# Additions

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OC **7** pp. 471–493.

- I One deduces from the analysis of n° 34 of Book I the expression of the ratio of the circumference to the radius, given by Wallis, *in infinite product*. Analysis of the remarkable method by which this great geometer is arrived there, a method which contains the germs of the theories of the interpolations and of the definite integrals.
- II Direct demonstration of the expression of $\Delta^n s^i$, found in n° 40 of Book I, by the passages from the postive to the negative and from the real to the imaginary.
- III Demonstration of the formula $(p)$ from n° 42 of Book I or of the expression of the finite differences of the powers, when one stops this expression at the term where the quantity raised to the power becomes negative.

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I.

We have integrated, by a very convergent approximation, in nº 34 of Book I, the equation in the finite differences

$$0 = (n' + s + 1)y_{s+1} - (n + s)y_s.$$

It is easy to conclude from our analysis the expression of the ratio of the circumference to the radius, in infinite products, given by Wallis. In fact, this analysis has led us, in the section cited, to the general expression

$$(a) \qquad \frac{(n+\mu)(n+\mu+1)\cdots(n+s+1)}{(n+\mu+1)(n'+\mu+2)\cdots(n'+s)} = \frac{\int u^{2n'-2n+1}du(1-u^2)^{n+s-1}}{\int u^{2n'-2n+1}du(1-u^2)^{n+\mu-1}},$$

the integrals being taken from $u = 0$ to $u = 1$. By making first $n' = 0$, $n = \frac{1}{2}$, $\mu = 1$ and observing that $\int du(1-u^2)^{\frac{1}{2}} = \frac{1}{4}\pi$, $\pi$ being the ratio of the semi-circumference to the radius, one will have

$$\frac{4}{\pi} = \frac{3.5\ldots(2s-1)}{4.6\ldots 2s\int du(1-u^2)^{s-\frac{1}{2}}}.$$

By supposing therefore generally

$$\frac{1}{\int du(1-u^2)^s} = y_s,$$

one will have

$$\frac{4}{\pi} = \frac{3.5\ldots(2s-1)}{4.6\ldots 2s}y_{s-\frac{1}{2}} = \frac{3.5\ldots(2s+1)}{4.6\ldots(2s+2)}y_{s+\frac{1}{2}} = \cdots,$$

this which gives

$$y_{s-\frac{1}{2}} = \frac{2s+1}{2s+2}y_{s+\frac{1}{2}}$$

If one makes next, in formula $(a)$, $n' = -\frac{1}{2}$, $n = 0$ and $\mu = 1$, it gives

$$\frac{3.5\ldots(2s-1)}{2.4\ldots(2s-2)} = y_{s-1};$$

whence one draws

$$y_{s-1} = \frac{2s}{2s+1}y_x,$$

an equation which coincides with the preceding between $y_{s-\frac{1}{2}}$ and $y_{s+\frac{1}{2}}$ by changing $s$ into $s + \frac{1}{2}$, so that this equation holds, $s$ being entire or equal to an entire plus $\frac{1}{2}$.

The two expressions of $y_{s-1}$ and of $\frac{4}{\pi}$ give

$$\frac{4}{\pi} = \frac{3.3}{2.4}\cdot\frac{5.5}{4.6}\cdots\frac{(2s-1)(2s-1)}{(2s-2)2s}\frac{y_{s-\frac{1}{2}}}{y_{s-1}};$$

the equations in the differences in $y_s$ and $y_{s-\frac{1}{2}}$ give

$$\frac{y_{s-\frac{1}{2}}}{y_{s-1}} = \frac{(2s+1)^2}{2s(2s+2)}\frac{y_{s+\frac{1}{2}}}{y_s} = \frac{(2s+1)^2}{2s(2s+2)}\frac{(2s+3)^2}{(2s+2)(2s+4)}\frac{y_{s+\frac{3}{2}}}{y_{s+1}} = \cdots$$

