BOOK II
CHAPTER VII
DE L'INFLUENCE DES INÉGALITÉS INCONNUES QUI PEUVENT EXISTER ENTRE DES CHANCES QUE L'ON SUPPOSE PARFAITMENT ÉGALES

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Théorie Analytique des Probabilités
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ON THE INFLUENCE OF THE UNKNOWN INEQUALITIES WHICH ARE ABLE TO EXIST AMONG THE CHANCES THAT ONE SUPPOSES PERFECTLY EQUAL

Examination of the cases in which this influence is favorable or contrary. It is contrary to the one who, in the game of *heads* and *tails*, wagers to bring forth *heads* an odd number of times, in an even number of trials. Means to correct this influence. N° 34.

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§34. I have already considered this influence in §1, where we have seen that these inequalities increase the probability of the events composed of the repetition of simple events. I will resume here this important object in the applications of the analysis of probabilities. [402]

There results from the section cited, that if in the game of *heads* and *tails*, there exists an unknown difference between the possibilities to bring forth one or the other; by naming α this difference, so that $\frac{1\pm\alpha}{2}$ is the possibility to bring forth *heads*, and consequently $\frac{1\mp\alpha}{2}$ that to bring forth *tails*, the one of the two signs $+$ and $-$ that we must adopt being unknown; the probability to bring forth *heads* n times consecutively, will be

$$\frac{(1+\alpha)^n+(1-\alpha)^n}{2^{n+1}}$$

or

$$(1)\qquad \frac{1}{2^n}\left(1+\frac{n.\overline{n-1}}{1.2}\alpha^2+\frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{1.2.3.4}\alpha^4+\text{etc.}\right).$$

The game of *heads* and *tails* consists, as we know, in casting into the air a very thin coin, which falls again necessarily on one of its two opposite faces that we name *heads* and *tails*. We are able to diminish the value of α, by rendering these two faces the most equal as it is possible. But it is physically impossible to obtain a perfect equality; and then, the one who wagers to bring forth *heads* twice consecutively, or *tails* twice consecutively, has the advantage over the one who wagers that in two trials, *heads* and *tails* will alternate; its probability being $\frac{1+\alpha^2}{2}$.

We are able to diminish the influence of the inequality of the two faces of the coin, by submitting them themselves to the chances of hazard. Let us designate by A this coin, and let us imagine a second coin B similar to the first. Let us suppose that after having projected this second coin, we project the coin A in order to form a first trial, and let us determine the probability that in n consecutive similar trials, the coin A will present the same faces as the coin B. If we name p the probability to bring forth *heads* with the coin A, and q the probability to bring forth *tails*; if we designate next by p' and q' the same probabilities for the coin B, $pp'+qq'$ will be the probability that in one trial, the coin A will present the same faces as the coin B; thus $(pp'+qq')^n$ will be the probability that that will take place constantly in n trials. Let [403]

$$p=\frac{1+\alpha}{2},\qquad q=\frac{1-\alpha}{2},$$
$$p'=\frac{1+\alpha'}{2},\qquad q'=\frac{1-\alpha'}{2};$$

we will have

$$(pp'+qq')^n=\frac{1}{2^n}(1+\alpha\alpha')^n.$$

But as we are ignorant of what the faces are that the inequalities α and α' favor, the preceding probability is able to be equally either $\frac{1}{2^n}(1+\alpha\alpha')^n$, or $\frac{1}{2^n}(1-\alpha\alpha')^n$, according as α

or α' are of like sign or of contrary signs; the true value of this probability is therefore, α and α' being supposed positives,

$$\frac{1}{2^{n+1}}\left[(1+\alpha\alpha')^n+(1-\alpha\alpha')^n\right]$$

or

$$\frac{1}{2^n}\left(1+\frac{n.\overline{n-1}}{1.2}\alpha^2\alpha'^2+\frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{1.2.3.4}\alpha^4\alpha'^4+\text{etc.}\right).$$

If we compare this formula to formula (1), we see that it is approached more than it, to $\frac{1}{2^n}$, or to the probability which would hold, if the faces of the coins were perfectly equal. Thus the inequality of these faces, is thence corrected in great part: it would even be it in totality, [404]
if α' were null, or if the two faces of the coin B were perfectly equal.

p representing the probability of *heads*, with the coin A, and q that of *tails*; the probability to bring forth *heads* an odd number of times in n trials will be

$$\frac{1}{2}\left[(p+q)^n\mp(p-q)^n\right],$$

the $-$ sign holding if n is even, and the $+$ sign holding if n is odd. Making $p=\frac{1+\alpha}{2}$, $q=\frac{1-\alpha}{2}$, the preceding function becomes

$$\frac{1}{2}(1\mp\alpha^n).$$

If n is odd and equal to $2i+1$, this function is

$$\frac{1}{2}(1+\alpha^{2i+1});$$

but as we are able to suppose equally α positive or negative, it is necessary to take the half of the sum of its two values relative to these suppositions; that which gives $\frac{1}{2}$ for its true value; the inequality of the faces of the coin changes therefore not at all the probability $\frac{1}{2}$ to bring forth *heads* an odd number of times. But if n is even and equal to $2i$, this probability becomes

(2) $$\frac{1}{2}(1-\alpha^{2i}),$$

$\pm\alpha$ being the unknown inequality of the probability between *heads* and *tails*; there is therefore disadvantage to wager to bring forth *heads* or *tails* an odd number of times in $2i$ trials, and consequently, there is advantage to wager to bring forth one or the other, an even number of times.

