BOOK II
CHAPTER IV

DE LA PROBABILITÉ DES ERREURS DES RÉSULTES MOYENS D'UN GRAND NOMBRE D'OBSERVATIONS ET DES RÉSULTATS MOYENS LES PLUS AVANTAGEUX.

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ON THE PROBABILITY OF THE ERRORS OF THE MEAN RESULTS OF A GREAT NUMBER OF OBSERVATIONS, AND ON THE MOST ADVANTAGEOUS MEAN RESULTS

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§18. Let us consider now the mean results of a great number of observations of which we know the law of the facility of errors. Let us suppose first that for each observation, the errors are able to be equally [304]

$$-n,\ -n+1,\ -n+2,\ \ldots,\ -1,\ 0,\ 1,\ 2,\ \ldots,\ n-2,\ n-1,\ n.$$

The probability of each error will be $\frac{1}{2n+1}$. If we name s the number of observations, the coefficient of $c^{l\varpi\sqrt{-1}}$ in the development of the polynomial

$$\left\{ \begin{array}{c} c^{-n\varpi\sqrt{-1}} + c^{-(n-1)\varpi\sqrt{-1}} + c^{-(n-2)\varpi\sqrt{-1}} \ldots\ldots \\ \cdots + c^{-\varpi\sqrt{-1}} + 1 + c^{\varpi\sqrt{-1}} \cdots + c^{n\varpi\sqrt{-1}} \end{array} \right\}^s$$

will be the number of combinations in which the sum of the errors is l. This coefficient is the term independent of $c^{\varpi\sqrt{-1}}$ and of its powers, in the development of the same polynomial multiplied by $c^{-l\varpi\sqrt{-1}}$, and it is clearly equal to the term independent of ϖ in the same development multiplied by $\frac{c^{l\varpi\sqrt{-1}}+c^{-l\varpi\sqrt{-1}}}{2}$ or by cos $l\varpi$, we will have therefore for the expression of this coefficient,

$$\frac{1}{\pi}\int d\varpi \cos l\varpi(1 + 2\cos\varpi + 2\cos 2\varpi \cdots + 2\cos n\varpi)^s,$$

the integral being taken from $\varpi = 0$ to $\varpi = \pi$.

We have seen, in §36 of the first book, that this integral is

$$\frac{(2n+1)^s\sqrt{3}}{\sqrt{n(n+1)2s\pi}} c^{-\frac{\frac{3}{2}l^2}{n(n+1)s}};$$

the total number of combinations of the errors is $(2n+1)^s$; by dividing the preceding quantity by that here, we will have [305]

$$\frac{\sqrt{3}}{\sqrt{n(n+1)2s\pi}} c^{-\frac{\frac{3}{2}l^2}{n(n+1)s}},$$

for the probability that the sum of the errors of the s observations will be l.

If we make

$$l = 2t\sqrt{\frac{n(n+1)s}{6}},$$

the probability that the sum of the errors will be comprehended within the limits $+2T\sqrt{\frac{n(n+1)s}{6}}$ and $-2T\sqrt{\frac{n(n+1)s}{6}}$ will be equal to

$$\frac{2}{\pi}\int dt c^{-t^2},$$

the integral being taken from $t = 0$ to $t = T$. This expression holds further in the case of n infinite. Then by naming $2a$ the interval comprehended between the limits of the errors of each observation, we will have $n = a$, and the preceding limits would become $\pm\frac{2Ta\sqrt{s}}{\sqrt{6}}$: thus the probability that the sum of the errors will be comprehended within the limits $\pm ar\sqrt{s}$ is

$$2\sqrt{\frac{3}{2\pi}}\int dr c^{-\frac{3}{2}r^2};$$

this is also the probability that the mean error will be comprehended within the limits $\pm\frac{ar}{\sqrt{s}}$; because we have the mean error, by dividing by s the sum of the errors.

The probability that the sum of the inclination of the orbits of s comets, will be comprehended within some given limits, by supposing all the inclinations equally possible, from zero to a right angle, is evidently the same as the preceding probability; the interval $2a$ of the limits of the errors of each observation is, in this case, the interval $\frac{\pi}{2}$ of the limits of the possible inclinations; then [306]
the probability that the sum of the inclinations must be comprehended within the limits $\pm\frac{\pi r\sqrt{s}}{4}$ is $2\sqrt{\frac{3}{2\pi}}\int drc^{-\frac{3}{2}r^2}$; that which accords with that which we have found in §13.

Let us suppose generally that the probability of each positive or negative error, is expressed by $\phi\left(\frac{x}{n}\right)$, x and n being some infinite numbers. Then, in the function

$$1+2\cos\varpi+2\cos 2\varpi+2\cos 3\varpi\cdots+2\cos n\varpi,$$

each term, such as $2\cos x\varpi$, must be multiplied by $\phi\left(\frac{x}{n}\right)$; now we have

$$2\phi\left(\frac{x}{n}\right)\cos x\varpi=2\phi\left(\frac{x}{n}\right)-\frac{x^2}{n^2}\phi\left(\frac{x}{n}\right)n^2\varpi^2+\text{etc.}$$

By making therefore

$$x'=\frac{x}{n},\qquad dx'=\frac{1}{n},$$

the function

$$\phi\left(\frac{0}{n}\right)+2\phi\left(\frac{1}{n}\right)\cos\varpi+2\phi\left(\frac{2}{n}\right)\cos 2\varpi\cdots+2\phi\left(\frac{n}{n}\right)\cos n\varpi,$$

becomes

$$2n\int dx'\phi(x')-n^3\varpi^2\int x'^2dx'\phi(x')+\text{etc.};$$

the integrals must be extended from $x'=0$ to $x'=1$. Let then

$$k=2\int dx'\phi(x'),\quad k''=\int x'^2dx'\phi(x'),\quad \text{etc.}$$

The preceding series becomes

$$nk\left(1-\frac{k''}{k}n^2\varpi^2+\text{etc.}\right).$$

Now the probability that the sum of the errors of the s observations will be comprehended within the limits $\pm l$, is, as it easy to be assured of it by the preceding reasonings,

$$\frac{2}{\pi}\iint d\varpi\,dl\cos l\varpi\begin{Bmatrix}\phi\left(\frac{0}{n}\right)+2\phi\left(\frac{1}{n}\right)\cos\varpi+2\phi\left(\frac{2}{n}\right)\cos 2\varpi+\cdots\\ \cdots+2\phi\left(\frac{n}{n}\right)\cos n\varpi\end{Bmatrix}^s,$$ [307]

the integral being taken from ϖ null to $\varpi=\pi$; this probability is therefore

$$2\frac{(nk)^s}{\pi}\iint d\varpi\,dl\cos l\varpi\left(1-\frac{k''}{k}n^2\varpi^2-\text{etc.}\right)^s. \qquad (u)$$

Let us suppose

$$\left(1 - \frac{k''}{k} n^2 \varpi^2 - \text{etc.}\right)^s = c^{-t^2};$$

by taking the hyperbolic logarithms, we will have very nearly, when s is a great number,

$$s \frac{k''}{k} n^2 \varpi^2 = t^2;$$

that which gives

$$\varpi = \frac{t}{n} \sqrt{\frac{k}{k'' s}}.$$

If we observe next that nk or $2 \int dx\, \phi\left(\frac{x}{n}\right)$ expressing the probability that the error of an observation is comprehended within the limits $\pm n$, this quantity must be equal to unity; the function (u) will become

$$\frac{2}{n\pi} \sqrt{\frac{k}{k'' s}} \iint dl\, dt\, c^{-t^2} \cos\left(\frac{lt}{n} \sqrt{\frac{k}{k'' s}}\right);$$

the integral relative to t needing to be taken from t null to $t = \pi n \sqrt{\frac{k'' s}{k}}$, or to $t = \infty$, n being supposed infinite; now we have, by §25 of the first Book,

$$\int dt \cos\left(\frac{lt}{n} \sqrt{\frac{k}{k'' s}}\right) c^{-t^2} = \frac{\sqrt{\pi}}{2} c^{-\frac{l^2}{4n^2} \frac{k}{k'' s}};$$

by making therefore

$$\frac{l}{n} = 2t' \sqrt{\frac{k'' s}{k}};$$

the function (u) becomes [308]

$$\frac{2}{\sqrt{\pi}} \int dt'\, c^{-t'^2}.$$

Thus by naming, as above, $2a$ the interval comprehended between the limits of the errors of each observation, the probability that the sum of the errors of the s observations, will be comprehended within the limits $\pm ar\sqrt{s}$, is

$$\sqrt{\frac{k}{k'' s}} \int dr\, c^{-\frac{kr^2}{4k''}},$$

if $\phi\left(\frac{x}{n}\right)$ is constant, then $\frac{k}{k''} = 6$, and this probability becomes

$$2\sqrt{\frac{3}{2\pi}} \int dr c^{-\frac{3}{2} r^2},$$

that which is conformed to that which we have found above.

If $\phi\left(\frac{x}{n}\right)$ or $\phi(x')$ is a rational and entire function of x', we will have, by the method of §15, the probability that the sum of the errors will be comprehended within the limits $\pm ar\sqrt{s}$, expressed by a series of powers s, $2s$, etc. of quantities of the form $s - \mu \pm r\sqrt{s}$, in which μ increases in arithmetic progression, these quantities being continued until they become negatives. By comparing this series to the preceding expression of the same probability, we will obtain in a quite close manner, the value of the series; and we will arrive thus with respect to this kind of series, to some theorems analogous to those that we have given in §42 of the first Book, on the finite differences of the powers of a variable.

If the law of facility of the errors is expressed by a negative exponential which is able to be extended to infinity, and generally if the errors are able to be extended to infinity; then a becomes infinite, and the application of the preceding method can offer some difficulties. In all these cases, we will make

$$\frac{x}{h} = x', \qquad \frac{1}{h} = dx',$$

h being any finite quantity whatsoever, and by following exactly the preceding analysis, we will find for the probability that the sum of the errors of the s observations is comprehended within the limits $\pm hr\sqrt{s}$, [309]

$$\sqrt{\frac{k}{k''s}} \int dr\, c^{-\frac{kr^2}{4k''}},$$

an expression in which we must observe that $\phi\left(\frac{x}{h}\right)$ or $\phi(x')$ expresses the probability of the error $\pm x$, and that we have

$$k = 2\int dx'\phi(x'), \quad k'' = \int x'^2 dx'\phi(x'),$$

the integrals being taken from $x' = 0$ to $x' = \infty$.

§19. Let us determine presently the probability that the sum of the errors of a very great number of observations will be comprehended within some given limits, setting aside the sign of these errors, that is, by taking them all positively. For this, let us consider the series

$$\phi\left(\frac{n}{n}\right) c^{-n\varpi\sqrt{-1}} + \phi\left(\frac{n-1}{n}\right) c^{-(n-1)\varpi\sqrt{-1}} \cdots + \phi\left(\frac{0}{n}\right) \cdots$$
$$\cdots + \phi\left(\frac{n-1}{n}\right) c^{(n-1)\varpi\sqrt{-1}} + \phi\left(\frac{n}{n}\right) c^{n\varpi\sqrt{-1}},$$

$\phi\left(\frac{x}{n}\right)$ being the ordinate on the curve of probability of errors, corresponding to the error $\pm x$, and x being in the same way as n, considered as formed of an infinite number of units. If we raise this series to the power s, after having changed the sign of the negative exponentials; the coefficient of any one exponential, such as $c^{(l+\mu s)\varpi\sqrt{-1}}$, will be the probability that the sum of the errors taken setting aside the sign, is $l + \mu s$; this probability is therefore

$$\frac{1}{2\pi} \int d\varpi\, c^{-(l+\mu s)\varpi\sqrt{-1}} \left\{ \begin{array}{r} \phi\left(\frac{0}{n}\right) + 2\phi\left(\frac{1}{n}\right) c^{\varpi\sqrt{-1}} + 2\phi\left(\frac{2}{n}\right) c^{2\varpi\sqrt{-1}} \\ \cdots + 2\phi\left(\frac{n}{n}\right) c^{n\varpi\sqrt{-1}} \end{array} \right\}^s,$$

the integral relative to ϖ being taken from $\varpi = -\pi$ to $\varpi = \pi$; because in this interval, the integral $\int d\varpi\, c^{-r\varpi\sqrt{-1}}$, or [310]

$$\int d\varpi(\cos r\varpi - \sqrt{-1}\sin r\varpi)$$

disappears, whatever be r, provided that it is not null.

