BOOK II
CHAPTER VIII
DES DURÉES MOYENNES DE LA VIE, DES MARIAGES ET DES ASSOCIATIONS QUELCONQUE

Pierre Simon Laplace*

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ON THE MEAN DURATION OF LIFE, OF MARRIAGES AND OF ANY ASSOCIATIONS WHATSOEVER

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. July 26, 2021

§35. Let us suppose that we have followed with respect to a very great number n of infants, the law of mortality, from their birth to their total extinction; we will have their mean life, by making a sum of the durations of all their lives, and by dividing it by the number n. If this number were infinite, we would have exactly the duration of mean life. Let us seek the probability that the mean life of n infants, will deviate from this one, only within some given limits. [408]

Let us designate by $\phi\left(\frac{x}{a}\right)$, the probability to die at age x, a being the limit of x; a and x being supposed to contain an infinite number of parts taken for unity. We will consider the power

$$\left\{\begin{aligned}\phi\left(\frac{0}{a}\right)+\phi\left(\frac{1}{a}\right)c^{-\varpi\sqrt{-1}}+\phi\left(\frac{2}{a}\right)c^{-2\varpi\sqrt{-1}}\cdots+\phi\left(\frac{x}{a}\right)c^{-x\varpi\sqrt{-1}}\\ \cdots+\phi\left(\frac{a}{a}\right)c^{-a\varpi\sqrt{-1}}\end{aligned}\right\}^n,$$

it is clear that the coefficient of $c^{-(l+n\mu)\varpi\sqrt{-1}}$, in the development of this power, is the probability that the sum of the ages to which the n infants will arrive, will be $l+n\mu$; by multiplying therefore by $c^{(l+n\mu)\varpi\sqrt{-1}}$ the preceding power, the term independent of the powers of $c^{\pm\varpi\sqrt{-1}}$ in the product, will be this probability which consequently, is equal to

$$\frac{1}{2\pi}\int d\varpi c^{l\varpi\sqrt{-1}}\left\{c^{\mu\varpi\sqrt{-1}}\left[\begin{aligned}\phi\left(\frac{0}{a}\right)+\phi\left(\frac{1}{a}\right)c^{-\varpi\sqrt{-1}}\cdots+\phi\left(\frac{x}{a}\right)c^{-x\varpi\sqrt{-1}}\\ \cdots+\phi\left(\frac{a}{a}\right)c^{-a\varpi\sqrt{-1}}\end{aligned}\right]\right\}^n, \quad (1)$$

the integral being taken from $\varpi=-\pi$ to $\varpi=\pi$.

If we take in this integral, the hyperbolic logarithm of the quantity under the $\int$ sign, raised to the power n, we will have, by developing the exponentials into series, this logarithm equal to [409]

$$n\mu\varpi\sqrt{-1}+n\log\left\{\int\phi\left(\frac{x}{a}\right)-\varpi\sqrt{-1}\int x\phi\left(\frac{x}{a}\right)-\frac{\varpi^2}{2}\int x^2\phi\left(\frac{x}{a}\right)+\text{etc.}\right\}; \quad (2)$$

the $\int$ sign referring here to all the values of x, from $x=0$ to $x=a$. If we make $\frac{x}{a}=x'$, and if we observe that the variation of x being unity, we have $adx'=1$; we will have

$$\begin{gathered}\int\phi\left(\frac{x}{a}\right)=a\int dx'\phi(x'),\\ \int x\phi\left(\frac{x}{a}\right)=a^2\int x'dx'\phi(x'),\\ \int x^2\phi\left(\frac{x}{a}\right)=a^3\int x'^2dx'\phi(x'),\\ \text{etc.,}\end{gathered}$$

the integrals relative to x' being taken from $x'=0$ to $x'=1$. Let us name k, k', k'', etc. these successive integrals; the probability that the duration of life of an infant, will be comprehended within the limits zero and a, is $\int \phi\left(\frac{x}{a}\right)$ or $a\int dx'\phi(x')$; now this probability is certitude itself; we have therefore $ak = 1$. This premised, the function (2) becomes

$$n\mu\varpi\sqrt{-1} + n\log\left(1 - \frac{k'}{k}a\varpi\sqrt{-1} - \frac{k''}{k}\cdot\frac{a^2\varpi^2}{2} + \text{etc.}\right)$$

or

$$\left(\frac{n\mu}{a} - \frac{nk'}{k}\right)a\varpi\sqrt{-1} - n\frac{(kk'' - k'^2)}{2k^2}\cdot a^2\varpi^2 - \text{etc.}$$

