THE ANALYTIC THEORY OF PROBABILITIES
Third Edition
Book II

Pierre-Simon Laplace

THÉORIE
ANALYTIQUE
DES PROBABILITÉS

BY THE

MARQUIS DE LAPLACE,

Peer of France, Grand Officer of the Legion of Honor; one of the forty of the French Academy; of the Academy of Sciences; Member of the Bureau of Longitudes of France; of the Royal Societies of London et of Göttingen, of the Academies of Sciences of Russia, of Denmark, of Sweden, de Prussia, of Holland, of Italy, etc.

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REVIEWED AND AUGMENTED BY THE AUTHOR.

TRANSLATION

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Translator's Preface

We present in this volume a translation of Book II of the *Theorie Analytique des Probabilités* (TAP) based upon the third edition of 1820 with the *Supplements* included.[1] This edition was republished twice—in volume 7 of the national edition of [**6**] and again as volume 7 of the *Oeuvres Complète de Laplace* in 1878 [**7**].

The first two supplements were issued sometime between 1812 and 1820, the third in 1820 and the fourth, written by his son, after 1820. Not all printings of the 1820 publication included the last supplement. It was inserted in those copies that had not yet been distributed from the printer.

Book II is essentially a consolidation of Laplace's previous work on probability. He takes up many problems already considered by earlier researchers or himself. But he chooses to solve the most difficult problems. The most important features are the development of the central limit theorem and the exposition on the method of least squares.

One can consult the histories of Isaac Todhunter [**9**], Steven Stigler [**8**], or those of Anders Hald [**2, 3, 4**] for assistance in reading Laplace. The advantage of Todhunter is that he progresses through the memoirs of Laplace for the most part in chronological order. Stigler discusses the highlights. Hald, on the other hand, reorganizes the material by theme.

Reading the work is immensely frustrating. There are numerous typographical errors—not all listed in the errata nor necessarily corrected in the two later editions. New errors are introduced. Todhunter in comparing the 1820 edition with the 1847 remarks that "in the second supplement the misprints of the original were generally reproduced." By comparison of the various editions of the text, I believe that it is as nearly error-free as can be reasonably expected. All corrections have been introduced silently.

Beyond these difficulties is the frequent poor choice of notation with its legibility problems. I have compromised on reproducing his notation verbatim. Laplace uses a period to indicate multiplication. Throughout, this has been suppressed unless it seems necessary for comprehension.

Other compromises may be cited. We will restrict discussion to just one other.

Where we normally use a subscript for indexing, Laplace often uses a superscript. For example, he may write $x^{(i)}$ for the i^{th} term in a sequence. The second power of

[1]The text used is located at `https://gdz.sub.uni-goettingen.de/id/PPN585523401`.

this object is printed as

$$x^{(i)^2}$$

which means, of course,

$$[x^{(i)}]^2$$

We print, on the other hand,

$$x^{(i)2}$$

for which there should be no ambiguity as to its meaning. Similarly, when primes are used for indexing as in x', strictly speaking we should have for its second power

$$x'^2 = [x']^2$$

for which we print

$$x^{\prime 2}$$

The page numbers in the margin refer to the original 1820 edition and to the original printings of the supplements.

Commentary on particular aspects of the text will be included in a third volume.

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The probability of a future event, deduced from an observed event, is the quotient of the division of the probability of the event composed of these two events, and determined *à priori*, by the probability of the observed event, determined similarly *à priori*.

If an observed event is able to result from n different causes, their probabilities are respectively, as the probabilities of the event, deduced from their existence; and the probability of each of them, is a fraction of which the numerator is the probability of the event under the hypothesis of the existence of the cause, and of which the denominator is the sum of the similar probabilities, relative to all the causes. If these diverse causes considered *à priori*, are unequally probable; it is necessary, in place of the probability of the event, resulting from each cause, to employ the product of this probability by that of the cause itself.

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Two urns A and B each contain a very great number n of white and black balls, the number of whites being equal to the one of the blacks, in the totality $2n$ of balls; we draw at the same time one ball from each urn, and we place again into one urn the ball extracted from the other. By repeating this operation any number r times, we demand the probability that there will be x white balls in urn A.

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Comparison of the moral advantage of the placement of one same capital, on one head, with the one of the placement on two heads. We can at the same time, by some similar placements, increase its own advantage, and assure in the future the lot of the persons who interest us. .... §43 page 198

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We have extracted a ticket from an urn which contains the number n of them; a witness of this drawing, of whom the veracity and the probability that he is not mistaken at all, are supposed known, announces the exit of the n^o^ i; we demand the probability of this exit.
.... §44 page 201

We have extracted a ball from an urn which contains $n-1$ black balls, and one white ball. A witness to the drawing announces that the extracted ball is white; we demand the probability of this exit. If the number n is very great, that which renders extraordinary the exit of the white ball, the probability of the error or of the falsehood of the witness, becomes quite near to certitude; that which shows how the extraordinary facts weaken the belief due to the testimonies.
.... §45 page 203

Urn A contains n white balls, urn B contains the same number of black balls; we have extracted a ball from one of these urns, and we have put it into the other urn from which we have next extracted a ball. A witness of the first drawing has seen a white ball exit. A witness of the second drawing announces that he has seen similarly a white ball extracted. We demand the probability of this double exit. In order that this double exit take place, it is necessary that a white ball extracted from urn A in the first drawing, put next into urn B, has been extracted from it in the second drawing, that which is a quite extraordinary event, when the number n of black balls with which we have mixed it, is very considerable. The probability of this event becomes then very small; whence it follows that the probability of the fact, resulting from the collection of many testimonies, decreases in measure as this fact becomes more extraordinary. . . . . §46 page 204

Two witnesses attest to the exit of the n^o i from an urn which contains the number n of them, and of which we have extracted only one ticket. We demand the probability of this exit.

One of the witnesses attests to the exit of the n^o i and the other attests to the exit of the n^{o}i′; to determine the probability of the exit of the n^o i. . . . . §47 page 206

One or many traditional successions of r witnesses transmit the exit of the n^o i, from an urn which contains the number n of them; to determine the probability of this exit. . . . . §48 page 208

We know the respective veracities of two witnesses, of whom at least one, and perhaps both, attest to the exit of the n^o i, from one urn which contains the number n of them; to determine the probability of this exit . . . . §49 page 209

The judgments of the tribunals are able to be assimilated to the witnesses. *To determine the probability of the goodness of these judgments.*
. . . . §50 page 210

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CHAPTER 1

General principles of this Theory

§1. We have seen in the Introduction, that the probability of an event, is the ratio of the number of cases which are favorable to it, to the number of all possible cases; when nothing supports belief that one of these cases must arrive rather than the others, that which renders them for us, equally possible. The just estimation of these diverse cases, is one of the most delicate points of the analysis of chances. [179]

If all the cases are not equally possible, we will determine their respective possibilities; and then the probability of the event will be the sum of the probabilities of each favorable case. In fact, let us name p the probability of the first of these cases. This probability is relative to the subdivision of all the cases, into some others equally possible. Let N be the sum of all the cases thus subdivided, and n the sum of those cases which are favorable to the first case; we will have $p = \frac{n}{N}$. We will have similarly $p' = \frac{n'}{N}$, $p'' = \frac{n''}{N}$, etc.; by marking with one stroke, with two strokes, etc., the letters p and n, relatively to the second case, to the third, etc. Now, the probability of the event of which there is concern, is, by the same definition of probability, equal to

$$\frac{n + n' + n'' + \text{etc.}}{N};$$

it is therefore equal to $p + p' + p'' + \text{etc.}$

When an event is composed of two simple events, the one independent of the other; [180]
it is clear that the number of all possible cases, is the product of the two numbers which express all the possible cases relative to each simple event; because each of the cases relative to one of these events, can be combined with all the cases relative to the other event. By the same reason, the number of cases favorable to the compound event, is the product of the two numbers which express the cases favorable to each simple event; the probability of the compound event, is therefore then the product of the probabilities of each simple event. Thus the probability to bring forth twice consecutively, one ace with one die, is one thirty-sixth, when we suppose the faces of the die perfectly equal; because the number of all possible cases in two coups[1], is thirty-six, each case of the first cast being able to be combined with the six cases of the second; and among all these cases, one alone gives two aces consecutively.

[1] *Translator's note*: *Two coups*, that is, two rolls of a die. The word *coup* can take on many meanings depending upon context. A statistical experiment often consists of a sequence of individual *coups* where they are typically called trials. Similarly in the case of a game among players, these are the sub-games or rounds which comprise it. In general, the word coup can be replaced by the word trial, round or attempt.

In general, if p, p', p'', etc. are the respective possibilities of any number of simple events independent of one another; the product $p.p'.p''$, etc. will be the probability of an event composed of these events.

If the simple events are linked among them, in a manner that the supposition of the arrival of the first, influences the probability of the arrival of the second; we will have the probability of the compound event, by determining, 1° the probability of the first event; 2° the probability that this event having arrived, the second will take place.

In order to demonstrate this principle in a general manner, let us name p the number of all the possible cases, and let us suppose that in this number, there are of them p' favorable to the first event. Let us suppose next that in the number p', there are q favorable to the second event; it is clear that $\frac{q}{p}$ will be the probability of the compound event. But the probability of the first event is $\frac{p'}{p}$, the probability that this event having arrived, the second will take place, is $\frac{q}{p'}$; because then one of the cases
[181] p' needing to exist, we must consider only these cases. Now we have

$$\frac{q}{p} = \frac{p'}{p} \cdot \frac{q}{p'};$$

that which is the translation into analysis, of the principle enunciated above.

In considering how a compound event, the observed event, join to a future event; the probability of this last event, deduced from the observed event, is evidently the probability that the observed event taking place, the future event will take place similarly; now, by the principle that we have just exposed, this probability multiplied by that of the observed event, determined *à priori*, or independently from that which is already arrived, is equal to that of the compound event, determined *à priori*; we have therefore this new principle, relative to the probability of future events, deduced from observed events.

The probability of a future event, deduced from an observed event, is the quotient of the division of the probability of the event composed of these two events, and determined *à priori*, by the probability of the observed event, determined similarly *à priori*.

Thence proceeds further this other principle relative to the probability of causes, deduced from observed events.

If an observed event can result from n different causes; their probabilities are respectively, as the probabilities of the event, deduced from their existence; and the probability of each of them, is a fraction of which the numerator is the probability of the event, under the hypothesis of the existence of the cause, and of which the denominator is the sum of the similar probabilities, relative to all the causes.

Let us consider, indeed, as a compound event, the observed event, resulting from one of these causes. The probability of this compound event, a probability that we will designate by E, will be, by that which precedes, equal to the product of the probability of the observed event, determined *à priori*, and that we will name F, by the probability that this event taking place, the cause of which there is concern,

exists, a probability which is that of the cause, deduced from the observed event, and [182]
that we will name P.[2] will have therefore

$$P = \frac{E}{F}.$$

The probability of the compound event, is the product of the probability of the cause, by the probability that this cause taking place, the event will arrive, a probability that we will designate by H.[3] All the causes being supposed *à priori*, equally possible, the probability of each of them is $\frac{1}{n}$; we have therefore

$$E = \frac{H}{n}.$$

The probability of the observed event, is the sum of all the E relative to each cause; by designating therefore by $S.\frac{H}{n}$, the sum of all the values of $\frac{H}{n}$, we will have

$$F = S.\frac{H}{n};$$

the equation $P = \frac{E}{F}$ will become therefore

$$P = \frac{H}{S.H};$$

that which is the principle enunciated above, when all the causes are *à priori* equally possible. If this is not, by naming p the probability *à priori* of the cause that we just considered; we will have $E = Hp$; and, by following the preceding reasoning, we will find

$$P = \frac{Hp}{S.Hp};$$

that which gives the probabilities of the diverse causes, when they are not all, equally possible *à priori*.

In order to apply the preceding principle to an example, let us suppose that an
urn contains three balls of which each is able to be only white or black; that after [183]
having drawn a ball, we restore it to the urn in order to proceed to a new drawing, and that after m drawings, we have brought forth only white balls. It is clear that we can make *à priori*, only four hypotheses; because the balls can be, either all white, or two whites and one black, or two blacks and one white, or finally all black. If we consider these hypotheses as so many causes of the observed event; the probabilities of the event, relative to these causes, will be

$$1, \quad \frac{2^m}{3^m}, \quad \frac{1}{3^m}, \quad 0.$$

[2] *Translator's note*: In modern notation, $E = \Pr(\text{Event and Cause}_i)$, $F = \Pr(\text{Event})$, and $P = \Pr(\text{Cause}_i|\text{Event})$, $i = 1, 2, \dots n$..

[3] *Translator's note*: $H = \Pr(\text{Event}|\text{Cause}_i)$, $i = 1, 2, \dots n$.