The ratio $\frac{y_{s-\frac{1}{2}}}{y_{s-1}}$ is greater than unity; it diminishes without ceasing, in measure as $s$ increases, and, in the case of $s$ infinite, it becomes unity. In fact, this ratio is equal to

$$\frac{\int du(1-u^2)^{s-1}}{\int du(1-u^2)^{s-\frac{1}{2}}}.$$

Now the element $du(1-u^2)^{s-1}$ is greater than the element $du(1-u^2)^{s-\frac{1}{2}}$, or $du(1-u^2)^{s-1}(1-u^2)^{\frac{1}{2}}$; the integral of the numerator of the preceding fraction surpasses therefore that of the denominator; this fraction is therefore greater than unity. When $s$ is infinite, these integrals have a sensible value only when $u$ is infinitely small; because, $u$ being finite, the factor $(1-u^2)^{s-1}$ becomes a fraction having an infinitely great exponent; one can therefore then suppose $(1-u^2)^{\frac{1}{2}} = 1$, this which renders the ratio $\frac{s-\frac{1}{2}}{y_{s-1}}$ equal to unity.

This ratio is equal to the product of an infinite sequence of fractions, of which the first is $\frac{(2s+1)^2}{2s(2s+2)}$, and of which the others are deduced from it, by increasing successively $s$ by one unit; it becomes $\frac{y_s}{s-\frac{1}{2}}$, by changing $s$ into $s+\frac{1}{2}$, and the fraction $\frac{(2s+1)^2}{2s(2s+2)}$ becomes $\frac{(2s+2)^2}{(2s+1)(2s+3)}$; now one has, whatever be $s$,

$$\frac{(2s+1)^2}{2s(2s+2)} > \frac{(2s+2)^2}{(2s+1)(2s+3)};$$

one has therefore this inequality

$$\frac{y_{s-\frac{1}{2}}}{y_{s-1}} > \frac{y_s}{y_{s-\frac{1}{2}}}.$$

By changing $s$ into $s-\frac{1}{2}$, one will have

$$\frac{y_{s-1}}{y_{s-\frac{3}{2}}} > \frac{y_{s-\frac{1}{2}}}{y_{s-1}}.$$

The two inequalities give

$$\frac{y_{s-\frac{1}{2}}}{y_{s-1}} > \sqrt{\frac{y_s}{y_{s-1}}} < \sqrt{\frac{y_{s-\frac{1}{2}}}{y_{s-\frac{3}{2}}}}.$$

Substituting instead of the ratios $\frac{y_s}{y_{s-1}}$ and $\frac{y_{s-\frac{1}{2}}}{y_{s-\frac{3}{2}}}$ their values given by the equations in the differences in $y_s$, one will have

$$\frac{y_{s-\frac{1}{2}}}{y_{s-1}} > \sqrt{1+\frac{1}{2s}} < \sqrt{1+\frac{1}{2s-1}};$$

one will have

$$\text{(A)} \qquad \begin{cases} \frac{4}{\pi} > \frac{3.3}{2.4} \cdot \frac{5.5}{4.6} \cdots \frac{(2s-1)(2s-1)}{(2s-2)(2s} \sqrt{1 + \frac{1}{2s}}, \\ \frac{4}{\pi} < \frac{3.3}{2.4} \cdot \frac{5.5}{4.6} \cdots \frac{(2s-1)(2s-1)}{(2s-2)(2s} \sqrt{1 + \frac{1}{2s-1}}. \end{cases}$$

Wallis published in 1657, in his *Arithmetica infinitorum*, this beautiful theorem, one of the most curious in Analysis, by itself and by the manner in which the inventor is arrived there. His method containing the principles of the theory of definite integrals, that the geometers have specially cultivated in these last times, I think that they will see with pleasure a succinct exposition in the actual language of Analysis.