If we make

$$\mu = \frac{ak'}{k} = \frac{a^2k'}{ak} = a^2k',$$

the first power of ϖ disappears; and moreover, n being supposed a very great number, we can be arrested at the second power of ϖ; the function (1) becomes thus, by passing again from the logarithms to the numbers,

$$\frac{1}{2\pi}\int d\varpi\, c^{l\varpi\sqrt{-1} - n\frac{(kk''-k'^2)}{2k^2}\cdot a^2\varpi^2}.$$

If we make [410]

$$\beta^2 = \frac{k^2}{2(kk'' - k'^2)}, \quad t = \frac{a\varpi\sqrt{n}}{2\beta} - \frac{\beta l\sqrt{-1}}{a\sqrt{n}},$$

this integral becomes, by taking it from $t = -\infty$ to $t = \infty$,

$$\frac{\beta}{a\sqrt{n\pi}}c^{-\frac{\beta^2 l^2}{a^2 n}}.$$

By multiplying it by dl, and making $l = ar\sqrt{n}$, we will have

$$\frac{2}{\sqrt{\pi}}\int \beta dr\, c^{-\beta^2 r^2}$$

for the probability that the sum of the ages to which the n infants will arrive, will be comprehended within the limits $na^2k' \pm ar\sqrt{n}$.

The quantity a^2k' or $\int x\phi\left(\frac{x}{a}\right)$ is the sum of the products of each age, by the probability of arriving there; it is therefore the true duration of mean life; thus the probability that the sum of the ages to which the n infants cease to live, divided by their number, is comprehended within these limits

True duration of mean life, more or less $\frac{ar}{\sqrt{n}}$, has for expression

$$\frac{2}{\sqrt{\pi}}\int \beta dr\, c^{-\beta^2 r^2}.$$

The mean value of r, positive or negative, is by §20,

$$\pm\frac{1}{\sqrt{\pi}}\int \beta r\, dr\, c^{-\beta^2 r^2},$$

the integral being taken from $r = 0$ to r infinity. By multiplying it by $\frac{a}{\sqrt{n}}$, we will have the mean error to fear positive or negative, when we take for mean duration of life, the sum of the ages that the n infants considered above have lived, divided by n, a quotient that we will designate by G; this error is therefore [411]

$$\pm\frac{a}{2\beta\sqrt{n\pi}}.$$

We have very nearly,

$$a^2k' = G;$$

and as $ak = 1$, we will have

$$\frac{k'}{k} = \frac{G}{a}.$$

If we name next H, the sum of the squares of the ages that the n infants have lived, divided by n; we will find, by the analysis of §19,

$$\frac{k''}{k}a^2 = H;$$

these values give

$$\beta^2 = \frac{a^2}{2(H - G^2)}:$$

the mean error to fear positive or negative, with respect to the duration of life, becomes thus

$$\pm\frac{\sqrt{H - G^2}}{\sqrt{2n\pi}}.$$

It is clear that these results hold equally relatively to the mean duration of that which remains to live, when instead of departing from the epoch of birth, we depart from any epoch of life.

