BOOK
CHAPTER IX
DES BÉNÉFICES DÉPENDANTS DE LA PROBABILITÉ DES ÉVÉNEMENTS
FUTURS.

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ON THE BENEFITS DEPENDING ON THE PROBABILITY OF FUTURE EVENTS.

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§38. Let us imagine that the arrival of an event procures the benefit ν, and that its non- arrival causes the loss μ. A person A awaits the arrival of a number s of similar events, all equally probable, but independent of one another; we demand what is his advantage.

Let q be the probability of the arrival of each event, and consequently $1 - q$ that of its non-arrival; if we develop the binomial $(q + \overline{1 - q})^s$, the term

$$\frac{1.2.3\ldots s}{1.2.3\ldots i.1.2.3\ldots(s-i)}q^i(1-q)^{s-i}$$

of this development, will be the probability that out of s events, i will arrive. In this case, the benefit of A is $i\nu$, and his loss is $(s-i)\mu$; the difference is $i(\nu+\mu) - s\mu$; by multiplying it by its probability expressed by the preceding term, and taking the sum of these products for all the values of i, we will have the advantage of A, which, consequently is equal to

$$-s\mu(q + \overline{1 - q})^s + (\nu + \mu)S\frac{1.2.3\ldots s}{1.2.3\ldots i.1.2.3\ldots \overline{s - i}}q^i(1-q)^{s-i},$$

the sign S extending to all the values of i. We have

$$S\frac{i.1.2.3\ldots s}{1.2.3\ldots i.1.2.3\ldots \overline{s - i}}q^i(1-q)^{s-i}$$
$$= \frac{d}{dt}S\frac{1.2.3\ldots s}{1.2.3\ldots i.1.2.3\ldots \overline{s - i}}q^i t^i(1-q)^{s-i} = \frac{d}{dt}(qt + \overline{1 - q})^s,$$

provided that we suppose $t = 1$, after the differentiation, that which reduces this last mem- ber to qs; the advantage of A is therefore $s(q\nu - \overline{1 - q}.\mu)$. This advantage is null, if $\nu q = \mu(1 - q)$; that is, if the benefit of the arrival of the event, multiplied by its probability, is equal to the loss caused by its non-arrival, multiplied by its probability. The advantage becomes negative, and is changed into disadvantage, if the second product surpasses the first. In all cases, the advantage or the disadvantage of A is proportional to the number s of the events.

We will determine by the analysis of §16, the probability that the real benefit of A will be comprehended within the given limits, if s is a large number. Following this analysis, the sum of the diverse terms of the binomial $(q + \overline{1 - q})^s$ comprehended between the two terms distant by $l + 1$, on both sides of the greatest, is

$$\frac{2}{\sqrt{\pi}}\int dt\, c^{-t^2}\frac{1}{\sqrt{2s\pi q(1-q)}}c^{-\frac{i^2}{2sq(1-q)}},$$

the integral being taken from $t = 0$ to $t = \frac{l}{\sqrt{2sq(1-q)}}$. The exponent of q in the greatest term is very nearly by the same section, equal to sq; and the exponents of q, corresponding to the extreme terms comprehended within the preceding interval, are respectively $sq - l$ and $sq + l$. The benefits corresponding to these three terms are

$$s(q\nu - \overline{1 - q}.\mu) - l(\nu + \mu),$$
$$s(q\nu - \overline{1 - q}.\mu),$$
$$s(q\nu - \overline{1 - q}.\mu) + l(\nu + \mu);$$

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by making therefore $l = r\sqrt{s}$, the probability that the real benefit of A will not exceed the limits $s(q\nu - \overline{1-q}.\mu) \pm r\sqrt{s}(\nu + \mu)$ is equal to

$$\frac{2}{\sqrt{\pi}} \frac{\int dr\, c^{-\frac{r^2}{2q(1-q)}}}{\sqrt{2q(1-q)}} + \frac{1}{\sqrt{2s\pi q(1-q)}} c^{-\frac{r^2}{2q(1-q)}},$$

the integral being taken from $r = 0$, and the last term being able to be neglected. We see by this formula that if $q\nu - \overline{1-q}.\mu$ is not null, the real benefit increases without ceasing, [421] and becomes infinitely great and certain, in the case of an infinite number of events.