We have, by developing with respect to the powers of ϖ,

$$\log\left\{c^{-\mu s\varpi\sqrt{-1}}\left[\phi\left(\frac{0}{n}\right)+2\phi\left(\frac{1}{n}\right)\mathrm{c}^{\varpi\sqrt{-1}}\cdots+2\phi\left(\frac{n}{n}\right)\mathrm{c}^{n\varpi\sqrt{-1}}\right]^s\right\}$$

$$= s\log\left\{\begin{array}{l} \quad\phi\left(\frac{0}{n}\right)+2\phi\left(\frac{1}{n}\right)+2\phi\left(\frac{2}{n}\right)\cdots+2\phi\left(\frac{n}{n}\right) \\ +2\varpi\sqrt{-1}\left[\phi\left(\frac{1}{n}\right)+2\phi\left(\frac{2}{n}\right)\cdots+n\phi\left(\frac{n}{n}\right)\right] \\ \qquad -\varpi^2\left[\phi\left(\frac{1}{n}\right)+2^2\phi\left(\frac{2}{n}\right)\cdots+n^2\phi\left(\frac{n}{n}\right)\right] \\ -\text{etc.}\end{array}\right\}-\mu s\varpi\sqrt{-1} \tag{1}$$

By making therefore

$$\frac{x}{n}=x', \qquad \frac{1}{n}=dx',$$

we have

$$2\int dx'\phi(x')=k, \qquad \int x'dx'\phi(x')=k', \qquad \int x'^2dx'\phi(x')=k'',$$
$$\int x'^3dx'\phi(x')=k''', \qquad \int x'^4dx'\phi(x')=k^{\text{iv}}, \qquad \text{etc.,}$$

the integrals being taken from x' null to $x'=1$; the second member of equation (1) becomes

$$s\log nk+s\log\left(1+\frac{2k'}{k}n\varpi\sqrt{-1}-\frac{k''}{k}n^2\varpi^2-\text{etc.}\right)-\mu s\varpi\sqrt{-1}.$$

the error of each observation needing to fall necessarily within the limits $\pm n$, we have $nk=1$; the preceding quantity becomes thus

$$s\left(\frac{2k'}{k}-\frac{\mu}{n}\right)n\varpi\sqrt{-1}-\frac{(kk''-2k'^2)sn^2\varpi^2}{k^2}-\text{etc.};$$

by making therefore

$$\frac{\mu}{n}=\frac{2k'}{k},$$

and neglecting the powers of ϖ superior to the square, this quantity is reduced to its second term, and the preceding probability becomes [311]

$$\frac{1}{2\pi}\int d\varpi c^{-lw\sqrt{-1}-\frac{(kk''-2k'^2)}{k^2}s.n^2.\varpi^2}.$$

Let

$$\beta=\frac{k}{\sqrt{kk''-2k'^2}}, \qquad \varpi=\frac{\beta t}{n\sqrt{s}}, \qquad \frac{l}{n}=r\sqrt{s},$$

the preceding integral becomes

$$\frac{1}{2\pi}\frac{c^{-\frac{\beta^2r^2}{4}}}{n\sqrt{s}}\int\beta dtc^{-\left(t+\frac{l\beta\sqrt{-1}}{2n\sqrt{s}}\right)^2}.$$

This integral must be taken from $t = -\infty$ to $t = \infty$; and then the preceding quantity becomes

$$\frac{\beta}{2\sqrt{\pi}n\sqrt{s}} c^{-\frac{\beta^2 r^2}{4}}.$$

By multiplying it by dl or by $ndr\sqrt{s}$, the integral

$$\frac{1}{2\sqrt{\pi}} \int \beta dr c^{-\frac{\beta^2 r^2}{4}}$$

will be the half-probability that the value of l, and, consequently, the sum of the errors of the observations is comprehended within the limits $\frac{2k'}{k} as \pm ar\sqrt{s}$, $\pm a$ being the limits of the errors of each observation, limits that we designate by $\pm n$, when we imagine them partitioned into an infinite number of parts.

We see thus that the sum of the errors, the most probable, setting aside the sign, is that which corresponds to $r = 0$. This sum is $\frac{2k'}{k} as$. In the case where $\phi(x)$ is constant, $\frac{2k'}{k} = \frac{1}{2}$, the sum of the errors, the most probable, is therefore then the half of the greatest sum possible, a sum which is equal to sa. But if $\phi(x)$ is not constant and diminishes in measure as the error x increases, then $\frac{2k'}{k}$ is less than $\frac{1}{2}$, and the sum of the errors, setting aside the sign, is below the half of the greatest sum [312]
possible.

We can, by the same analysis, determine the probability that the sum of the squares of the errors, will be $l + \mu s$; it is easy to see that this probability has for expression the integral

$$\frac{1}{2\pi} \int d\varpi \, c^{-(l+\mu s)\varpi\sqrt{-1}} \left\{ \begin{matrix} \phi\left(\frac{0}{n}\right) + 2\phi\left(\frac{1}{n}\right) c^{\varpi\sqrt{-1}} + 2\phi\left(\frac{2}{n}\right) c^{2^2\varpi\sqrt{-1}} \\ \cdots + 2\phi\left(\frac{n}{n}\right) c^{n^2\varpi\sqrt{-1}} \end{matrix} \right\}^s,$$

taken from $\varpi = -\pi$ to $\varpi = \pi$. By following exactly the preceding analysis, we will have

$$\mu = \frac{2n^2 k''}{k},$$

and by making

$$\beta' = \frac{k}{\sqrt{kk^{\text{iv}} - 2k''^2}},$$

the probability that the sum of the squares of the errors of the s observations will be comprehended within the limits $\frac{2k''}{k} a^2 s \pm a^2 r\sqrt{s}$, will be

$$\frac{1}{\sqrt{\pi}} \int \beta' dr \, c^{-\frac{\beta'^2 r^2}{4}}.$$

The most probable sum is that which corresponds to r null; it is therefore $\frac{2k''}{k} a^2 s$. If s is a very great number, the result of the observations will deviate very little from this value, and consequently it will make known very nearly the factor $\frac{a^2 k''}{k}$.

§20. When we wish to correct an element already known quite nearly, by the collection of a great number of observations, we form equations of condition in the following manner. Let z be the correction of the element, and β the observation; the analytic expression of the latter will be a function of the element. By substituting, instead of the element, its approximate value, plus the correction z; by reducing into series with respect to z, and neglecting the square of z, this function will take the form $h + pz$; by equating it to the observed quantity β, we will have [313]

$$\beta = h + pz;$$

z would be therefore determined, if the observation was rigorous; but as it is susceptible of error, by naming ϵ this error, we have exactly, to the quantities near of order z^2,

$$\beta + \epsilon = h + pz;$$

and by making $\beta - h = \alpha$, we have

$$\epsilon = pz - \alpha.$$

Each observation furnishes a similar equation, that we can represent for the $(i+1)^{\text{st}}$ observation, by this one

$$\epsilon^{(i)} = p^{(i)}z - \alpha^{(i)}.$$

By uniting all these equations, we have

$$S\epsilon^{(i)} = zSp^{(i)} - S\alpha^{(i)}, \tag{1}$$

the sign S being related to all the values of i, from $i = 0$ to $i = s - 1$, s being the total number of observations. By supposing null the sum of the errors, this equation gives

$$z = \frac{S\alpha^{(i)}}{Sp^{(i)}};$$

it is that which we name ordinarily *mean result of the observations*.

We have seen, in §18, that the probability that the sum of the errors of s observations will be comprehended within the limits $\pm ar\sqrt{s}$, is

$$\sqrt{\frac{k}{k''\pi}} \int dr\, c^{-\frac{kr^2}{4k''}}.$$

Let us name $\pm u$ the error of the result z; by substituting into equation (1), $\pm ar\sqrt{s}$ instead of S$\epsilon^{(i)}$, and $\frac{S\alpha^{(i)}}{Sp^{(i)}} \pm u$ instead of z, it gives

$$r = \frac{uSp^{(i)}}{a\sqrt{s}};$$

the probability that the error of the result z, will be comprehended within the limits $\pm u$ is therefore, [314]

$$\sqrt{\frac{k}{k''s\pi}} Sp^{(i)} \int \frac{du}{a}\, c^{-\frac{ku^2(Sp^{(i)})^2}{4k''a^2s}}.$$

Instead of supposing null the sum of the errors, we are able to suppose null any linear function of these errors, that we will represent thus,

$$m\epsilon + m^{(1)}\epsilon^{(1)} + m^{(2)}\epsilon^{(2)} \cdots + m^{(s-1)}\epsilon^{(s-1)}, \tag{m}$$

m, $m^{(1)}$, $m^{(2)}$, etc. being positive or negative whole numbers. By substituting into this function (m), instead of ϵ, $\epsilon^{(1)}$, $\epsilon^{(2)}$, etc., their values given by the equations of condition, it becomes

$$zSm^{(i)}p^{(i)} - Sm^{(i)}\alpha^{(i)};$$

by equating therefore to zero, the function (m), we have

$$z = \frac{Sm^{(i)}\alpha^{(i)}}{Sm^{(i)}p^{(i)}}.$$

Let u be the error of this result, so that we have

$$z = \frac{Sm^{(i)}\alpha^{(i)}}{Sm^{(i)}p^{(i)}} + u;$$

the function (m) becomes

$$uSm^{(i)}p^{(i)}.$$

Let us determine the probability of the error u, when the observations are in great number.