We are able to diminish this advantage, by changing the wager to bring forth *heads* an odd number of times in $2i$ trials, into the wager to bring forth in the same number of trials, an odd number of resemblances between the faces of the two coins A and B, projected as we have said above. In fact, the probability of a resemblance at each trial is, as we have seen, $pp' + qq'$, and the probability of a dissemblance is $pq' + p'q$. Let us name P the first of these two quantities, and Q the second; the probability to bring forth an odd number of [405] resemblances in $2i$ trials, will be

$$\frac{1}{2}\left[(P+Q)^{2i} - (P-Q)^{2i}\right].$$

If we make, as previously,

$$p = \frac{1+\alpha}{2}, \quad q = \frac{1-\alpha}{2}, \quad p' = \frac{1+\alpha'}{2}, \quad q' = \frac{1-\alpha'}{2};$$

we will have

$$P = \frac{1+\alpha\alpha'}{2}, \quad Q = \frac{1-\alpha\alpha'}{2};$$

the preceding function becomes thus,

$$\frac{1}{2}(1 - \alpha^{2i}\alpha'^{2i}).$$

This function remains the same, whatever change that we make in the signs of α and of α'; it is therefore the true probability to bring forth an odd number of resemblances; but α and α' being small fractions, we see that it is nearer $\frac{1}{2}$, more than formula (2); the disadvantage of an odd number is therefore thence diminished.

We see by that which precedes, that we are able to diminish the influence of the unknown inequalities among the chances that we suppose equals, by submitting them themselves to chance. For example, if we put into an urn the tickets 1, 2, 3, $\dots n$ following this order, and if next after having agitated the urn in order to mix well the tickets, we draw one from it; if there is among the probabilities to exit of the tickets a small difference depending on the order according to which they have been placed in the urn; we will diminish it considerably, by putting into a second urn, these tickets, according to their order of exit from the first urn, and by agitating next this second urn, in order to well mix the tickets. Then the order according to which we have placed the tickets in the first urn, will have extremely little influence on the extraction of the first ticket which will exit from the second urn. We would diminish further this influence, by considering in the same manner a third urn, a fourth, etc.

Let us consider two players A and B playing together, in a manner that at each trial, the one who loses, gives a token to his adversary, and that the game endures until one of them has won all the tokens of the other. Let p and q be their respective skills; a and b their [406] numbers of tokens at commencement. There results from formula (H) of §10, by making i infinity, that the probability of A, to win the game, is

$$\frac{p^b(p^a - q^a)}{p^{a+b} - q^{a+b}}.$$

If we make in this expression,

$$p = \frac{1 \pm \alpha}{2}, \quad q = \frac{1 \mp \alpha}{2},$$

we will have, by taking the superior sign, the probability relative to the case where A is stronger than B; and by taking the inferior sign, we will have the probability relative to the case where A is less strong than B. If we are ignorant of who is the strongest of the players, the half-sum of these two probabilities will be the probability of A, that we find thus equal to

$$\frac{\frac{1}{2}\left[(1+\alpha)^a-(1-\alpha)^a\right]\left[(1+\alpha)^b+(1-\alpha)^b\right]}{(1+\alpha)^{a+b}-(1-\alpha)^{a+b}}; \tag{3}$$

by changing a into b, and reciprocally, we will have the probability of B. If we suppose α infinitely small or null; these probabilities become $\frac{a}{a+b}$ and $\frac{b}{a+b}$; they are therefore proportionals to the numbers of tokens of the players; thus for equality of the game, their stakes must be in this ratio. But then the inequality which is able to exist between them, is favorable to the player who has the smallest number of tokens; because if we suppose a less than b, it is easy to see that expression (3) is greater than $\frac{a}{a+b}$. If the players agree to double, to triple, etc. their tokens; the advantage of A increases without ceasing, and in the case of a and b infinite, its probability becomes $\frac{1}{2}$ or the same as that of B.

P being the probability of an event composed of two simple events of which p and $1-p$ are the respective probabilities; if we suppose that the value of p is susceptible of an unknown inequality z which is able to be extended from $-\alpha$ to $+\alpha$; by naming ϕ the probability of $p+z$, ϕ being a function of z; we will have for the true probability of the [407] composite event,

$$\frac{\int P'\phi\,dz}{\int \phi\,dz},$$

P' being that which P becomes when we change p into $p+z$, and the integrals being taken from $z=-\alpha$ to $z=\alpha$.

If we have no other data in order to determine z, but one observed event, formed from the same simple events; by naming Q the probability of this event, $p+z$ and $1-p-z$ being the probabilities of the simple events; the preceding expression gives, by changing ϕ into Q, for the probability of the composite event,

$$\frac{\int P'Q\,dz}{\int Q\,dz},$$

the integrals being taken here from $z=-p$ to $z=1-p$; that which is conformed to that which we have found in the preceding chapter.