We are able to extend by the following analysis, this result, to the case where the probability of the s events are different, in the same way as the benefits and the losses which are attached. Let q, $q^{(1)}$, $q^{(2)}$, ..., $q^{(s-1)}$ be the respective probabilities of these events; ν, $\nu^{(1)}$, $\nu^{(2)}$,..., $\nu^{(s-1)}$ the benefits which their arrivals procure. We can, in order to simplify, set aside the losses which their non-arrivals cause, by comprehending in the benefit which the arrival of each event procures, the quantity that A would lose by its non-arrival, and by subtracting next from the total advantage of A, the sum of these last quantities; because it is easy to see that that changes not at all the position of A.

This premised, let us consider the product

$$\left(1 - q + qc^{\nu\varpi\sqrt{-1}}\right)\left(1 - q^{(1)} + q^{(1)}c^{\nu^{(1)}\varpi\sqrt{-1}}\right)\cdots$$
$$\cdots\left(1 - q^{(s-1)} + q^{(s-1)}c^{\nu^{(s-1)}\varpi\sqrt{-1}}\right).$$

It is clear that the probability that the sum of the benefits will be $f + l'$, is equal to the coefficient of $c^{(f+l')\varpi\sqrt{-1}}$ in the development of this product; it is therefore equal to

$$\frac{1}{2\pi}\int d\varpi\, c^{-(f+l')\varpi\sqrt{-1}}\left(1 - q + qc^{\nu\varpi\sqrt{-1}}\right)\cdots \qquad (a)$$
$$\cdots\left(1 - q^{(s-1)} + q^{(s-1)}c^{\nu^{(s-1)}\varpi\sqrt{-1}}\right),$$

the integral being taken from $\varpi = -\pi$ to $\varpi = \pi$, and the numbers ν, $\nu^{(1)}$, etc. being supposed, as we can make it, whole numbers. Let us take the logarithm of the product

$$c^{-f\varpi\sqrt{-1}}\left(1 - q + qc^{\nu\varpi\sqrt{-1}}\right)\cdots\left(1 - q^{(s-1)} + q^{(s-1)}c^{\nu^{(s-1)}\varpi\sqrt{-1}}\right); \qquad (b)$$

by developing it according to the powers of ϖ, it becomes

$$(Sq^{(i)}\nu^{(i)} - f)\varpi\sqrt{-1} - \frac{\varpi^2}{2}Sq^{(i)}(1 - q^{(i)})\nu^{(i)2} - \text{etc.}$$

the sign S corresponding to all the values of i, from $i = 0$ to $i = s - 1$. The supposition of f equal to $Sq^{(i)}\nu^{(i)}$ makes the first power of ϖ disappear; and the size of s, a very [422] great number, renders insensible the terms depending on the powers of ϖ, superior to the

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square; by passing again therefore from the logarithms to the numbers in the preceding development, the product (b) becomes very nearly

$$c^{-\frac{\varpi^2}{2} Sq^{(i)}(1-q^{(i)})\nu^{(i)2}};$$

that which changes the integral (a) into this one

$$\frac{1}{2\pi} \int d\varpi \, c^{-l'\varpi\sqrt{-1}-\frac{\varpi^2}{2} Sq^{(i)}(1-q^{(i)})\nu^{(i)2}}.$$