The respective probabilities of these hypotheses, deduced from the observed event, will be therefore, by the third principle,

$$\frac{3^m}{3^m+2^m+1}, \quad \frac{2^m}{3^m+2^m+1}, \quad \frac{1}{3^m+2^m+1}, \quad 0.$$

We see, besides, that it is useless to have regard to the hypotheses which exclude the event, because the probability resulting from these hypotheses, being null, their omission changes not at all the expressions of the other probabilities.

If we wish to have the probability to bring forth only some black balls in the following m' drawings, we will determine *à priori*, the probabilities to bring forth first m white balls, next m' black balls. These probabilities are, relatively to the preceding hypotheses,

$$0, \quad \frac{2^m}{3^{m+m'}}, \quad \frac{2^{m'}}{3^{m+m'}}, \quad 0;$$

and as, *à priori*, the four hypotheses are equally possible, the probability of the compound event will be the quarter of the sum of the four preceding probabilities, or

$$\frac{1}{4}\,\frac{2^m+2^{m'}}{3^{m+m'}}.$$

The probabilities of the observed event, determined *à priori*, under the preceding four hypotheses, being respectively

[184]

$$\frac{3^m}{3^m}, \quad \frac{2^m}{3^m}, \quad \frac{1}{3^m}, \quad 0,$$

the quarter of their sum, or

$$\frac{1}{4}\left(\frac{3^m+2^m+1}{3^m}\right),$$

will be the probability of the observed event, determined *à priori*; by dividing therefore the probability of the compound event, by this probability, we will have by the second principle,

$$\frac{2^m+2^{m'}}{3^{m'}(3^m+2^m+1)},$$

for the probability to bring forth m' black balls in the m' following drawings.

We are able further to determine this probability, by the following principle.

The probability of a future event is the sum of the products of the probability of each cause, deduced from the observed event, by the probability that this cause existing, the future event will take place.

Here the probabilities of each cause, deduced from the observed event, are, as we have seen,

$$\frac{3^m}{3^m+2^m+1}, \quad \frac{2^m}{3^m+2^m+1}, \quad \frac{1}{3^m+2^m+1}, \quad 0;$$

the probabilities of the future event, relative to these causes, are respectively

$$0, \quad \frac{1}{3^{m'}}, \quad \frac{2^{m'}}{3^{m'}}, \quad 1;$$

the sum of their respective products, or

$$\frac{2^m + 2^{m'}}{3^{m'}(3^m + 2^m + 1)},$$

will be the probability of the future event, deduced from the observed event; that which is conformed to that which precedes.

If we suppose four balls in the urn, and that having brought forth a white ball [185] at the first drawing, we seek the probability to bring forth only black balls in the following m' drawings; we will find, by the principles exposed above, this probability equal to

$$\frac{3 + 2^{m'+1} + 3^{m'}}{10.4^{m'}}.$$

If the number of white balls equals the one of the blacks; the probability to bring forth only black balls in m' drawings, is $\frac{1}{2^{m'}}$. It surpasses the preceding, when m' is equal or less than 5; but it becomes inferior to it, when m' surpasses 5, although the white ball extracted first from the urn, indicates a superiority in the number of white balls. The explication of this paradox, holds in this that this indication excludes not at all the superiority of the number of black balls; it renders it only less probable; whereas the supposition of a perfect equality between the number of the whites and the one of the blacks, excludes this superiority; now this superiority, however small that its probability be, must render the probability to bring forth consecutively, m' black balls, greater than the case of equality of the colors, when m' is considerable.

The inequality which is able to exist among some things that we suppose perfectly similar, is able to have on the results of the calculus of probabilities, a sensible influence which merits a particular attention. Let us consider the game of *heads* and *tails*, and let us suppose that it is equally easy to bring forth *heads* as *tails*; then the probability to bring forth *heads* at the first coup, is $\frac{1}{2}$, and that to bring it forth two times consecutively, is $\frac{1}{4}$. But if there exists in the coin an inequality which makes one of the faces appear rather than the other, without us knowing the face that this inequality favors; the probability to bring forth *heads* at the first coup, will remain always $\frac{1}{2}$; because, in the ignorance in which one is, of the face that this inequality favors; as much as the probability of the simple event is increased, if this inequality is favorable to it, so much is it diminished, if this inequality is contrary to it. But the probability to bring forth *heads* two times consecutively, is increased, notwithstanding this ignorance; because this probability is equal to that to bring forth *heads* at the first coup, multiplied by the probability that having brought it forth at the [186] first coup, we will bring it forth at the second; now its arrival at the first coup, is a motive to believe that the inequality of the coin, favors it; it increases therefore the probability to bring it forth at the second; thus the product of the two probabilities is increased by this inequality. In order to submit this object to calculation, let us suppose that the inequality of the coin increases by the quantity α, the probability of the simple event that it favors. If this event is *heads*, the probability will be $\frac{1}{2} + \alpha$, and the probability to bring it forth two times consecutively will be $\left(\frac{1}{2} + \alpha\right)^2$. If the event favored is *tails*, the probability of *heads* will be $\frac{1}{2} - \alpha$, and the probability to bring it forth two times consecutively will be $\left(\frac{1}{2} - \alpha\right)^2$. As we have in advance, no reason to believe that the inequality favors the one of the simple events rather than the other, it is clear that in order to have the probability of the compound event *heads-heads*, it is necessary to add the two preceding probabilities, and to take the half of their sum, that which gives $\frac{1}{4} + \alpha^2$ for this probability: it is also the probability of *tails-tails*. We will find by the same reasoning, that the probability of the compound event *heads-tails* or *tails-heads* is $\frac{1}{4} - \alpha^2$; consequently, it is less than that of the repetition of the same simple event.

The preceding considerations can be extended to any events whatsoever. p representing the probability of a simple event, and $1 - p$ that of the other event; if we designate by P, the probability of a result relative to these events, and if we suppose that p is really $p \pm \alpha$, α being an unknown quantity, as well as the sign which affects it; the probability P of the result will be

$$P + \frac{1}{1.2}\alpha^2.\frac{ddP}{dp^2} + \frac{1}{1.2.3.4}\alpha^4.\frac{d^4P}{dp^4} + \text{ etc.}$$

By making $P = p^n$, that is by supposing that the result relative to the events, be n times the repetition of the first; the probability P will become

$$p^n + \frac{n(n-1)}{1.2}\alpha^2 p^{n-2} + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\alpha^4 p^{n-4} + \text{etc.}$$

[187] Thus the unknown error that we are able to suppose in the probability of the simple events, increases always the probability of the events composed of the repetition of the same event.

§2. The probability of events serves to determine the expectation[4] and the fear of the persons interested in their existence. The word *espérance* has diverse meanings; it expresses generally the advantage of the one who awaits any good, under a supposition that is only likely. In the theory of chances, this advantage is the product of the expected sum, by the probability to obtain it; it is the partial sum which must return, when we no longer wish to incur the risks of the event, by supposing that the apportionment of the entire sum is made proportional to the probabilities. This manner to apportion it, is alone equitable, when we set aside all strange circumstance, because with an equal degree of probability, we have an equal right with respect to the expected sum. We will name this advantage *mathematical expectation*, in order to distinguish it from moral expectation which depends, as it does, on the expected good and on the probability to obtain it, but which is regulated further on a thousand variable circumstances that it is nearly always impossible to define, and yet more, to subject to the calculus. These circumstances, it is true, making only to increase or to decrease the value of the expected good, we can consider the moral expectation itself as the product of this value, by the probability to obtain it; but we must then distinguish in the expected good, its relative value, from its absolute value: the latter

[4] espérance

is independent of the motives which make it desired, whereas the first increases with these motives.

We are not able to give a general rule in order to estimate this relative value; however it is natural to suppose the value relative to an infinitely small sum, in direct ratio to its absolute value, in inverse ratio of the total good of the interested person. In fact, it is clear that a franc has very little value for the one who possesses a great number of them, and that the most natural manner to estimate its relative value, is to suppose it in inverse ratio to this number.

Such are the general principals of the analysis of probabilities. We will now apply [188] them to the most delicate and the most difficult questions of this analysis. But in order to put in order in this matter, we will treat first the questions in which the probabilities of the simple events, are given; we will consider next those in which these possibilities are unknown, and must be determined by the observed events.

CHAPTER 2

On the probability of events composed of simple events of which the respective possibilities are given

§3. If we develop the product $(1+p)(1+p')(1+p'')$.etc. composed of n factors; [189]
this development will contain all the possible combinations of the n letters p, p', p'', $\dots$, $p^{(n-1)}$, taken one by one, two by two, three by three, etc.[1] to n; and each combination will have for coefficient unity. Thus the combination $pp'p''$ resulting from the product $(1+p)(1+p')(1+p'')$, multiplied by the term 1 of the development of the other factors; its coefficient is evidently unity. Now, in order to have the total number of combinations of n letters taken x by x; we will observe that each of these combinations become p^x, when we suppose p', p'', etc. equal to p. Then the product of the n preceding factors is changed into the binomial $(1+p)^n$; now the coefficient of p^x in the development of this binomial, is

$$\frac{n(n-1)(n-2)\dots(n-x+1)}{1.2.3\dots x};$$

this quantity expresses therefore the number of combinations of n letters taken x by x. We will have the total number of combinations of these letters, taken one by one, two by two, etc. to n by n, by making $p=1$, in the binomial $(1+p)^n$, and by subtracting unity from it; that which gives 2^n-1 for this number.

Let us suppose that in each combination, we have regard not only to the number of letters, but further to their situation; we will determine the number of combinations, by observing that, in the combination of two letters pp', we are able to put p' in the second place, and next in the first; that which gives the two combinations pp', [190]
$p'p$. By introducing next a new letter p'' in each of these combinations, we are able to put it in the first, in the second or in the third place; that which gives 2.3 combinations. By continuing thus, we see that in a combination of x letters, we are able to give $1.2.3\dots x$ different situations; whence it follows that the total number of combinations of n letters, taken x by x, being by that which precedes,

$$\frac{n(n-1)(n-2)\dots(n-x+1)}{1.2.3\dots x},$$

the total number of combinations, when we have regard to the different situation of the letters, will be this same function, by suppressing its denominator.[2]

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[1] *Translator's note*: one by one, two by two, etc. In other words, one at a time, two at a time. In modern notation, these are the combinations $\binom{n}{1}$, $\binom{n}{2}$, etc.

[2] *Translator's note*: That is, the number of permutations.

We are able easily, by means of these formulas, to determine the benefits of lotteries. Let us suppose that the number of tickets[3] of a lottery, be n, and that there exits r of them at each drawing; we wish to have the probability that a combination of s of these tickets, will exit in the first drawing.

The total number of combinations of tickets, taken r by r, is by that which precedes,

$$\frac{n(n-1)(n-2)\dots(n-r+1)}{1.2.3\dots r},$$

In order to have among these combinations, the number of those in which the s tickets are comprehended, we will observe that if we subtract these tickets from the total of the tickets, and if we combine $r-s$ by $r-s$, there remains $n-s$, the number of these combinations will be the sought number; because it is clear that by adding the s tickets to each of these combinations, we will have the combinations r by r of the tickets in which are these s tickets.[4] This number is therefore

$$\frac{(n-s)(n-s-1)\dots(n-r+1)}{1.2.3\dots(r-s)};$$

by dividing it by the total number of combinations r by r of the n tickets, we will have for the sought probability

$$\frac{r(r-1)(r-2)\dots(r-s+1)}{n(n-1)(n-2)\dots(n-s+1)}.$$

[191] By dividing this quantity by $1.2.3\dots s$, we will have by that which precedes, the probability that the s tickets will exit in a determined order among them. We will have the probability that the first s tickets of the drawing, will be those of the proposed combination, by observing that this probability reverts to that of bringing forth this combination, by supposing that there exits only s tickets at each drawing, that which reverts to making $r=s$ in the preceding function which becomes thus

$$\frac{1.2.3\dots s}{n(n-1)\dots(n-s+1)}.$$

Finally, we will have the probability that the s chosen tickets will exit first in a determined order, by reducing the numerator of this fraction, to unity.

The quotients of the stakes divided by these probabilities, are those which the lottery must render to the players: the excess of these quotients over that which it gives, is its benefit. In fact, if we name p the probability of the player, m his stake, and x that which the lottery must render to him, for equality of the game; $x-m$ will be the stake of the lottery; because having received the stake m, and rendering x to the player; it puts into the game only $x-m$. Now for equality of the game, the mathematical hope[5] of each player must be equal to his fear: his hope is the product

[3] *Translator's note*: The word is "numéro," or number used in the sense of a label. I have therefore chosen to render it as ticket. Laplace later uses "billet," in this case referring specifically to a lottery ticket.

[4] *Translator's note*: Laplace is here expressing the quantity $\binom{s}{s}\binom{n-s}{r-s}$.

