

TROISIÈME SUPPLÉMENT.
APPLICATION DES FORMULES GÉODÉSIQUES DE PROBABILITÉ A LA
MÉRIDIENNE DE FRANCE

Pierre Simon Laplace*

Sequel to Second Supplement
OC 7 pp. 581–616

§1. The part of the meridian which extends from Perpignan to Formentera is supported on a base measured near Perpignan. Its length is around 466 km, and its last extremity is joined to the base of Perpignan by a chain of twenty-six triangles. We are able to fear that so great a length, which has not been verified at all by the measure of a second base toward its other extremity, is susceptible to a sensible error arising from the errors of the twenty-six triangles employed to measure it. It is therefore interesting to determine the probability that this error not exceed 40^m or 50^m. Mr. Damoiseau, lieutenant-colonel of the Artillery, who has just gained the prize proposed by the Academy of Turin, on the return of the comet of 1759, has well wished, at my request, to apply to this part of the meridian my formulas of probability. Here the meridian does not cut all the triangles, as we have proposed for more simplicity; but it is easy to see that we are able to apply, to the angles formed by the prolongations of the sides of the triangles with the meridian, that which I have said respecting the angles that these sides would form if they were cut by the meridian. Mr. Damoiseau has found thus that departing from the latitude of the signal of Busgarach, a little more to the north than Perpignan, to Formentera, that which comprehends an arc of the meridian of about 466,006^m, and by taking for unity the base of Perpignan, we have (second Supplement, §1) [581]

$$p^2 - pq + q^2 + p_1^{(1)2} - p^{(1)}q^{(1)} + q^{(1)2} + \dots \\ + p^{(25)2} - p^{(25)}q^{(25)} + q^{(25)2} = 48350,606.$$

The probability that an error in the measure of this arc is comprehended within the limits $\pm s$ becomes, by the formulas of the same section, [582]

$$\frac{2 \int dt e^{-t^2}}{\sqrt{\pi}},$$

the integral being taken from t null to the value of t equal to

$$\frac{s}{2\theta} \frac{\sqrt{n+1}}{\sqrt{48350,606}},$$

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. January 10, 2014

$n + 1$ being the number of triangles employed, and θ^2 being the sum of the squares of the errors observed in the sum of the three angles of each triangle; π is the ratio of the circumference to the diameter. By taking for unity the sexagesimal second, we find

$$\theta^2 = 118,178$$

But, the number of triangles employed being only 26, it is preferable to determine by a great number of triangles this constant θ^2 which depends on the unknown law of the partial observations. For this, we have made use of the one hundred seven triangles which have served to measure the meridian from Dunkirk to Formentera. The set of the sums of observed errors of the three angles of each triangle is, in taking all of them positively, equal to 173,82. The sum of the squares of these errors is 445,217. By multiplying it by $\frac{26}{107}$, we will have, for the value of θ^2 ,

$$\theta^2 = 108,184.$$

This value, which differs little from the preceding, must be preferred. It is necessary to reduce θ into parts of the radius taken for unity, that which we will make by dividing it by the number of sexagesimal seconds that this radius contains. We will have thus

$$t = s\ 689,797;$$

s is a fraction of the base of Perpignan taken for unity. This base is 11706^m,40. By [583] supposing therefore the error of 60^m, we will have

$$t = \frac{60 \times 689,797}{11706,40}.$$

This put, we find, for the probabilities of the errors of the arc of the meridian of which there is question are comprehended within the limits $\pm 60^m$, $\pm 50^m$, $\pm 40^m$, the following fractions:

$$\frac{1743695}{1743696}, \quad \frac{32345}{32436}, \quad \frac{1164}{1165}.$$

There are odds one against one that the error falls within the limits 8^m,0757.¹

If the Earth were a spheroid of revolution and if the angles of all the triangles were exact, we would have exactly the inclination of the last side of the chain of the triangle on its meridian, by supposing given this inclination relative to the base. The probability that the error of the first of these inclinations, proceeding from the errors

¹The comparison of this Memoir with the one which was published in the *Connaissance des Temps* for 1822 permits the identification of errors in this supplement. In the *Connaissance des Temps*, Laplace gives $\pm 8^m,0937$. In reality, it would be $\pm 8^m,0940$. We have $\frac{2}{\pi} \int_0^t c^{-t^2} dt = \frac{1}{2}$ for $t = 0,476936$ thereby giving even odds. From the previous relation here,

$$t = \frac{s\ 689,797}{11706,40},$$

where $s = \pm 8^m,0940$. The number given by Laplace differs slightly, without doubt, because the value of t , deduced from the formulas of Laplace, has not been calculated with as much precision as that employed later.

of the observed angles of the triangles, is comprehended within the limits $\pm \frac{2}{3}\theta t$ is, by that which precedes,

$$\frac{2 \int dt e^{-t^2}}{\sqrt{\pi}},$$

the integral being taken from t null: these limits become, by substituting for θ it preceding value, $\pm t6''$, 8997, the seconds being sexagesimal. Thence it follows that there are odds one against one that the error falls within the limits $\pm 3''$, 2908. If the azimuthal observations were made with a great precision, we would determine by this means the probability that they indicate an eccentricity in the terrestrial parallels. If we measured, on the side of Spain, a base of verification equal to the base of Perpignan, and if we joined it by two triangles to the chain of triangles of the meridian, we find, by the calculation, that there are odds one against one that the difference, between this base and its value concluded from the base of Perpignan, will not surpass a third of a meter: that is, to quite nearly, the difference of the measure of the base of Perpignan to its value concluded from the base of Melun.

We have seen, in the section cited, that, the angles of the triangles having been measured by means of the repeating circle, we are able to suppose the probability of an error x in the observed sum of the three angles of each triangle proportional to the exponential e^{-kx^2} , k being a constant. Thence it follows that the probability of this error is [584]

$$\frac{dx \sqrt{k} e^{-kx^2}}{\sqrt{\pi}}.$$

By multiplying this differential by x and integrating from x null to x infinity, the double of this integral will be the mean of all the errors taken positively. By designating therefore by ϵ this mean error, we will have

$$\epsilon = \frac{1}{\sqrt{k\pi}}.$$

We will have the mean value of the squares of these errors by multiplying by x^2 the preceding differential and by integrating it from $x = -\infty$ to x infinity. By naming therefore ϵ' this value, we will have

$$\epsilon' = \frac{1}{2k};$$

thence we deduce

$$\epsilon' = \frac{\epsilon^2 \pi}{2}.$$

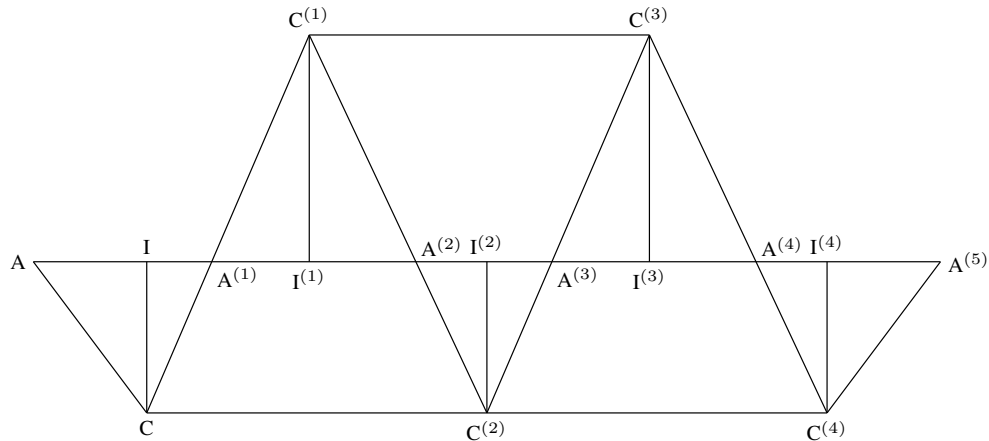
We are able thus to obtain θ^2 by means of the errors, taken all to plus, of the observed sums of the angles of each triangle. In the one hundred seven triangles of the meridian, the sum of the errors is 173, 82; we are able thus to take, for ϵ , $\frac{173,82}{107}$; that which gives, for $26\epsilon'$ or for θ^2 ,

$$\theta^2 = 13\pi \left(\frac{173,82}{107} \right)^2 = 107,78;$$

this differs very little from the value 108,134 given by the sum of the squares of the errors of the observed sum of the angles of each of the one hundred seven triangles. This accord is remarkable.

We are able to estimate the relative exactitude of the instruments of which we make use in the geodesic observations, by the value of ϵ' concluded from a great number of triangles. This value, concluded from the one hundred seven triangles of the meridian, is $\frac{445,217}{107}$ or 4,1609. The same value, concluded from the forty-three triangles employed by La Condamine in the measure of the three degrees of the equator, is $\frac{1718}{43}$ or 39,953, and, consequently, nearly ten times greater than the preceding. The equally probable errors, relative to the instruments employed in these two operations, are proportionals to the square roots of the values of ϵ' . Thence it follows that the limits $\pm 8^m, 0937$, between which we have just seen that there are odds one against one that the error of the arc measured from Perpignan to Formentera falls, would have been $\pm 25^m, 022$ with the instruments employed by La Condamine. These limits would have surpassed $\pm 40^m$ with the instruments employed by La Caille and Cassini in their measure of the meridian. We see thus how the introduction of the repeating circle in the geodesic operations has been advantageous. [585]

§2. In order to give a very simple example of the application of the geodesic formulas, I will consider the straight line $AA^{(5)}$, of which we have determined the length by a chain of triangles $CC^{(1)}C^{(2)}$, $C^{(1)}C^{(2)}C^{(3)}$, ...