We are able to determine easily, by means of the tables of mortality, formed from year to year, the mean duration of that which remains to live to a person of whom the age is an entire number of years A. For that, we will add all the numbers of the table which follow the one which corresponds to age A; we will divide the sum by this last number, and we will add $\frac{1}{2}$ to the quotient. In fact, if we designate by (1), (2), (3), etc., the numbers of the table, corresponding to the year A and to the following years; the number of individuals who die in the first year, in departing from year A, will be (1)$-$(2); but in this short interval, the mortality is able to be supposed constant; $\frac{1}{2}\left[(1) - (2)\right]$ is therefore the sum of the durations of their life, in departing from age A. Similarly $\frac{3}{2}\left[(2) - (3)\right]$, $\frac{5}{2}\left[(3) - (4)\right]$, etc. are the sums of the durations of life, by departing from the same age, of those who [412]
die in the second, third, etc. years counted from year A. The reunion of all these sums is $\frac{(1)}{2}$+(2)+(3)+(4)+etc.; and by dividing it by (1), we will have the mean duration of that which remains to live to the person of age A. We will form thus a table of the mean durations of that which remains to live at the different ages. We will be able likewise to

conclude these durations from one another, by observing that if F designates this duration for age A, and F' the corresponding duration at age $A+1$, we have

$$F = \frac{(2)}{(1)}\left(F' + \frac{1}{2}\right) + \frac{1}{2}.$$

§36. Let us determine now the mean duration of life, which would hold, if one of the causes of mortality were to be extinguished. Let U be the number of infants who out of the number n of births, would survive yet to the age x under this hypothesis, u being the one of the infants living to this age out of the same number of births, in the case where that cause of mortality subsists. Let us name $z\triangle x$, the probability that one individual of age x, will perish of this malady in the very short interval of time $\triangle x$; $uz\triangle x$ will be very nearly by §25, the number of individuals u, who will perish of this malady in the interval of time $\triangle x$, if this number is considerable. Similarly if we designate by $\phi\triangle x$ the probability that one individual of age x will perish by the other causes of mortality in the interval $\triangle x$; $u\phi\triangle x$ will be the number of individuals who will perish by these causes, in the interval of time $\triangle x$; this will be therefore the value of $-\triangle u$; I affect $\triangle u$ with the $-$ sign, because u diminishes in measure as x increases; we have therefore

$$-\triangle u = u\triangle x(\phi + z).$$

We will have similarly

$$-\triangle U = U\phi\triangle x.$$

By eliminating ϕ from these two equations, we will have

$$\frac{\triangle U}{U} = \frac{\triangle u}{u} + z\triangle x.$$

$\triangle x$ being a very small quantity, we can transform the characteristic $\triangle$ into the differential [413]
characteristic d, and then the preceding equation becomes

$$\frac{dU}{U} = \frac{du}{u} + z\,dx;$$

whence we deduce by integrating, and observing that at age zero $U = u = n$,

$$U = uc^{\int z\,dx}, \tag{3}$$

the integral being taken from x null. We can obtain this integral, by means of the registers of mortality, in which we take account of the age of the dead individuals, and of the causes of their death. In fact, $uz\triangle x$ being by that which precedes, the number of those who, arrived to age x, have perished in the interval of time $\triangle x$, by the malady of which there is concern; we will have very nearly the integral $\int z\,dx$, by supposing $\triangle x$ equal to one year, and by taking from the birth of the n infants that we have considered, until the year x, the sum of the fractions which have for numerator the number of individuals who the malady has made perish each year, and for denominator, the number of the n infants who survive yet to the middle of the same year. Thus we will be able to transform by means of equation (3), a table of ordinary mortality, into that which would hold, if the malady of which there is concern, did not exist.

Smallpox has this in particular, namely, that the same individual is never twice attacked, or at least this case is so rare, that, if it exists, we are able to set it aside. Let us imagine that out of a very great number n of infants, u arrive to age x, and that in the number u, y have not had smallpox at all. Let us imagine further that out of this number y, $iy\,dx$ take this malady in the instant dx, and that out of this number, $iry\,dx$ perish from this malady. By designating, as above, by ϕ the probability to perish at age x, by some other causes; we will have evidently

$$du = -u\phi\,dx - iry\,dx.$$

We will have next

$$dy = -y\phi\,dx - iy\,dx.$$

In fact, y diminishes by the number of those who, in the instant dx, take smallpox, and [414]
this number is by the supposition, $iy\,dx$. y diminishes further by the number of individuals comprehended in y, who perish by some other causes, and this number is $y\phi\,dx$.