The integral must be taken from $\varpi = -\pi$ to $\varpi = \pi$, and $Sq^{(i)}(1-q^{(i)})\nu^{(i)2}$ being a great number of order s; it is clear that this integral is able to be extended without sensible error, to the positive and negative infinite values of ϖ. By making therefore

$$\varpi\sqrt{\frac{Sq^{(i)}(1-q^{(i)})\nu^{(i)2}}{2}} + \frac{l'\sqrt{-1}}{\sqrt{2Sq^{(i)}(1-q^{(i)})\nu^{(i)2}}} = t,$$

and integrating from $t = -\infty$ to $t = \infty$, the integral (a) becomes

$$\frac{1}{\sqrt{2\pi Sq^{(i)}(1-q^{(i)})\nu^{(i)2}}} c^{-\frac{l'^2}{2Sq^{(i)}(1-q^{(i)})\nu^{(i)2}}}.$$

If we multiply this quantity by $2dl'$, and if next we integrate it from $l' = 0$, this integral will be the expression of the probability that the benefit of A will be comprehended within the limits $f \pm l'$, or $Sq^{(i)}\nu^{(i)} \pm l'$; by making thus

$$l' = r\sqrt{2Sq^{(i)}(1-q^{(i)})\nu^{(i)2}},$$

the probability that the benefit of A will be comprehended within the limits

$$Sq^{(i)}\nu^{(i)} \pm r\sqrt{2Sq^{(i)}(1-q^{(i)})\nu^{(i)2}},$$

is

$$\frac{2}{\sqrt{\pi}} \int dr \, c^{-r^2},$$

the integral being taken from $r = 0$.

Now it is necessary, by that which precedes, to change within the preceding limits, $\nu^{(i)}$ [423] into $\nu^{(i)} + \mu^{(i)}$, and to subtract $S\mu^{(i)}$ from it; the probability that the real benefit of A will be comprehended within the limits

$$S[q^{(i)}\nu^{(i)} - (1-q^{(i)})\mu^{(i)}] \pm r\sqrt{2Sq^{(i)}(1-q^{(i)})(\nu^{(i)} + \mu^{(i)})^2}$$

is therefore

$$\frac{2}{\sqrt{\pi}} \int dr \, c^{-r^2}.$$

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We see by this formula, that as little as the mathematical expectation of each event, surpasses zero; by multiplying the events to infinity, the first term of the expression of the limits being of order s, while the second is only of order $\sqrt{s}$, the real benefit is increased without ceasing, and becomes at the same time infinitely great and certain, in the case of an infinite number of events.

§39. Let us consider now the case where, at each event, the person A has any number of chances to hope or to fear. Let us suppose, for example, that an urn contains some balls of diverse colors; that we draw a ball from this urn, by replacing it into the urn after the drawing, and that the benefit of A is ν, if the extracted ball is of the first color; that it is ν', if the extracted ball is of the second color, that it is ν'' if the extracted ball is of the third color, and so forth; the benefits becoming negative, when A is forced to give instead of receive. Let us name a, a', a'', etc. the probabilities that the ball extracted at each drawing, will be of the first, or of the second, or of the third, etc. color, and let us suppose that we have thus s drawings; we will have first

$$a + a' + a'' + \text{etc.} = 1.$$

By multiplying next the terms of the first member of this equation, respectively by $c^{\nu\varpi\sqrt{-1}}$, $c^{\nu'\varpi\sqrt{-1}}$, $c^{\nu''\varpi\sqrt{-1}}$, etc., the term independent of the powers of $c^{\varpi\sqrt{-1}}$, in the development of the function

$$c^{-(l+s\mu)\varpi\sqrt{-1}} \left[ac^{\nu\varpi\sqrt{-1}} + a'c^{\nu'\varpi\sqrt{-1}} + a''c^{\nu''\varpi\sqrt{-1}} + \text{etc.} \right]^{s},$$

will be, by that which precedes, the probability that in s drawings, the benefit of A will be   [424] $s\mu + l$; this probability is therefore equal to