I will suppose all these triangles equal and isosceles, and such that their bases $CC^{(2)}$, $C^{(1)}C^{(3)}$, ... are parallels to the line $AA^{(5)}$. We will have, by lowering onto this line [586]

the perpendiculars $CI, C^{(1)}I^{(1)}, \dots$

$$\begin{aligned} II^{(1)} &= CC^{(1)} \cos A^{(1)}, \\ C^{(1)}C^{(2)} &= \frac{CC^{(1)} \sin C^{(1)}CC^{(2)}}{\sin C^{(1)}C^{(2)}C}, \\ I^{(1)}I^{(2)} &= C^{(1)}C^{(2)} \cos A^{(2)}, \\ C^{(2)}C^{(3)} &= \frac{C^{(1)}C^{(2)} \sin C^{(2)}C^{(1)}C^{(3)}}{\sin C^{(2)}C^{(3)}C^{(1)}}, \end{aligned}$$

and generally

$$\begin{aligned} I^{(i)}I^{(i+1)} &= C^{(i)}C^{(i+1)} \cos A^{(i+1)} \\ C^{(i+1)}C^{(i+2)} &= \frac{C^{(i)}C^{(i+1)} \sin C^{(i+1)}C^{(i)}C^{(i+2)}}{\sin C^{(i+1)}C^{(i+2)}C^{(i)}}. \end{aligned}$$

Let $\alpha^{(1)}$ and $\beta^{(1)}$ be the errors of the angles opposed to the sides $CC^{(1)}$ and $C^{(1)}C^{(2)}$ in the first triangle. Let $\alpha^{(2)}$ and $\beta^{(2)}$ be the errors of the angles opposed to the sides $C^{(1)}C^{(2)}$ and $C^{(1)}C^{(3)}$ of the second triangle, and thus consecutively. By designating by δ a variation relative to these errors, we will have

$$\begin{aligned} \frac{\delta I^{(i)}I^{(i+1)}}{I^{(i)}I^{(i+1)}} &= \frac{\delta C^{(i)}C^{(i+1)}}{C^{(i)}C^{(i+1)}} - \delta A^{(i+1)} \tan A^{(i+1)}, \\ \frac{\delta C^{(i)}C^{(i+1)}}{C^{(i)}C^{(i+1)}} &= \frac{\delta C^{(i)}C^{(i-1)}}{C^{(i)}C^{(i-1)}} + \beta^{(i)} \cot C^{(i+1)}C^{(i-1)}C^{(i)} \\ &\quad - \alpha^{(i)} \cot C^{(i)}C^{(i+1)}C^{(i-1)}. \end{aligned}$$

We have further, by supposing the angles $A^{(i)}$ relative to the acute angles that the sides of the triangles form with the line $AA^{(1)}, \dots$,

$$\delta A^{(i+1)} + \delta A^{(i)} + \delta C^{(i-1)}C^{(i)}C^{(i+1)} = 0;$$

we will suppose here that the errors $\alpha^{(i)}$ and $\beta^{(i)}$ of the angles $C^{(i+1)}C^{(i-1)}C^{(i)}$, $C^{(i)}C^{(i+1)}C^{(i-1)}$ of the triangle $C^{(i-1)}C^{(i)}C^{(i+1)}$ are those which remain, when we have subtracted from each angle of the triangle the third of the sum of the errors of the three angles. Then we have

$$\delta C^{(i-1)}C^{(i)}C^{(i+1)} = -\alpha^{(i)} - \beta^{(i)},$$

that which gives

$$\delta A^{(i+1)} = -\delta A^{(i)} + \alpha^{(i)} + \beta^{(i)};$$

we will have therefore

[587]

$$\begin{aligned} \delta A^{(i+1)} &= \alpha^{(i)} - \alpha^{(i-1)} + \alpha^{(i-2)} - \dots \mp \alpha^{(1)} \\ &\quad + \beta^{(i)} - \beta^{(i-1)} + \beta^{(i-2)} - \dots \pm \delta A^{(1)}, \end{aligned}$$

the superior sign having place if i is even, and the inferior if i is odd.

We will have next, by observing that

$$\cot C^{(i)}C^{(i-1)}C^{(i+1)} = \cot C^{(i)}C^{(i+1)}C^{(i-1)} = \cot A^{(i)}$$

and that $A^{(i)} = A^{(1)}$.

$$\frac{\delta C^{(i)}C^{(i+1)}}{C^{(i)}C^{(i+1)}} = \frac{\delta CC^{(1)}}{CC^{(1)}} + (\beta^{(i)} + \beta^{(i-1)} + \dots + \beta^{(1)} - \alpha^{(i)} - \alpha^{(i-1)} - \dots - \alpha^{(1)}) \cot A^{(1)};$$

we will have therefore

$$\begin{aligned} \frac{\delta I^{(i)}I^{(i+1)}}{I^{(i)}I^{(i+1)}} &= \frac{\delta CC^{(1)}}{CC^{(1)}} + (\beta^{(i)} + \beta^{(i-1)} + \dots + \beta^{(1)} - \alpha^{(i)} - \alpha^{(i-1)} - \dots - \alpha^{(1)}) \cot A^{(1)} \\ &\quad - (\alpha^{(i)} - \alpha^{(i-1)} + \dots \mp \alpha^{(1)} + \beta^{(i)} - \beta^{(i-1)} + \dots \mp \beta^{(1)} \pm \delta A^{(i)}) \tan A^{(1)}. \end{aligned}$$

Let us suppose now that we have measured a base AC situated in a manner that the angle $CAC^{(1)}$ is equal to the angle $CA^{(1)}A$. The first of these angles determines the position of the line $AA^{(1)}$ with respect to the base, and it is supposed known. By naming α and β the errors of the angles $CC^{(1)}A$ and $CAC^{(1)}$, we will have

$$\begin{aligned} \delta A^{(1)} &= \alpha + \beta, \\ \frac{\delta CC^{(1)}}{CC^{(1)}} &= \beta \cot CAC^{(1)} - \alpha \cot CC^{(1)}A. \end{aligned}$$

Let us make

$$\begin{aligned} \cot CAC^{(1)} &= \cot A + h, \\ \cot CC^{(1)}A &= \cot A + h'; \end{aligned}$$

we will have, by designating by b the base AC and by a the straight line $II^{(i)}$,

$$\begin{aligned} h &= \frac{b}{2a \sin A} - \frac{1}{\sin 2A}, \\ h' &= \frac{a}{2b \sin A \cos^2 A} - \frac{1}{\sin 2A}. \end{aligned}$$

we will have next

[588]

$$\begin{aligned} \delta A^{(i+1)} &= \alpha^{(i)} - \alpha^{(i-1)} + \dots \pm \alpha + \beta^{(i)} - \beta^{(i-1)} + \dots \pm \beta, \\ \frac{\delta I^{(i)}I^{(i+1)}}{I^{(i)}I^{(i+1)}} &= (\beta^{(i)} + \beta^{(i-1)} + \dots + \beta - \alpha^{(i)} - \alpha^{(i-1)} - \dots - \alpha) \cot A \\ &\quad - (\alpha^{(i)} - \alpha^{(i-1)} + \dots \pm \alpha + \beta^{(i)} - \beta^{(i-1)} + \dots \pm \beta) \tan A + h\beta - h'\alpha. \end{aligned}$$

The variation of the total length $II^{(i+1)}$ will be therefore

$$\begin{aligned} \delta II^{(i+1)} &= [(i+1)(\beta - \alpha) + i(\beta^{(i)} - \alpha^{(i)} + \dots + (\beta^{(i)} - \alpha^{(i)}))]a \cot A \\ &\quad + (i+1)h\alpha\beta - (i+1)h'\alpha\alpha \\ &\quad - (\alpha^{(i)} + \alpha^{(i-2)} + \alpha^{(i-4)} + \dots + \beta^{(i)} + \beta^{(i-2)} + \beta^{(i-4)} + \dots)a \tan A. \end{aligned}$$

The quantity

$$p^2 - pq + q^2 + p^{(1)2} - p^{(1)}q^{(1)} + q^{(1)2} + \dots + p^{(i)2} - p^{(i)}q^{(i)} + q^{(i)2}$$

becomes thus, by neglecting the terms of order i ,

$$\frac{(i+1)(i+2)(2i+3)}{2} a^2 \cot^2 A + 3(h+h')(i+1)^2 a^2 \cot A + (h^2 + hh' + h'^2)(i+1)^2 a^2.$$

Let us name Q this quantity; the probability that the error of the line $II^{(i+1)}$ is comprehended within the limits $\pm s$ will be, by that which precedes,

$$\frac{2 \int dt e^{-t^2}}{\sqrt{\pi}},$$

the integral being taken from t null to

$$t = \frac{3s}{2\theta} \sqrt{\frac{i+1}{Q}},$$

θ^2 being the sum of the squares of the errors of the sum of the three angles of the $i+1$ triangles.

Let us suppose that we have, as for the part of the meridian of which we have spoken previously, twenty-six triangles, that which gives $i = 25$. Let us suppose further that the length $II^{(i+1)}$ is that of this part of the meridian or of 466006^m; then we will have

$$a = \frac{466006}{26}.$$

By taking for unity the base measured near to Perpignan, which is of 11706^m, 40 and by supposing right-angled the isosceles triangles $CC^{(1)}C^{(2)}$, $C^{(1)}C^{(2)}C^{(3)}$, ... that which gives $\tan A = \cot A = 1$, we find [589]

$$Q = 48207, 6.$$

We have seen previously that the twenty-six triangles which join the base of Perpignan to Formentera give

$$Q = 48350, 6;$$

these two values of Q are not very different, and as the equally probable errors are proportionals to the square roots of these values, we see that we are able to wager one against one that the errors of the entire measure are contained within the limits $\pm 8^m$, 1. Under this relation, the case that we examine represents perfectly the measure of the arc of the meridian from the base of Perpignan to Formentera.