[5] *Translator's note*: hope, the *espérance* or expectation.

of the stake $x - m$ of his adversary, by the probability p to obtain it: his fear is the product of his stake m, by the probability $1 - p$ of the loss. We have therefore

$$p(x - m) = (1 - p)m;$$

that is that for the equality of the game, the stakes must be reciprocal to the probabilities to win. This equation gives

$$x = \frac{m}{p};$$

thus that which the lottery must render, is the quotient of the stake divided by the probability of the player to win.

§4. *A lottery being composed of n numbered tickets of which r exit at each drawing, we require the probability that after i drawings, all the tickets will have exited.*

Let us name $z_{n,q}$ the number of cases in which, after i drawings, the totality of [192]
the tickets 1, 2, 3, $\ldots q$ will have exited. It is clear that this number is equal to the number $z_{n,q-1}$ of cases in which the tickets 1, 2, 3, $\ldots q - 1$ have exited, less the number of cases in which these tickets being brought out, the ticket q is not drawn; now this last number is evidently the same as the one of the cases in which the tickets 1, 2, 3, $\ldots q - 1$ would be extracted, if we remove the ticket q from the n tickets of the lottery, and this number is $z_{n-1,q-1}$; we have therefore

$$z_{n,q} = z_{n,q-1} - z_{n-1,q-1}. \tag{i}$$

Now the number of all possible cases in a single drawing, being $\frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots r}$, the one of all possible cases in i drawings, is

$$\left(\frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots r}\right)^i.$$

The number of all the cases in which the ticket 1 will not exit in these i drawings, is the number of all possible cases, when we subtract this ticket from the n tickets in the lottery; and this number is

$$\left(\frac{(n-1)(n-2)\ldots(n-r)}{1.2.3\ldots r}\right)^i;$$

the number of cases in which the ticket 1 will exit in i drawings, is therefore

$$\left(\frac{n(n-1)\ldots(n-r+1)}{1.2.3\ldots r}\right)^i - \left(\frac{(n-1)(n-2)\ldots(n-r)}{1.2.3\ldots r}\right)^i,$$

or

$$\triangle\left(\frac{(n-1)(n-2)\ldots(n-r)}{1.2.3\ldots r}\right)^i;$$

this is the value of $z_{n,1}$. This premised, equation (i) will give, by making successively $q = 2$, $q = 3$, etc.,

$$z_{n,2} = \triangle^2 \left(\frac{(n-2)(n-3)\dots(n-r-1)}{1.2.3\dots r} \right)^i,$$

$$z_{n,3} = \triangle^3 \left(\frac{(n-3)(n-4)\dots(n-r-2)}{1.2.3\dots r} \right)^i,$$

etc.;

[193] and generally,

$$z_{n,q} = \triangle^q \left(\frac{(n-q)(n-q-1)\dots(n-r-q+1)}{1.2.3\dots r} \right)^i.$$

Thus the probability that the tickets 1, 2, 3, $\dots q$ will exit in i drawings, being equal to $z_{n,q}$ divided by the number of all possible cases, it will be

$$\frac{\triangle^q[(n-q)(n-q-1)\dots(n-r-q+1)]^i}{[n(n-1)(n-2)\dots(n-r+1)]^i}$$

If we make in this expression $q = n$, we will have, s being here the variable which must be supposed null in the result,

$$\frac{\triangle^n[s(s-1)\dots(s-r+1)]^i}{[n(n-1)\dots(n-r+1)]^i}$$

for the expression of the probability that all the tickets of the lottery will exit in i drawings.

If n and i are very great numbers, we will have by the formulas of §40 of the first Book, the value of this probability, by means of a highly convergent series. Let us suppose, for example, that only one ticket exits at each drawing, the preceding probability becomes

$$\frac{\triangle^n s^i}{n^i}.$$

Let us propose to determine the number i of drawings in which this probability is $\frac{1}{k}$, n and i being very great numbers. By following the analysis of the section cited, we will determine first a by the equation

$$0 = \frac{i+1}{a} - s - \frac{nc^a}{c^a - 1};$$

that which gives

$$a = \frac{i+1}{n+s} \left\{ \frac{1 - c^{-a}}{1 - \frac{sc^{-a}}{n+s}} \right\}.$$

We have next by §40 of the first Book, when c^{-a} is a very small quantity of the order $\frac{1}{i}$, as that takes place in the present question; we have, I say, to the quantities nearly of order $\frac{1}{i^2}$, s being supposed null in the result of the calculation, [194]

$$\frac{\triangle^n s^i}{n^i} = \frac{\left(\frac{i}{i+1}\right)^{i+\frac{1}{2}} c^{na-i}(1-c^{-a})^{n-i}}{\sqrt{1-\left(\frac{i+1}{n}\right)c^{-a}}}.$$

Now we have, to the quantities nearly of the order $\frac{1}{i^2}$,

$$\left(\frac{i}{i+1}\right)^{i+\frac{1}{2}} = c^{-1};$$

by supposing next $c^{-a} = z$, we have

$$(1-c^{-a})^{n-i} = c^{(i-n)z}\left[1+\left(\frac{i-n}{2}\right)z^2\right];$$

moreover, the equation which determines a, gives

$$i+1-na = (i+1)z;$$

whence we deduce

$$c^{na-i-1} = c^{-iz}(1-z);$$

we will have therefore, to the quantities nearly of order $\frac{1}{i^2}$,

$$\frac{\triangle^n s^i}{n^i} = c^{-nz}\left[1+\left(\frac{i-2n+1}{2n}\right)z+\left(\frac{i-n}{2}\right)z^2\right].$$

In order to determine z, let us take up again the equation

$$a = \frac{i+1}{n} - \left(\frac{i+1}{n}\right)c^{-a};$$

we will have by formula (p) of §21 of the second Book of the *Mécanique céleste*,

$$z = c^{-a} = q + \left(\frac{i+1}{n}\right)q^2 + \frac{3\left(\frac{i+1}{n}\right)^2}{1.2}q^3 + \frac{4^2\left(\frac{i+1}{n}\right)^3}{1.2.3}q^4 + \text{etc.};$$

q being supposed equal to $c^{-\left(\frac{i+1}{n}\right)}$. This value of z gives

$$c^{-nz} = c^{-nq}[1-(i+1)q^2];$$

consequently, [195]

$$\frac{\triangle^n s^i}{n^i} = c^{-nq}\left[1+\left(\frac{i+1-2n}{2n}\right)q-\left(\frac{n+i+2}{2}\right)q^2\right].$$

By equating this quantity to the fraction $\frac{1}{k}$, we will have

$$q = \frac{\log k}{n}\left[1+\left(\frac{i+1-2n}{2n^2}\right)-\left(\frac{n+i+2}{2n^2}\right)\log k\right];$$

now we have

$$i+1 = -n\log q;$$

we will have therefore very nearly for the expression of the number i of drawings, according to which the probability that all the tickets will have exited is $\frac{1}{k}$,

$$i = (\log n - \log\log k)(n - \tfrac{1}{2} + \tfrac{1}{2}\log k) + \tfrac{1}{2}\log k;$$

we must observe that all these logarithms are hyperbolic.

Let us suppose the lottery composed of ten thousand tickets, or $n = 10000$, and $k = 2$, this formula gives

$$i = 95767,4$$

for the expression of the number of drawings, in which we can wager one against one, that the ten thousand tickets of the lottery will exit; it is therefore odds a little less than one against one that they will exit in 95767 drawings, and odds a little more than one against one that they will exit in 95768 drawings.

We will determine by a similar analysis, the number of drawings in which we are able to wager one against one, that all the tickets of the lottery of France will exit. This lottery is, as one knows, composed of 90 tickets of which five exit at each drawing. The probability that all the tickets will exit in i drawings, is then by that which precedes,

$$\frac{\triangle^n[s'(s'-1)(s'-2)(s'-3)(s'-4)]^i}{[n(n-1)(n-2)(n-3)(n-4)]^i},$$

[196] n being here equal to 90, and s' needing to be supposed null in the result of the calculation. If we make $s = s' - 2$, this function becomes

$$\frac{\triangle^n[s(s^2-1)(s^2-4)]^i}{[(n-2)(\overline{n-2}^2-1)(\overline{n-2}^2-4)]^i};$$

or by developing in series,

$$\frac{(\triangle^n s^{5i} - 5i\triangle^n s^{5i-2} + \text{etc.})}{(n-2)^{5i}}\left(1 + \frac{5i}{(n-2)^2} + \text{etc.}\right),$$

s needing to be supposed equal to -2 in the result of the calculation.

We have by §40 of the first Book, by neglecting the terms of order $\frac{1}{i^2}$, and supposing c^{-a} very small of order $\frac{1}{i}$,

$$\frac{\triangle^n s^{5i}}{(n-2)^{5i}} = \frac{\left(\frac{5i+1}{a}\right)^{5i}\left(\frac{5i}{5i+1}\right)^{5i} c^{(n-2)a-5i}(1-c^{-a})^n}{(n-2)^{5i}\sqrt{1 + \frac{1}{5i} - \frac{na^2c^{-a}}{5i(1-c^{-a})^2}}},$$

a being given by the equation

$$a = \frac{(5i+1)(1-c^{-a})}{(n-2)\left(1 + \frac{2c^{-a}}{n-2}\right)}.$$

We have thus, by neglecting the terms of order $\frac{1}{i^2}$,

$$\frac{\triangle^n s^{5i}}{(n-2)^{5i}} = \frac{\left(1+\frac{2c^{-a}}{n-2}\right)^{5i}}{(1-c^{-a})^{5i}}(1-c^{-a})^n c^{1-(5i+1)c^{-a}-\frac{10ic^{-a}}{n-2}} \times \left(\frac{5i}{5i+1}\right)^{5i}\left(1-\frac{1}{10i}+\frac{na^2c^{-a}}{10i}\right);$$

now we have

$$\left(1+\frac{2c^{-a}}{n-2}\right)^{5i} = c^{\frac{10ic^{-a}}{n-2}},$$

$$(1-c^{-a})^{-5i} = c^{5ic^{-a}}\left(1+\frac{5i}{2}c^{-2a}\right),$$

$$\left(\frac{5i}{5i+1}\right)^{5i} = c^{-1}\left(1+\frac{1}{10i}\right);$$

we will have therefore to the quantities nearly of order $\frac{1}{i^2}$,

$$\frac{\triangle^n s^i}{(n-2)^{5i}} = (1-c^{-a})^n\left(1-c^{-a}+\frac{5i}{2}c^{-2a}+\frac{na^2c^{-a}}{10i}\right).$$

By substituting for a its value, and observing that i is very little different from $n-2$, [197]
in the present case, as we will see hereafter; we have very nearly,

$$\frac{na^2c^{-a}}{10i} = \frac{5i+12}{2(n-2)}c^{-a}.$$

I keep for greater exactitude, the term $\frac{12c^{-a}}{2(n-2)}$, although of order $\frac{1}{i^2}$, because of the size of its factor 12; we will have therefore

$$\frac{\triangle^n s^{5i}}{(n-2)^{5i}} = (1-c^{-a})^n\left(1+\frac{5i-2n+16}{2(n-2)}c^{-a}+\frac{5i}{2}c^{-2a}\right).$$

If we change in this equation $5i$ into $5i-2$, we will have that of $\frac{\triangle^n s^{5i-2}}{(n-2)^{5i-2}}$; but the value of a will no longer be the same. Let a' be this new value, we will have

$$a' = \frac{(5i-1)(1-c^{-a'})}{(n-2)\left(1+\frac{2c^{-a'}}{n-2}\right)},$$

that which gives, very nearly,

$$a' = a - \frac{2}{n-2}.$$

In that case we have

$$1-c^{-a'} = 1-c^{-a}-\frac{2c^{-a}}{n-2};$$

whence we deduce, by neglecting the quantities of order $\frac{1}{i}$,

$$(1-c^{-a'})^n = (1-c^{-a})^n;$$

consequently we have, by neglecting the quantities of order $\frac{1}{i}$,

$$\frac{\triangle^n s^{5i-2}}{(n-2)^{5i-2}} = (1-c^{-a})^n.$$

We will have therefore, to the quantities nearly of order $\frac{1}{i^2}$,

$$\begin{aligned}&\frac{\triangle^n[s(s^2-1)(s^2-4)]^i}{[n(n-1)(n-2)(n-3)(n-4)]^i}\\ &\quad= (1-c^{-a})^n\left[1+\frac{(5i-2n+16)}{2(n-2)}c^{-a}+\frac{5i}{2}c^{-2a}\right].\end{aligned}$$

[198] This quantity must, by the condition of the problem, be equal to $\frac{1}{2}$, that which gives

$$1-c^{-a} = \sqrt[n]{\frac{1}{2}}\left[1-\frac{(5i-2n+16)}{2n(n-2)}c^{-a}-\frac{5i}{2n}c^{-2a}\right];$$

whence we deduce

$$c^{-a} = \left(1-\sqrt[n]{\frac{1}{2}}\right)\left[1+\frac{(5i-2n+16)}{2n(n-2)}+\frac{5i}{2n}c^{-a}\right];$$

consequently we have by hyperbolic logarithms,

$$a = \log\left(\frac{\sqrt[n]{2}}{\sqrt[n]{2}-1}\right)-\frac{(5i-2n+16)}{2n(n-2)}-\frac{5i}{2n}c^{-a};$$

now we have, to the quantities nearly of order $\frac{1}{i^2}$,

$$a = \frac{5i+1}{(n-2)\sqrt[n]{2}};$$

we will have therefore

$$i = \frac{n-2}{5}\sqrt[n]{2}\left[1-\frac{1}{2n}-\frac{16}{10in}-\frac{1}{2}(\sqrt[n]{2}-1)\right]\log\left(\frac{\sqrt[n]{2}}{\sqrt[n]{2}-1}\right).$$

By substituting for n its value 90, we find

$$i = 85, 53;$$

so that there is odds a little less than one to one that all the tickets will exit in 85 drawings, and odds a little more than one to one that they will exit in 86 drawings.