For this, let us consider the product

$$\int \phi\left(\frac{x}{a}\right) c^{mx\varpi\sqrt{-1}} \times \int \phi\left(\frac{x}{a}\right) c^{m^{(1)}x\varpi\sqrt{-1}} \cdots \times \int \phi\left(\frac{x}{a}\right) c^{m^{(s-1)}xn\varpi\sqrt{-1}},$$

the $\int$ sign extending to all the values of x, from the extreme negative value of x, to its positive extreme value. $\phi\left(\frac{x}{a}\right)$ is, as in the preceding sections, the probability of an error x, in each observation; x being supposed, in the same way as a, formed from an infinity of parts taken for unity. It [315]
is clear that the coefficient of any exponential $c^{l\varpi\sqrt{-1}}$, in the development of this product, will be the probability that the sum of the errors of the observations, multiplied respectively by m, $m^{(1)}$, etc., that is, the function (m), will be equal to l; by multiplying therefore the preceding product by $c^{-l\varpi\sqrt{-1}}$, the term independent of $c^{\varpi\sqrt{-1}}$ and of its powers, in this new product, will express this probability. If we suppose, as we will do here, the probability of the positive errors, the same as that of the negative errors; we will be able, in the sum $\int \phi\left(\frac{x}{a}\right) c^{mx\varpi\sqrt{-1}}$, to reunite the terms multiplied, one by $c^{mx\varpi\sqrt{-1}}$, and the other by $c^{-mx\varpi\sqrt{-1}}$; then this sum takes the form $2\int \phi\left(\frac{x}{a}\right)\cos mx\varpi$. It is likewise of it of all the similar sums. Thence it follows that the probability that the function (m) will be equal to l, is equal to

$$\frac{1}{2\pi}\int d\varpi \left\{ \begin{array}{l} \mathrm{c}^{-l\varpi\sqrt{-1}} \times 2\int \phi\left(\frac{x}{a}\right)\cos mx\varpi \\ \quad \times 2\int \phi\left(\frac{x}{a}\right)\cos m^{(i)}x\varpi \cdots \times 2\int \phi\left(\frac{x}{a}\right)\cos m^{(s-1)}x\varpi \end{array} \right\}; \qquad (i)$$

the integral being taken from $\varpi = -\pi$ to $\varpi = \pi$. We have by reducing the cosines into series,

$$\int \phi\left(\frac{x}{a}\right)\cos\ mx\varpi = \int \phi\left(\frac{x}{a}\right) - \frac{1}{2}m^2a^2\varpi^2 \int \frac{x^2}{a^2}\phi\left(\frac{x}{a}\right) + \text{etc.}$$

If we make $\frac{x}{a} = x'$ and if we observe that the variation of x being unity, we have $dx' = \frac{1}{a}$; we will have

$$\int \phi\left(\frac{x}{a}\right) = a\int dx'\phi(x').$$

Let us name, as in the preceding sections, k the integral $2\int dx'\phi(x')$, taken from x' null to its extreme positive value; let us name similarly k'' the integral $\int x'^2 dx'\phi(x')$, taken within the same limits, and so forth; we will have

$$2\int\phi\left(\frac{x}{a}\right)\cos mx\varpi = ak\left(1-\frac{k''}{k}m^2a^2\varpi^2+\frac{k^{\mathrm{iv}}}{12k}m^4a^4\varpi^4-\text{etc.}\right).$$

The logarithm of the second member of this equation is [316]

$$-\frac{k''}{k}m^2a^2\varpi^2+\frac{kk^{\mathrm{iv}}-6k''^2}{12k^2}m^4a^4\varpi^4-\text{etc.}+\log ak;$$

ak or $2a\int dx'\phi(x')$ expresses the probability that the error of each observation, will be comprehended within its limits, that which is certain; we have therefore $ak=1$; that which reduces the preceding logarithm to

$$-\frac{k''}{k}m^2a^2\varpi^2+\frac{kk^{\mathrm{iv}}-6k''^2}{12k^2}m^4a^4\varpi^4-\text{etc.}$$

Thence it is easy to conclude that the product

$$2\int\phi\left(\frac{x}{a}\right)\cos mx\varpi\times 2\int\phi\left(\frac{x}{a}\right)\cos m^{(i)}x\varpi\cdots\times 2\int\phi\left(\frac{x}{a}\right)\cos m^{(s-1)}x\varpi,$$

is

$$\left(1+\frac{kk^{\mathrm{iv}}-6k''^2}{12k^2}a^4\varpi^4Sm^{(i)4}+\text{etc.}\right)c^{-\frac{k''}{k}a^2\varpi^2Sm^{(i)2}};$$

the preceding integral (i) is reduced therefore to

$$\frac{1}{2\pi}\int d\varpi\left\{1+\frac{kk^{\mathrm{iv}}-6k''^2}{12k^2}a^4\varpi^4Sm^{(i)4}+\text{etc.}\right\}\times c^{-lw\sqrt{-1}-\frac{k''}{k}a^2\varpi^2Sm^{(i)2}}.$$

By making $sa^2\varpi^2=t^2$, this integral becomes

$$\frac{1}{2a\pi\sqrt{s}}\int dt\left\{1+\frac{kk^{\mathrm{iv}}-6k''^2}{12k^2}\cdot\frac{Sm^{(i)4}}{s^2}t^4+\text{etc.}\right\}\times c^{-\frac{lw\sqrt{-1}}{a\sqrt{s}}-\frac{k''}{k}\cdot\frac{Sm^{(i)2}}{s}t^2};$$

$Sm^{(i)2}, Sm^{(i)4}$, etc. are evidently quantities of order s; thus $\frac{Sm^{(i)4}}{s^2}$ is of order $\frac{1}{s}$; by neglecting therefore the terms of this last order, vis-à-vis of unity, the last integral is reduced to

$$\frac{1}{2a\pi\sqrt{s}}\int dtc^{-\frac{lt\sqrt{-1}}{a\sqrt{s}}-\frac{k''}{k}\cdot\frac{Sm^{(i)2}}{s}t^2}.$$

The integral relative to ϖ needing to be taken from $\varpi=-\pi$ to $\varpi=\pi$, the integral relative to t must be taken from $t=-a\pi\sqrt{s}$ to $t=a\pi\sqrt{s}$; and in these cases, the exponential under the [317]

$\int$ sign is insensible at these two limits, either because s is a great number, or because a is here supposed divided into an infinity of parts taken for unity; we are able therefore to take the integral from $t = -\infty$ to $t = \infty$. Let us make

$$t' = \sqrt{\frac{k'' S m^{(i)2}}{ks}} \left\{ t + \frac{l\sqrt{-1}k\sqrt{s}}{2ak'' S m^{(i)2}} \right\},$$

the preceding integral function becomes

$$\frac{c^{-\frac{kl^2}{4k''a^2 S m^{(i)2}}}}{2a\pi\sqrt{\frac{k''}{k} S m^{(i)2}}} \int dt' c^{-t'^2}.$$

The integral relative to t' must be taken, as the integral relative to t, from $t' = -\infty$ to $t' = \infty$; that which reduces the preceding quantity to this one,

$$\frac{c^{-\frac{kl^2}{4k''a^2 S m^{(i)2}}}}{2a\sqrt{\pi}\sqrt{\frac{k''}{k} S m^{(i)2}}}.$$

If we make $l = ar\sqrt{s}$, and if we observe that the variation of l being unity, we have $adr = 1$, we will have

$$\frac{\sqrt{s}}{2\sqrt{\frac{k''\pi}{k} S m^{(i)2}}} \int dr\, c^{-\frac{kr^2 s}{4k'' S m^{(i)2}}},$$

for the probability that the function (m) will be comprehended within the limits zero and $ar\sqrt{s}$, the integral being taken from r null.

We have need here to know the probability of the error u, of the element determined by making null the function (m). This function being supposed equal to l or to $ar\sqrt{s}$; we will have, by that which precedes,

$$uSm^{(i)}p^{(i)} = ar\sqrt{s};$$

by substituting this value into the preceding integral function, it becomes [318]

$$\frac{Sm^{(i)}p^{(i)}}{2a\sqrt{\frac{k''\pi}{k} S m^{(i)2}}} \int du\, c^{-\frac{ku^2 (Sm^{(i)}p^{(i)})^2}{4k''a^2 S m^{(i)2}}};$$

this is the expression of the probability that the value of u will be comprehended within the limits zero and u: it is also the expression of the probability that u will be comprehended within the limits zero and $-u$. If we make

$$u = 2at\sqrt{\frac{k''}{k}} \cdot \frac{\sqrt{Sm^{(i)2}}}{Sm^{(i)}p^{(i)}},$$

the preceding probability becomes

$$\frac{1}{\sqrt{\pi}} \int dt\, c^{-t^2}.$$

Now the probability remaining the same, t remains the same, and the interval of the two limits of u, are tightened so much more as $a\sqrt{\frac{k''}{k}} \cdot \frac{\sqrt{Sm^{(i)2}}}{Sm^{(i)}p^{(i)}}$ is smaller. This interval remaining the same, the value of t, and consequently the probability that the error of the element falls within this interval, is so much greater, as the same quantity $a\sqrt{\frac{k''}{k}}\cdot\frac{\sqrt{Sm^{(i)2}}}{Sm^{(i)}p^{(i)}}$ is smaller; it is necessary therefore to choose the system of factors $m^{(i)}$, which renders this quantity a *minimum*; and as a, k, k'' are the same in all these systems, it is necessary to choose the system which renders $\frac{\sqrt{Sm^{(i)2}}}{Sm^{(i)}p^{(i)}}$ a *minimum*.

We are able to arrive to the same result, in this manner. Let us resume the expression of the probability that u will be comprehended within the limits zero and u. The coefficient of du in the differential of this expression, is the ordinate of the curve of probabilities of the errors u of the element, errors represented by the abscissa u of this curve, that we can extend to infinity, on each side of the ordinate which corresponds to u null. This premised, each error, either positive, or negative, must be considered as a disadvantage or a real loss, in any game whatsoever; now, by the principles of the theory of probabilities, exposed at the beginning of this Book, we evaluate [319]
this disadvantage, by taking the sum of all the products of each disadvantage by its probability; the mean value of the positive error to fear, is therefore the sum of the products of each error by its probability; it is consequently equal to the integral

$$\frac{\int u\,du\,Sm^{(i)}p^{(i)}c^{-\frac{ku^2(Sm^{(i)}p^{(i)})^2}{4k''a^2Sm^{(i)2}}}}{2a\sqrt{\frac{k''\pi}{k}}Sm^{(i)2}},$$

taken from u null to u infinity; thus this error is

$$a\sqrt{\frac{k''}{k\pi}}\cdot\frac{\sqrt{Sm^{(i)2}}}{Sm^{(i)}p^{(i)}}.$$

This quantity taken with the $-$ sign, gives the mean error to fear less. It is clear that the system of factors $m^{(i)}$ that it is necessary to choose, must be such that these errors are some *minima*, and consequently such that $\frac{\sqrt{Sm^{(i)2}}}{Sm^{(i)}p^{(i)}}$ be a *minimum*.

If we differentiate this function with respect to $m^{(i)}$, we will have by equating its differential to zero, by the condition of the *minimum*,

$$\frac{m^{(i)}}{Sm^{(i)2}}=\frac{p^{(i)}}{Sm^{(i)}p^{(i)}}.$$

This equation holds whatever be i; and as the variation of i does not change the fraction $\frac{Sm^{(i)2}}{Sm^{(i)}p^{(i)}}$ at all; by naming μ this fraction, we will have

$$m=\mu p,\quad m^{(1)}=\mu p^{(1)},\quad \ldots\quad m^{(s-1)}=\mu p^{(s-1)};$$

and we can, whatever be p, $p^{(1)}$, etc., take μ such that the numbers m, $m^{(1)}$, etc. are whole numbers, as the preceding analysis supposes them. Then we have

$$z=\frac{Sp^{(i)}\alpha^{(i)}}{Sp^{(i)2}},$$

and the mean error to fear becomes [320]

$$\pm\frac{a\sqrt{\frac{k''}{k\pi}}}{Sp^{(i)2}}:$$

it is under all the hypotheses that we can make on the factors m, $m^{(1)}$, etc., the smallest possible mean error.

If we make the values of m, $m^{(1)}$, etc. equal to ± 1, the mean error to fear will be smaller when the sign $\pm$ will be determined in a manner that $m^{(i)}p^{(i)}$ is positive, that which returns to supposing $1 = m = m^{(1)} =$ etc., and to prepare the equations of condition, so that the coefficient of z in each of them, is positive; this is that which we do in the ordinary method. Then the mean result of the observations is

$$z = \frac{S\alpha^{(i)}}{Sp^{(i)}},$$

and the mean error to fear positive or negative,[1] is

$$\pm\frac{a\sqrt{\frac{k''s}{k\pi}}}{Sp^{(i)}};$$

but this error surpasses the preceding which, as we have seen, is the smallest possible. We are able to be convinced of it besides in this manner. It suffices to show that we have the inequality

$$\frac{\sqrt{s}}{Sp^{(i)}} > \frac{1}{\sqrt{Sp^{(i)2}}},$$

or

$$sSp^{(i)2} > (Sp^{(i)})^2.$$

In fact, $2pp^{(1)}$ is less than $p^2 + p^{(1)2}$, since $(p^{(1)} - p)^2$ is a positive quantity; we can therefore, in the second member of the preceding inequality, substitute, for $2pp^{(1)}$, $p^2 + p^{(1)2} - f$, f being a positive quantity. By making some similar substitutions for all the similar products, this second member will be equal to the first, less a positive quantity.