§3. Let us suppose now that we measure, toward the last extremity of the line $II^{(i+1)}$, a base $C^{(i+1)}A^{(i+2)}$ equal to the base CA , and put in a manner that the angle $C^{(i+1)}C^{(i)}A^{(i+2)}$ is equal to the angle $CC^{(1)}A$, and that the angle $C^{(i)}A^{(i+2)}C^{(i+1)}$

is equal to the angle $CAC^{(1)}$. In designating by $\alpha^{(i+1)}$ and $\beta^{(i+1)}$ the errors of the angles $C^{(i+1)}C^{(i)}A^{(i+2)}$ and $C^{(i)}A^{(i+2)}C^{(i+1)}$, the equation

$$C^{(i+1)}A^{(i+2)} = C^{(i+1)}C^{(i)} \frac{\sin C^{(i+1)}C^{(i)}A^{(i+2)}}{\sin C^{(i)}A^{(i+2)}C^{(i+1)}}$$

will give

$$\frac{\delta C^{(i+1)}A^{(i+2)}}{C^{(i+1)}A^{(i+2)}} = \frac{\delta C^{(i)}C^{(i+1)}}{C^{(i)}C^{(i+1)}} + \alpha^{(i+1)} \cot CC^{(1)}A - \beta^{(i+1)} \cot CAC^{(1)},$$

that which gives

$$\begin{aligned} \frac{\delta C^{(i+1)}A^{(i+2)}}{C^{(i+1)}A^{(i+2)}} = & (\beta^{(1)} + \beta^{(2)} + \dots + \beta^{(i)} - \alpha^{(1)} - \alpha^{(2)} - \dots - \alpha^{(i)} \cot A \\ & + \beta(h + \cot A) - \alpha(h' + \cot A) \\ & + \alpha^{(i+1)}(h' + \cot A) - \beta^{(i+1)}(h + \cot A). \end{aligned}$$

That which we have designated in §2 of the second Supplement by $l, l^{(1)}, \dots, m, m^{(1)}, \dots$ becomes [590]

$$\begin{aligned} l &= -(1 + h')b, & m &= (1 + h)b, \\ l^{(i)} &= -b, & m^{(i)} &= b, \\ \dots\dots, & & \dots\dots, & \\ l^{(i)} &= -b, & m^{(i)} &= b, \\ l^{(i+1)} &= (1 + h')b, & m^{(i+1)} &= -(1 + h)b; \end{aligned}$$

the quantity that we have designated by $Sf^{(i)2}$ in the section cited or by

$$l^2 - ml + m^2 + l^{(1)2} - m^{(1)}l^{(1)} + m^{(1)2} + \dots$$

becomes here

$$3(i + 2)b^2 + 6(h + h')b^2 + 2(h^2 + hh' + h'^2)b^2.$$

The quantity that we have named $Sr^{(i)}f^{(i)}$ in the same section, or

$$l(p - \frac{1}{2}q) + m(q - \frac{1}{2}p) + l^{(1)}(p^{(1)} - \frac{1}{2}q^{(1)}) + m^{(1)}(q^{(1)} - \frac{1}{2}p^{(1)}) + \dots,$$

becomes, by neglecting the terms which do not have i for coefficient,

$$\frac{3(i + 1)(i + 2)}{2}ab + 3(i + 1)(h + h')ab + (i + 1)(h^2 + hh' + h'^2)ab;$$

by representing therefore, as above, by λ the excess of the measured base $C^{(i+1)}A^{(i+2)}$ on the calculated base, and by s the excess of the true length of the line $II^{(i+1)}$ over that calculated length, we will have

$$s = \frac{\lambda Sr^{(i)}f^{(i)}}{Sf^{(i)2}} = \frac{(i + 1)a\lambda}{2b};$$

it is necessary, consequently, to add to the calculated length of the line $II^{(i+1)}$ the product of λ by the ratio of the half of this line to the base b ; that which reverts to calculating the first half of the line $II^{(i+1)}$ with the base AC , and the second half with the base $A^{(i+2)}C^{(i+1)}$. This process would be generally exact, whatever was the magnitude and the disposition of the triangles which unite the two bases, if the parts of $Sr^{(i)}f^{(i)}$ and of $Sf^{(i)2}$ corresponding to these halves were respectively equal. This is the process that we adopted in the Commission which fixed the length of the meter; [591] and, in the ignorance where we were then of the true theory of these corrections, it was most convenient; but it did not make known the correction of the diverse parts of the total arc $II^{(i+1)}$. For this, it is necessary to correct the angles of each triangle, or to determine the corrections $\alpha, \beta, \alpha^{(1)}, \beta^{(1)}, \dots$ which result from the excess λ of the second base observed over that base calculated after the first. I have given, in the second Supplement, these corrections, by supposing the law of errors of the observations of the angles proportional to the exponential $c^{-k(\alpha+\frac{1}{3}T)^2}$, k being a constant, T being the sum of the errors of the three angles of the triangle, $\alpha + \frac{1}{3}T, \beta + \frac{1}{3}T$ and $\frac{1}{3}T - \alpha - \beta$ being the errors of each of the angles. We have seen, in the Supplement cited, that the supposition of this law of probability must be admitted when the angles have been measured with the repeating circle, and that then we have

$$\alpha^{(s)} = \frac{l^{(s)} - \frac{1}{2}m^{(s)}}{F}\lambda, \quad \beta^{(s)} = \frac{m^{(s)} - \frac{1}{2}l^{(s)}}{F}\lambda,$$

by designating by F the sum of all the quantities $l^2 - ml + m^2, l^{(1)2} - m^{(1)}l^{(1)} + m^{(1)2}, \dots$ I will demonstrate here that these corrections have place, whatever be the law of probability of the errors.

For this, I designate this law by $\phi(\alpha+\frac{1}{3}T)^2$: by supposing it the same for the positive errors and for the negative errors, its expression must contain only some even powers of these errors. The law of probability of the simultaneous values of α and β will be thus proportional to the product

$$\phi(\alpha+\frac{1}{3}T)^2\phi(\beta+\frac{1}{3}T)^2\phi(\frac{1}{3}T-\alpha-\beta)^2.$$

If we develop this product, with respect to the powers of α and of β , by arresting ourselves at the squares and at the products of these quantities, we will have

$$[\phi(\frac{1}{9}T)]^3 + (\alpha^2 + \alpha\beta + \beta^2)\phi(\frac{1}{9}T^2) \\ \times \left\{ 2\phi(\frac{1}{9}T^2)\phi'(\frac{1}{9}T^2) - \frac{4}{9}T^2[\phi'(\frac{1}{9}T^2)]^2 + \frac{4}{9}T^2\phi(\frac{1}{9}T^2)\phi''(\frac{1}{9}T^2) \right\}$$

$\phi'(x)$ expressing $\frac{d\phi(x)}{dx}$, and $\phi''(x)$ expressing $\frac{d^2\phi(x)}{dx^2}$. T being able to be supposed to vary from $-\infty$ to $T = \infty$, we will multiply the preceding function by dT and we will integrate within these limits; we will have thus for the probability of the simultaneous values of α and β a quantity of the form [592]

$$H - H'(\alpha^2 + \alpha\beta + \beta^2).$$

This probability will be therefore proportional to

$$1 - \frac{H'}{H}(\alpha^2 + \alpha\beta + \beta^2).$$

The probability of the simultaneous existence of $\alpha, \beta, \alpha^{(1)}, \beta^{(1)}, \dots$ will be proportional to the product of the quantities

$$1 - \frac{H'}{H}(\alpha^2 + \alpha\beta + \beta^2),$$

$$1 - \frac{H'}{H}(\alpha^{(1)2} + \alpha^{(1)}\beta^{(1)} + \beta^{(1)2}),$$

.....

The logarithm of this product is, s being an indeterminate number,

$$-\frac{H'}{H}S(\alpha^{(s)2} + \alpha^{(s)}\beta^{(s)} + \beta^{(s)2}) - \dots;$$

this product is at its maximum if the preceding term is at its minimum, or if the function

$$S(\alpha^{(s)2} + \alpha^{(s)}\beta^{(s)} + \beta^{(s)2})$$

is the smallest possible, the quantities $\alpha, \beta, \alpha^{(1)}, \dots$ satisfying besides the equation

$$\lambda = l\alpha + m\beta + l^{(1)}\alpha^{(1)} + m^{(1)}\beta^{(1)} + \dots$$

We are able to give to this function the form

$$\frac{1}{4}S \left\{ \left(2\beta^{(s)} + \alpha^{(s)} - \frac{3m^{(s)}\lambda}{2F} \right)^2 + \frac{3}{4} \left[\alpha^{(s)} - \frac{(l^{(s)} - \frac{1}{2}m^{(s)})\lambda}{F} \right]^2 \right\} + \frac{3}{4} \frac{\lambda^2}{F};$$

this function is evidently at its minimum if we suppose

[593]

$$2\beta^{(s)} + \alpha^{(s)} - \frac{3m^{(s)}\lambda}{2F} = 0, \quad \alpha^{(s)} - \frac{(l^{(s)} - \frac{1}{2}m^{(s)})\lambda}{F} = 0;$$

whence we deduce generally

$$\alpha^{(s)} = (l^{(s)} - \frac{1}{2}m^{(s)}) \frac{\lambda}{F}, \quad \beta^{(s)} = (m^{(s)} - \frac{1}{2}l^{(s)}) \frac{\lambda}{F}.$$

In the case that we just considered, we have

$$\alpha = -\frac{\lambda b}{F}(\frac{3}{2} + h' + \frac{1}{2}h), \quad \beta = \frac{\lambda b}{F}(\frac{3}{2} + h + \frac{1}{2}h'),$$

$$\alpha^{(1)} = \alpha^{(2)} = \dots = \alpha^{(i)} = -\frac{\frac{3}{2}b\lambda}{F}, \quad \beta^{(1)} = \beta^{(2)} = \dots = \beta^{(i)} = \frac{\frac{3}{2}b\lambda}{F},$$

$$\alpha^{(i+1)} = \frac{\lambda b}{F}(\frac{3}{2} + h' + \frac{1}{2}h), \quad \beta^{(i+1)} = -\frac{\lambda b}{F}(\frac{3}{2} + h + \frac{1}{2}h');$$

thus by these corrections all the triangles other than those which have one of the bases for one of their sides will remain right-angled.