A quite simple and very close way to obtain the value of i, is to suppose $\frac{\triangle^n s^i}{n^i}$, or the series

$$1-n\left(\frac{n-1}{n}\right)^i+\frac{n(n-1)}{2}\left(\frac{n-2}{n}\right)^i-\text{etc.}$$

equal to the development

$$1-n\left(\frac{n-1}{n}\right)^i+\frac{n(n-1)}{1.2}\left(\frac{n-1}{n}\right)^{2i}-\text{etc.}$$

of the binomial $\left[1-\left(\frac{n-1}{n}\right)^i\right]^n$. In reality, the two series have the first two terms equal [199]
respectively. Their third terms are also, more of less, equal between them; for we have quite nearly $\left(\frac{n-2}{n}\right)^i$ equal to $\left(\frac{n-1}{n}\right)^{2i}$. In fact, their hyperbolic logarithms are, by neglecting the terms of order $\frac{i}{n^2}$, both equal to $-\frac{i}{n}$. We will see in the same way, that the fourth terms, the fifth, etc., are very little different, when n and i are very great numbers; but the difference increases without ceasing, in measure as the terms move away from the first, that which must in the end, produce in them an evident difference between the series themselves. In order to estimate it, let us determine the value of i concluded from the equality of the two series. By equating to $\frac{1}{k}$, the binomial $\left[1-\left(\frac{n-1}{n}\right)^i\right]^n$, we will have

$$i=\frac{\log\left(1-\sqrt[n]{\frac{1}{k}}\right)}{\log\left(\frac{n-1}{n}\right)},$$

these logarithms being able to be, at will, hyperbolic or tabulated. Let $\sqrt[n]{\frac{1}{k}}=1-z$. We will have by taking the hyperbolic logarithms of each member of this equation,

$$\frac{1}{n}\log k=-\log(1-z)=z+\frac{z^2}{2}+\text{etc.},$$

that which gives very nearly,

$$z=\frac{\log k}{n}\left(1-\frac{\log k}{2n}\right);$$

we will have therefore in hyperbolic logarithms,

$$\log\left(1-\sqrt[n]{\frac{1}{k}}\right)=\log z=\log\log\, k-\log\, n-\frac{\log\, k}{2n}.$$

We have next [200]

$$\log\frac{n-1}{n}=-\frac{1}{n}-\frac{1}{2n^2}-\text{etc.}$$

The preceding expression for i becomes thus very nearly,

$$i=n(\log n-\log\log k)\left(1-\frac{1}{2n}\right)+\tfrac{1}{2}\log k;$$

the excess of the value found previously for i, over this one, is

$$\frac{\log k}{2}(\log n-\log\log k);$$

this excess becomes infinite, when n is infinite; but a very great number is necessary in order to render it very evident; and in the case of $n=10000$ and of $k=2$, it is still only three units.

If we consider likewise the development

$$1 - n\left(\frac{n-5}{n}\right)^i + \text{etc.}$$

of the expression $\frac{\triangle^n[s'(s'-1)(s'-2)(s'-3)(s'-4)]^i}{[n(n-1)(n-2)(n-3)(n-4)]^i}$, as the one of the binomial $\left[1-\left(\frac{n-5}{n}\right)^i\right]^n$, we will have in order to determine the number i of coups in which we can wager one against one, that all the tickets will exit, the equation

$$\left[1-\left(\frac{n-5}{n}\right)^i\right]^n = \frac{1}{2};$$

that which gives

$$i = \frac{\log\left(\frac{\sqrt[n]{2}}{\sqrt[n]{2}-1}\right)}{\log\left(\frac{n}{n-5}\right)}.$$

These logarithms can be tabulated. By making $n = 90$, we find

$$i = 85,204,$$

[201] that which differs very little from the value $i = 85,53$ that we have found above.

§5. An urn being supposed to contain the number x of balls, we draw from it a part or the totality, and we demand the probability that the number of extracted balls will be even.

The sum of the cases in which this number is unity, equals evidently x; since each of the balls can equally be extracted. The sum of the cases in which this number equals 2, is the sum of the combinations of x balls taken two by two, and this sum is, by §3, equal to $\frac{x(x-1)}{1.2}$. The sum of the cases in which the same number equals 3, is the sum of the combinations of balls taken three by three, and this sum is $\frac{x(x-1)(x-2)}{1.2.3}$, and so forth. Thus the successive terms of the development of the function $(1+1)^x - 1$, will represent all the cases in which the number of extracted balls, is successively 1, 2, 3, etc. to x; whence it is easy to conclude that the sum of all the cases relative to the odd numbers, is $\frac{1}{2}(1+1)^x - \frac{1}{2}(1-1)^x$, or 2^{x-1}; and that the sum of all the cases relative to the even numbers, is $\frac{1}{2}(1+1)^x + \frac{1}{2}(1-1)^x - 1$, or $2^{x-1} - 1$. The union of these two sums is the number of all the possible cases; this number is therefore $2^x - 1$; thus the probability that the number of extracted balls will be even, is $\frac{2^{x-1}-1}{2^x-1}$, and the probability that this number will be odd, is $\frac{2^{x-1}}{2^x-1}$; there is therefore advantage to wager with equality, on an odd number.

If the number x is unknown, and if we know only that it can not exceed n, and that this number and all the lesser are equally possible; we will have the number of all the possible cases relative to the odd numbers, by making the sum of all the values of 2^{x-1}, from $x = 1$ to $x = n$, and it is easy to see that this sum is $2^n - 1$. We will likewise have the sum of all the possible cases relative to the even numbers, by
[202] summing the function $2^{x-1} - 1$, from $x = 1$ to $x = n$, and we find this sum equal to

$2^n - n - 1$; the probability of an even number is therefore then $\frac{2^n-n-1}{2^{n+1}-n-2}$, and that of an odd number is $\frac{2^n-1}{2^{n+1}-n-2}$.

Let us suppose now that the urn contains the number x of white balls, and the same number of black balls; we ask the probability that by drawing any even number of balls, we will bring forth as many white balls as black balls, all the even numbers being able to be brought forth equally.

The number of cases in which one white ball from the urn can be combined with a black ball, is evidently $x.x$. The number of cases in which two white balls can be combined with two black balls, is $\frac{x(x-1)}{1.2}\frac{x(x-1)}{1.2}$, and so forth. The number of cases in which we will bring forth as many white balls as black balls, is therefore the sum of the squares of the terms of the development of the binomial $(1+1)^x$, less unity. In order to have this sum, we will observe that it is equal to a term independent of a, in the development of $\left(1+\frac{1}{a}\right)^x(1+a)^x$. This function is equal to $\frac{(1+a)^{2x}}{a^x}$. The term independent of a, in its development, is thus the coefficient of the middle term of the binomial $(1+a)^{2x}$; this coefficient is $\frac{1.2.3\ldots 2x}{(1.2.3\ldots x)^2}$; the number of cases in which we can draw from the urn as many white balls as black balls, is therefore

$$\frac{1.2.3\ldots 2x}{(1.2.3\ldots x)^2} - 1.$$

The number of all possible cases is the sum of the odd terms in the development of the binomial $(1+1)^{2x}$, less the first, or unity. This sum is $\frac{1}{2}(1+1)^{2x} + \frac{1}{2}(1-1)^{2x}$; the number of possible cases is therefore $2^{2x-1} - 1$, which gives for the expression of the probability sought

$$\frac{\frac{1.2.3\ldots 2x}{(1.2.3\ldots x)^2} - 1}{2^{2x-1} - 1}.$$

In the case where x is a large number, this probability is reduced by §33 of the first [203]
Book, to $\frac{2}{\sqrt{x\pi}}$, π being the semi-circumference of which 1 is the radius.

§6. Let us consider a number $x + x'$ of urns, of which the first contains p white balls and q black balls; the second, p' white balls and q' black balls; the third, p'' white balls and q'' black balls, and so forth. Let us suppose that we draw successively one ball from each urn. It is clear that the number of all the possible cases in the first drawing, is $p+q$; in the second drawing, each of the cases of the first being able to be combined with the $p'+q'$ balls of the second urn, we will have $(p+q)(p'+q')$ for the number of all the possible cases relative to the first two drawings. In the third drawing, each of these cases can be combined with the $p''+q''$ balls of the third urn; that which gives $(p+q)(p'+q')(p''+q'')$ for the number of all the possible cases relative to the three drawings, and thus of the rest. This product for the totality of the urns, will be composed of $x+x'$ factors; and the sum of all the terms of its development, in which the letter p, with or without accent, is repeated x times, and consequently the letter q, x' times, will express the number of cases in which we can draw from the urns, x white balls and x' black balls.

If p', p'', etc. are equal to p, and if q', q'', etc. are equal to q; the preceding product becomes $(p+q)^{x+x'}$. The term multiplied by $p^x q^{x'}$ in the development of this binomial is

$$\frac{(x+x')(x+x'-1)\dots(x+1)}{1.2.3\dots x'}p^x q^{x'}$$

or

$$\frac{1.2.3\dots(x+x')}{1.2.3\dots x.1.2.3\dots x'}p^x q^{x'}.$$

Thus this quantity expresses the number of cases in which we can bring forth x white balls and x' black balls. The number of all the possible cases being $(p+q)^{x+x'}$, the probability to bring forth x white balls and x' black balls is

$$\frac{1.2.3\dots(x+x')}{1.2.3\dots x.1.2.3\dots x'}\left(\frac{p}{p+q}\right)^x \left(\frac{q}{p+q}\right)^{x'},$$

[204] where we must observe that $\frac{p}{p+q}$ is the probability of drawing a white ball from one of the urns, and that $\frac{q}{p+q}$ is the probability of drawing from it a black ball.

It is clear that it is perfectly equal to draw x white balls and x' black balls, from $x+x'$ urns which each contain p white balls and q black balls, or one alone of these urns, provided that we replace into the urn the ball extracted at each drawing.

Let us consider now a number $x+x'+x''$ urns of which the first contains p white balls, q black balls, and r red balls, of which the second contains p' white balls, q' black balls and r' red balls, and so forth. Let us suppose that we draw one ball from each of these urns. The number of all the possible cases will be the product of the $x+x'+x''$ factors,

$$(p+q+r)(p'+q'+r')(p''+q''+r'').\text{etc.}$$

The number of cases in which we will bring forth x white balls, x' black balls, and x'' red balls, will be the sum of all the terms of the development of this product, in which the letter p will be repeated x times; the letter q, x' times, and the letter r, x'' times. If all the accented letters p', q', etc., are equal to their non-accented correspondents, the preceding product is changed into the trinomial $(p+q+r)^{x+x'+x''}$. The term of its development which has for factor $p^x q^{x'} r^{x''}$, is

$$\frac{1.2.3\dots(x+x'+x'')}{1.2.3\dots x.1.2.3\dots x'.1.2.3\dots x''}p^x q^{x'} r^{x''};$$

thus the number of all the possible cases being $(p+q+r)^{x+x'+x''}$, the probability to bring forth x white balls, x' black balls, and x'' red balls, will be

$$\frac{1.2.3\dots(x+x'+x'')}{1.2.3\dots x.1.2.3\dots x'.1.2.3\dots x''}\left(\frac{p}{p+q+r}\right)^x \left(\frac{q}{p+q+r}\right)^{x'} \left(\frac{r}{p+q+r}\right)^{x''},$$

whence we must observe that $\frac{p}{p+q+r}$, $\frac{q}{p+q+r}$, $\frac{r}{p+q+r}$ are the respective probabilities of drawing from each urn one white ball, one black ball, and one red ball.