Now, if from the first of the two preceding equations, multiplied by y, we subtract the second multiplied by u, and if we divide the difference by y^2, we will have

$$d\frac{u}{y} = i\frac{u}{y}\,dx - ir\,dx;$$

that which gives, by integrating from x null, and observing that at this origin, $u = y = n$,

$$\frac{u}{y} = \left(1 - \int ir\,dx\,c^{-\int i\,dx}\right)c^{\int i\,dx}; \tag{4}$$

this equation will make known the number of individuals of age x, who have not at all yet had smallpox. We have next

$$z\,dx = \frac{iry\,dx}{u},$$

$nz\,dx$ being, as above, those who perish in the time dx, of the malady that we consider. By substituting instead of $\frac{y}{n}$, its preceding value; we will have, after having integrated,

$$c^{\int z\,dx} = \frac{1}{1 - \int ir\,dx\,c^{-\int i\,dx}};$$

equation (3) will give therefore

$$U = \frac{u}{1 - \int ir\,dx\,c^{-\int i\,dx}}. \tag{5}$$

This value of U supposes that we knew by observation i and r. If these numbers were constants, it would be easy to determine them; but as they are able to vary from age to age, the elements of formula (3) are easier to know, and this formula seems to me more proper to determine the law of mortality which would hold, if smallpox was extinct. By applying to it the data that we have been able to procure with respect to the mortality caused by this malady, at the diverse ages of life; we find that its extinction by means of the vaccine, [415] would increase more than three years, the duration of mean life, if besides this duration was not at all restrained by the diminution related to the subsistances, due to a greater increase of population.

§37. Let us consider presently the mean duration of marriages. For that let us imagine a great number n of marriages among n young men of age a, and n young women of age a'; and let us determine the number of these marriages subsisting after x years elapsed from their origin. Let us name ϕ the probability that a young man who is married at age a, will arrive to age $a+x$; and ψ the probability that a young woman who is married at age a', will arrive to age $a'+x$. The probability that their marriage will subsist after its x^{th} year, will be $\phi\psi$; therefore if we develop the binomial $(\phi\psi+\overline{1-\phi\psi})^n$, the term $H(\phi\psi)^i(1-\phi\psi)^{n-i}$ of this development, will express the probability that out of n marriages, i will subsist after x years. The greatest term of the development is, by §16, the one in which i is equal to the greatest whole number contained in $\overline{n+1}.\phi\psi$; and, by the same section, it is extremely probable that the number of the marriages subsisting will deviate only very little positive or negative from this number. Thus, by designating by i, the number of subsisting marriages, we will be able to suppose very nearly,

$$i = n\phi\psi.$$

$n\phi$ is quite near the number of the n husbands surviving to the age $a+x$. The tables of mortality will make it known in a quite close manner, if they have been formed out of the numerous lists of mortality; because if we designate by p' the number of men surviving to age a, out of the collection of these lists, and by q' the number of the surviving to age $a+x$, we will have quite nearly, by §29,

$$n\phi = \frac{nq'}{p'}.$$

If we name similarly p'' the number of women surviving to age a' and by q'' the number of the survivors to age $a'+x$, we will have very nearly, [416]

$$n\psi = \frac{nq''}{p''};$$

therefore

$$i = \frac{nq'q''}{p'p''}.$$

We will form thus from year to year, a table of values of i. By making next a sum of all the numbers of this table, and by dividing it by n; we will have the mean duration of the marriages made at age a for the young men, and at the age a' for the young women.