The probability of the error $\pm u$ of the line $II^{(i+1)}$, corrected by the second base, will be, by the section cited in the second Supplement,

$$\frac{2 \int dt e^{-t^2}}{\sqrt{\pi}},$$

the integral being taken from t null to

$$t = \frac{3u}{2\theta} \sqrt{\frac{i+1}{Q \frac{(Sr^{(i)} f^{(i)})^2}{Sf^{(i)^2}}}},$$

which becomes here

$$t = \frac{3u}{2\theta} \sqrt{\frac{i+1}{Q'}},$$

by designating by Q' the function

$$\frac{(i+1)(i+2)(i+3)}{4} a^2 + \frac{3}{2}(i+1)^2 (h+h') a^2 + \frac{1}{2}(i+1)^2 (h^2 + hh' + h'^2) a^2.$$

The equally probable errors being proportionals to the square roots of Q and of Q' , [594] we see that they are diminished and nearly reduced to half by the measure of a second base.

The probability of an error $\pm \lambda$ in the measure of a second base is, by the second Supplement,

$$\frac{2 \int dt e^{-t^2}}{\sqrt{\pi}},$$

the integral being taken from t null to

$$t = \frac{3u}{2\theta} \sqrt{\frac{i+1}{Sf^{(i)^2}}};$$

and $f^{(i)^2}$ is equal to

$$3(i+1)b^2 + 6(h+h')b^2 + 2(h^2 + hh' + h'^2)b^2.$$

In the present case where $i = 25$, this quantity becomes

$$86,8030b^2;$$

the equally probable errors in the measures of the arc $II^{(i+1)}$ and of a new base equal to the first are therefore in the ratio of \sqrt{Q} to $\sqrt{86,8030}$; whence it follows that there are odds one against one that the error of a new base will be comprehended within the limits $\pm 0^m, 34236$, or to very nearly $\pm \frac{1}{3}^m$. These are the same limits which result from the angles of the twenty-six triangles which reunite the base of Perpignan to Formentera. Thus, under this relation again, the hypothetical case, which we have just examined, accords with that which this chain of triangles gives.

§4. I will consider now the zenithal distances of the vertices of the triangles and the leveling which results from it. From one same vertex such as $C^{(2)}$, we are able to observe the four points C , $C^{(1)}$, $C^{(3)}$, $C^{(4)}$. Let us name f the distance $CC^{(1)}$ and h the base $CC^{(2)}$ of the isosceles triangle; all the triangles being supposed equal, if we name $x^{(i)}$ the height of $C^{(i)}$ above of the level of the sea, the observed distance from $C^{(i-2)}$ to the zenith of $C^{(i)}$ being designated by θ , the true distance will be quite nearly, the triangles being able to be supposed horizontal, [595]

$$\theta + \frac{hu}{R} + \frac{h\epsilon}{R},$$

u being the factor by which we must multiply the angle $\frac{h}{R}$ in order to have the terrestrial refraction at the point $C^{(i)}$, R being the radius of the Earth and ϵ being the error of u . I take account here only of this error, as being much greater than that of θ . If we name similarly θ' the zenithal distance of $C^{(i)}$, observed from $C^{(i-2)}$, the true distance will be

$$\theta' + \frac{hu}{R} + \frac{h\epsilon'}{R},$$

ϵ' being the error of u in this observation. We will have

$$\theta + \theta' + \frac{2hu}{R} + \frac{h}{R}(\epsilon + \epsilon') = \pi + \frac{h}{R};$$

we will have next

$$x^{(i)} - x^{(i-2)} = \frac{h}{2}(\theta - \theta') + \frac{h^2}{2R}(\epsilon - \epsilon').$$

If we name similarly θ'' the zenithal distance of $C^{(i-1)}$ observed from $C^{(i)}$, the true distance will be

$$\theta'' + \frac{fu}{R} + \frac{f\epsilon''}{R},$$

ϵ'' being the error of u in this observation. By naming further θ''' and ϵ''' the same quantities relative to the zenithal distance of $C^{(i)}$, observed from $C^{(i-1)}$, we will have

$$\begin{aligned} \theta'' + \theta''' + \frac{2fu}{R} + \frac{f}{R}(\epsilon'' + \epsilon''') &= \pi + \frac{f}{R}, \\ x^{(i)} - x^{(i-1)} &= \frac{f}{2}(\theta'' - \theta''') + \frac{f^2}{2R}(\epsilon'' - \epsilon'''). \end{aligned}$$

As I myself propose here only to examine what degree of confidence we must accord to this kind of leveling, I will make $h = f$, that which reverts to supposing all the triangles equilateral. I will take, moreover, $\frac{h^2}{2R}$ for unit of distance: by making next [596]

$\epsilon - \epsilon' = \lambda^{(i)}$, $\epsilon'' - \epsilon''' = \gamma^{(i)}$, one will have two equations of the form

$$(A) \quad \begin{cases} x^{(i)} - x^{(i-1)} = \gamma^{(i)} + p^{(i)}, \\ x^{(i)} - x^{(i-2)} = \lambda^{(i)} + q^{(i)}. \end{cases}$$

The first of these equations extends from $i = 1$ to $i = n + 1$, n being the number of triangles. The second equation extends from $i = 2$ to $i = n + 1$. It is necessary now to conclude from this system of equations the most advantageous value of $x^{(n+1)} - x^{(0)}$, the elevation $x^{(0)}$ of the point C above the sea being supposed known. For this, we will multiply the first of the equations (A) by $f^{(i)}$ and the second by $g^{(i)}$, $f^{(i)}$ and $g^{(i)}$ being indeterminate constants. In the system of these equations added all together, the coefficient of $x^{(i)}$ will be $f^{(i)} - f^{(i+1)} + g^{(i)} - g^{(i+2)}$. By equating it to zero and observing that $g^{(i+2)} - g^{(i)} = \Delta g^{(i+1)} + \Delta g^{(i)}$, Δ being the characteristic of finite differences, we will have, by integrating,

$$f^{(i)} = a - g^{(i)} - g^{(i+1)},$$

a being a constant. But, the values of $g^{(i)}$ beginning to take place only when $i = 2$, this expression of $f^{(i)}$ is able to serve only when $i = 2$. In order to have the value of $f^{(1)}$, we will observe that the equating to zero of the coefficient of $x^{(1)}$ gives

$$f^{(1)} = f^{(2)} + g^{(3)};$$

substituting, instead of $f^{(2)}$, $a - g^{(2)} - g^{(3)}$, we will have

$$f^{(1)} = a - g^{(2)}.$$

Next, the preceding expression of $f^{(i)}$ extends only to $i = n$; but, relatively to $i = n + 1$, we must observe that the coefficient of $x^{(n+1)}$ must be unity, that which gives

$$f^{(n+1)} + g^{(n+1)} = 1$$

or

$$f^{(n+1)} = 1 - g^{(n+1)};$$

the equality to zero of the coefficient of $x^{(n)}$ gives $f^{(n)} = f^{(n+1)} - g^{(n)}$, or $f^{(n)} = 1 - g^{(n)} - g^{(n+1)}$. By comparing this expression to this one $f^{(n)} = a - g^{(n)} - g^{(n+1)}$, we will have $a = 1$. The error of the value of $x^{(n+1)}$ will be thus [597]

$$\begin{aligned} & f^{(1)}\gamma^{(1)} + f^{(2)}\gamma^{(2)} + \dots + f^{(n+1)}\gamma^{(n+1)} \\ & + g^{(2)}\lambda^{(2)} + g^{(3)}\lambda^{(3)} + \dots + g^{(n+1)}\lambda^{(n+1)} \end{aligned}$$

The values of $\gamma^{(1)}$, $\gamma^{(2)}$, \dots , $\lambda^{(1)}$, $\lambda^{(2)}$, \dots being evidently subject to the same law of probability, if we name s this error and if we make

$$\begin{aligned} H &= f^{(1)2} + f^{(2)2} + \dots + f^{(n+1)2} \\ &+ g^{(1)2} + g^{(3)2} + \dots + g^{(n+1)2}, \end{aligned}$$

the probability of the error s will be proportional, by §20 of Book II, to an exponential of the form

$$c^{-\frac{Ks^2}{H}},$$

K being a constant dependent on the law of probability of $\gamma^{(i)}$ and $\lambda^{(i)}$.