The result [321]

$$z = \frac{Sp^{(i)}\alpha^{(i)}}{Sp^{(i)2}},$$

to which corresponds the *minimum* of mean error to fear, is the one which the method of least squares of the errors of the observations gives; because, the sum of these squares being

$$(pz - \alpha)^2 + (p^{(1)}z - \alpha^{(1)})^2 \cdots + (p^{(s-1)}z - \alpha^{(s-1)})^2,$$

the condition of the *minimum* of this function, by making z vary, gives for this variable, the preceding expression; this method must therefore be employed in preference, whatever be the law of facility of the errors, a law on which the ratio $\frac{k''}{k}$ depends.

This ratio is $\frac{1}{6}$, if $\phi(x)$ is a constant; it is less than $\frac{1}{6}$, if $\phi(x)$ is variable, and such that it diminishes in measure as x increases, as it is natural to suppose. By adopting the mean law of errors, that we have given in §15, and according to which $\phi(x)$ is equal to $\frac{1}{2a}\log\frac{a}{x}$, we have $\frac{k''}{k} = \frac{1}{18}$. As for the limits $\pm a$, we are able to take for these limits, the deviations from the mean result, which would cause to reject an observation.

But we can, by the same observations, determine the factor $a\sqrt{\frac{k''}{k}}$ of the expression of the mean error. In fact, we have seen, in the preceding section, that the sum of the squares of the errors of

[1]*en plus ou en moins*: more or less, i.e. positive or negative.

the observations, is very nearly $2s\frac{a^2k''}{k}$, and that if they are in great number, it becomes extremely probable that the observed sum will not deviate from this value, by a sensible quantity; we are able therefore to equate them; now the observed sum is equal to $S\epsilon^{(i)2}$, or to $S(p^{(i)}z-\alpha^{(i)})^2$, by substituting for z its value $\frac{Sp^{(i)}\alpha^{(i)}}{Sp^{(i)2}}$; we find thus,

$$2s\frac{a^2k''}{k}=\frac{Sp^{(i)2}.S\alpha^{(i)2}-(Sp^{(i)}\alpha^{(i)})^2}{Sp^{(i)2}}.$$

The preceding expression of the mean error to fear respecting the result z, becomes then [322]

$$\pm\frac{\sqrt{Sp^{(i)2}.S\alpha^{(i)2}-(Sp^{(i)}\alpha^{(i)})^2}}{Sp^{(i)2}\sqrt{2s\pi}},$$

an expression in which there is nothing which is not given by the observations and by the coefficients of the equations of condition.

§21. Let us suppose now that we have two elements to correct by the collection of a great number of observations. By naming z and z' the respective corrections of these elements, we will form, as in the preceding section, some equations of condition, which will be comprehended under this general form

$$\epsilon^{(i)}=p^{(i)}z+q^{(i)}z'-\alpha^{(i)},$$

$\epsilon^{(i)}$ being, as in that section, the error of the $(i+1)^{\text{st}}$ observation. If we multiply respectively by m, $m^{(1)}$, …, $m^{(s-1)}$ these equations, and if we add together these products, we will have a first final equation

$$Sm^{(i)}\epsilon^{(i)}=z.Sm^{(i)}p^{(i)}+z'.Sm^{(i)}q^{(i)}-Sm^{(i)}\alpha^{(i)}.$$

By multiplying further the same equations respectively by n, $n^{(1)}$, …, $n^{(s-1)}$ and adding these products, we will have a second final equation

$$Sn^{(i)}\epsilon^{(i)}=z.Sn^{(i)}p^{(i)}+z'.Sn^{(i)}q^{(i)}-Sn^{(i)}\alpha^{(i)},$$

the sign S extending here, as in the preceding section, to all the values of i, from $i=0$ to $i=s-1$.

If we suppose null the two functions $Sm^{(i)}\epsilon^{(i)}$, $Sn^{(i)}\epsilon^{(i)}$, functions which we will designate respectively by (m) and (n); the two preceding final equations will give the corrections z and z' of the two elements. But these corrections are susceptible of errors relative to that of which the supposition that we have just made, is itself susceptible. Let us imagine therefore that the functions (m) and (n), instead of being nulls, are respectively l and l', and let us name u and u' the errors corresponding to the corrections z and z', determined by that which precedes; the two final equations [323]
will become

$$l=u.Sm^{(i)}p^{(i)}+u'.Sm^{(i)}q^{(i)},$$
$$l'=u.Sn^{(i)}p^{(i)}+u'.Sn^{(i)}q^{(i)}.$$

It is necessary now to determine the factors m, $m^{(1)}$, etc.; n, $n^{(1)}$, etc., in a manner that the mean error to fear respecting each element, is a *minimum*. For this, let us consider the product

$$\int\phi\left(\frac{x}{a}\right)c^{-(m\varpi+n\varpi')x\sqrt{-1}}\times\int\phi\left(\frac{x}{a}\right)c^{-(m^{(1)}\varpi+n^{(1)}\varpi')x\sqrt{-1}}\cdots$$
$$\cdots\times\int\phi\left(\frac{x}{a}\right)c^{-(m^{(s-1)}\varpi+n^{(s-1)}\varpi')x\sqrt{-1}},$$

the sign $\int$ referring to all the values of x, from $x=-a$ to $x=a$; $\phi\left(\frac{x}{a}\right)$ being, as in the preceding section, the probability of the error x, in the same way as of the error $-x$. The preceding function becomes, by reuniting the two exponentials relative to x and to $-x$,

$$2\int\phi\left(\frac{x}{a}\right)\cos(mx\varpi+nx\varpi')\times 2\int\phi\left(\frac{x}{a}\right)\cos(m^{(1)}x\varpi+n^{(1)}x\varpi')$$
$$\cdots\times 2\int\phi\left(\frac{x}{a}\right)\cos(m^{(s-1)}x\varpi+n^{(s-1)}x\varpi'),$$

the sign $\int$ extending here to all the values of x, from $x=0$ to $x=a$; x being supposed, in the same way as a, divided into an infinity of parts taken for unity. Presently, it is clear that the term independent of the exponentials, in the product of the preceding function, by $c^{-l\varpi\sqrt{-1}-l'\varpi'\sqrt{-1}}$, is the probability that the sum of the errors of each observation, multiplied respectively by m, $m^{(1)}$, etc., or the function (m), will be equal to l, at the same time as the function (n), sum of the errors of each observation, multiplied respectively by n, $n^{(1)}$, etc., will be equal to l'; this probability is therefore

$$\frac{1}{4\pi^2}\iint d\varpi d\varpi' c^{-l\varpi\sqrt{-1}-l'\varpi'\sqrt{-1}}\left\{\begin{matrix} 2\int\phi\left(\frac{x}{a}\right)\cos(m\varpi+n\varpi')x\cdots\cdots \\ \ldots\times 2\int\phi\left(\frac{x}{a}\right)\cos(m^{(s-1)}\varpi+n^{(s-1)}\varpi')x\end{matrix}\right\}$$

the integrals being taken from ϖ and ϖ' equal to $-\pi$, to ϖ and ϖ' equal to π. This premised; [324]

By following exactly the analysis of the preceding section, we find that the preceding function is reduced to very nearly

$$\frac{1}{4\pi^2}\iint d\varpi d\varpi' c^{-l\varpi\sqrt{-1}-l'\varpi'\sqrt{-1}-\frac{k''}{k}a^2[\varpi^2 Sm^{(i)2}+2\varpi\varpi'.Sm^{(i)}n^{(i)}+\varpi'^2.Sn^{(i)2}]},$$

k and k'' having here the same signification as in the section cited. We see further, by the same section, that the integrals are able to be extended from $a\varpi=-\infty$, $a\varpi'=-\infty$, to $a\varpi=\infty$ and $a\varpi'=\infty$. If we make

$$t=a\varpi+\frac{a\varpi'.Sm^{(i)}n^{(i)}}{Sm^{(i)2}}+\frac{kl\sqrt{-1}}{2k''a.Sm^{(i)2}}$$
$$t'=a\varpi'-\frac{k}{2k''a}\cdot\frac{(lSm^{(i)}n^{(i)}-l'Sm^{(i)2})\sqrt{-1}}{Sm^{(i)2}.Sn^{(i)2}-(Sm^{(i)}n^{(i)})^2};$$

if we make next

$$E=Sm^{(i)2}.Sn^{(i)2}-(Sm^{(i)}n^{(i)})^2;$$

the preceding double integral becomes

$$c^{-\frac{k}{4k''a^2E}[l^2Sn^{(i)2}-2ll'Sm^{(i)}n^{(i)}+l'^2Sm^{(i)2}]}$$
$$\times\iint\frac{dt\,dt'}{4\pi^2a^2}c^{-\frac{k''t^2}{k}Sm^{(i)2}-\frac{k''t'^2E}{kSm^{(i)2}}}.$$

By taking the integrals within the positive and negative infinite limits, as those relative to $a\varpi$ and $a\varpi'$, we will have

$$\frac{1}{\frac{4k''\pi}{k}a^2\sqrt{E}}c^{-\frac{k}{4k''a^2}\cdot\frac{l^2Sn^{(i)2}-2ll'Sm^{(i)}n^{(i)}+l'^2Sm^{(i)2}}{E}}. \qquad (o)$$

It is necessary now, in order to have the probability that the values of l and of l' will be comprehended within the given limits, to multiply this quantity by $dl\,dl'$, and to integrate next within these limits. By naming X this quantity, the probability of which there is concern will be therefore $\iint X\,dl\,dl'$. But in order to have the probability that the errors u and u' of the corrections of the elements will be comprehended within the given limits, it is necessary to substitute into this integral, instead of l and l', their values in u and u'. Now if we differentiate the expressions of l and of l', by supposing l' constant, we have [325]

$$dl = duSm^{(i)}p^{(i)} + du'Sm^{(i)}q^{(i)},$$
$$0 = duSn^{(i)}p^{(i)} + du'Sn^{(i)}q^{(i)};$$

that which gives

$$dl = \frac{du[Sm^{(i)}p^{(i)}.Sn^{(i)}q^{(i)} - Sn^{(i)}p^{(i)}.Sm^{(i)}q^{(i)}]}{Sn^{(i)}q^{(i)}}.$$

If we differentiate next the expression of l', by supposing u constant, we have

$$dl' = du'Sn^{(i)}q^{(i)};$$

we will have therefore

$$dl\,dl' = [Sm^{(i)}p^{(i)}.Sn^{(i)}q^{(i)} - Sn^{(i)}p^{(i)}.Sm^{(i)}q^{(i)}]du\,du'.$$

By making next

$$\begin{aligned}
F &= Sn^{(i)2}(Sm^{(i)}p^{(i)})^2 - 2Sm^{(i)}n^{(i)}.Sm^{(i)}p^{(i)}.Sn^{(i)}p^{(i)} + Sm^{(i)2}.(Sn^{(i)}p^{(i)})^2,\\
G &= Sn^{(i)2}.Sm^{(i)}p^{(i)}.Sm^{(i)}q^{(i)} + Sm^{(i)2}.Sn^{(i)}p^{(i)}.Sn^{(i)}q^{(i)}\\
&\quad - Sm^{(i)}n^{(i)}.[Sn^{(i)}p^{(i)}.Sm^{(i)}q^{(i)} + Sm^{(i)}p^{(i)}.Sn^{(i)}q^{(i)}],\\
H &= Sn^{(i)2}(Sm^{(i)}q^{(i)})^2 - 2Sm^{(i)}n^{(i)}.Sm^{(i)}q^{(i)}.Sn^{(i)}q^{(i)} + Sm^{(i)2}.(Sn^{(i)}q^{(i)})^2,\\
I &= Sm^{(i)}p^{(i)}.Sn^{(i)}q^{(i)} - Sn^{(i)}p^{(i)}.Sm^{(i)}q^{(i)},
\end{aligned}$$

the function (o) becomes

$$\iint \frac{k}{4k''\pi} \cdot \frac{1}{\sqrt{E}} \cdot \frac{du\,du'}{a^2} c^{-\frac{k(Fu^2+2Guu'+Hu'^2)}{4k''a^2E}}.$$