$$\frac{1}{2\pi} \int d\varpi\, c^{-l\varpi\sqrt{-1}} \left[c^{-\mu\varpi\sqrt{-1}} \left(ac^{\nu\varpi\sqrt{-1}} + a'c^{\nu'\varpi\sqrt{-1}} + \text{etc.} \right) \right]^{s}; \qquad (c)$$

the integral relative to ϖ being taken from $\varpi = -\pi$ to $\varpi = \pi$. If we develop with respect to the powers of ϖ, the hyperbolic logarithm of the quantity raised to the power s, under the $\int$ sign, and if we observe that $a + a' + a'' + \text{etc.} = 1$, we will have for this logarithm,

$$\varpi\sqrt{-1}[a\nu + a'\nu' + a''\nu'' + \text{etc.} - \mu]$$
$$- \frac{\varpi^2}{2} \left[\begin{array}{c} a\nu^2 + a'\nu'^2 + a''\nu''^2 + \text{etc.} \\ -(a\nu + a'\nu' + a''\nu'' + \text{etc.})^2 \end{array} \right] - \text{etc.}$$

We will make the first power of ϖ disappear, by making

$$\mu = a\nu + a'\nu' + a''\nu'' + \text{etc.};$$

If we suppose next

$$2k^2 = a\nu^2 + a'\nu'^2 + a''\nu''^2 + \text{etc.} - (a\nu + a'\nu' + a''\nu'' + \text{etc.})^2,$$

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and if we observe that s being supposed a great number, we can neglect the powers of ϖ superior to the square; we will have, by passing again from the logarithms to the numbers,

$$\left[c^{-\mu\varpi\sqrt{-1}} \left(ac^{\nu\varpi\sqrt{-1}} + a'c^{\nu'\varpi\sqrt{-1}} + a''c^{\nu''\varpi\sqrt{-1}} + \text{etc.} \right) \right]^s = c^{-sk^2\varpi^2};$$

that which changes the integral (c) into this one,

$$\frac{1}{2\pi} \int d\varpi \, c^{-l\varpi\sqrt{-1}-sk^2\varpi^2},$$

which becomes by integrating as in the preceding section,

$$\frac{1}{2k\sqrt{s\pi}} c^{-\frac{l^2}{4sk^2}}.$$

By multiplying it by $2dl$, and integrating the product from $l = 0$, we will have the probability that the real benefit of A, will be comprehended within the limits [425]

$$s(a\nu + a'\nu' + a''\nu'' + \cdots) \pm l;$$

by making therefore
$$l = 2kr'\sqrt{s},$$

this probability will be, by taking the integral from $r' = 0$,

$$\frac{2}{\sqrt{\pi}} \int dr' \, c^{-r'^2}.$$

We have supposed in that which precedes, the probabilities of the events, known; let us examine the case where they are unknown. Let us suppose that out of m similar events awaited, n have arrived, and that A awaits s similar events of which each procures to him by its arrival, the benefit ν, the non-arrival causing to him the loss μ. If we represent by $\frac{n}{m}s + z$, the number of events which will arrive out of the s events awaited, the probability that z will be contained within the limits $\pm kt$ will by §30,

$$\frac{2}{\sqrt{\pi}} \int dt \, c^{-t^2},$$

the integral being taken from $t = 0$, k^2 being equal to

$$\frac{2ns(m-n)(m+s)}{m^3}.$$

But $\frac{n}{m}s + z$ being the number of arrived events, the real benefit of A is

$$\left(\frac{n\nu}{m} - \frac{(m-n)\mu}{m} \right) s + z(\nu + \mu);$$

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the preceding integral is therefore the probability that the real benefit of A will be comprehended within the limits

$$\left(\frac{n\nu}{m} - \frac{(m-n)\mu}{m}\right) s \pm kt(\nu + \mu).$$

k is of order $\sqrt{s}$, if m and n are of order equal or greater than s; thus howsoever small that the mathematical expectation be, relative to each event, the real benefit becomes at infinity, certain and infinitely great, when the number of past events, is supposed infinite, as the one of future events.