It is necessary to determine the constants of H , in a manner that H is a minimum. Now we have

$$H = (1 - g^{(2)})^2 + (1 - g^{(2)} - g^{(3)})^2 + \dots + (1 - g^{(n)} - g^{(n+1)})^2 \\ + (1 - g^{(n+1)})^2 + g^{(2)2} + g^{(3)2} + \dots + g^{(n+1)2};$$

by equating to zero the coefficient of the differential of $g^{(i)}$, we have

$$(1) \quad g^{(i+1)} + 3g^{(i)} + g^{(i-1)} = 2.$$

This equation holds from $i = 3$ to $i = n$. The equality to zero of the coefficient of $dg^{(2)}$ gives

$$g^{(3)} + 3g^{(2)} = 2,$$

and the equating to zero of the coefficient of $dg^{(n+1)}$ gives

$$3g^{(n+1)} + g^{(n)} = 2,$$

that which reverts to considering the general equation (1) as holding from $i = 2$ to $i = n + 1$, and to supposing null $g^{(1)}$ and $g^{(n+2)}$. The integration of equation (1) in the finite differences gives [598]

$$g^{(i)} = \frac{2}{5} + Al^{i-1} + A'l^{i-1},$$

l and l' being the two roots $-\frac{3}{2} - \frac{1}{2}\sqrt{5}$, $-\frac{3}{2} + \frac{1}{2}\sqrt{5}$ of the equation

$$y^3 + 3y + 1 = 0;$$

A and A' are two arbitraries such that $g^{(i)}$ becomes null when $i = 1$ and when $i = n + 2$. We have therefore

$$Al^{n+1} + A'l^{n+1} = -\frac{2}{5}, \\ A + A' = -\frac{2}{5}.$$

l^{n+1} is an extremely great quantity when n is a great number and, l'^{n+1} being $\frac{1}{l^{n+1}}$, we see that A is then an excessively small quantity and that thus $A' = -\frac{2}{5}$. We have next

$$f^{(i)} = \frac{1}{5} - Al^{i-1}(1+l) - A'l^{i-1}(1+l').$$

Thence it is easy to conclude that we have, very nearly and without fear $\frac{1}{25}$ of error.

$$H = \frac{n+1}{5},$$

and that thus the exponential proportional to the probability of error s is

$$e^{-\frac{5Ks^2}{n+1}};$$

we are able therefore thus to determine this probability.

We have concluded the value of $x^{(n+1)}$ of the system of equations (A) by the following process.

The system of equations (A) gives

$$x^{(1)} - x^{(0)} = p^{(1)} + \gamma^{(1)};$$

whence we deduce

$$x^{(1)} = p^{(1)} + x^{(0)} + \gamma^{(1)}.$$

We have next the two equations

[599]

$$\begin{aligned} x^{(2)} - x^{(1)} &= p^{(2)} + \gamma^{(2)}, \\ x^{(2)} - x^{(0)} &= q^{(2)} + \lambda^{(2)}; \end{aligned}$$

that which gives

$$x^{(2)} = \frac{1}{2}x^{(1)} + \frac{1}{2}x^{(0)} + \frac{1}{2}(p^{(2)} + q^{(2)}) + \frac{1}{2}\gamma^{(2)} + \frac{1}{2}\lambda^{(2)}.$$

We have the two equations

$$\begin{aligned} x^{(3)} - x^{(2)} &= p^{(3)} + \gamma^{(3)}, \\ x^{(3)} - x^{(1)} &= q^{(3)} + \lambda^{(3)}; \end{aligned}$$

that which gives

$$x^{(3)} = \frac{1}{2}x^{(2)} + \frac{1}{2}x^{(1)} + \frac{1}{2}(p^{(3)} + q^{(3)}) + \frac{1}{2}\gamma^{(3)} + \frac{1}{2}\lambda^{(3)}.$$

By continuing thus, we will have $x^{(n+1)}$. The quantities $\gamma^{(m)}$ and $\lambda^{(m)}$ commence to be introduced into this expression only with the two values of $x^{(m)} - x^{(m-1)}$ and of $x^{(m)} - x^{(m-2)}$. Let us designate by $k^{(r)}$ the coefficient of $\gamma^{(m)}$ in the expression of $x^{(m+r)}$; this expression is

$$x^{(m+r)} = \frac{1}{2}x^{(m+r-1)} + \frac{1}{2}x^{(m+r-2)} + \frac{1}{2}(p^{(m+r)} + q^{(m+r)}) + \frac{1}{2}\gamma^{(m+r)} + \frac{1}{2}\lambda^{(m+r)};$$

by substituting for $x^{(m+r)}$, $x^{(m+r-1)}$, $x^{(m+r-2)}$ the parts of their values relative to $\gamma^{(m)}$, the comparison of the coefficients of this quantity will give

$$k^{(r)} = \frac{1}{2}k^{(r-1)} + \frac{1}{2}k^{(r-2)};$$

whence we deduce, by integrating,

$$k^{(r)} = A + A' \left(-\frac{1}{2}\right)^{r-1},$$

A and A' being two arbitrarities. In order to determine them, we will observe that, r being null, we have $k^{(0)} = \frac{1}{2}$, and that, r being 1, we have

$$k^{(1)} = \frac{1}{2}k^{(0)} = \frac{1}{4};$$

thence we deduce

$$A = \frac{1}{3}, \quad A' = -\frac{1}{12};$$

thus, in the value of $x^{(n+1)}$, where $r = n + 1 - m$, we will have, for the coefficient $k^{(n+1-m)}$ of $\gamma^{(m)}$,

$$k^{(n+1-m)} = \frac{1}{3} - \frac{1}{12} \left(-\frac{1}{2}\right)^{n-m};$$

the coefficient of $\lambda^{(m)}$ in the same value will be evidently the same. Thus the expression of $x^{(n+1)}$ will be a known quantity, plus the series [600]

$$k^{(n)}\gamma^{(1)} + k^{(n-1)}(\gamma^{(2)} + \lambda^{(2)}) + \dots + k^{(0)}(\gamma^{(n+1)} + \lambda^{(n+1)}).$$

Let us designate by s this error and by H the sum of the squares of the coefficients of $\gamma^{(1)}, \gamma^{(2)}, \dots, \lambda^{(2)}, \lambda^{(3)}, \dots$; the probability of s will be proportional to $c^{-\frac{Ks^2}{H}}$. We have, very nearly,

$$H = \frac{2}{9}(n+1);$$

thus the probability of s is very nearly proportional to $c^{\frac{-9Ks^2}{2(n+1)}}$; the equally probable errors are therefore greater in this process than according to the most advantageous method, and nearly in the ratio of $\sqrt{5}$ to $\sqrt{\frac{9}{2}}$; this process approaches therefore much the exactitude of the most advantageous method, and, as the calculation of it is quite simple, we will determine the probability of the errors to which it exhibits, in the general case where the diverse triangles are neither equal nor equilateral.

If we represent by $m^{(i)}$ the square of $C^{(i-1)}C^{(i)}$ divided by $2R$, and by $n^{(i)}$ the square of $C^{(i-2)}C^{(i)}$ divided similarly by $2R$, the system of equations (A) will be changed into the following:

$$(A') \quad \begin{cases} x^{(i)} - x^{(i-1)} = p^{(i)} + m^{(i)}\gamma^{(i)}, \\ x^{(i)} - x^{(i-2)} = q^{(i)} + n^{(i)}\lambda^{(i)}. \end{cases}$$

The process that we have just examined gives, by following the preceding analysis, the coefficient of $\gamma^{(i)}$ in the expression of $x^{(n+1)}$ equal to

$$\frac{1}{3}m^{(i)} - \frac{1}{12}m^{(i)} \left(-\frac{1}{2}\right)^{n-i}.$$

Similarly, the coefficient of $\lambda^{(i)}$, in the same expression, is

$$\frac{1}{3}n^{(i)} - \frac{1}{12}n^{(i)} \left(-\frac{1}{2}\right)^{n-i};$$

thence it follows that the value of H is, very nearly,

$$\frac{1}{9}S(m^{(i)2} + n^{(i)2}),$$

the integral sign S extending to all the values of i to $i = n + 1$; the probability of an error s , in the expression of $x^{(n+1)}$, is therefore proportional to [601]

$$c^{\frac{-9Ks^2}{S(m^{(i)2} + n^{(i)2})}}.$$

If we apply to the equations (A') the analysis that we have given above for the case of the most advantageous method, we will find, by multiplying them respectively by $f^{(i)}$ and $g^{(i)}$, the following equation

$$f^{(i)} = 1 - g^{(i)} - g^{(i+1)},$$

and this equation will hold from $i = 1$ to $i = n + 1$, by supposing $g^{(i)}$ and $g^{(i+2)}$ nulls. We will have next the general equation

$$m^{(i)2}g^{(i+1)} + (n^{(i)2} + m^{(i)2} + n^{(i-1)2})g^{(i)} + m^{(i-1)2}g^{(i-1)} = m^{(i)2} + m^{(i-1)2}.$$