[205] We see generally that if the urns contain each the same number of colors, p being the number of balls of the first color; q the one of the balls of the second color; r, s, etc., those of the balls of the third, the fourth, etc.; $x+x'+x''+x'''+\text{etc.}$ being the

number of urns; the probability to bring forth x balls of the first color, x' balls of the second, x'' of the third, x''' of the fourth, etc., will be

$$\frac{1.2.3\ldots(x+x'+x''+x'''+\text{etc.})}{1.2.3\ldots x.1.2.3\ldots x'.1.2.3\ldots x''.1.2.3\ldots x'''.\text{etc.}}\left(\frac{p}{p+q+r+s+\text{etc.}}\right)^{x}$$
$$\times\left(\frac{q}{p+q+r+s+\text{etc.}}\right)^{x'}\left(\frac{r}{p+q+r+s+\text{etc.}}\right)^{x''}\left(\frac{s}{p+q+r+s+\text{etc.}}\right)^{x'''}.\text{etc.}$$

§7. Let us determine now the probability of drawing from the preceding urns, x white balls, before bringing forth either x' black balls, or x'' red balls, etc. It is clear that n expressing the number of the colors, this must happen at the latest after $x+x'+x''+\text{etc.}-n+1$ drawings. Because when the number of extracted white balls is equal or less than x, the one of the extracted black balls, less than x', the one of the extracted red balls, less than x'', etc.; the total number of the extracted balls, and consequently, the number of drawings is equal or less than $x+x'+x''+\text{etc.}-n+1$; we can therefore consider here only $x+x'+x''+\text{etc.}-n+1$ urns.

In order to have the number of cases in which we can bring forth x white balls at the $(x+i)^{\text{th}}$ drawing, it is necessary to determine all the cases in which $x-1$ white balls will have come forth at the drawing $x+i-1$. This number is the term multiplied by p^{x-1} in the development of the polynomial $(p+q+r+\text{etc.})^{x+i-1}$, and this term is

$$\frac{1.2.3\ldots(x+i-1)}{1.2.3\ldots(x-1)1.2.3\ldots i}p^{x-1}(q+r+\text{etc.})^{i}.$$

By combining it with the p white balls of the urn $x+i$, we will have a product which it will be necessary further to multiply by the number of all the possible cases relative to the $x'+x''+\text{etc.}-n-i+1$ following drawings, and this number is

$$(p+q+r+\text{etc.})^{x'+x''+\text{etc.}-n-i+1};$$

we will have therefore [206]

$$\frac{1.2.3\ldots(x+i-1)}{1.2.3\ldots(x-1)1.2.3\ldots i}p^{x}(q+r+\text{etc.})^{i}(p+q+r+\text{etc.})^{x'+x''+\text{etc.}-n-i+1}; \qquad (a)$$

for the number of cases in which the event can happen precisely at the drawing $x+i$. It is necessary however to exclude from it the cases in which q is raised to the power x', those in which r is raised to the power x'', etc.; because in all these cases, it has already happened in the drawing $x+i-1$, either x' black balls, or x'' red balls, or etc. Thus in the development of the polynomial $(q+r+\text{etc.})^{i}$, it is necessary to have regard only to the terms multiplied by $q^{f}r^{f'}s^{f''}.\text{etc.}$, in which f is less than x', f' is less than x'', f'' is less than x''', etc. The term multiplied by $q^{f}r^{f'}s^{f''}.\text{etc.}$, in this development, is

$$\frac{1.2.3\ldots i}{1.2.3\ldots f.1.2.3\ldots f'.1.2.3\ldots f''.\text{etc.}}q^{f}r^{f'}s^{f''}.\text{etc.}$$

All the terms that we must consider in the function (a) are therefore represented by

$$\begin{aligned}&\frac{1.2.3\dots(x+f+f'+\text{etc.}-1)}{1.2.3\dots(x-1).1.2.3\dots f.1.2.3\dots f'.\text{etc.}}p^x q^f r^{f'}.\text{etc.}\\ &\times(p+q+r+\text{etc.})^{x'+x''+\text{etc.}-f-f'-\text{etc.}-n+1};\end{aligned} \qquad (b)$$

because i is equal to $f+f'+$ etc. Thus by giving in this last function, to f all the integral values from $f=0$ to $f=x'-1$, to f' all the values from $f'=0$ to $f'=x''-1$, and so forth, the sum of all these terms will express the number of cases in which the proposed event can happen in $x+x'+\text{etc.}-n+1$ drawings. It is necessary to divide this sum by the number of all the possible cases, that is by $(p+q+r+\text{etc.})^{x+x'+x''+\text{etc.}-n+1}$. If we designate by p' the probability of drawing a white ball from any one of the urns; by q' that of drawing from it a black ball; by r' that of drawing a red ball, etc., we will have

$$p'=\frac{p}{p+q+r+\text{etc.}},\quad q'=\frac{q}{p+q+r+\text{etc.}},\quad r'=\frac{r}{p+q+r+\text{etc.}},\quad \text{etc.};$$

[207] the function (b) divided by $(p+q+r+\text{etc.})^{x+x'+x''+\text{etc.}-n+1}$; will become thus,

$$\frac{1.2.3\dots(x+f+f'+\text{etc.}-1)}{1.2.3\dots\overline{x-1}.1.2.3\dots f.1.2.3\dots f'.\text{etc.}}p'^x q'^f r'^{f'}.\text{etc.}$$

The sum of the terms which we will obtain by giving to f all the values from $f=0$ to $f=x'-1$, to f' all the values from $f'=0$ to $f'=x''-1$, etc., will be the sought probability to bring forth x white balls before x' black balls, or x'' red balls, or, etc.

We can, after this analysis, determine the lot of a number n of players A, B, C, etc., of whom p', q', r', etc. represent the respective skills, that is, their probabilities to win a coup, when in order to win the game, there lack x coups to player A, x' coups to player B, x'' coups to player C, and so forth; because it is clear that relatively to player A, this reverts to determining the probability to bring forth x white balls before x' black balls, or x'' red balls, etc.; by drawing successively a ball from a number $x+x'+x''+\text{etc.}-n+1$ from urns which contain each p white balls, q black balls, r red balls, etc., p, q, r, etc. being respectively equal to the numerators of the fractions p', q', r', etc. reduced to the same denominator.

§8. The preceding problem can be resolved in a quite simple manner, by the analysis of the generating functions. Let us name $y_{x,x',x'',\text{ etc.}}$ the probability of player A to win the game. At the following coup, this probability is changed into $y_{x-1,x',x'',\text{ etc.}}$, if A wins this coup, and the probability for this is p'. The same probability is changed into $y_{x,x'-1,x'',\text{ etc.}}$, if the coup is won by player B, and the probability for this is q'; it is changed into $y_{x,x',x''-1,\text{ etc.}}$ if the coup is won by player C, and the probability for this is r', and so forth; we have therefore the equation in the partial differences

$$y_{x,x',x'',\text{ etc.}}=p'y_{x-1,x',x'',\text{ etc.}}+q'y_{x,x'-1,x'',\text{ etc.}}+r'y_{x,x',x''-1,\text{ etc.}}+\text{etc.}$$

Let u be a function of t, t', t'', etc., such that $y_{x,x',x'',\text{ etc.}}$ is the coefficient of
[208] $t^x t'^{x'} t''^{x''}.\text{etc.}$ in its development; the preceding equation in the partial differences

will give, by passing from the coefficients to the generating functions,

$$u = u(p't + q't' + r't'' + \text{etc.});$$

whence we deduce

$$1 = p't + q't' + r't'' + \text{etc.};$$

consequently,

$$\frac{1}{t} = \frac{p'}{1 - q't' - r't'' - \text{etc.}};$$

that which gives

$$\frac{u}{t^x} = \frac{up'^x}{(1 - q't' - r't'' - \text{etc.})^x} = up'^x \left\{ \begin{aligned} &1 + x(q't' + r't'' + \text{etc.}) \\ &\quad + \frac{x(x+1)}{1.2}(q't' + r't'' + \text{etc.})^2 \\ &\quad + \frac{x(x+1)(x+2)}{1.2.3}(q't' + r't'' + \text{etc.})^3 \\ &\quad + \text{etc.} \end{aligned} \right\}.$$

Now the coefficient of $t^0 t'^{x'} t''^{x''}.\text{etc.}$ in $\frac{u}{t^x}$ is $y_{x,x',x'',\text{ etc.}}$; and the same coefficient in any term of the last member of the preceding equation, such as $ku.p'^x t'^{l'} t''^{l''}$, etc., is $kp'^x y_{0,x'-l',x''-l'',\text{ etc.}}$; the quantity $y_{0,x'-l',x''-l'',\text{ etc.}}$ is equal to unity, since then player A lacks no coup. Moreover, it is necessary to reject all the values of $y_{0,x'-l',x''-l'',\text{ etc.}}$ in which l' is equal or greater than x', l'' is equal or greater than x'', and so forth, because these terms are not able to be given by the equation in the partial differences, the game being finite, when any one of the players B, C, etc. have no more coups to play; it is necessary therefore to consider in the last member of the preceding equation, only the powers of t' less than x', only the powers of t'' less than x'', etc. The preceding expression of $\frac{u}{t^x}$ will give thus, by passing again from the generating functions to the coefficients,

$$y_{x,x',x'',\text{ etc.}} = p'^x \left\{ \begin{aligned} &1 + x(q' + r' + \text{etc.}) \\ &\quad + \frac{x(x+1)}{1.2}(q' + r' + \text{etc.})^2 \\ &\quad + \frac{x(x+1)(x+2)}{1.2.3}(q' + r' + \text{etc.})^3 \\ &\quad + \text{etc.} \end{aligned} \right\},$$

provided that we reject the terms in which the power of q' surpasses $x' - 1$, those in [209]
which the power of r' surpasses $x'' - 1$, etc. The second member of this equation is developed into one sequence of terms comprehended in the general formula

$$\frac{1.2.3\ldots(x + f + f' + \text{etc.} - 1)}{1.2.3\ldots(x-1).1.2.3\ldots f.1.2.3\ldots f'.\text{etc.}} p'^x q'^f r'^{f'}.\text{etc.}$$

The sum of these terms relative to all the values of f, from f null to $f = x' - 1$, to all the values of f', from f' null to $f' = x'' - 1$, etc., will be the probability $y_{x,x',x'',\text{ etc.}}$; that which is conformed to that which precedes.

In the case of two players A and B, we will have for the probability of player A,

$$p'^x\left\{1+xq'+\frac{x(x+1)}{1.2}q'^2\cdots+\frac{x(x+1)(x+2)\dots(x+x'-2)}{1.2.3\dots(x'-1)}q'^{x'-1}\right\}.$$

By changing p' into q', and x into x', and reciprocally, we will have

$$q'^{x'}\left\{1+x'p'+\frac{x'(x'+1)}{1.2}p'^2\cdots+\frac{x'(x'+1)(x'+2)\dots(x+x'-2)}{1.2.3\dots(x-1)}p'^{x-1}\right\}$$

for the probability that player B will win the game. The sum of these two expressions must be equal to unity, that which we see evidently by giving them the following forms. The first expression can, by §37 of the first Book, be transformed into this one

$$p'^{x+x'-1}\left\{\begin{aligned}1+\frac{(x+x'-1)}{1}\cdot\frac{q'}{p'}+\frac{(x+x'-1)(x+x'-2)}{1.2}\cdot\frac{q'^2}{p'^2}\\ \dots+\frac{(x+x'-1)\dots(x+1)}{1.2.3\dots(x'-1)}\cdot\frac{q'^{x'-1}}{p'^{x'-1}}\end{aligned}\right\};$$

and the second can be transformed into this one,

$$q'^{x+x'-1}\left\{\begin{aligned}1+\frac{(x+x'-1)}{1}\cdot\frac{p'}{q'}+\frac{(x+x'-1)(x+x'-2)}{1.2}\cdot\frac{p'^2}{q'^2}\\ \dots+\frac{(x+x'-1)\dots(x'+1)}{1.2.3\dots(x-1)}\cdot\frac{p'^{x-1}}{q'^{x-1}}\end{aligned}\right\}.$$

The sum of these expressions is the development of the binomial $(p'+q')^{x+x'-1}$,
[210] and consequently it is equal to unity; because A or B needing to win each coup, the sum $p'+q'$ of their probabilities for this, is unity.