Wallis considers the series of fractions of which the general term is $\frac{1}{\int dx \left(1 - x^{\frac{1}{n}}\right)^s}$, $n$ and $s$ being some entire numbers, by commencing with zero. By expanding the binomial contained under the integral sign and integrating each term of the expansion, he obtains, for one same value of $n$, the numerical values of the preceding fraction, corresponding to $s = 0$, $s = 1$, $s = 2$, $\ldots$, this which gives to him a horizontal series, of which $s$ is the index. By supposing successively $n = 0$, $n = 1$, $n = 2$, $\ldots$, he has so many horizontal series. Thence, he forms a Table in double entry, of which $s$ is the horizontal index and $n$ the vertical index.

In this Table, the horizontal and vertical series are the same, so that, by designating by $y_{n,s}$ the term corresponding to the indices $n$ and $s$, one has this fundamental equation

$$y_{n,s} = y_{s,n}.$$

Wallis observes next that the first series is unity; that the second is formed of the natural numbers; that the third is formed of the triangular numbers, and so forth; in a manner that the general term $y_{n,s}$ of the horizontal series corresponding to $n$ is

$$\frac{(s+1)(s+2)\cdots(s+n)}{1.2.3\ldots n};$$

this fraction being equal to

$$\frac{(n+1)(n+2)\cdots(s+n)}{1.2.3\ldots s},$$

one sees clearly that $y_{n,s}$ is equal to $y_{s,n}$.

Now, if one arrives to interpolate in the preceding Table the term corresponding to $n$ and $s$ equal to $\frac{1}{2}$, one will have the ratio of the square of the diameter to the surface of the circle; because the term of which there is concern is $\frac{1}{\int dx (1-x^2)^{\frac{1}{2}}}$, or $\frac{4}{\pi}$. Wallis seeks therefore to make this interpolation. It is easy in the case where one of the two numbers $n$ and $s$ is an entire number. Thus, by making successively $s$ equal to an entire number less $\frac{1}{2}$ in the function $\frac{(s+1)(s+2)\cdots(s+n)}{1.2.3\ldots n}$, he obtains all the terms of the horizontal series, corresponding to the values of $s$, $-\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\ldots$; and by making $n$ equal to an entire number less $\frac{1}{2}$ in the function $\frac{(n+1)(n+2)\cdots(n+s)}{1.2.3\ldots s}$, he obtains all the terms of the vertical series, corresponding to the values of $n$, $-\frac{1}{2}$, $\frac{3}{2}$, $\ldots$ But the difficulty consists in finding the terms corresponding to $n$ and $s$, both equal to some entire numbers less $\frac{1}{2}$.

Wallis observes for this that the equation

$$y_{n,s} = \frac{(s+1)(s+2)\cdots(s+n)}{1.2.3\ldots n}$$

gives

$$y_{n,s-1} = \frac{s(s+1)\cdots(s+n-1)}{1.2.3\ldots n},$$

and that thus one has

$$(a) \qquad y_{n,s} = \frac{s+n}{s} y_{n,s-1};$$

so that each term of a horizontal series is equal to the preceding, multiplied by the fraction $\frac{s+n}{s}$; whence it follows that all the terms of a horizontal series, departing from $s = -\frac{1}{2}$, $s$ increasing successively by unity, are the products of $y_{n,-\frac{1}{2}}$ by the fractions $\frac{2n+1}{1}$, $\frac{2n+3}{3}$, $\frac{2n+5}{5}$, ..., and, departing from $s = 1$, these terms are the products of $y_{n,0}$ by the fractions $\frac{n+1}{1}$, $\frac{n+2}{2}$, $\frac{n+3}{3}$, ... He supposes that the same laws subsist in the case of $n$ fractional and equal to $\frac{1}{2}$, so that one has all the terms, departing from $s = -\frac{1}{2}$, by multiplying $y_{\frac{1}{2},-\frac{1}{2}}$ by the series of fractions $\frac{2}{1}$, $\frac{4}{3}$, $\frac{6}{5}$,... by designating therefore by $\square$ the term corresponding to $n = \frac{1}{2}$ and $s = \frac{1}{2}$, a term which, as one has seen, is equal to $\frac{4}{\pi}$, one has