This equation holds from $i = 2$ to $i = n + 1$. By combining it with the equations $g^{(1)} = 0$, $g^{(n+2)} = 0$, we will have the values of $f^{(1)}$, $f^{(2)}$, ..., $f^{(n+1)}$; $g^{(1)}$, $g^{(2)}$, ..., $g^{(n+2)}$; we will have next

$$H = S(f^{(i)2}m^{(i)2} + g^{(i)2}n^{(i)2}),$$

the sign S comprehending all the values of $f^{(i)}m^{(i)2}$ and of $g^{(i)2}n^{(i)2}$; the probability of an error s in the value of $x^{(n+1)}$ will be proportional to

$$c^{-\frac{Ks^2}{H}},$$

§5. It is necessary now to determine the value of K . For this, we will observe that the factor u is determined, by that which precedes, by means of the equation

$$u = \frac{\pi - \theta - \theta' + \frac{h}{R}}{\frac{2h}{R}},$$

and that the error of this expression is $\frac{\epsilon + \epsilon'}{2}$. Each double station furnishes a value of u , and the mean of these values is the value that it is necessary to adopt. If we name i the number of these values, the error to fear will be $S\frac{\epsilon + \epsilon'}{2i}$, the sign S corresponding to the i quantities $\frac{\epsilon + \epsilon'}{2i}$ related to each double station. Let s be the sum $S\frac{\epsilon + \epsilon'}{2i}$; the probability of s will be, by §20 of Book II, proportional to an exponential of the form [602]

$$c^{-\frac{K's^2}{i}},$$

and, if we name q the sum of the squares of the differences of each partial value to its mean value, we will have

$$K' = \frac{i}{2q}.$$

We have, by that which precedes, the probability of the error of a value s' of the function $S \frac{\epsilon - \epsilon'}{2i}$ proportional to the exponential

$$e^{-\frac{4Ks'^2}{i}},$$

the sign S extending to i quantities of the form $\frac{\epsilon - \epsilon'}{2}$. Now, the errors ϵ and $-\epsilon$ being supposed equally probable, it is clear that the same values of $S \frac{\epsilon + \epsilon'}{2i}$ and of $S \frac{\epsilon - \epsilon'}{2i}$ are equally probable; we have therefore

$$4K = K',$$

that which gives

$$K = \frac{i}{8q}.$$

The forty-five first values of u , given in the second Volume of the *Base du Système métrique* (p. 771), and which are founded on some observations made in the month of the year where we observe most often, give, for its mean value,

$$u = 0,07818,$$

and the sum q of the squares of the differences of these values to the mean is 0,04900629; [603] i being here equal to 45, we have

$$K = \frac{45}{0,39205032} = 114,781.$$

If we suppose the number n of triangles equal to 25 and if we make all the sides equal to 20000^m, we will have 240000^m for the distance from $x^{(26)}$ to $x^{(0)}$; this is nearly the distance from Paris to Dunkirk. In this case, the quantity $\frac{f^2}{2R}$, taken for unit of distance, is 31^m, 416. Thence we conclude that the odds are one against one that the error on the height $x^{(26)}$ is comprehended within the limits $\pm 3^m$, 1839. There are odds nine against one that it is comprehended within the limits $\pm 7^m$, 761; we are not able therefore then to respond with a sufficient probability that this error will not exceed $\pm 8^m$.

The chain of triangles that we have just considered is much more favorable to the determination of the height of its last point than that of which Delambre has made use, in the Work cited, in order to determine the height of the Pantheon above the level of the sea. By considering this last chain, we see that we are not able to respond, with a sufficient probability, that the error respecting this height will not exceed $\pm 16^m$.

§6. We see, by that which precedes, that the great triangles, which are very proper to the measure of terrestrial degrees, are too small in order to determine the respective heights of the diverse stations. Thus, in the case of a chain of equilateral triangles of which f is the length of each side, the equally probable errors of the difference of level of two extreme stations being proportional to $\frac{f^2 \sqrt{n+1}}{2R}$, n being the number of triangles, if we name a the distance of these two stations, we will have, by supposing $n + 1$ even,

$$a = \frac{1}{2}(n + 1)f;$$

$\frac{f^2\sqrt{n+1}}{2R}$ will be therefore proportional to $\frac{1}{(n+1)^{\frac{3}{2}}}$; the equally probable errors will be [604]

therefore proportional to this fraction. Thus, by quadrupling the number of triangles, they will become eight times smaller; but then the errors due to the observations of the angles become comparable to the errors due to the variability of the terrestrial refractions. Let us examine how we are able to have regard at the same time to these two kinds of errors.

Let us consider a sequence of points $C, C^{(1)}, C^{(2)}, \dots$. Let $h^{(0)}$ be the distance from C to $C^{(1)}$; $h^{(1)}$ the distance from $C^{(1)}$ to $C^{(2)}$; $h^{(2)}$ the distance from $C^{(2)}$ to $C^{(3)}$, and thus consecutively. Let us imagine that from the point $C^{(i)}$ we observe $C^{(i+1)}$, and reciprocally. The zenithal distance from $C^{(i+1)}$, observed from $C^{(i)}$ will be, by that which precedes,

$$\theta + \frac{h^{(i)}u}{R} + \frac{h^{(i)}\epsilon}{R} + \alpha,$$

ϵ being the error of u and α being that of the observed angle θ . The zenithal distance of $C^{(i)}$, observed from $C^{(i+1)}$, will be

$$\theta' + \frac{h^{(i)}u}{R} + \frac{h^{(i)}\epsilon'}{R} + \alpha',$$

ϵ' and α' being the errors of u and of θ' in the observation made at the point $C^{(i+1)}$. We will have therefore the two equations

$$\begin{aligned} \theta + \theta' + \frac{2h^{(i)}u}{R} + \frac{h^{(i)}u}{R}(\epsilon + \epsilon') + \alpha + \alpha' &= \pi + \frac{h^{(i)}}{R}, \\ x^{(i+1)} - x^{(i)} &= \frac{\theta - \theta'}{2}h^{(i)} + \frac{h^{(i)2}}{2R}(\epsilon - \epsilon') + \frac{1}{2}h^{(i)}(\alpha - \alpha'). \end{aligned}$$

Let us designate as above $\epsilon - \epsilon'$ by $\gamma^{(i)}$, and let us make $\alpha - \alpha'$ equal to $\lambda^{(i)}$; we will have, for the elevation $x^{(n+1)} - x^{(0)}$ of the point $C^{(n+1)}$ above C , an expression of the form

$$x^{(n+1)} - x^{(0)} = M + S \frac{h^{(i)2}}{2R} \gamma^{(i)} + S \frac{1}{2} h^{(i)} \lambda^{(i)},$$

the integral sign S corresponding to all the values of i , from $i = 0$ to $i = n$. The error of this value of $x^{(n+1)}$ is [605]

$$S \frac{h^{(i)2} \gamma^{(i)}}{2R} + S \frac{h^{(i)}}{2} \lambda^{(i)}.$$

It is necessary now to determine the probability of this error that we will designate by s . Let there be generally

$$s = Sm^{(i)}\gamma^{(i)} + Sn^{(i)}\lambda^{(i)};$$

the probability of s will be, by the analysis of §20 of Book II of the *Théorie analytique des Probabilités*, proportional to

$$\begin{aligned} &\int d\varpi dx dy \phi(x)\psi(y)c^{-s\varpi\sqrt{-1}} \\ &\times [\cos(m^{(0)}x + n^{(0)}y)\varpi \cos(m^{(1)}x + n^{(1)}y)\varpi \cos(m^{(2)}x + n^{(2)}y)\varpi \dots]; \end{aligned}$$

$\phi(x)$ is the law of probability of a value x of $\gamma^{(0)}$; $\psi(y)$ is the law of probability of a value y of $\lambda^{(0)}$. The negative and positive errors are supposed equally probable: the integrals relative to x and y are taken from negative infinity to positive infinity, and the integral relative to $\varpi = -\pi$ to $\varpi = \pi$. By making

$$\begin{aligned} 2 \int dx \phi(x) &= k, & \int x^2 dx \phi(x) &= k'', \\ 2 \int dy \psi(y) &= \bar{k}, & \int y^2 dy \psi(y) &= \bar{k}'', \end{aligned}$$

the integrals being taken from x and y null to x and y equal to infinity, the analysis of the section cited will give the probability of s proportional to

$$c \frac{-s^2}{\frac{4k''}{k} Sm^{(i)2} + \frac{4\bar{k}''}{k} Sn^{(i)2}}$$

It is easy to conclude generally from the same analysis that, if we make

$$s = Sm^{(i)}\gamma^{(i)} + Sn^{(i)}\lambda^{(i)} + Sr^{(i)}\delta^{(i)} + \dots,$$

$\gamma^{(i)}$, $\lambda^{(i)}$, $\delta^{(i)}$, ... being of the errors deriving from different sources, the probability of s is proportional to the exponential [606]

$$c \frac{-s^2}{\frac{4k''}{k} Sm^{(i)2} + \frac{4\bar{k}''}{k} Sn^{(i)2} + \frac{4\bar{k}''}{k} Sr^{(i)2} + \dots}$$

by designating by $\pi(x)$ the probability of an error x due to the third source of error, and making

$$2 \int dx \pi(x) = \bar{k}, \quad \int x^2 dx \pi(x) = \bar{k}'',$$

the integrals being taken from x null to x infinity; and thus of the other errors.