The problem which we just resolved, is the one which we name the *problem of points* in the analysis of chances. The chevalier de Meré proposed it to Pascal, with some other problems on the game of dice. Two players of whom the skills are equal, have put into the game the same sum; they must play until one of them has beat a given number of given times, his adversary; but they agree to quit the game, when there lack yet x points to the first player in order to attain this given number, and when there lack x' points to the second player. We demand in what way they must share the sum put into the game. Such is the problem that Pascal resolved by means of his arithmetic triangle. He proposed it to Fermat who gave the solution to it by way of combinations; that which occasioned between these two great geometers a discussion, after which Pascal recognized the goodness of the method of Fermat, for any number of players. Unhappily we have only one part of their correspondence, in which we see the first elements of the theory of probabilities, and their application to one of the most curious problems of this theory.[6]

The problem proposed by Pascal to Fermat, reverts to determining the respective probabilities of the players in order to win the game; because it is clear that the stake must be shared between the players, proportionally to their probabilities. These probabilities are the same as those of two players A and B, who must attain a given

[6]For this correspondence, see F.N. David, *Games, gods and gambling*, [**1**].

number of points, x being the number of those which are lacking to player A, and x' being the number of those which are lacking to player B, by imagining an urn containing two balls of which one is white and the other black, both bearing the no. 1, the white ball being for player A, and the black ball for player B. We draw successively one of these balls, and we return it into the urn after each drawing. By naming $y_{x,x'}$ the probability that player A will attain first, the given number of points,
or, that which reverts to the same, that he will have x points before B has x', we will [211]
have

$$y_{x,x'} = \frac{1}{2}y_{x-1,x'} + \frac{1}{2}y_{x,x'-1};$$

because if the ball that we extract is white, $y_{x,x'}$ is changed into $y_{x-1,x'}$, and if the ball extracted is black, $y_{x,x'}$ is changed into $y_{x,x'-1}$, and the probability of each of these events is $\frac{1}{2}$; we have therefore the preceding equation.

The generating function of $y_{x,x'}$ in this equation in the partial differences, is, by §20 of the first Book,

$$\frac{M}{1 - \frac{1}{2}t - \frac{1}{2}t'},$$

M being an arbitrary function of t'. In order to determine it, we will observe that $y_{0,0}$ can not take place, since the game ceases, when one or the other of the variables x and x' is null; M must therefore have t' for factor. Moreover $y_{0,x'}$ is unity, whatever be x'; the probability of player A is changing then into certitude: now the generating function of unity, is generally $\frac{t'^i}{1-t'}$, because the coefficients of the powers of t' in the development of this function, are all equal to unity; in the present case, $y_{0,x'}$ being able to hold when x' is either 1, or 2, or 3, etc., i must be equal to unity; the generating function of $y_{0,x'}$ is therefore equal to $\frac{t'}{1-t'}$; this is the coefficient of t^0 in the development of the generating function of $y_{x,x'}$ or in

$$\frac{M}{1 - \frac{1}{2}t - \frac{1}{2}t'};$$

we have therefore

$$\frac{M}{1 - \frac{1}{2}t'} = \frac{t'}{1 - t'},$$

that which gives

$$M = \frac{t'(1 - \frac{1}{2}t')}{(1 - t')};$$

consequently the generating function of $y_{x,x'}$ is

$$\frac{t'(1 - \frac{1}{2}t')}{(1 - t')(1 - \frac{1}{2}t - \frac{1}{2}t')}.$$

By developing it with respect to the powers of t, we have [212]

$$\frac{t'}{1 - t'}\left(1 + \frac{1}{2}\cdot\frac{t}{1 - \frac{1}{2}t'} + \frac{1}{2^2}\cdot\frac{t^2}{(1 - \frac{1}{2}t')^2} + \frac{1}{2^3}\cdot\frac{t^3}{(1 - \frac{1}{2}t')^3} + \text{etc.}\right).$$

The coefficient of t^x in this series, is

$$\frac{1}{2^x}\cdot\frac{t'}{(1-t')(1-\frac{1}{2}t')^x};$$

$y_{x,x'}$ is therefore the coefficient of $t'^{x'}$ in this last quantity: now we have

$$\frac{t'}{(1-t')(1-\frac{1}{2}t')^x} = \frac{t'+\frac{1}{2}x\,t'^2+\frac{1}{2^2}\,\frac{x(x+1)}{2}\,t'^3\cdots+\frac{1}{2^{x'-1}}\,\frac{x(x+1)(x+2)\ldots(x+x'-2)}{1.2.3\ldots(x'-1)}\,t'^{x'}+\text{etc.}}{1-t'}.$$

By reducing into series the denominator of this last fraction, and multiplying the numerator by this series, we see that the coefficient of $t'^{x'}$ in this product, is that which this numerator becomes when we make $t'=1$; we have therefore

$$y_{x,x'}=\frac{1}{2^x}\left\{\begin{array}{c}1+x\cdot\frac{1}{2}+\frac{x(x+1)}{1.2}\cdot\frac{1}{2^2}+\frac{x(x+1)(x+2)}{1.2.3}\cdot\frac{1}{2^3}\\ \cdots+\frac{x(x+1)\ldots(x+x'-2)}{1.2.3\ldots(x'-1)}\cdot\frac{1}{2^{x'-1}}\end{array}\right\};$$

a result conformed to that which precedes.

Let us imagine presently that there is in the urn a white ball bearing the no. 1, and two black balls, of which one bears the no. 1, and the other bears the no. 2, the white ball being favorable to A, and the black balls to his adversary: each ball diminishing by its value, the number of points which lack to the player to which it is favorable. $y_{x,x'}$ being always the probability that player A will attain first the given number, we will have the equation in the partial differences

$$y_{x,x'}=\frac{1}{3}\,y_{x-1,x'}+\frac{1}{3}\,y_{x,x'-1}+\frac{1}{3}\,y_{x,x'-2};$$

[213] because in the following drawing, if the white balls exits, $y_{x,x'}$ becomes $y_{x-1,x'}$; if the black ball numbered 1 exits, $y_{x,x'}$ becomes $y_{x,x'-1}$; and if the black ball numbered 2 exits, $y_{x,x'}$ becomes $y_{x,x'-2}$, and the probability of each of these events is $\frac{1}{3}$.

The generating function of $y_{x,x'}$ is

$$\frac{M}{1-\frac{1}{3}t-\frac{1}{3}t'-\frac{1}{3}t'^2},$$

M being an arbitrary function of t', which must, by that which precedes, have for factor t', and in the present case, be equal to

$$\frac{t'}{1-t'}\cdot\left(1-\frac{1}{3}t'-\frac{1}{3}t'^2\right);$$

so that the generating function of $y_{x,x'}$ is

$$\frac{t'(1-\frac{1}{3}t'-\frac{1}{3}t'^2)}{(1-t')(1-\frac{1}{3}t-\frac{1}{3}t'-\frac{1}{3}t'^2)}:$$

The coefficient of t^x in the development of this function, is

$$\frac{1}{3^x}\cdot\frac{t'}{1-t'}\cdot\frac{1}{(1-\frac{1}{3}t'-\frac{1}{3}t'^2)^x};$$

and there results from this that we just said, that the coefficient of $t'^{x'}$ in the development of this last quantity, is equal to

$$\frac{1}{3^x}\cdot\left\{\begin{array}{l} t'+\dfrac{xt'^2(1+t')}{3}+\dfrac{x(x+1)}{1.2}\cdot\dfrac{t'^3(1+t')^2}{3^2}\\ \quad+\dfrac{x(x+1)(x+2)}{1.2.3}\cdot\dfrac{t'^4(1+t')^3}{3^3}+\text{etc.}\end{array}\right\};$$

by rejecting from the development in this series, all the powers of t' superior to $t'^{x'}$, and supposing in this that we conserve, $t'=1$, this will be the expression of $y_{x,x'}$.

It is easy to translate this process into a formula. Thus by supposing x' even and equal to $2r+2$, we find

$$\begin{aligned} y_{x,x'} = & \frac{1}{3^x}\left\{1+x\cdot\frac{2}{3}+\frac{x(x+1)}{1.2}\left(\frac{2}{3}\right)^2\cdots\frac{x(x+1)\dots(x+r-1)}{1.2.3\dots r}\left(\frac{2}{3}\right)^r\right\}\\ & +\frac{x(x+1)\dots(x+r)}{1.2.3\dots(r+1)\,3^{x+r+1}}\left\{1+(r+1)+\frac{(r+1)r}{1.2}\cdots+\frac{(r+1)r\dots 2}{1.2.3\dots r}\right\}\\ & +\frac{x(x+1)\dots(x+r+1)}{1.2.3\dots(r+2)\,3^{x+r+2}}\left\{1+(r+2)\cdots+\frac{(r+2)(r+1)\dots 4}{1.2.3\dots(r-1)}\right\}\\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ & +\frac{x(x+1)\dots(x+2r)}{1.2.3\dots(2r+1)\,3^{x+2r+1}}. \end{aligned}$$

If we suppose x' odd and equal to $2r+1$, we will have [214]

$$\begin{aligned} y_{x,x'} = & \frac{1}{3^x}\left\{1+x\cdot\frac{2}{3}+\frac{x(x+1)}{1.2}\left(\frac{2}{3}\right)^2\cdots+\frac{x(x+1)\dots(x+r-1)}{1.2.3\dots r}\left(\frac{2}{3}\right)^r\right\}\\ & +\frac{x(x+1)\dots(x+r)}{1.2.3\dots(r+1)\,3^{x+r+1}}\left\{1+(r+1)+\frac{(r+1)r}{1.2}\cdots+\frac{(r+1)r\dots 3}{1.2.3\dots(r-1)}\right\}\\ & +\frac{x(x+1)\dots(x+r+1)}{1.2.3\dots(r+2)\,3^{x+r+2}}\left\{1+(r+2)+\frac{(r+2)(r+1)}{1.2}\cdots+\frac{(r+2)(r+1)\dots 5}{1.2.3\dots(r-2)}\right\}\\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ & +\frac{x(x+1)\dots(x+2r-1)}{1.2.3\dots 2r\,3^{x+2r}}. \end{aligned}$$

Thus in the case of $x=2$ and $x'=5$, we have

$$y_{2,5}=\frac{350}{729}.$$

Let us imagine further that there are in the urn two white balls distinguished as the two black balls, by the nos. 1 and 2; the probability of player A will be given by the equation in the partial differences

$$y_{x,x'} = \frac{1}{4}y_{x-1,x'} + \frac{1}{4}y_{x-2,x'} + \frac{1}{4}y_{x,x'-1} + \frac{1}{4}y_{x-1,x'-2}.$$

The generating function of $y_{x,x'}$ is then, by §20 of the first Book,

$$\frac{M+Nt}{1-\frac{1}{4}t-\frac{1}{4}t'-\frac{1}{4}t^2-\frac{1}{4}t'^2};$$

M and N being two arbitrary functions of t'. In order to determine them, we will observe that $y_{0,x'}$ is always equal to unity, and that it is necessary to exclude in M the null power of t'; we have therefore

$$M = \frac{t'}{1-t'}\left(1-\frac{1}{4}t'-\frac{1}{4}t'^2\right).$$

In order to determine N, let us seek the generating function of $y_{1,x'}$. If we observe that $y_{0,x'}$ is equal to unity, and that player A having no more need but of one point, he wins the game, either that he brings forth the white ball numbered 1, or the white ball numbered 2; the preceding equation in the partial differences will give

$$y_{1,x'} = \frac{1}{2} + \frac{1}{4}y_{1,x'-1} + \frac{1}{4}y_{1,x'-2}.$$

[215] Let us suppose $y_{1,x'} = 1 - y'_{x'}$; we will have

$$y'_{x'} = \frac{1}{4}y'_{x'-1} + \frac{1}{4}y'_{x'-2}.$$

The generating function of this equation is

$$\frac{m+nt'}{1-\frac{1}{4}t'-\frac{1}{4}t'^2},$$

m and n being two constants. In order to determine them, we will observe that $y_{1,0} = 0$, and that consequently $y'_0 = 1$, that which gives $m = 1$. The generating function of $y'_{x'}$ is therefore

$$\frac{1+nt'}{1-\frac{1}{4}t'-\frac{1}{4}t'^2}.$$

We have next evidently $y_{1,1} = \frac{1}{2}$, that which gives $y'_1 = \frac{1}{2}$; y'_1 is the coefficient of t' in the development of the preceding function, and this coefficient is $n + \frac{1}{4}$; we have therefore $n + \frac{1}{4} = \frac{1}{2}$, or $n = \frac{1}{4}$. The generating function of unity is $\frac{1}{1-t'}$, because here all the powers of t' can be admitted; we have thus

$$\frac{1}{1-t'} - \frac{1+\frac{1}{4}t'}{1-\frac{1}{4}t'-\frac{1}{4}t'^2}, \quad \text{or} \quad \frac{\frac{1}{2}t'}{(1-t')(1-\frac{1}{4}t'-\frac{1}{4}t'^2)},$$

for the generating function of $y_{1,x'}$. This same function is the coefficient of t in the development of the generating function of $y_{x,x'}$, a function which, by that which precedes, is

$$\frac{\frac{t'}{1-t'}(1-\frac{1}{4}t'-\frac{1}{4}t'^2)+Nt}{1-\frac{1}{4}t-\frac{1}{4}t'-\frac{1}{4}t^2-\frac{1}{4}t'^2};$$

this coefficient is

$$\frac{\frac{1}{4}t'}{(1-t')(1-\frac{1}{4}t'-\frac{1}{4}t'^2)}+\frac{N}{1-\frac{1}{4}t'-\frac{1}{4}t'^2};$$

by equating it to

$$\frac{\frac{1}{2}t'}{(1-t')(1-\frac{1}{4}t'-\frac{1}{4}t'^2)};$$

we will have

$$N=\frac{\frac{1}{4}t'}{1-t'}.$$

The generating function of $y_{x,x'}$ is thus [216]

$$\frac{t'(1-\frac{1}{4}t'-\frac{1}{4}t'^2)+\frac{1}{4}tt'}{(1-t')(1-\frac{1}{4}t-\frac{1}{4}t'-\frac{1}{4}t^2-\frac{1}{4}t'^2)}.$$