Let us integrate first this function from $u' = -\infty$ to $u' = \infty$. If we make

$$t = \frac{\sqrt{\frac{kH}{4k''}}\left(u' + \frac{Gu}{H}\right)}{a\sqrt{E}},$$

and if we take the integral from $t = -\infty$ to $t = \infty$, we will have by considering in it only the variation of u', [326]

$$\int \sqrt{\frac{k}{4k''\pi}} \cdot \frac{du}{a} \cdot \frac{1}{\sqrt{H}} \cdot c^{-\frac{ku^2}{4k''a^2}\frac{FH-G^2}{EH}}.$$

Now we have

$$\frac{FH-G^2}{E} = I^2;$$

the preceding integral becomes therefore

$$\int \frac{I}{\sqrt{H}}\frac{du}{a}\sqrt{\frac{k}{4k''\pi}}c^{-\frac{k}{4k''}\frac{I^2u^2}{a^2H}}.$$

We will have, by the preceding section, the mean error to fear positive or negative, respecting the correction of the first element, by multiplying the quantity under the sign $\int$ by $\pm u$, and taking the integral from $u = 0$ to $u = \infty$, that which gives, for this error,

$$\pm\frac{a\sqrt{H}}{I\sqrt{\frac{k\pi}{k''}}},$$

the $+$ sign indicating the mean positive error to fear, and the $-$ sign the mean negative error to fear.

Let us determine presently the factors $m^{(i)}$ and $n^{(i)}$, in a manner that this error is a *minimum*. By making $m^{(i)}$ alone vary, we have

$$\begin{aligned} d\log\frac{\sqrt{H}}{I} &= dm^{(i)}\frac{[-p^{(i)}Sn^{(i)}q^{(i)} + q^{(i)}Sn^{(i)}p^{(i)}]}{I} \\ &+ dm^{(i)}\frac{\left\{\begin{aligned} &q^{(i)}Sn^{(i)2}.Sm^{(i)}q^{(i)} - n^{(i)}.Sm^{(i)}q^{(i)}.Sn^{(i)}q^{(i)} \\ &\quad - q^{(i)}.Sm^{(i)}n^{(i)}.Sn^{(i)}q^{(i)} + m^{(i)}(Sn^{(i)}q^{(i)})^2\end{aligned}\right\}}{H}\end{aligned}$$

It is easy to see that this differential disappears, if we suppose in the coefficients of $dm^{(i)}$,

$$m^{(i)} = \mu p^{(i)}, \quad n^{(i)} = \mu q^{(i)},$$

μ being an arbitrary coefficient independent of i, and by means of which we can render $m^{(i)}$ and $n^{(i)}$ whole numbers; the preceding supposition renders therefore null the differential of $\frac{\sqrt{H}}{I}$, taken with [327]
respect to $m^{(i)}$. We will see in the same manner, that this supposition renders null the differential of the same quantity, taken with respect to $n^{(i)}$. Thus this supposition renders a *minimum* the mean error to fear respecting the correction of the first element; and we will see in the same manner, that it renders further a *minimum*, the mean error to fear respecting the correction of the second element, an error that we obtain by changing in the expression of the preceding, H into F. Under this supposition, the corrections of the two elements are

$$\begin{aligned} z &= \frac{Sq^{(i)2}.Sp^{(i)}\alpha^{(i)} - Sp^{(i)}q^{(i)}.Sq^{(i)}\alpha^{(i)}}{Sp^{(i)2}.Sq^{(i)2} - (Sp^{(i)}q^{(i)})^2}, \\ z' &= \frac{Sp^{(i)2}.Sq^{(i)}\alpha^{(i)} - Sp^{(i)}q^{(i)}.Sp^{(i)}\alpha^{(i)}}{Sp^{(i)2}.Sq^{(i)2} - (Sp^{(i)}q^{(i)})^2}.\end{aligned}$$

It is easy to see that these corrections are those that the method of least squares of the errors of the observations gives, or of the *minimum* of the function

$$S(p^{(i)}z + q^{(i)}z' - \alpha^{(i)})^2;$$

whence it follows that this method holds generally, whatever be the number of elements to determine; because it is clear that the previous analysis can be extended to any number of elements.

By substituting for $a\sqrt{\frac{k''}{k\pi}}$, the quantity $\sqrt{\frac{S\epsilon^{(i)2}}{2s\pi}}$, to which we can, by §20, suppose it equal, ϵ, $\epsilon^{(1)}$, etc. being that which remains in the equations of condition, after having substituted there the corrections given by the method of least squares of the errors; the mean error to fear respecting the first element, is

$$\pm\frac{\sqrt{\frac{S\epsilon^{(i)2}}{2s\pi}}\sqrt{Sq^{(i)2}}}{\sqrt{Sp^{(i)2}.Sq^{(i)2}-(Sp^{(i)}q^{(i)})^2}}.$$

The mean positive or negative error to fear respecting the second element, is

$$\pm\frac{\sqrt{\frac{S\epsilon^{(i)2}}{2s\pi}}\sqrt{Sp^{(i)2}}}{\sqrt{Sp^{(i)2}.Sq^{(i)2}-(Sp^{(i)}q^{(i)})^2}}.$$

whence we see that the first element is more or less well determined as the second, according as [328]
$Sq^{(i)2}$ is smaller or greater than $Sp^{(i)2}$.

If the r first equations of condition do not contain q at all, and if the $s-r$ last do not contain p at all; then $Sp^{(i)}q^{(i)}=0$, and the preceding formulas coincide with that of the preceding section.

We are able to obtain thus the mean error to fear respecting each element determined by the method of least squares of the errors, whatever be the number of elements, provided that we consider a great number of observations. Let z, z', z'', z''', etc., be the corrections of each element, and let us represent generally the equations of condition, by the following:

$$\epsilon^{(i)}=p^{(i)}z+q^{(i)}z'+r^{(i)}z''+t^{(i)}z'''+\text{etc.}-\alpha^{(i)}.$$

In the case of a single element, the mean error to fear is, as we have seen,

$$\pm\sqrt{\frac{S\epsilon^{(i)2}}{2s\pi}}\cdot\frac{1}{\sqrt{Sp^{(i)2}}}. \qquad (a)$$

When there are two elements, we will have the mean error to fear respecting the first element, by changing, in the function (a), $Sp^{(i)2}$ into $Sp^{(i)2}-\frac{(Sp^{(i)}q^{(i)})^2}{Sq^{(i)2}}$, that which gives for this error,

$$\pm\frac{\sqrt{\frac{S\epsilon^{(i)2}}{2s\pi}}\sqrt{Sq^{(i)2}}}{\sqrt{Sp^{(i)2}.Sq^{(i)2}-(Sp^{(i)}q^{(i)})^2}}. \qquad (a')$$

When there are three elements, we will have the error to fear respecting the first element, by changing in this expression (a'), $Sp^{(i)2}$ into $Sp^{(i)2}-\frac{(Sp^{(i)}r^{(i)})^2}{Sr^{(i)2}}$, $Sp^{(i)}q^{(i)}$ into $Sp^{(i)}q^{(i)}-\frac{Sp^{(i)}r^{(i)}.Sq^{(i)}r^{(i)}}{Sr^{(i)2}}$, and $Sq^{(i)2}$ into $Sq^{(i)2}-\frac{(Sq^{(i)}r^{(i)})^2}{Sr^{(i)2}}$; this which gives for this error,

$$\pm\frac{\sqrt{\frac{S\epsilon^{(i)2}}{2s\pi}}\sqrt{Sq^{(i)2}.Sr^{(i)2}-(Sq^{(i)}r^{(i)})^2}}{\sqrt{\begin{matrix}Sp^{(i)2}.Sq^{(i)2}.Sr^{(i)2}-Sp^{(i)2}(Sq^{(i)}r^{(i)})^2-Sq^{(i)2}(Sp^{(i)}r^{(i)})^2\\ -Sr^{(i)2}(Sp^{(i)}q^{(i)})^2+2Sp^{(i)}q^{(i)}.Sp^{(i)}r^{(i)}.Sq^{(i)}r^{(i)}\end{matrix}}}. \qquad (a'')$$

In the case of four elements, we will have the mean error to fear respecting the first element, [329]
by changing in this expression (a''), $Sp^{(i)2}$ into $Sp^{(i)2}-\frac{(Sp^{(i)}t^{(i)})^2}{St^{(i)2}}$, $Sp^{(i)}q^{(i)}$ into $Sp^{(i)}q^{(i)}-$

$\frac{Sp^{(i)}t^{(i)}.Sq^{(i)}t^{(i)}}{St^{(i)2}}$, etc. By continuing thus, we will have the mean error to fear respecting the first element, whatever be the number of elements. By changing in the expression of this error, that which is relative to the first element, into that which is relative to the second, and reciprocally; we will have the mean error to fear respecting the second element, and thus of the others.

Thence results a simple way to compare among them diverse astronomical tables, on the side of precision. These tables can always be supposed reduced to the same form, and then they differ only by the epochs, the mean movements, and the coefficients of their arguments; because, if one of them, for example, contains an argument which is not found at all in the others, it is clear that that returns to supposing in the latter, this coefficient null. Now, if we compared these tables to the totality of the good observations, by rectifying them through this comparison, these tables thus rectified, would satisfy, by that which precedes, the condition that the sum of the squares of the errors that they would permit to yet subsist, be a *minimum*. The tables which would approach most to fulfill this condition, would merit therefore preference; whence it follows that by comparing these diverse tables, to a considerable number of observations, the presumption of exactitude must be in favor of that in which the sum of the squares of the errors is smaller than in the others.

§22. To here we have supposed the facilities of the positive errors, the same as those of the negative errors. Let us consider now the general case in which these facilities are able to be different. Let us name a the interval in which the errors of each observation are able to be extended, and let us suppose it divided into an infinite number $n + n'$ of equal parts and taken for unity, n being the number of the parts which correspond to the negative errors, and n' being the number of the parts which correspond to the positive errors. On each point of the interval a, let us raise an ordinate [330]
which expresses the probability of the corresponding error, and let us designate by $\phi\left(\frac{x}{n+n'}\right)$, the ordinate corresponding to the error x. This premised, let us consider the series

$$\phi\left(\frac{-n}{n+n'}\right)c^{-qn\varpi\sqrt{-1}} + \phi\left[\frac{-(n-1)}{n+n'}\right]c^{-q(n-1)\varpi\sqrt{-1}}\dots$$
$$\dots+\phi\left(\frac{-1}{n+n'}\right)c^{-q\varpi\sqrt{-1}} + \phi\left(\frac{0}{n+n'}\right) + \phi\left(\frac{1}{n+n'}\right)c^{q\varpi\sqrt{-1}}\dots$$
$$\dots+\phi\left(\frac{n'-1}{n+n'}\right)c^{q(n'-1)\varpi\sqrt{-1}} + \phi\left(\frac{n'}{n+n'}\right)c^{qn'\varpi\sqrt{-1}}.$$