In order to determine, in the present question, the constants $\frac{4k''}{k}$ and $\frac{4\bar{k}''}{k}$, I will suppose first the second null or very small, relatively to the first, as we are able to do in the great triangulations of the meridian. In this case, the probability of an error s will be, by making $m^{(i)} = 1$, proportional to

$$c \frac{-s^2}{\frac{4k''}{k} n},$$

n being the number of intervals which separate the stations. The probability of a value s' of $S \frac{\epsilon - \epsilon'}{2}$ or of $S \frac{\epsilon + \epsilon'}{2}$, that which corresponds to an error $2s'$ in the value of $S(\epsilon - \epsilon')$, will be proportional to

$$c \frac{-4s'^2}{\frac{4k''}{k} n};$$

but, by that which precedes, this probability is proportional to

$$c \frac{-s'^2}{2q n};$$

we have therefore

$$\frac{2q}{i} = \frac{k''}{k} \quad \text{or} \quad \frac{4k''}{k} = \frac{8q}{i} = \frac{1}{114,781}.$$

If we suppose now $\frac{k''}{k}$ null and $n^{(i)} = 1$, the probability of a value s' of the sum $S^{\frac{\alpha' - \alpha}{2}}$ will be proportional to [607]

$$c^{\frac{-4s'^2}{4k''n}},$$

and the probability of a same value s' of $S(\alpha + \alpha' + \alpha'')$ will be proportional to

$$c^{\frac{-2s'^2}{12k''n}}.$$

If we suppose this law of probability the same as for the errors of the sum of the three angles of a spherical triangle, in the geodesic measures, and which, by §1 of the second Supplement, is able to be supposed proportional to

$$c^{\frac{-(i+2)s'^2}{2\theta^2n}},$$

θ^2 being the sum of the squares of the excess observed in the sum of the errors of the three angles in i triangles, we will have

$$\frac{4\bar{k}''}{\bar{k}} = \frac{4\theta^2}{3(i+2)}.$$

We have, by that which we have seen,

$$\frac{i+2}{\theta^2} = \frac{109}{445,217};$$

hence,

$$\frac{4\bar{k}''}{\bar{k}} = \frac{4}{3} \frac{445,217}{109},$$

a quantity that it is necessary to divide by the square of the number of sexagesimal seconds that this radius contains, and then we have

$$\frac{4\bar{k}''}{\bar{k}} = \frac{1,2801}{10^{10}}.$$

Let us suppose the distances of the consecutive stations equal to 1200 m; we will find that there are odds one against one that the error respecting the value of $x^{(n+1)}$ is not above $\pm 0^m, 08555$ when $n = 200$. There are odds one thousand against one that the error is not above $\pm 0^m, 413$. [608]

General method of the Calculation of probabilities, when there are many sources of errors.

The consideration of the two independent sources of error which exist in the operations of the leveling has led me to examine the general case of the observations subject to many sources of errors. Such are astronomical observations. The greater part are made by means of two instruments, the meridian lunette and the circle, of which the errors must not be supposed to have the same law of probability. In the equations of condition that we deduce from these observations, in order to obtain the elements of the celestial movements, these errors are multiplied by some different coefficients for each source of error and for each equation. The most advantageous systems of factors by which it is necessary to multiply these equations, in order to have the final equations which determine the elements, are no longer, as in the case of a unique source of errors, the coefficients of each element in the equations of condition. The facility with which the analysis that I have given in Book II of my *Théorie des Probabilités* is applied to this general case will show the advantages of this analysis.

Let us suppose first that we have a system of equations of condition represented by this here

$$p^{(i)}y = a^{(i)} + m^{(i)}\gamma^{(i)} + n^{(i)}\lambda^{(i)} + \dots,$$

y being an element of which we seek the most advantageous value. If we multiply the preceding equation by a factor $f^{(i)}$, the reunion of all these products will give for y the expression

$$y = \frac{Sa^{(i)}f^{(i)}}{Sp^{(i)}f^{(i)}} + \frac{Sm^{(i)}f^{(i)}\gamma^{(i)} + Sn^{(i)}f^{(i)}\lambda^{(i)} + \dots}{Sp^{(i)}f^{(i)}}$$

The error of y will be

[609]

$$\frac{Sm^{(i)}f^{(i)}\gamma^{(i)} + Sn^{(i)}f^{(i)}\lambda^{(i)} + \dots}{Sp^{(i)}f^{(i)}}$$

By designating by s this error, its probability will be proportional, by the preceding section, to the exponential

$$c^{\frac{-s^2(Sp^{(i)}f^{(i)})^2}{\frac{4k''}{k}Sm^{(i)2}f^{(i)2} + \frac{4\bar{k}''}{k}Sn^{(i)2}f^{(i)2} + \dots}}$$

It is necessary to determine $f^{(i)}$ in a manner that

$$\frac{\frac{4k''}{k}Sm^{(i)2}f^{(i)2} + \frac{4\bar{k}''}{k}Sn^{(i)2}f^{(i)2} + \dots}{(Sp^{(i)}f^{(i)})^2}$$

is a minimum, because it is clear that then the same error s becomes less probable than in each other system of factors. If we name A the numerator of this fraction, and if we make $f^{(i)}$ vary by a quantity dq , we will have, through the condition of the minimum, by equating to zero the differential of this fraction,

$$0 = \frac{\frac{k''}{k}m^{(i)2}f^{(i)} + \frac{\bar{k}''}{k}n^{(i)2}f^{(i)} + \dots}{A} - \frac{p^{(i)}}{Sp^{(i)}f^{(i)}}$$

that which gives for $f^{(i)}$ an expression of this form

$$f^{(i)} = \frac{\mu p^{(i)}}{\frac{k''}{k} m^{(i)2} + \frac{\bar{k}''}{k} n^{(i)2} + \dots}$$

We are able to make here $\mu = 1$, because, this quantity being independent of i , it affects equally all the multipliers $f^{(i)}$; thus the quantity $f^{(i)}$, by which we must multiply each equation of condition in order to have the most advantageous result, is [610]

$$\frac{p^{(i)}}{\frac{k''}{k} m^{(i)2} + \frac{\bar{k}''}{k} n^{(i)2} + \dots}$$

and the probability of an error s of this result is proportional to the exponential

$$\frac{e^{-\frac{s^2}{4}}}{c} S \frac{p^{(i)2}}{\frac{k''}{k} m^{(i)2} + \frac{\bar{k}''}{k} n^{(i)2} + \dots}$$

We will have, by the same analysis and by §22 of Book II, the factors by which we must multiply the equations of condition in order to have the most advantageous results, whatever be the number of elements to determine and the number of kinds of errors; we will have similarly the laws of probability of the errors of these results.

Let us suppose that we have, between two elements x and y , the equation of condition

$$l^{(i)} x + p^{(i)} y = a^{(i)} + m^{(i)} \gamma^{(i)} + n^{(i)} \lambda^{(i)} + r^{(i)} \delta^{(i)} + \dots,$$

$\gamma^{(i)}, \lambda^{(i)}, \delta^{(i)}, \dots$ being some errors of which the sources are different. By multiplying first this equation by a system $f^{(i)}$ of factors, the reunion of these products will give the final equation

$$xSl^{(i)} f^{(i)} + ySp^{(i)} f^{(i)} = Sa^{(i)} f^{(i)} + Sm^{(i)} f^{(i)} \gamma^{(i)} + Sn^{(i)} f^{(i)} \lambda^{(i)} + \dots,$$

By multiplying next the equation of condition by another system $g^{(i)}$ of factors, the reunion of the products will give a second final equation

$$xSl^{(i)} g^{(i)} + ySp^{(i)} g^{(i)} = Sa^{(i)} g^{(i)} + Sm^{(i)} g^{(i)} \gamma^{(i)} + \dots.$$

We deduce from these two final equations

$$x = \frac{Sa^{(i)} f^{(i)} Sp^{(i)} g^{(i)} - Sa^{(i)} g^{(i)} Sp^{(i)} f^{(i)}}{L} + \frac{Sm^{(i)} f^{(i)} \gamma^{(i)} Sp^{(i)} g^{(i)} - Sm^{(i)} g^{(i)} \gamma^{(i)} Sp^{(i)} f^{(i)} + \dots}{L},$$

L being equal to

$$Sl^{(i)} f^{(i)} Sp^{(i)} g^{(i)} - Sl^{(i)} g^{(i)} Sp^{(i)} f^{(i)}.$$

[611]

The coefficient of $\gamma^{(i)}$ in this value is

$$\frac{m^{(i)} f^{(i)} Sp^{(i)} g^{(i)} - m^{(i)} g^{(i)} Sp^{(i)} f^{(i)}}{L}.$$

By changing $m^{(i)}$ into $n^{(i)}$, $r^{(i)}$, \dots , we will have the coefficients corresponding to $\lambda^{(i)}$, $\delta^{(i)}$, \dots . By naming therefore s the value of the part of x dependent on the errors $\gamma^{(i)}$, $\lambda^{(i)}$, $\delta^{(i)}$, \dots , the probability of this value will be, by that which precedes, proportional to the exponential

$$c^{-\frac{s^2}{H}},$$

by making

$$H = \frac{SM^{(i)}f^{(i)2}(Sp^{(i)}g^{(i)})^2 - 2SM^{(i)}f^{(i)}g^{(i)}Sp^{(i)}f^{(i)}Sp^{(i)}g^{(i)} + SM^{(i)}g^{(i)2}(Sp^{(i)}f^{(i)})^2}{L^2},$$

$M^{(i)}$ being equal to

$$\frac{4k''}{k}m^{(i)2} + \frac{4\bar{k}''}{\bar{k}}n^{(i)2} + \frac{4\bar{\bar{k}}''}{\bar{\bar{k}}}r^{(i)2} + \dots$$

It is necessary now to determine $f^{(i)}$ and $g^{(i)}$ in a manner that H is a minimum. For this, we will make $f^{(i)}$ vary, and we will equate to zero the coefficient of its differential; that which will give, by naming P the numerator of the expression of H ,

$$\begin{aligned} 0 &= M^{(i)}f^{(i)}(Sp^{(i)}g^{(i)})^2 - M^{(i)}g^{(i)}Sp^{(i)}f^{(i)}Sp^{(i)}g^{(i)} \\ &\quad - p^{(i)}SM^{(i)}f^{(i)}g^{(i)}Sp^{(i)}g^{(i)} + p^{(i)}SM^{(i)}g^{(i)2}Sp^{(i)}f^{(i)} \\ &\quad - \frac{P}{L}(l^{(i)}Sp^{(i)}g^{(i)} - p^{(i)}Sl^{(i)}g^{(i)})^2. \end{aligned}$$

