z}_x$ the probability of B winning before or at trial $x$, we will have

$$\overset{1}{z}_x = mr\overset{1}{z}_{x-1} - \left[r^2\frac{m(m-1)}{1.2} - pqm\right] \overset{1}{z}_{x-2} + \cdots + \overset{1}{C}.$$

In order to integrate these equations, it is necessary to have the roots of the equation

$$(\Lambda) \qquad f^m = mrf^{m-1} - \left[r^2\frac{m(m-1)}{1.2} - pqm\right] f^{m-2} + \cdots;$$

now here is how one can determine them.

We have seen previously how one could have the roots of the equation

$$y^m = mpqy^{m-2} - \frac{m(m-3)}{1.2}p^2q^2y^{m-4} + \cdots.$$

Let $y = f - r$, and we will have

$$\begin{aligned} f^m =& mrf^{m-1} - \left[r^2\frac{m(m-1)}{1.2} - pqm\right] f^{m-2} \\ &+ \left[r^3\frac{m(m-1)(m-3)}{1.2.3} - pqrm(m-2)\right] f^{m-3} \\ &- \cdots, \end{aligned}$$

an equation which is the same as equation $(\Lambda)$; the different values of $f$ are consequently equal to those of $y$, augmented by the quantity $r$; now the integration of the differential equation in $z_x$ has nothing troublesome.