§40. We will now determine the benefits of the establishments founded on the proba- [426] bilities of human life. The most simple way to calculate these benefits, is to reduce them to actual capitals. Let us take for example, life pensions. A person of age A wishes to constitute on his head, a life pension h; we demand the capital that he must for it, give to the funds of the establishment which makes this pension for him.

If we name y_0 the number of individuals of age A, in the table of mortality of which we make use, and y_x the number of individuals of age $A + x$; the probability to pay the pension at the end of the year $A + x$, will be $\frac{y_x}{y_0}$; consequently, the value of the payment will be $\frac{hy_x}{y_0}$. But if we designate by r the annual interest on unity, so that the capital 1 becomes $1 + r$ after one year; it will become $(1+r)^x$ after x years; thus, the payment $(1+r)^x$ made at the end of the xth year, reduced to actual capital, becomes unity, or this same payment divided by $(1+r)^x$; the payment $\frac{hy_x}{y_0}$ reduced to actual capital, is therefore $\frac{hy_x}{y_0(1+r)^x}$. The sum of all the payments made during the duration of life of the person who constitutes the pension, and multiplied by their probability, is equivalent therefore to an actual capital represented by the finite integral

$$\sum \frac{hy_x}{y_0(1+r)^x},$$

the characteristic Σ needing to embrace all the values of the function that it affects.

We are able to determine this integral by forming all these values according to the table of mortality, and by adding them together: we will deduce next the capitals from one another, by observing that if we name F the capital relative to the age A, and F' the capital relative to age $A + 1$, we have

$$F = \frac{y_1}{y_0} \cdot \frac{F' + h}{1 + r}.$$

But this process is simplified, when the law of mortality is known, and especially when it is given by a rational and entire function of x, that which is always possible, by considering the numbers of the table of mortality, as ordinates of which the corresponding ages are [427] the abscissas, and by making a parabolic curve pass through the extremities of the two extreme ordinates, and of many intermediate ordinates. The differences which exist among the diverse tables of mortality, permit regarding this mean, as exact also as these tables, and even to be satisfied with a small number of ordinates.

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Let us make

$$\frac{1}{1+r} = p, \quad \frac{y_x}{y_0} = u;$$

let us resume formula (16) of §11 of the first Book, which gives

$$\sum p^x u = \frac{p^x}{pc^{\frac{du}{dx}} - 1} + f,$$

f being an arbitrary constant. It is necessary in the development of the first term of the second member of this equation, with respect to the ratio to the powers of $\frac{du}{dx}$, to change any power $\left(\frac{du}{dx}\right)^i$ into $\frac{d^i u}{dx^i}$, and to multiply by u, the first term, which is independent of $\frac{du}{dx}$. We have thus

$$\sum p^x u = f - \frac{p^x u}{1-p} - \frac{p^{x+1}\frac{du}{dx}}{(1-p)^2} - \frac{(p+1)p^{x+1}}{1.2.(1-p)^3} \cdot \frac{d^2 u}{dx^2} + \text{etc.}$$

In order to determine f, we will observe that the integral $\sum p^x u$ is null, when $x = 1$, and that it is terminated, when $x = n+1$, $A+n$ being the limit of life; because then it embraces the terms corresponding to all the numbers 1, 2, 3, ... n. Let us designate therefore by (u), $\left(\frac{du}{dx}\right)$, etc.; u', $\left(\frac{du'}{dx}\right)$, etc., the values of u, $\frac{du}{dx}$, etc. corresponding to $x = 1$, and to $x = n+1$; we will have

$$\sum \frac{hp^x y_x}{y_0} = h \left\{ \begin{array}{l} \dfrac{p}{1-p}[(u) - p^n(u')] \\[2mm] + \dfrac{p^2}{(1-p)^2}\left[\left(\dfrac{du}{dx}\right) - p^n\left(\dfrac{du'}{dx}\right)\right] \\[2mm] + \dfrac{(p+1)p^2}{1.2.(1-p)^3}\left[\left(\dfrac{d^2 u}{dx^2}\right) - p^n\left(\dfrac{d^2 u'}{dx^2}\right)\right] \\[2mm] +\text{etc.} \end{array} \right\}. \qquad (o)$$