$$\square = \frac{2}{1} y_{\frac{1}{2},-\frac{1}{2}},$$

this which gives

$$y_{\frac{1}{2},-\frac{1}{2}} = \frac{1}{2}\square.$$

Departing from $y_{\frac{1}{2},0}$, or from unity, he obtains the successive terms of the series, corresponding to $s$ entire, by multiplying successively unity by the fractions $\frac{3}{2}$, $\frac{5}{4}$, $\frac{7}{6}$, ... He forms thus the horizontal series according to which correspond to $n = \frac{1}{2}$, and to $s$ successively equal to $-\frac{1}{2}$, 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, ...,

$$(i) \qquad \frac{1}{2}\square, \quad 1, \quad \square, \quad \frac{3}{2}, \quad \frac{4}{3}\square, \quad \frac{3}{2}\cdot\frac{3}{4}, \quad \frac{4}{3}\cdot\frac{6}{5}\square, \quad \cdots,$$

a series which represents this here,

$$\frac{1}{\int dx(1-x^2)^{-\frac{1}{2}}}, \quad \frac{1}{\int dx(1-x^2)^{0}}, \quad \frac{1}{\int dx(1-x^2)^{\frac{1}{2}}}, \quad \cdots$$

The series $(i)$ gives generally, $s$ being an entire number,

$$\begin{aligned} y_{\frac{1}{2},s-\frac{1}{2}} &= \frac{4}{3}\cdot\frac{6}{5}\cdots\frac{2s}{2s-1}\square, \\ y_{\frac{1}{2},s-1} &= \frac{3}{2}\cdot\frac{5}{4}\cdots\frac{2s-1}{2s-2}; \end{aligned}$$

whence one draws

$$\text{(B)} \qquad \square = \frac{3.3}{2.4}\cdot\frac{5.5}{4.6}\cdots\frac{(2s-1)(2s-1)}{(2s-2)2s}\,\frac{y_{\frac{1}{2},s-\frac{1}{2}}}{y_{\frac{1}{2},s-1}}.$$

Wallis considers next that, in the series $(i)$, the ratio of each term to the one which precedes it by one unit is greater than unity and diminished without ceasing, so that one has

$$\frac{y_{\frac{1}{2},s}}{y_{\frac{1}{2},s-1}} > \frac{y_{\frac{1}{2},s+1}}{y_{\frac{1}{2},s}}.$$

This results in fact with the equation

$$y_{\frac{1}{2},s} = \frac{2s+1}{2s} y_{\frac{1}{2},s-1}.$$

He supposes that this holds equally for all the consecutive terms of the series, so that one has the two inequalities

$$\frac{y_{\frac{1}{2},s-\frac{1}{2}}}{y_{\frac{1}{2},s-1}} > \frac{y_{\frac{1}{2},s}}{y_{\frac{1}{2},s-\frac{1}{2}}} < \frac{y_{\frac{1}{2},s-1}}{y_{\frac{1}{2},s-\frac{3}{2}}};$$

whence it follows, as one has done above,

$$\frac{y_{\frac{1}{2},s-\frac{1}{2}}}{y_{\frac{1}{2},s-1}} > \sqrt{1+\frac{1}{2s}} < \sqrt{1+\frac{1}{2s-1}};$$

thence, it changes formula (B) into formula (A).

This manner to proceed by way of induction must appear and appeared, in fact, extraordinary to the geometers accustomed to the rigor of the ancients. Thus we see that some great contemporary geometers of Wallis were not very satisfied with it, and Fermat, in his correspondence with Digby, made some objections not very worthy of him against this method which he had not studied sufficiently deeply. It must be, without doubt, employed with an extreme circumspection: Wallis himself said, in responding to Fermat, that it is thus that he is served by it, and, in order to confirm the exactitude, he supported it on a calculation by which lord Brouncker had found, by means of formula (A), the ratio of the circumference to the diameter, contained between the limits

$$3.141592653569,$$
$$3.141592653696,$$

limits which coincide in the first ten digits with this ratio that one has carried beyond one hundred decimals. Notwithstanding these confirmations, it is always useful to demonstrate in rigor that which one obtains by these means of invention. Wallis observes that the ancients had, without doubt, similar that they had not made known at all, being content to give their results supported on synthetic demonstrations. He regrets, with reason, that they had concealed from us their ways to arrive there, and he said to Fermat that one must be thankful to him not to have imitated them, and to not have *destroyed the bridge after the flood having passed*. It is worthy of remark that Newton, who had profited from this method of induction of Wallis and of his results in order to discover his theorem on the binomial, has merited the reproaches that Wallis made to the ancients geometers, in seeking the means which had led to their discoveries.