Let us represent this series by $\int\phi\left(\frac{x}{n+n'}\right)c^{qx\varpi\sqrt{-1}}$, the $\int$ sign extending to all the values of x, from $x = -n$ to $x = n'$. The term independent of $c^{\varpi\sqrt{-1}}$ and of its powers, in the development of the function

$$c^{-(l+\mu)\varpi\sqrt{-1}} \times \int\phi\left(\frac{x}{n+n'}\right)c^{qx\varpi\sqrt{-1}} \times \int\phi\left(\frac{x}{n+n'}\right)c^{q^{(1)}x\varpi\sqrt{-1}}\dots$$
$$\dots\times\int\phi\left(\frac{x}{n+n'}\right)c^{q^{(s-1)}x\varpi\sqrt{-1}},$$

will be, by §21, the probability that the function

$$q\epsilon + q^{(1)}\epsilon^{(1)}\dots + q^{(s-1)}\epsilon^{(s-1)} \qquad (m)$$

will be equal to $l + \mu$; this probability is therefore

$$\frac{1}{2\pi} \int d\varpi\, c^{-l\varpi\sqrt{-1}} c^{-\mu\varpi\sqrt{-1}} \times \int \phi\left(\frac{x}{n+n'}\right) c^{qx\varpi\sqrt{-1}} \times \text{etc.}, \tag{1}$$

the integral being taken from $\varpi = -\pi$ to $\varpi = \pi$. The logarithm of the function

$$c^{-\mu\varpi\sqrt{-1}} \int \phi\left(\frac{x}{n+n'}\right) c^{qx\varpi\sqrt{-1}} \times \int \phi\left(\frac{x}{n+n'}\right) c^{q^{(1)}x\varpi\sqrt{-1}} \times \text{etc.}, \tag{2}$$

is

$$-\mu\varpi\sqrt{-1} + \log\left[\int \phi\left(\frac{x}{n+n'}\right) c^{qx\varpi\sqrt{-1}}\right] + \text{etc.},$$

n and n' being supposed infinite numbers, if we make [331]

$$\frac{x}{n+n'} = x', \quad \frac{1}{n+n'} = dx';$$

if moreover we suppose

$$k = \int dx'\phi(x'), \quad k' = \int x'dx'\phi(x'), \quad k'' = \int x'^2 dx'\phi(x'), \quad \text{etc.},$$

the integrals being taken from $x' = -\frac{n}{n+n'}$ to $x' = \frac{n'}{n+n'}$, we will have

$$\int \phi\left(\frac{x}{n+n'}\right) c^{qx\varpi\sqrt{-1}} = (n+n')k \left\{ \begin{array}{l} 1 + \frac{k'}{k} \cdot q.(n+n').\varpi\sqrt{-1} \\ \quad - \frac{k''}{2k} \cdot q^2.(n+n')^2.\varpi^2 + \text{etc.} \end{array} \right\}.$$

The error of each observation needing to fall within the limits $-n$ and $+n'$, and the probability that this will hold being $\int \phi\left(\frac{x}{n+n'}\right)$, or $(n+n')k$, this quantity must be equal to unity. Thence it is easy to conclude that the logarithm of the function (2) is, by making $\mu' = \frac{\mu}{n+n'}$,

$$\left(\frac{k'}{k} Sq^{(i)} - \mu'\right)(n+n')\varpi\sqrt{-1} - \frac{kk'' - k'^2}{2k^2} \cdot Sq^{(i)2}(n+n')^2.\varpi^2 + \text{etc.},$$

the sign S embracing all the values of i, from i null to $i = s-1$. We will make the first power of ϖ vanish, by making

$$\mu' = \frac{k'}{k} \cdot Sq^{(i)},$$

and if we consider only the second power, that which we can do by that which precedes, when s is a very great number, we will have, for the logarithm of the function (2),

$$-\frac{kk'' - k'^2}{2k^2} Sq^{(i)2}(n+n')^2\varpi^2.$$

By passing again from the logarithms to the numbers, the function (2) is transformed into the following

$$c^{-\frac{kk''-k'^2}{2k^2}(n+n')^2\varpi^2 Sq^{(i)2}};$$

the integral (1) becomes thus [332]

$$\frac{1}{2\pi}\int d\varpi\, c^{-lw\sqrt{-1}}\, c^{-\frac{kk''-k'^2}{2k^2}(n+n')^2\varpi^2 \mathrm{S} q^{(i)2}}.$$

Let us suppose

$$l=(n+n')r\sqrt{Sq^{(i)2}},$$
$$t=\sqrt{\frac{(kk''-k'^2)Sq^{(i)2}}{2k^2}}(n+n')\varpi-\frac{r\sqrt{-1}}{2}\sqrt{\frac{2k^2}{kk''-k'^2}}.$$

The variation of l being unity, we will have

$$1=(n+n')dr\sqrt{Sq^{(i)2}};$$

the preceding integral becomes thus, after having integrated it from $t=-\infty$ to $t=\infty$,

$$\frac{kdr}{\sqrt{2(kk''-k'^2)\pi}}c^{-\frac{k^2r^2}{2(kk''-k'^2)}}$$

Thus the probability that the function (m) will be comprehended within the limits

$$\frac{ak'}{k}Sq^{(i)}\pm ar\sqrt{Sq^{(i)2}},$$

is equal to

$$\frac{2}{\sqrt{\pi}}\int\frac{kdr}{\sqrt{2(kk''-k'^2)}}c^{-\frac{k^2r^2}{2(kk''-k'^2)}},$$

the integral being taken from r null.

$\frac{ak'}{k}$ is the abscissa of the ordinate which passes through the center of gravity of the area of the curve of the probabilities of the errors of each observation; the product of this abscissa by $Sq^{(i)}$ is therefore the mean result toward which the function (m) converges without ceasing. If we suppose $1=q=q^{(1)}=$ etc.; the function (m) becomes the sum of the errors, and then $Sq^{(i)}$ becomes s; therefore by dividing the sum of the errors by s, in order to have the mean error; this error converges without ceasing toward the abscissa of the center of gravity, in a manner that by taking on both sides [333]
any interval whatsoever as small as we will wish, the probability that the mean error will fall within that interval, will finish, by multiplying indefinitely the observations, by differing from certainty, only by a quantity less than every given magnitude.

§23. We just investigated the mean result that observations numerous and not yet made, must indicate with most advantage, and the law of probability of the errors of this result. Let us consider presently the mean result of observations already made, and of which we know the respective deviations. For this, let us imagine a number s of observations of the same kind, that is, such that the law of errors is the same for all. Let us name A the result of the first; $A+q$, the one of the second; $A+q^{(1)}$ the one of the third, and so forth; q, $q^{(1)}$, $q^{(2)}$, etc. being positive and increasing quantities, that which we can always obtain by a convenient disposition of the observations. Let us designate further by $\phi(z)$, the probability of the error z for each observation, and let us suppose that $A+x$ be the true result. The error of the first observation is then $-x$; $q - x$, $q^{(1)} - x$, etc. are the errors of the second, of the third, etc. The probability of the simultaneous existence of all these errors, is the product of their respective probabilities; it is therefore

$$\phi(-x)\phi(q-x)\phi(q^{(1)}-x).\text{etc.}$$

Now, x being susceptible of an infinity of values; by considering them as so many causes of the observed event, the probability of each of them will be, by §1,

$$\frac{dx\,\phi(-x)\phi(q-x)\phi(q^{(1)}-x).\text{etc.}}{\int dx\,\phi(-x)\phi(q-x)\phi(q^{(1)}-x).\text{etc.}},$$

the integral of the denominator being taken for all the values of which x is susceptible. Let us name $\frac{1}{H}$ this denominator. This premised, let us imagine a curve of which x is the abscissa, and of which the ordinate y is

$$H\phi(-x)\phi(q-x)\phi(q^{(1)}-x).\text{etc.};$$

this curve will be that of the probabilities of the values of x. The value that it is necessary to choose [334]
for the mean result, is that which renders the mean error to fear, a *minimum*. Each error, either positive, or negative, needing to be considered as a disadvantage, or a real loss in the game; we have the mean disadvantage, by taking the sum of the products of each disadvantage, by its probability; the mean value of the error to fear, is therefore the sum of the products of each error, setting aside the sign, by its probability. Let us determine the abscissa that it is necessary to choose in order that this sum be a *minimum*. For this, let us give to the abscissas, for origin, the first extremity of the preceding curve, and let us name x' and y' the coordinates of the curve, by departing from this origin. Let l be the value that it is necessary to choose. It is clear that if the true result were x', the error of the result l would be, setting aside the sign, $l - x'$, as much as x' would be less than l; now y' is the probability that x' is the true result; the sum of the errors to fear, setting aside the sign, multiplied by their probability, is therefore for all the values of x', less than l, $\int(l-x')y'dx'$, the integral being taken from $x' = 0$ to $x' = l$. We will see in the same manner, that for the values of x' superior to l, the sum of the errors to fear, multiplied by their probability, is $\int(x'-l)y'dx'$, the integral being taken from $x' = l$ to the abscissa x' corresponding to the last extremity of the curve; the entire sum of the errors to fear, setting aside the sign, multiplied by their respective probabilities, is therefore

$$\int(l-x')y'dx' + \int(x'-l)y'dx'.$$

The differential of this function, taken with respect to l, is

$$dl\int y'dx' - dl\int y'dx';$$

because we have the differential of $\int(l-x')y'dx'$, by differentiating first the value of l under the $\int$ sign, and by adding to this differential, the increase which results from the variation of the limit of the integral, a limit which is changed into $l + dl$. This increase is equal to the element $(l-x')y'dx'$, to the limit where $x' = l$; it is therefore null, and $dl\int y'dx'$ is the differential of the integral $\int(l-x')y'dx'$. We will see in the same manner, that $-dl\int y'dx'$ is the differential of the [335] integral $\int (x'-l)y'dx'$. The sum of these differentials is null relatively to the abscissa l, for which the mean error to fear is a *minimum*; we have therefore, relatively to this abscissa,

$$\int y'dx' = \int y'dx',$$

the first integral being taken from $x'=0$ to $x'=l$, and the second being taken from $x'=l$ to the extreme value of x'.

It follows thence that the abscissa which renders the mean error to fear, a *minimum*, is that of which the ordinate divides the area of the curve into two equal parts. This point enjoys further the property to be the one on the side of which it is as probable that the true result falls, as the other side; and by this reason, it is able further to be named *middle of probability*. Some celebrated geometers have taken for the middle that it is necessary to choose, the one which renders the observed result, the most probable, and consequently the abscissa which corresponds to the greatest ordinate of the curve; but the middle that we adopt, is evidently indicated by the theory of probabilities.

If we put $\phi(x)$ under the form of an exponential, and if we designate it by $c^{-\psi(x^2)}$, so that it is able equally to agree to the positive and negative errors; we will have

$$y = Hc^{-\psi(x^2)-\psi(x-q)^2-\psi(x-q^{(1)})^2-\text{etc.}} \tag{1}$$

If we make $x=a+z$, and if we develop the exponent of c with respect to the powers of z, y will take this form

$$y = Hc^{-M-2Nz-Pz^2-Qz^3-\text{etc.}},$$

an expression in which we have

$$\begin{aligned} M =& \psi(a^2)+\psi(a-q)^2+\psi(a-q^{(1)})^2+\text{etc.},\\ N =& a\psi'(a^2)+(a-q)\psi'(a-q)^2+(a-q^{(1)})\psi'(a-q^{(1)})^2+\text{etc.},\\ P =& \psi'(a^2)+\psi'(a-q)^2+\psi'(a-q^{(1)})^2+\text{etc.}+2a^2\psi''(a^2)\\ &+2(a-q)^2\psi''(a-q)^2+a(a-q^{(1)})^2\psi''(a-q^{(1)})^2+\text{etc.},\\ &\text{etc.,}\end{aligned}$$

$\psi'(t)$ being the coefficient of dt in the differential of $\psi(t)$, $\psi''(t)$ being the coefficient of dt in the [336]
differential of $\psi'(t)$, and so forth.