If we develop into series the function

$$\frac{t'(1-\frac{1}{4}t'-\frac{1}{4}t'^2)+\frac{1}{4}tt'}{1-\frac{1}{4}t-\frac{1}{4}t'-\frac{1}{4}t^2-\frac{1}{4}t'^2}-t';$$

we will have

$$\frac{(2+t)tt'}{4}\left\{\begin{aligned}&1+\frac{1}{4}t'(1+t')+\frac{1}{4^2}t'^2(1+t')^2+\frac{1}{4^3}t'^3(1+t')^3+\text{etc.}\\ &+\frac{t(1+t)}{4}\left[1+\frac{2}{4}t'(1+t')+\frac{3}{4^2}t'^2(1+t')^2+\frac{4}{4^3}t'^3(1+t')^3+\text{etc.}\right]\\ &+\frac{t^2(1+t)^2}{4^2}\left[1+\frac{3}{4}t'(1+t')+\frac{3.4}{1.2.4^2}t'^2(1+t')^2+\frac{3.4.5}{1.2.3.4^3}t'^3(1+t')^3+\text{etc.}\right]\\ &+\frac{t^3(1+t)^3}{4^3}\left[1+\frac{4}{4}t'(1+t')+\frac{4.5}{1.2.4^2}t'^2(1+t')^2+\frac{4.5.6}{1.2.3.4^3}t'^3(1+t')^3+\text{etc.}\right]\\ &+\text{etc.}\end{aligned}\right\}.$$

If we reject from this series, all the powers of t other than t^x, and all the powers of t' superior to $t'^{x'}$, and if in that which remains, we make $t=1$, $t'=1$, we will have the expression of $y_{x,x'}$ when x is equal or greater than unity: when x is null, we have $y_{0,x'}=1$. It is easy to translate this process into a formula, as we have done for the preceding case.

Let us name $z_{x,x'}$ the probability of player B; the generating function of $z_{x,x'}$ will be that which the generating function of $y_{x,x'}$ becomes when we change in it t into t', and reciprocally; that which gives for this function,

$$\frac{t(1-\frac{1}{4}t-\frac{1}{4}t^2)+\frac{1}{4}tt'}{(1-t)(1-\frac{1}{4}t-\frac{1}{4}t'-\frac{1}{4}t^2-\frac{1}{4}t'^2)}.$$

By adding the two generating functions, their sum is reduced to

$$\frac{t}{1-t}+\frac{t'}{1-t'}+\frac{tt'}{(1-t)(1-t')},$$

in which the coefficient of $t^x t'^{x'}$ is unity; thus we have

$$y_{x,x'}+z_{x,x'}=1;$$

[217] that which is clear besides, since the game must be necessarily won by one of the players.

§9. Let us imagine in an urn, r balls marked with the n° 1, r balls marked with n° 2, r balls marked with n° 3, and so forth to the n° n. These balls being well mixed in the urn, we draw them successively; we require the probability that there will exit at least one of these balls, at the rank[7] indicated by its label[8], or that there will exit at least two of them, or at least three, etc.

Let us seek first the probability that there will exit at least one of them. For this, we will observe that each ball can exit at its rank, only in the first n drawings; we can therefore here set aside the following drawings; now the total number of balls being rn, the number of their combinations n by n, by having regard for the order that they observe among themselves, is, by that which precedes,

$$rn(rn-1)(rn-2)\dots(rn-n+1);$$

this is therefore the number of all possible cases in the first n drawings.

Let us consider one of the balls marked with the n° 1, and let us suppose that it exits at its rank, or the first. The number of combinations of the $rn-1$ other balls taken $n-1$ by $n-1$, will be

$$(rn-1)(rn-2)\dots(rn-n+1);$$

this is the number of cases relative to the assumption that we just made; and as this assumption can be applied to r balls marked with n° 1, we will have

$$r(rn-1)(rn-2)\dots(rn-n+1)$$

for the number of cases relative to the hypothesis that one of the balls marked with the n° 1 will exit at its rank. The same result holds for the hypothesis that any one

[7] *Translator's note*: This means that a ball marked with 1 will be drawn first, a ball marked with 2 will be drawn second, and so on. In other words, balls will be drawn consecutively by number.

[8] *Translator's note*: The word here is *numéro*, number. However, this refers to the use of a number as a label. In order to distinguish it from *nombre*, number or quantity, I choose to render it as such.

of the $n-1$ other kinds of balls will exit at the rank indicated by its label: by adding therefore all the results relative to these diverse hypotheses, we will have

$$rn(rn-1)(rn-2)\dots(rn-n+1), \qquad (a)$$

for the number of cases in which one ball at least will exit at its rank, provided [218]
however that we remove from them the cases which are repeated.

In order to determine these cases, let us consider one of the balls of the n° 1, exiting first, and one of the balls of the n° 2, exiting second. This case is comprehended twice in the preceding number; for it is comprehended one time in the number of the cases relative to the assumption that one of the balls labeled[9] 1, will exit at its rank, and a second time, in the number of cases relative to the assumption that one of the balls labeled 2, will exit at its rank; and as this extends to any two balls exiting at their rank, we see that it is necessary to subtract from the number of the cases preceding, the number of all the cases in which two balls exit at their rank.

The number of combinations of two balls of different labels is $\frac{n(n-1)}{1.2}r^2$; for the number of the labels being n, their combinations two by two are in number $\frac{n(n-1)}{1.2}$, and in each of these combinations, we can combine the r balls marked with one of the labels, with the r balls marked with the other label. The number of combinations of the $rn-2$ balls remaining, taken $n-2$ by $n-2$, by having regard for the order that they observe among themselves, is

$$(rn-2)(rn-3)\dots(rn-n+1);$$

thus the number of cases relative to the assumption that two balls exit at their rank is

$$\frac{n(n-1)}{1.2}r^2(rn-2)(rn-3)\dots(rn-n+1);$$

by subtracting it from the number (a), we will have

$$\begin{aligned} &rn(rn-1)(rn-2)\dots(rn-n+1)\\ -&\frac{n(n-1)}{1.2}r^2(rn-2)(rn-3)\dots(rn-n+1); \end{aligned} \qquad (a')$$

for the number of all the cases in which one ball at least will exit at its rank, provided that we subtract again from this function, the repeated cases, and that we add to them those which are lacking.

These cases are those in which three balls exit at their rank. By naming k this [219]
number, it is repeated three times in the first term of the function (a'); for it can result, in this term, from the three assumptions of each of the three balls exiting at its rank. The number k is likewise comprehended three times in the second term of the function; for it can result from each of the assumptions relative to any two of the three balls exiting at their rank; thus this second term being affected with the $-$ sign, the number k in not found in the function (a'); it is necessary therefore to add it to it in order that it contain all the cases in which one ball at least exits at its rank. The

[9] *Translator's note*: The word is *numérotées*, numbered. I have chosen to render it as labeled for the same reason as above.

number of combinations of n labels taken three by three, is $\frac{n(n-1)(n-2)}{1.2.3}$, and as we can combine the r balls of one of these labels of each combination, with the r balls of the second label, and with the r balls of the third label, we will have the total number of combinations in which three balls exit at their rank, by multiplying $\frac{n(n-1)(n-2)}{1.2.3}r^3$ by $(rn-3)(rn-4)\ldots(rn-n+1)$, a number which expresses that of the combinations of the $rn-3$ remaining balls, taken $n-3$ by $n-3$, by having regard for the order that they observe among themselves. If we add this product to the function (a'), we will have

$$\begin{aligned} &nr(rn-1)(rn-2)\ldots(rn-n+1)\\ -&\frac{n(n-1)}{1.2}r^2(rn-2)(rn-3)\ldots(rn-n+1)\\ +&\frac{n(n-1)(n-2)}{1.2.3}r^3(rn-3)(rn-4)\ldots(rn-n+1); \end{aligned} \qquad (a'')$$

this function expresses the number of all cases in which one ball at least exits at its rank, provided that we subtract from it again the repeated cases. These cases are those in which four balls exit at their rank. By applying here the preceding reasonings, we will see that it is necessary again to subtract from the function (a'') the term

$$\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}r^4(rn-4)(rn-5)\ldots(rn-n+1).$$

By continuing thus, we will have for the expression of the cases in which one ball at least exits at its rank

[220]

$$\begin{aligned} &nr(rn-1)(rn-2)\ldots(rn-n+1)\\ -&\frac{n(n-1)}{1.2}r^2(rn-2)(rn-3)\ldots(rn-n+1)\\ +&\frac{n(n-1)(n-2)}{1.2.3}r^3(rn-3)(rn-4)\ldots(rn-n+1)\\ -&\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}r^4(rn-4)(rn-5)\ldots(rn-n+1)\\ &+\text{etc.} \end{aligned} \qquad \text{(A)}$$

the series being continued as far at it can be. In this function, each combination is not repeated; thus the combination of s balls exiting at their rank, is found here only one time; for this combination is comprehended s times in the first term of the function, since it can result from each of the s balls exiting at its rank; it is subtracted $\frac{s(s-1)}{1.2}$ times in the second term, since it can result from two by two combinations of the s balls exiting at their rank; it is added $\frac{s(s-1)(s-2)}{1.2.3}$ times in the third term, since it can result from the combinations of s letters taken three by three, and so forth; it is therefore, in the function (A), comprehended a number of times equal to

$$s-\frac{s(s-1)}{1.2}+\frac{s(s-1)(s-2)}{1.2.3}-\text{etc.};$$

and consequently equal to $1-(1-1)^s$, or to unity. By dividing the function (A) by the number $rn(rn-1)(rn-2)\dots(rn-n+1)$ of all possible cases, we will have for the expression of the probability that one ball at least will exit at its rank,

$$\begin{aligned} 1 &- \frac{(n-1)r}{1.2(rn-1)} + \frac{(n-1)(n-2)r^2}{1.2.3(rn-1)(rn-2)} \\ &- \frac{(n-1)(n-2)(n-3)r^3}{1.2.3.4(rn-1)(rn-2)(rn-3)} + \text{etc.} \end{aligned} \tag{B}$$

Let us seek now the probability that at least s balls will exit at their rank. The number of cases in which s balls exit a their rank, is, by that which precedes,

$$\frac{n(n-1)(n-2)\dots(n-s+1)}{1.2.3\dots s} r^s (rn-s)(rn-s-1)\dots(rn-n+1), \tag{b}$$

provided that we subtract from this function, the cases which are repeated. These [221]
cases are those in which $s+1$ balls exit at their rank, for they can result in the function, from $s+1$ balls taken s by s; these cases are therefore repeated $s+1$ times in this function; consequently it is necessary to subtract them s times. Now the number of cases in which $s+1$ balls exit at their rank, is

$$\frac{n(n-1)(n-2)\dots(n-s)}{1.2.3\dots(s+1)} r^{s+1}(rn-s-1)(rn-s-2)\dots(rn-n+1).$$

By multiplying it by s, and subtracting it from the function (b), we will have

$$\begin{aligned} &\frac{n(n-1)(n-2)\dots(n-s+1)}{1.2.3\dots s} r^s (rn-s)(rn-s-1)\dots(rn-n+1) \\ &\times \left\{1 - \frac{s(n-s)r}{(s+1)(rn-s)}\right\}. \end{aligned} \tag{b'}$$

In this function, many cases are again repeated, namely, those in which $s+2$ balls exit at their rank; for they result in the first term, from $s+2$ balls exiting at their rank, and taken s by s; they result, in the second term, from $s+2$ balls exiting at their rank, and taken $s+1$ by $s+1$, and moreover multiplied by the factor s, by which we have multiplied the second term. They are therefore comprehended in this function, the number of times $\frac{(s+2)(s+1)}{1.2} - s(s+2)$; thus it is necessary to multiply by unity less this number of times, the number of cases in which $s+2$ balls exit at their rank. This last number is

$$\frac{n(n-1)(n-2)\dots(n-s-1)}{1.2.3\dots(s+2)} r^{s+2}(rn-s-2)(rn-s-3)\dots(rn-n+1);$$

the product in question will be therefore

$$\frac{n(n-1)\dots(n-s-1)}{1.2.3\dots(s+2)} r^{s+2}(rn-s-2)\dots(rn-n+1)\frac{s(s+1)}{1.2}.$$