It is easy to see that we satisfy this equation by supposing

$$f^{(i)} = \frac{l^{(i)}}{M^{(i)}}, \quad g^{(i)} = \frac{p^{(i)}}{M^{(i)}};$$

and we must conclude from it that we would satisfy, by the same supposition, the corresponding equation that would give $dH = 0$, by making $g^{(i)}$ vary. We see that the same values of $f^{(i)}$ and $g^{(i)}$ satisfy the similar equations which result from the consideration of the element y . [612]

If we have, among the elements x , y , z , \dots , some equations of condition represented by the general equation

$$l^{(i)}x + p^{(i)}y + q^{(i)}z + \dots = a^{(i)} + m^{(i)}\gamma^{(i)} + n^{(i)}\lambda^{(i)} + r^{(i)}\delta^{(i)} + \dots,$$

$\gamma^{(i)}$, $\lambda^{(i)}$, $\delta^{(i)}$, \dots being the errors of diverse kinds, we will find by the preceding analysis that the factors by which we must multiply respectively this equation, in order to form the final equations which give the values of the most advantageous elements, are, for the first final equation, represented by

$$\frac{l^{(i)}}{\frac{k''}{k}m^{(i)2} + \frac{\bar{k}''}{\bar{k}}n^{(i)2} + \frac{\bar{\bar{k}}''}{\bar{\bar{k}}}r^{(i)2} + \dots}$$

They are represented, for the second final equation, by

$$\frac{p^{(i)}}{\frac{k''}{k}m^{(i)2} + \frac{\bar{k}''}{k}n^{(i)2} + \frac{\bar{\bar{k}}''}{k}r^{(i)2} + \dots}$$

and thus consecutively. By applying therefore to the equations thus multiplied the analysis of §2 of the first Supplement, we will have the values of the most advantageous elements and the laws of probabilities of their errors.

In order to give an example of this application, let us consider only two elements x and y . If we make

$$M^{(i)} = \frac{k''}{k}m^{(i)2} + \frac{\bar{k}''}{k}n^{(i)2} + \frac{\bar{\bar{k}}''}{k}r^{(i)2} + \dots$$

We will multiply the previous equation of condition by $\frac{p^{(i)}}{M^{(i)}}$, and we will deduce from [613] it

$$xS \frac{l^{(i)}p^{(i)}}{M^{(i)}} + yS \frac{p^{(i)2}}{M^{(i)}} = S \frac{a^{(i)}p^{(i)}}{M^{(i)}} + S \frac{p^{(i)}}{M^{(i)}} (m^{(i)}\gamma^{(i)} + n^{(i)}\lambda^{(i)} + \dots);$$

but the condition of the most advantageous method gives

$$0 = S \frac{l^{(i)}}{M^{(i)}} (m^{(i)}\gamma^{(i)} + n^{(i)}\lambda^{(i)} + \dots),$$

$$0 = S \frac{p^{(i)}}{M^{(i)}} (m^{(i)}\gamma^{(i)} + n^{(i)}\lambda^{(i)} + \dots);$$

we will have therefore

$$y = \frac{S \frac{a^{(i)}p^{(i)}}{M^{(i)}} - xS \frac{p^{(i)}l^{(i)}}{M^{(i)}}}{S \frac{p^{(i)2}}{M^{(i)}}}.$$

Substituting this value of y into the general equation of condition, and making

$$l_1^{(i)} = l^{(i)} - p^{(i)} \frac{S \frac{l^{(i)}p^{(i)}}{M^{(i)}}}{S \frac{p^{(i)2}}{M^{(i)}}},$$

$$a_1^{(i)} = a^{(i)} - p^{(i)} \frac{S \frac{l^{(i)}p^{(i)}}{M^{(i)}}}{S \frac{p^{(i)2}}{M^{(i)}}},$$

we will have

$$x = \frac{S \frac{a_1^{(i)}l_1^{(i)}}{M^{(i)}}}{S \frac{l_1^{(i)2}}{M^{(i)}}};$$

and the probability of an error s of this value will be proportional to

$$c \frac{-s^2}{4} S \frac{l_1^{(i)2}}{M^{(i)}}.$$

This analysis supposes knowledge of the constants $\frac{k''}{k}$ and $\frac{\bar{k}''}{k}$. But we are able to obtain from it, by the same observations, some very close values, in the following [614]

manner.

Let us imagine that we have determined the elements x, y, z, \dots by the method according to which we form the final equations, by multiplying each equation of condition successively by the corresponding coefficient of each element. If we substitute the values of the elements thus determined into the equation of condition

$$l^{(i)}x + p^{(i)}y + \dots - a^{(i)} = m^{(i)}\gamma^{(i)} + n^{(i)}\lambda^{(i)} + \dots,$$

we will have an equation of this form

$$R^{(i)} = m^{(i)}\gamma^{(i)} + n^{(i)}\lambda^{(i)} + \dots$$

Let us suppose, for greater simplicity, that we have only two kinds of errors $\gamma^{(i)}$ and $\lambda^{(i)}$: we will multiply first the preceding equation by $m^{(i)}$. By raising next each member to the square and taking the sum of all the equations thus formed, we will have

$$Sm^{(i)2}R^{(i)2} = S(m^{(i)4}\gamma^{(i)2} + 2m^{(i)3}n^{(i)}\gamma^{(i)}\lambda^{(i)} + n^{(i)2}m^{(i)2}\lambda^{(i)2}).$$

The mean value of $m^{(i)4}\gamma^{(i)2}$ is evidently

$$\frac{m^{(i)4} \int \gamma^2 d\gamma \phi(\gamma)}{\int d\gamma \phi(\gamma)},$$

the integrals being taken from $\gamma = -\infty$ to γ infinity, that which gives $\frac{2k''m^{(i)4}}{k}$. We have similarly $\frac{2\bar{k}''}{k}m^{(i)2}n^{(i)2}$ for the mean value of $m^{(i)2}n^{(i)2}\lambda^{(i)2}$. We find in the same manner that the mean value of $2m^{(i)3}n^{(i)}\gamma^{(i)}\lambda^{(i)}$ is null; we have therefore, by substituting instead of the quantities their mean values, that which we are able to make with so much more precision as the number of observations is greater,

$$Sm^{(i)2}R^{(i)2} = \frac{2k''}{k}Sm^{(i)4} + \frac{2\bar{k}''}{k}m^{(i)2}n^{(i)2}.$$

We will have similarly

[615]

$$Sn^{(i)2}R^{(i)2} = \frac{2k''}{k}Sm^{(i)2}n^{(i)2} + \frac{2\bar{k}''}{k}Sn^{(i)4};$$

whence we deduce

$$\frac{4k''}{k} = \frac{2Sn^{(i)4}Sm^{(i)2}R^{(i)2} - 2Sm^{(i)2}n^{(i)2}Sn^{(i)2}R^{(i)2}}{Sm^{(i)4}Sn^{(i)4} - (Sm^{(i)2}n^{(i)2})^2},$$

$$\frac{4\bar{k}''}{k} = \frac{2Sm^{(i)4}Sn^{(i)2}R^{(i)2} - 2Sm^{(i)2}n^{(i)2}Sm^{(i)2}R^{(i)2}}{Sm^{(i)4}Sn^{(i)4} - (Sm^{(i)2}n^{(i)2})^2};$$

be designating therefore by $2P$ and $2Q$ the numerators of these two expressions, the factors by which we must multiply the equation of condition will be

$$\frac{l^{(i)}}{m^{(i)2}P + n^{(i)2}Q},$$

$$\frac{p^{(i)}}{m^{(i)2}P + n^{(i)2}Q},$$

.....

The concern now is to show that these values of $\frac{4k''}{k}$, $\frac{4\bar{k}''}{k}$ are quite close. For this, let us consider only one element x : the equation of condition

$$l^{(i)}x = a^{(i)} + m^{(i)}\gamma^{(i)} + n^{(i)}\lambda^{(i)}$$

will give

$$x = \frac{Sa^{(i)}l^{(i)}}{Sl^{(i)2}} + \frac{Sl^{(i)}m^{(i)}\gamma^{(i)} + Sl^{(i)}n^{(i)}\lambda^{(i)}}{Sl^{(i)2}}.$$

Substituting this value in the equation of condition, we will have

$$R^{(i)} = \frac{l^{(i)}Sa^{(i)}l^{(i)} - a^{(i)}Sl^{(i)2}}{Sl^{(i)2}},$$

$$R^{(i)} + l^{(i)}\frac{S(l^{(i)}m^{(i)}\gamma^{(i)} + l^{(i)}n^{(i)}\lambda^{(i)})}{Sl^{(i)2}} = m^{(i)}\gamma^{(i)} + n^{(i)}\lambda^{(i)};$$

but it is easy to see that the values of $Sl^{(i)}m^{(i)}\gamma^{(i)}$ and of $Sl^{(i)}n^{(i)}\lambda^{(i)}$ are nulls by [616] the supposition of the negative errors as probable as the positive errors: we are able therefore to make, as above,

$$R^{(i)} = m^{(i)}\gamma^{(i)} + n^{(i)}\lambda^{(i)},$$

that which it was necessary to establish.