Let us suppose the number s of observations, very great, and let us determine a by the equation $N=0$ which gives the condition of the *maximum* of y; then we have

$$y = Hc^{-M-Pz^2-Qz^3-\text{etc.}}.$$

M, P, Q, etc. are of order s; now, if z is very small of order $\frac{1}{\sqrt{s}}$, Qz^3 becomes of order $\frac{1}{\sqrt{s}}$, and the exponential $c^{-Qz^3-\text{etc.}}$ is able to be reduced to unity. Thus, in the interval from $z=0$ to $z=\frac{r}{\sqrt{s}}$, we are able to suppose

$$r = Hc^{-M-Pz^2}.$$

Farther on, and when z is of order $s^{-\frac{m}{2}}$, m being smaller than unity, Pz^2 becomes of order s^{1-m}; consequently c^{-Pz^2} becomes in the same way as y, insensible; so that we can, in all extent of the curve, suppose

$$y = Hc^{-M-Pz^2}.$$

The value of a given by the equation $N = 0$, or

$$0 = a\psi'(a^2) + (a-q)\psi'(a-q)^2 + (a-q^{(1)})\psi'(a-q^{(1)})^2 + \text{etc.},$$

is then the abscissa x corresponding to the ordinate which divides the area of the curve into equal parts. The condition that the entire area of the curve must represent certitude or unity, gives

$$\frac{1}{H} = \int dz\, c^{-M-Pz^2},$$

the integral being taken from $z = -\infty$ to $z = \infty$, that which gives

$$H = \frac{c^M\sqrt{P}}{\sqrt{\pi}}.$$

The mean positive or negative error to fear, by taking a for the mean result of the observations, is $\pm\int zydz$, the integral being taken from z null to z infinity, that which gives for this error [337]

$$\pm\frac{1}{2\sqrt{\pi}P}.$$

But the entire ignorance which we have of the law $c^{-\psi(x^2)}$ of the errors of each observation, does not permit forming the equation

$$0 = a\psi'(a^2) + (a-q)\psi'(a-q)^2 + \text{etc.}$$

Thus knowledge of the values of q, $q^{(1)}$, etc., shedding *a posteriori*, no light on the mean result a of the observations; it is necessary to be held to the most advantageous result determined *a priori*, and that we have seen to be the one which the method of least squares of the errors furnishes.

Let us seek the function $\psi(x^2)$ which gives constantly the rule of the arithmetic means, admitted by the observers. For this, let us imagine that out of the s observations, the first i coincide, in the same way as the $s-i$ last. The equation $N = 0$ becomes then

$$0 = ia\psi'(a^2) + (s-i)(a-q)\psi'(a-q)^2.$$

The rule of the arithmetic mean gives

$$a = \frac{s-i}{s}q;$$

the preceding equation becomes thus

$$\psi'\left[\left(\frac{s-i}{s}\right)^2 q^2\right] = \psi'\left(\frac{i^2}{s^2}q^2\right).$$

This equation must hold whatever be $\frac{i}{s}$ and q, it is necessary that $\psi'(t)$ be independent of t, that which gives

$$\psi'(t) = k,$$

k being a constant. By integrating, we have

$$\psi(t) = kt - L,$$

L being an arbitrary constant; hence, [338]

$$c^{-\psi(x^2)} = c^{L-kx^2}.$$

Such is therefore the function which can alone, give generally the rule of the arithmetic means. The constant L must be determined in a manner that the integral $\int dx\, c^{L-kx^2}$, taken from $x = -\infty$ to $x = \infty$, be equal to unity; because it is certain that the error x of an observation must fall within these limits; we have therefore

$$c^L = \sqrt{\frac{k}{\pi}};$$

consequently the probability of the error x is $\sqrt{\frac{k}{\pi}}c^{-kx^2}$.

In truth, this expression gives infinity for the limit of the errors, that which is not admissible; but, seeing the rapidity with which this kind of exponential diminishes in measure as x increases, we are able to take k rather great, for which beyond the admissible limit of the errors, their probabilities are insensible, and can be supposed null.

The preceding law of errors gives for the general expression (1) of y,

$$y = \sqrt{\frac{sk}{\pi}}e^{-ksu^2};$$

by determining H in a manner that the entire integral $\int ydx$ be unity, and making

$$x = \frac{Sq^{(i)}}{s} + u.$$

The ordinate which divides the area of the curve into two equal parts, is that which corresponds to $u = 0$, and consequently to

$$x = \frac{Sq^{(i)}}{s};$$

this is therefore the value of x that it is necessary to choose for the mean result of the observations; now, this value is that which the rule of the arithmetic means gives; the preceding law of errors of each observation, gives therefore constantly the same results as this rule, and we have seen that it is [339]
the only law which enjoys this property.

By adopting this law, the probability of the error $\epsilon^{(i)}$ of the $(i+1)^{\text{st}}$ observation, is

$$\sqrt{\frac{k}{\pi}}e^{-k\epsilon^{(i)2}};$$

now we have seen in §20, that z being the correction of an element, this observation furnishes the equation of condition

$$\epsilon^{(i)} = p^{(i)}z - \alpha^{(i)}.$$

The probability of the value of $p^{(i)}z - \alpha^{(i)}$ is therefore

$$\sqrt{\frac{k}{\pi}} e^{-k(p^{(i)}z-\alpha^{(i)})^2};$$

the probability of the simultaneous existence of the s values $p.z - \alpha$, $p^{(1)}.z - \alpha^{(1)}$, $\ldots p^{(s-1)}.z - \alpha^{(s-1)}$, will be therefore

$$\left(\sqrt{\frac{k}{\pi}}\right)^{s-1} e^{-kS(p^{(i)}z-\alpha^{(i)})^2}.$$

This probability varies with z; we will have the probability of any value whatsoever of z, by multiplying this quantity by dz, and dividing the product by the integral of this product, taken from $z = -\infty$ to $z = \infty$. Let

$$z = \frac{Sp^{(i)}q^{(i)}}{Sp^{(i)2}} + u,$$

this probability becomes

$$du\sqrt{\frac{kSp^{(i)2}}{\pi}} e^{-ku^2 Sp^{(i)2}};$$

so that if we describe a curve of which the coefficient of du is the ordinate, and of which u is the abscissa, this curve extended from $u = -\infty$ to $u = \infty$, can be considered as the curve of the probabilities of the errors u, of which the result

$$z = \frac{Sp^{(i)}\alpha^{(i)}}{Sp^{(i)2}}$$

is susceptible. The ordinate which divides the area of the curve into two equal parts, is that which [340]
corresponds to $u = 0$, and consequently to z equal to $\frac{Sp^{(i)}\alpha^{(i)}}{Sp^{(i)2}}$; this result is therefore the one that it is necessary to choose; now, it is the same as the one which the method of least squares of the errors of observations gives; the preceding law of errors of each observation, leads therefore to the same results as this method.

The method of least squares of the errors becomes necessary, when there is concern to take a mean among many given results, each, by the collection of a great number of observations of diverse kinds. Let us suppose that one same element is given, 1° by the mean result of s observations of a first kind, and that it is by these observations, equal to A; 2° by the mean result of s' observations of a second kind, and that it is equal to $A + q$; 3° by the mean result of s'' observations of a third kind, and that it is equal to $A + q'$, and thus of the remaining. If we represent by $A + x$, the true element; the error of the result of the observations s will be $-x$; by supposing therefore β equal to

$$\sqrt{\frac{k}{k''}} \cdot \frac{\sqrt{Sp^{(i)2}}}{2a},$$

if we make use of the method of least squares of the errors in order, to determine the mean result; or to

$$\sqrt{\frac{k}{k''}} \frac{Sp^{(i)2}}{2a\sqrt{s}},$$

if we make use of the ordinary method; the probability of this error will be, by §20,

$$\frac{\beta}{\sqrt{\pi}} c^{-\beta^2 x^2}.$$

The error of the result of the s' observations will be $q - x$, and by designating by β' for these observations, that which we have named β for the s observations, the probability of this error will be

$$\frac{\beta'}{\sqrt{\pi}} c^{-\beta'^2 (x-q)^2}.$$

Similarly the error of the result of the s'' observations will be $q' - x$; and by naming for them, β'', that which we have named β for the s observations; the probability of this error will be [341]

$$\frac{\beta''}{\sqrt{\pi}} c^{-\beta''^2 (x-q')^2},$$

and so forth. The product of all these probabilities will be the probability that $-x$, $q - x$, $q' - x$, etc. will be the errors of the mean results of the observations s, s', s'', etc. By multiplying it by dx, and taking the integral from $x = -\infty$ to $x = \infty$, we will have the probability that the mean results of the observations s', s'', etc., will surpass respectively by q, q', etc., the mean result of the s observations.

If we take the integral within the determined limits, we will have the probability that the preceding condition being fulfilled, the error of the first result will be comprehended within these limits; by dividing this probability by that of the condition itself, we will have the probability that the error of the first result will be comprehended within some given limits, when we are certain that the condition effectively holds; this probability is therefore

$$\frac{\int dx c^{-\beta^2 x^2 - \beta'^2 (x-q)^2 - \beta''^2 (x-q')^2 - \text{etc.}}}{\int dx c^{-\beta^2 x^2 - \beta'^2 (x-q)^2 - \beta''^2 (x-q')^2 - \text{etc.}}},$$

the integral of the numerator being taken within the given limits, and that of the denominator being taken from $x = -\infty$ to $x = \infty$. We have

$$\begin{aligned} &\beta^2 x^2 + \beta'^2 (x-q)^2 + \beta''^2 (x-q')^2 + \text{etc.} \\ =&(\beta^2 + \beta'^2 + \beta''^2 + \text{etc.}) x^2 - 2x(\beta'^2 q + \beta''^2 q' + \text{etc.}) \\ +&\beta'^2 q^2 + \beta''^2 q'^2 + \text{etc.} \end{aligned}$$

Let

$$x = \frac{\beta'^2 q + \beta''^2 q' + \text{etc.}}{\beta^2 + \beta'^2 + \beta''^2 + \text{etc.}} + t;$$

the preceding probability will become

$$\frac{\int dt\, c^{-(\beta^2 + \beta'^2 + \beta''^2 + \text{etc.}) t^2}}{\int dt\, c^{-(\beta^2 + \beta'^2 + \beta''^2 + \text{etc.}) t^2}},$$

the integral of the numerator being taken within some given limits, and that of the denominator being taken from $t = -\infty$ to $t = \infty$. This last integral is [342]

$$\frac{\sqrt{\pi}}{\sqrt{\beta^2 + \beta'^2 + \beta''^2 + \text{etc.}}}.$$

By making therefore

$$t' = t\sqrt{\beta^2 + \beta'^2 + \beta''^2 + \text{etc.}};$$

the preceding probability becomes

$$\frac{1}{\sqrt{\pi}} \int dt' c^{-t'^2}.$$

The most probable value of t', is that which corresponds to t' null; whence it follows that the most probable value of x, is that which corresponds to $t = 0$, thus the correction of the first result, that the collection of all the observations s, s', s'', etc. give with most probability, is

$$\frac{\beta'^2 q + \beta''^2 q' + \text{etc.}}{\beta^2 + \beta'^2 + \beta''^2 + \text{etc.}}.$$

This correction added to the result A, gives for the result that it is necessary to choose,

$$\frac{A\beta^2 + (A + q)\beta'^2 + (A + q')\beta''^2 + \text{etc.}}{\beta^2 + \beta'^2 + \beta''^2 + \text{etc.}}.$$

The preceding correction is that which renders a *minimum*, the function

$$(\beta x)^2 + (\beta'.\overline{x - q})^2 + (\beta''.\overline{x - q'})^2 + \text{etc.}$$

Now the greatest ordinate of the curve of probabilities of the first result is, as we have just seen, $\frac{\beta}{\sqrt{\pi}}$; that of the curve of probabilities of the second result, is $\frac{\beta'}{\sqrt{\pi}}$, and so forth; the mean that it is necessary to choose among the diverse results, is therefore the one which renders a *minimum*, the sum of the squares of the error of each result, multiplied by the greatest ordinate of its curve of probability. Thus the law of the *minimum* of the squares of the errors, becomes necessary, when we [343] must take a mean among some given results, each, by a great number of observations.