If u or $\frac{y_x}{y_0}$ is constant and equal to unity, from $x = 1$ to $x = n$; then the life pension must [428] be paid certainly during the number n of years, and it becomes an annuity. In this case, $\frac{du}{dx}$ is null, and the preceding formula gives $\frac{hp(1-p^n)}{1-p}$ for the capital equivalent to the annuity h.

If $u = 1 - \frac{x}{n}$; then the probability of life decreases in arithmetic progression, and the preceding formula gives

$$\frac{hp}{1-p}\left(1 - \frac{1-p^n}{n(1-p)}\right)$$

for the capital equivalent to the life annuity h, and so forth.

Let us suppose now that we wish to constitute a life pension h, on many individuals of ages A, $A + a$, $A + a + a'$, etc., so that the pension remains to the survivors. Let us designate by y_x, y_{x+a}, $y_{x+a+a'}$, etc., the numbers of the table of mortality, corresponding to the ages A, $A + a$, $A + a + a'$, etc.; the probability that the first individual has, to live to the age $A + x$, being $\frac{y_x}{y_0}$; the probability that at this age, he will have ceased to live, is

$1 - \frac{y_x}{y_0}$. Similarly, the probability that the second individual has, to live to age $A + a + x$, or to the end of the xth year of the constitution of the pension, being $\frac{y_{x+a}}{y_a}$; the probability that he will have ceased to live then, is $1 - \frac{y_{x+a}}{y_a}$; the probability that the third individual will have ceased to live, at the same epoch of the constitution of the pension, is $1 - \frac{y_{x+a+a'}}{y_{a+a'}}$, and so forth. The probability that none of these individuals will exist at this epoch, is

$$\left(1 - \frac{y_x}{y_0}\right)\left(1 - \frac{y_{x+a}}{y_a}\right)\left(1 - \frac{y_{x+a+a'}}{y_{a+a'}}\right) \text{.etc.}$$

By subtracting this product, from unity; the difference will be the probability that one of these individuals at least, will be living at the end of the xth year of the constitution of the pension. Let us name u this probability; $\sum hp^x u$ will be the actual capital equivalent to the life annuity h. But we must observe by taking this integral, that the quantities y_x, y_{x+a}, etc. are nulls, when their indices x, $x + a$, etc. surpass the number n, $A + n$ being the limit of the life.   [429]

If y_x is a rational and entire function of x, and of exponentials, such as q^x, r^x, etc.; we will have easily, by the formulas of the first Book, the integral $h \sum p^x u$. But we are able in all cases, to form by means of a table of mortality, all the terms of this integral, by taking the sum, and constructing thus tables of life annuities, on one or many heads.

The preceding analysis similarly serves to determine the life annuity that one must make at an establishment, in order to assure to his heirs a capital after his death. The capital equivalent to the life annuity h, made on a person of age A, is by that which precedes, $hS\frac{p^x y_x}{y_0}$, the sign S comprehending all the terms inclusively, from $x = 1$ to the limit of the life of the person. Let us name hq this integral, and let us imagine that the establishment receives from this person the rent h, and gives to him in exchange, the capital hq. Let us imagine next that the same person places this capital at perpetual interest with the establishment itself; the annual interest of unity being r or $\frac{1-p}{p}$. It is clear that the establishment must render the capital hq, to the heirs of the person. But the person has made during his life, the rent h to the establishment, and the person has received from it the pension $\frac{hq(1-p)}{p}$; the rent that the person has made really is therefore $h\left[1 - \frac{q(1-p)}{p}\right]$; this is therefore that which the person must give annually to the establishment, in order to assure to his heirs the capital hq.