Let us resume formula (B) of Wallis. If one supposes

$$\frac{y_{\frac{1}{2},s-\frac{1}{2}}}{y_{\frac{1}{2},s-1}} = u_s,$$

this formula will give

$$u_{s-1} = \frac{(2s-1)^2}{(2s-2)2s} u_s$$

or

$$(l) \qquad 0 = 2s(2s-2)(u_s - u_{s-1}) + u_s.$$

Let there be

$$u_s = \mathrm{A}^{(0)} + \frac{\mathrm{A}^{(1)}}{s+1} + \frac{\mathrm{A}^{(2)}}{(s+1)(s+2)} + \frac{\mathrm{A}^{(3)}}{(s+1)(s+2)(s+3)} + \cdots,$$

and let us consider that which produces, in the second member of equation ($l$), the term

$$\frac{\mathrm{A}^{(r)}}{(s+1)\cdots(s+r)}.$$

By having regard only to this term in $u_s$, one will have

$$u_s - u_{s-1} = \frac{-r\mathrm{A}^{(r)}}{s(s+1)(s+2)\cdots(s+r)};$$

the term $2s(2s-2)(u_s - u_{s-1})$ of the equation ($l$) becomes thus

$$\frac{-4r\mathrm{A}^{(r)}(s-1)}{(s+1)\cdots(s+r)},$$

or

$$\frac{-4r\mathrm{A}^{(r)}}{(s+1)\cdots(s+r-1)} + \frac{4r(r+1)\mathrm{A}^{(r)}}{(s+1)\cdots(s+r)}.$$

The term of $u_s$ depending on $\mathrm{A}^{(r+1)}$ will produce some similar terms, and thus of the others. By comparing therefore in equation ($l$) the terms which have the same denominator $(s+1)\cdots(s+r)$, one will have

$$0 = 4r(r+1)\mathrm{A}^{(r)} - 4(r+1)\mathrm{A}^{(r+1)} + \mathrm{A}^{(r)},$$

this which gives

$$\mathrm{A}^{(r+1)} = \frac{(2r+1)^2\mathrm{A}^{(r)}}{4(r+1)}.$$

It is clear, by that which precedes, that $u_s$ is reduced to unity when $s$ is infinite, this which gives $\mathrm{A}^{(0)} = 1$. Thence one draws

$$u_s = 1 + \frac{1^2}{4(s+1)} + \frac{1^2.3^2}{4^2/1/2(s+1)(s+2)} + \frac{1^2.3^2.5^2}{4^3.1.2.3(s+1)(s+2)(s+3)} + \cdots = \frac{y_{s-\frac{1}{2}}}{y_{s-1}}.$$

The ratio of the mean term of the binomial $(1+1)^{2s}$ to the entire binomial is

$$\frac{(s+1)(s+2)\cdots 2s}{2^{2s}.1.2.3\ldots s}$$

or

$$\frac{1.3.5\ldots(2s-1)}{2.4.6\ldots 2s}.$$

By naming therefore T this mean term, formula (B) will give

$$\mathrm{T}^2 = \frac{1}{s\pi u_s}.$$

This theorem and the preceding expression of $u_s$ in series are due to Stirling, and one sees how they are attached to the theorem and to the analysis of Wallis. This value of $\mathrm{T}^2$ is able to serve to determine by approximation the ratio of the circumference to the diameter, this which was the object of Wallis; or, this ratio being supposed known, it gives the mean term of the binomial, this which was the object of Stirling.

II. (pp. 480–485) (omitted)

III. (pp. 485–493) (omitted)