§24. We have seen previously, that in all the manners to combine the equations of condition, in order to form the linear final equations, necessary to the determination of the elements; the most advantageous is that which results from the method of least squares of the errors of the observations, at least when the observations are in great number. If instead of considering the *minimum* of the squares of the errors, we considered the *minimum* of other powers of the errors, or even of every other function of the errors; the final equations would cease to be linear, and their resolution would become impractical, if the observations were in great number. However there is a case which merits a particular attention, in this that it gives the system in which the greatest error, setting aside the sign, is less than in every other system. This case is the one of the *minimum* of the infinite and even powers of the errors. Let us consider here only the correction of a single element; and z expressing this correction, let us represent, as previously, the equations of condition, by the following,

$$\epsilon^{(i)} = p^{(i)} z - \alpha^{(i)},$$

i being able to vary from zero to $s - 1$, s being the number of observations. The sum of the powers $2n$ of the errors will be $\mathrm{S}(\alpha^{(i)} - p^{(i)} z)^{2n}$, the sign S extending to all the values of i. We can suppose, in this sum, all the values of $p^{(i)}$ positive; because if one of them was negative, it would become positive by changing, as we are able to do, the signs of the two terms of the binomial raised to the power $2n$, to which it corresponds. We will suppose therefore the quantities $\alpha - pz$,

$\alpha^{(1)} - p^{(1)}z$, $\alpha^{(2)} - p^{(2)}z$, etc., disposed in a manner that the quantities p, $p^{(1)}$, $p^{(2)}$, etc. are positive and increasing. This premised, if $2n$ is infinite, it is clear that the greatest term of the sum $\mathrm{S}(\alpha^{(i)} - p^{(i)}z)^{2n}$, will be the entire sum, unless there was one or many other terms which were equal to it, and this is that which must hold in the case of the *minimum* of the sum. In fact, if there [344] was only a single greatest quantity, setting aside the sign, such as $\alpha^{(i)} - p^{(i)}z$, we would be able to diminish it by making z vary conveniently, and then the sum $S(\alpha^{(i)} - p^{(i)}z)^{2n}$ would diminish and would not be a *minimum*. It is necessary moreover that if $\alpha^{(i)} - p^{(i)}z$ and $\alpha^{(i')} - p^{(i')}z$ are, setting aside the sign, the two greatest quantities and equal between them, they be of contrary sign. In fact, the sum $(\alpha^{(i)} - p^{(i)}z)^{2n} + (\alpha^{(i')} - p^{(i')}z)^{2n}$ needing to be then a *minimum*, its differential $-2ndz[p^{(i)}(\alpha^{(i)} - p^{(i)}z)^{2n-1} + p^{(i')}(\alpha^{(i')} - p^{(i')}z)^{2n-1}]$ must be null, that which can be when n is infinite, only in the case where $\alpha^{(i)} - p^{(i)}z$ and $\alpha^{(i')} - p^{(i')}z$ are infinitely little different, and of contrary sign. If there are three greatest quantities, and equals among them, setting aside the sign, we will see in the same manner that their signs are not able to be the same.

Now, let us consider the sequence

$$\begin{gathered}\alpha^{(s-1)} - p^{(s-1)}z,\ \alpha^{(s-2)} - p^{(s-2)}z,\ \alpha^{(s-3)} - p^{(s-3)}z,\ \ldots\ \alpha - pz,\\ -\alpha + pz,\ \ldots\ -\alpha^{(s-3)} + p^{(s-3)}z,\ -\alpha^{(s-2)} + p^{(s-2)}z, -\alpha^{(s-1)} + p^{(s-1)}z.\end{gathered} \qquad (o)$$

If we suppose $x = -\infty$, the first term of the sequence surpasses the following, and continues to surpass them by making z increase, to the moment where it becomes equal to one of them. Then this one, by the increase of z, becomes greatest of all; and in measure as we make z increase, it continues always to surpass those which precede it. In order to determine this term, we will form the sequence of quotients

$$\frac{\alpha^{(s-1)} - \alpha^{(s-2)}}{p^{(s-1)} - p^{(s-2)}},\ \frac{\alpha^{(s-1)} - \alpha^{(s-3)}}{p^{(s-1)} - p^{(s-3)}},\ \ldots,\ \frac{\alpha^{(s-1)} - \alpha}{p^{(s-1)} - p},\ \frac{\alpha^{(s-1)} + \alpha}{p^{(s-1)} + p},\ \ldots,\ \frac{\alpha^{(s-1)} + \alpha^{(s-1)}}{p^{(s-1)} + p^{(s-1)}}.$$

Let us suppose that $\frac{\alpha^{(s-1)} - \alpha^{(r)}}{p^{(s-1)} - p^{(r)}}$ is the smallest of these quotients by having regard to the sign, that is by regarding a greater negative quantity, as smaller than another lesser negative quantity. If there are many least and equal quotients, we will consider the one which is related to the most distant term of the first in the sequence (o); this term will be the greatest of all, to the moment where, by the increase of z, it becomes equal to one of the following, which begins then to be the greatest. In [345] order to determine this new term, we will form a new sequence of quotients

$$\frac{\alpha^{(r)} - \alpha^{(r-1)}}{p^{(r)} - p^{(r-1)}},\ \frac{\alpha^{(r)} - \alpha^{(r-2)}}{p^{(r)} - p^{(r-2)}},\ \ldots,\ \frac{\alpha^{(r)} - \alpha}{p^{(r)} - p},\ \frac{p^{(r)} + \alpha}{p^{(r)} + p},\ \text{etc.};$$

the term of the sequence (o), to which the least of these quotients correspond, will be the new term. We will continue thus until one of the two terms which become equal and the greatest, is in the first half of the sequence (o), and the other in the second half. Let $\alpha^{(i)} - p^{(i)}z$ and $-\alpha^{(i')} + p^{(i')}z$ be these two terms; then the value of z which corresponds to the system of the *minimum* of the greatest of the errors, setting aside the sign, is

$$z = \frac{\alpha^{(i)} + \alpha^{(i')}}{p^{(i)} - p^{(i')}}.$$

If there are many elements to correct, the equations of condition which determine their corrections, contain many unknowns, and the investigation of the system of correction, in which the greatest error is, setting aside the sign, smaller than in every other system, becomes more complicated. I have considered this case in a general manner, in the third Book of the *Mécanique Céleste*. I will observe only here, that then the sum of the $2n$ powers of the errors of the observations is, as in the case of a single unknown, a *minimum*, when $2n$ is infinite; whence it is easy to conclude that in the system of which there is concern, it must have as many errors plus one, equals, and greatest setting aside the sign, as there are elements to correct. We imagine that the results corresponding to $2n$ equal to a great number must differ little from those which $2n$ infinite gives. It is not even necessary for this, that the $2n$ power be quite elevated, and I have recognized through many examples, that in the same case where this power does not surpass the square, the results differ little from those that the system of the *minimum* of the greatest errors gives, that which is a new advantage of the method of least squares of the errors of observations.

For a long time, geometers take an arithmetic mean among their observations; and, in order to determine the elements that they wish to know, they choose the most favorable circumstances for [346] this object, namely, those in which the errors of the observations alter the least that it is possible, the value of these elements. But Cotes is, if I do not deceive myself, the first who has given a general rule in order to make agree in the determination of an element, many observations, proportionally to their influence. By considering each observation as a function of the element, and regarding the error of the observation as an infinitely small differential; it will be equal to the differential of the function, taken with respect to that element. The more the coefficient of the differential of the element will be considerable, the less it will be necessary to make the element vary, in order that the product of its variation, by this coefficient, be equal to the error of the observation; this coefficient will express therefore the influence of the observation on the value of the element. This premised, Cotes represents all the values of the element, given by each observation, by the parts of an indefinite straight line, all these parts having a common origin. He imagines next, at their other extremities, weights proportional to the respective influences of the observations. The distance from the common origin of the parts, to the common center of gravity of all these weights, is the value that he chose for the element.

Let us take the equation of condition of §20,

$$\epsilon^{(i)} = p^{(i)}z - \alpha^{(i)},$$

$\epsilon^{(i)}$ being the error of the $(i+1)^{\text{st}}$ observation, and z being the correction of the element already known quite nearly; $p^{(i)}$ that we are always able to suppose positive, will express the influence of the corresponding observation. $\frac{\alpha^{(i)}}{p^{(i)}}$ being the value of z resulting from the observation, the rule of Cotes reverts to multiplying this value by $p^{(i)}$, to make a sum of all the products relative to the diverse values, and to divide it by the sum of all the $p^{(i)}$, that which gives

$$z = \frac{S\alpha^{(i)}}{Sp^{(i)}}.$$

This was indeed the correction adopted by the observers, before the usage of the method of least [347] squares of the errors of the observations.

However, we do not see that since this excellent geometer, we have employed his rule, until Euler who in his first piece on Jupiter and Saturn, appears to me the first to have served himself, of the equations of the condition, in order to determine the elements of the elliptic movement of these two planets. Near the same time, Tobie Mayer made use of it in his beautiful researches on the libration of the moon, and next in order to form his lunar Tables. Since, the best astronomers have followed this method, and the success of the Tables which they have constructed by its means, has verified the advantage of it.

When we have only one element to determine, this method leaves no difficulty; but when we must correct at the same time many elements, it is necessary to have as many final equations formed by the reunion of many equations of condition, and by means of which we determine by elimination, the corrections of the elements. But what is the most advantageous manner to combine the equations of condition, in order to form the final equations? It is here that the observers abandoned themselves to some arbitrary gropings which must have led them to some different results, although deduced from the same observations. In order to avoid these gropings, Mr. Legendre had the simple idea to consider the sum of the squares of the errors of the observations, and to render it a *minimum*; that which furnishes directly as many final equations, as there are elements to correct. This scholarly geometer is the first who has published this method; but we owe to Mr. Gauss the justice to observe that he had had, many years before this publication, the same idea of which he made a habitual usage, and that he had communicated to many astronomers. Mr. Gauss, in his Theory of elliptic movement, has sought to connect this method to the Theory of Probabilities, by showing that the same law of errors of the observations, which give generally the rule of the arithmetic mean among many observations, admitted by the observers, gives similarly the rule of the least squares of the errors of the observations; and it is that which we have seen in §23. But, as nothing proves that the first of these rules gives the most advantageous result, the same uncertainty exists with respect [348] to the second. The investigation of the most advantageous manner to form the final equations, is without doubt one of the most useful of the Theory of Probabilities: its importance in physics and astronomy, carries me to occupy myself with it. For this, I will consider that all the ways to combine the equations of condition, in order to form a linear final equation, reverted to multiplying them respectively by some factors which were null relatively to the equations that we employed not at all, and to make a sum of all these products; that which gives a first final equation. A second system of factors gives a second final equation, and so forth, until we have as many final equations, as elements to correct. Now, it is clear that it is necessary to choose the system of factors, such that the mean positive or negative error to fear respecting each element, be a *minimum*; the mean error being the sum of the products of each error by its probability. When the observations are in small number, the choice of these systems depends on the law of errors of each observation. But if we consider a great number of observations, that which holds most often in the astronomical researches, this choice becomes independent of this law; and we have seen in that which precedes, that analysis leads then directly to the results of the method of least squares of the errors of the observations. Thus this method which offered first only the advantage to furnish, without groping, the final equations necessary to the correction of the elements, gives at the same time the most precise corrections, at least when we wish to employ only final equations which are linear, an indispensable condition, when we consider at the same time a great number of observations; otherwise, the elimination of the unknowns and their determination would be impractical.