Let us seek now the probability that the error of the preceding value of i, will be comprehended within some given limits. Let us suppose in order to simplify the calculation, that the two spouses are of the same age, and that the probability of the life of the men is the same as that of the women; then we have

$$a' = a, \quad q'' = q', \quad p'' = p', \quad \phi = \psi;$$

and the preceding expression of i becomes

$$i = \frac{nq'^2}{p'^2}.$$

Let us imagine that the value of i is $\frac{nq'^2}{p'^2} + s$; s will be the error of this expression of i. We have seen in §30, that if we have observed that out of a very great number p of individuals of age a, q are arrived to the age $a+x$; the probability that out of p' other individuals of the age a, $\frac{p'q}{p} + z$ will arrive to the age $a+x$, is

$$\sqrt{\frac{p^3}{2qp'(p-q)(p+p')\pi}}\, c^{-\frac{p^3z^2}{2qp'(p-q)(p+p')}}.$$

If we suppose p and q infinite, we will have evidently

$$\phi = \frac{q}{p},$$

and if we make

$$\frac{p'q}{p} + z = q';$$

we will have [417]

$$\phi = \frac{q'}{p'} - \frac{z}{p'};$$

that which gives very nearly, by neglecting the square $\frac{nz^2}{p'^2}$,

$$n\phi^2 = \frac{nq'^2}{p'^2} - \frac{2nq'z}{p'^2};$$

thus the preceding probability of z, is at the same time the probability of this expression of $n\phi^2$. Let us suppose now $i = n\phi^2 + l$; by considering the binomial $(\phi^2 + \overline{1-\phi^2})^n$, the probability of this expression of i is by §16,

$$\frac{1}{\sqrt{\pi.2n\phi^2(1-\phi^2)}}\, c^{-\frac{l^2}{2n\phi^2(1-\phi^2)}}.$$

But the preceding value of i becomes, by substituting for $n\phi^2$ its value,

$$i = \frac{nq'^2}{p'^2} - \frac{2nq'z}{p'^2} + l;$$

the probability of this last expression of i is equal to the product of those of i and of z, found above; it is therefore equal to

$$\frac{c^{-\frac{z^2}{2p'\phi(1-\phi)}-\frac{l^2}{2n\phi^2(1-\phi^2)}}}{2\pi\sqrt{np'\phi^3(1-\phi)^2(1+\phi)}}.$$

Having supposed previously $i = \frac{nq'^2}{p'^2} + s$, we will have $s = l - \frac{2nq'z}{p'^2}$; by substituting therefore for l its value deduced from this equation, and observing that we have very nearly $\frac{q'}{p'} = \phi$; we will have for the probability that the value of s will be comprehended within some given limits, the integral expression

$$\frac{\iint dz\, ds\, c^{-\frac{z^2}{2\phi(1-\phi)p'}-\frac{\left(s+\frac{2nq'z}{p'^2}\right)^2}{2n\phi^2(1-\phi^2)}}}{2\pi\sqrt{np'\phi^3(1-\phi)^2(1+\phi)}},$$

the integral relative to z being able to be taken from $z = -\infty$ to $z = \infty$. Thence it is easy [418] to conclude by the methods exposed previously, that if we make

$$k^2 = \frac{p'}{2n\phi^2(1-\phi)[p' + (p' + 4n)\phi]};$$

the preceding integral becomes

$$\int \frac{k\, ds}{\sqrt{\pi}} c^{-k^2 s^2};$$

thus the probability that the error of the expression $i = \frac{nq'^2}{p'^2}$ will be $\pm s$, is

$$\frac{2}{\sqrt{\pi}} \int k\, ds\, c^{-k^2 s^2},$$

the integral being taken from s null.

The preceding analysis is applied equally to the mean duration of a great number of associations formed of three individuals, or of four individuals, etc. Let n be this number, and let us suppose that all the associates are of the same age a at the moment of association; let us designate by p the number of individuals from the table of mortality, of the age a, and by q the number of individuals of the age $a + x$; the number i of the associations existing after x years elapsed from the origin of the associations, will be quite nearly

$$i = \frac{nq^r}{p^r},$$

r being the number of individuals of each association. We will find by the same analysis, the probability that this number will be contained within the given limits. The sum of the values of i corresponding to all the values of x, divided by n, will be the mean duration of this kind of associations.