I will not insist further on these objects, in the same way with respect to those which are relative to the establishments of assurance of each kind, because they present no difficulties. I will observe only that all these establishments must, in order to prosper, reserve themselves a benefit, and multiply considerably their affairs; so that, their real benefit becoming near certain, they are exposed the least as it is possible, to some great losses which would be able to destroy them. In fact, if the number of the affairs is s, and if the advantage of the establishment in each of them, is b; then it becomes extremely probable that the real   [430] benefit of the establishment will be sb, s being supposed a very great number.

In order to show it, let us suppose that s persons of age A constitute, each on his head, a life annuity h; and let us consider one of these persons who we will designate by C. If C

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dies in the interval from the end of the year x elapsed from the constitution of his pension, to the end of the year $x + 1$; the establishment will have paid to him the pension h during x years, and the sum of these payments, reduced to actual capital, will be $h(p + p^2 + \cdots + p^x)$ or $\sum hp^{x+1}$; now the probability that C will die within this interval, is $\frac{y_x - y_{x+1}}{y_0}$, or $-\frac{\triangle y_x}{y_0}$; the value of the loss that the establishment must then support, is therefore $-\frac{\triangle y_x}{y_0} \sum hp^{x+1}$. The sum of all these losses is

$$-\sum \left(\frac{\triangle y_x}{y_0} \sum hp^{x+1} \right); \qquad (r)$$

this is the capital that C must deposit into the funds of the establishment, in order to receive from it the life pension h. We are able to observe here that we have

$$-\triangle y_x \sum p^{x+1} = -y_{x+1} \sum p^{x+1} + y_x \sum p^x + y_x p^x :$$

by integrating the second member of this equation, the function (r) is reduced to

$$-\frac{y_x}{y_0} \sum hp^x + \frac{\sum hy_x p^x}{y_0} \text{constant};$$

now $\sum p^x$ is reduced to zero, when $x = 1$, and when $x = n + 1$, y_x is null by that which precedes; the function (r), or the capital that C must pay to the establishment, is therefore $\frac{\sum hy_x p^x}{y_0}$; that which is conformed to that which precedes. But under the form of the function (r), we are able to apply to the benefit of the establishment, the analysis of §39. In fact, we have in this case, by the section cited,

$$a\nu + a'\nu' + a''\nu'' + \text{etc.} = -\sum \left(\frac{\triangle y_x}{y_0} \sum hp^{x+1} \right);$$

next a, a', etc. being the successive values of $-\frac{\triangle y_x}{y_0}$, we will have $\qquad$ [431]

$$a\nu^2 + a'\nu'^2 + \text{etc.} = \sum \left[-\frac{\triangle y_x}{y_0} \left(\sum hp^{x+1} \right)^2 \right],$$

so that

$$2k^2 = \sum \left[-\frac{\triangle y_x}{y_0} \left(\sum hp^{x+1} \right)^2 \right] - \left[\sum \frac{\triangle y_x}{y_0} \left(\sum hp^{x+1} \right) \right]^2.$$

In supposing that each of s persons who constitute the pension h on his head, deposit to the funds of the establishment, beyond the capital corresponding to this pension, a sum b, in order to defray the expense of the establishment; we will have by §39,

$$\frac{2}{\sqrt{\pi}} \int dr' c^{-r'^2},$$

for the probability that the real benefit of the establishment will be comprehended within the limits

$$sb \pm 2kr'\sqrt{s}.$$

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Thus in the case of an infinite number of affairs, the real benefit of the establishment, becomes certain and infinite. But then those who treat with it, have a mathematical disadvantage which must be compensated by a moral advantage, of which the estimation will be the object of the following chapter.