By adding it to the function (b'), we will have

$$\frac{n(n-1)\dots(n-s+1)}{1.2.3\dots s}r^s(rn-s)(rn-s-1)\dots(rn-n+1)$$
$$\times\left\{\begin{aligned}&1-\frac{s}{s+1}\cdot\frac{(n-s)r}{rn-s}\\&\quad+\frac{s}{s+2}\cdot\frac{(n-s)(n-s-1)r^2}{1.2(rn-s)(rn-s-1)}\end{aligned}\right\}; \qquad (b'')$$

[222] this is the number of all possible cases in which s balls exit at their rank, provided that we subtract from it again the cases which are repeated. By continuing to reason so, and by dividing the final function by the number of all possible cases; we will have for the expression of the probability that s balls at least will exit at their rank,

$$\frac{(n-1)(n-2)\dots(n-s+1)r^{s-1}}{1.2.3\dots s(rn-1)(rn-2)\dots(rn-s+1)}$$
$$\times\left\{\begin{aligned}&1-\frac{s}{s+1}\cdot\frac{(n-s)r}{rn-s}+\frac{s}{s+2}\cdot\frac{(n-s)(n-s-1)r^2}{1.2.(rn-s)(rn-s-1)}\\&\quad-\frac{s}{s+3}\cdot\frac{(n-s)(n-s-1)(n-s-2)r^3}{1.2.3(rn-s)(rn-s-1)(rn-s-2)}+\text{etc.}\end{aligned}\right\}. \qquad \text{(C)}$$

We will have the probability that none of the balls will exit at its rank, by subtracting formula (B) from unity; and we will find, for its expression,

$$\frac{[1.2.3\dots rn]-nr[1.2.3\dots(rn-1)]+\frac{n(n-1)}{1.2}r^2[1.2.3\dots(rn-2)]-\text{etc.}}{1.2.3\dots rn}.$$

We have, by §33 of the first Book, whatever be i,

$$1.2.3\dots i=\int x^i dx c^{-x},$$

the integral being taken from x null to x infinity. The preceding expression can therefore be put under this form

$$\frac{\int x^{rn-n}dx(x-r)^n c^{-x}}{\int x^{rn}dx c^{-x}}. \qquad (o)$$

Let us suppose the number rn of balls in the urn, very great; then by applying to the preceding integrals, the method of §24 of the first Book, we will find more or less nearly, for the integral of the numerator,

$$\frac{\sqrt{2\pi}X^{rn+2}\left(1-\frac{r}{X}\right)^{n+1}c^{-X}}{\sqrt{nX^2+n(r-1)(X-r)^2}},$$

X being the value of x which renders a *maximum*, the function $x^{rn-n}(x-r)^n c^{-x}$. The equation relative to this *maximum* gives for X, the two values

$$X=\frac{rn+r}{2}\pm\frac{\sqrt{r^2(n-1)^2+4rn}}{2}.$$

We can consider here only the greatest of these values which is, to the quantities [223]
nearly, of the order $\frac{1}{rn}$, equal to $rn + \frac{n}{n-1}$; then the integral of the numerator of the function (o) becomes nearly

$$\frac{\sqrt{2\pi}(rn)^{rn+\frac{1}{2}}c^{-rn}\left(1-\frac{1}{n}\right)^{n+1}\sqrt{r}}{\sqrt{(r-1)(1-\frac{1}{n})^2+1}}.$$

The integral of the denominator of the same function is, by §33, quite nearly,

$$\sqrt{2\pi}(rn)^{rn+\frac{1}{2}}c^{-rn};$$

the function (o) becomes thus

$$\frac{\left(1-\frac{1}{n}\right)^{n+1}\sqrt{r}}{\sqrt{(r-1)(1-\frac{1}{n})^2+1}}.$$

We can put it under the form

$$\frac{\left(1-\frac{1}{n}\right)^{n+1}}{\sqrt{(1-\frac{1}{n})^2+\frac{2}{rn}-\frac{1}{rn^2}}},$$

rn being supposed a very great number, this function is reduced quite nearly to this very simple form

$$\left(\frac{n-1}{n}\right)^n.$$

This is therefore the quite close expression of the probability that none of the balls of the urn will exit at its rank, when there is a great number of balls. The hyperbolic logarithm of this expression being

$$-1-\frac{1}{2n}-\frac{1}{3n^2}-\text{etc.};$$

we see that it always increases in measure as n increases; that it is null, when $n=1$,
and that it becomes $\frac{1}{c}$, when n is infinity, c being always the number of which the [224]
hyperbolic logarithm is unity.

Let us imagine now a number i of urns each containing the number n of balls, all of different colors; and that we draw successively all the balls from each urn. We can, by the preceding reasonings, determine the probability that one or more balls of the same color will exit at the same rank in the i drawings. In fact, let us suppose that the ranks of the colors are settled after the complete drawing of the first urn, and let us consider first the first color: let us suppose that it exits first in the drawings of the $i-1$ other urns. The total number of combinations of the $n-1$ other colors from each urn is, by having regard for their situation among them, $1.2.3\ldots(n-1)$; thus the total number of these combinations relative to $i-1$ urns, is $[1.2.3\ldots(n-1)]^{i-1}$; this is the number of cases in which the first color is drawn the first altogether from all these urns; and as there are n colors, we will have

$$n[1.2.3\ldots(n-1)]^{i-1}$$

for the number of cases in which one color at least will arrive at its rank in the drawings from the $i-1$ urns. But there are in this number, some repeated cases: thus the cases where two colors arrive at their rank in these drawings, are comprehended twice in this number; it is necessary therefore to subtract them from it. The number of these cases is, by that which precedes,

$$\frac{n(n-1)}{1.2}[1.2.3\ldots(n-2)]^{i-1};$$

by subtracting it from the preceding number, we will have the function

$$n[1.2.3\ldots(n-1)]^{i-1} - \frac{n(n-1)}{1.2}[1.2.3\ldots(n-2)]^{i-1}.$$

But this function contains itself repeated cases. By continuing to exclude from them, as we have done above relatively to a single urn; by dividing next the final function, by the number of all possible cases, and which is here $[1.2.3\ldots n]^{i-1}$; we will have, for
[225] the probability that one of the $n-1$ colors at least will exit at its rank in the $i-1$ drawings which follow the first,

$$\frac{1}{n^{i-2}} - \frac{1}{1.2[n(n-1)]^{i-2}} + \frac{1}{1.2.3[n(n-1)(n-2)]^{i-2}} - \text{etc.},$$

an expression in which it is necessary to take as many terms as there are units in n. This expression is therefore the probability that at least one of the colors will exit at the same rank in the drawings from the i urns.

§10. Let us consider two players A and B, of whom the skills are p and q, and of whom the first has a tokens, and the second, b tokens. Let us suppose that at each coup, the one who loses gives a token to his adversary, and that the game ends only when one of the players will have lost all his tokens; we demand the probability that one of the players, A for example, will win the game, before or at the n^{th} coup.

This problem can be resolved with facility by the following process which is in some way, mechanical. Let us suppose b equal or less than a, and let us consider the development of the binomial $(p+q)^b$. The first term p^b of this development will be the probability of A to win the game at coup b. We will subtract this term, from the development, and we will subtract similarly the last term q^b, if $b=a$; because then this term expresses the probability of B to win the game at coup b. Next we will multiply the rest by $p+q$. The first term of this product will have for factor p^bq, and, as the exponent b surpasses only by $b-1$ the exponent of q, there results from it that the game cannot be won by player A, at the coup $b+1$, that which is clear besides; because if A has lost a token in the first b coups, he must, in order to win the game win this token plus the b tokens of player B, that which requires $b+2$ coups. But if $a=b+1$, we will subtract from the product, its last term which expresses the probability of the player B to win the game at the coup $b+1$.

We will multiply anew this second remainder, by $p+q$. The first term of the product will have for factor $p^{b+1}q$, and as the exponent of p surpasses by b there the one of q, this term will express the probability of A to win the game at the coup $b+2$.

We will subtract similarly from the product, the last term, if the exponent of q there surpasses by a the one of p. [226]

We will multiply anew this third remainder, by $p+q$, and we will continue these multiplications up to the number of times $n-b$, by subtracting at each multiplication, the first term, if the exponent of p there surpasses by b, the one of q, and the last term, if the exponent of q there surpasses by a, the one of p. This premised, the sum of the first terms thus subtracted, will be the probability of A to win the game, before or at coup n; and the sum of the last terms subtracted will be the similar probability relative to player B.

In order to have an analytic solution of the problem, let $y_{x,x'}$ be the probability of player A to win the game, when he has x tokens, and when he has no more than x' coups to play in order to attain the n coups. This probability becomes at the following coup, either $y_{x+1,x'-1}$, or $y_{x-1,x'-1}$, according as player A wins or loses the coup; now the respective probabilities of these two events are p and q: we have therefore the equation in the partial differences,

$$y_{x,x'} = py_{x+1,x'-1} + qy_{x-1,x'-1}.$$

In order to integrate this equation, we will consider, as previously, a function u of t and of t' generator of $y_{x,x'}$, so that $y_{x,x'}$ be the coefficient of $t^x t'^{x'}$ in the development of this function. In passing again from the coefficients, to the generating functions, the preceding equation will give

$$u = u.\left(\frac{pt'}{t} + qtt'\right);$$

whence we deduce

$$1 = \frac{pt'}{t} + qtt';$$

consequently,

$$\frac{1}{t} = \frac{1}{2pt'} \pm \frac{\sqrt{\frac{1}{t'^2} - 4pq}}{2p};$$

that which gives

$$\frac{1}{t^x} = \frac{1}{(2p)^x}\left(\frac{1}{t'} \pm \sqrt{\frac{1}{t'^2} - 4pq}\right)^x;$$

therefore [227]

$$\frac{u}{t^x t'^{x'}} = \frac{u}{(2p)^x t'^{x'}}\left(\frac{1}{t'} \pm \sqrt{\frac{1}{t'^2} - 4pq}\right)^x.$$

This equation can be put under the following form,

$$\frac{u}{t^x t'^{x'}} = \frac{u}{2(2p)^x t'^{x'}} \times \left\{ \begin{array}{l} \left(\frac{1}{t'} + \sqrt{\frac{1}{t'^2} - 4pq}\right)^x + \left(\frac{1}{t'} - \sqrt{\frac{1}{t'^2} - 4pq}\right)^x \\ \pm \sqrt{\frac{1}{t'^2} - 4pq}\, \frac{\left(\frac{1}{t'} + \sqrt{\frac{1}{t'^2} - 4pq}\right)^x - \left(\frac{1}{t'} - \sqrt{\frac{1}{t'^2} - 4pq}\right)^x}{\sqrt{\frac{1}{t'^2} - 4pq}}. \end{array} \right\}$$

The preceding expression of $\frac{1}{t}$ gives

$$\pm\sqrt{\frac{1}{t'^2} - 4pq} = \frac{2p}{t} - \frac{1}{t'};$$

we have therefore

$$\begin{aligned} \frac{u}{t^x t'^{x'}} =& \frac{u}{2(2p)^x t'^{x'}} \left[\left(\frac{1}{t'} + \sqrt{\frac{1}{t'^2} - 4pq}\right)^x + \left(\frac{1}{t'} - \sqrt{\frac{1}{t'^2} - 4pq}\right)^x\right] \\ &+ \frac{u\left(\frac{1}{t} - \frac{1}{2pt'}\right)}{2(2p)^{x-1} t'^{x'}} \, \frac{\left(\frac{1}{t'} + \sqrt{\frac{1}{t'^2} - 4pq}\right)^x - \left(\frac{1}{t'} - \sqrt{\frac{1}{t'^2} - 4pq}\right)^x}{\sqrt{\frac{1}{t'^2} - 4pq}} : \end{aligned}$$

under this form, the ambiguity of the $\pm$ sign disappears.

Now if we pass again from the generating functions to their coefficients, and if we observe that $y_{0,x'}$ is null, because player A loses the game necessarily, when he has no more tokens; the preceding equation will give, by passing again from the generating functions to the coefficients,

$$\begin{aligned} y_{x,x'} &= \frac{1}{2^x p^{x-1}} \\ &\times [X^{(x-1)} y_{1,x+x'-1} + X^{(x-3)} y_{1,x+x'-3} \cdots + X^{(x-2r-1)} y_{1,x+x'-2r-1} + \text{etc.}], \end{aligned}$$

the series of the second member being arrested when $x - 2r - 1$ has a negative value.
[228] $X^{(x-1)}$, $X^{(x-3)}$, etc., are the coefficients of $\frac{1}{t'^{x-1}}$, $\frac{1}{t'^{x-3}}$, etc., in the development of the function

$$\frac{\left(\frac{1}{t'} + \sqrt{\frac{1}{t'^2} - 4pq}\right)^x - \left(\frac{1}{t'} - \sqrt{\frac{1}{t'^2} - 4pq}\right)^x}{\sqrt{\frac{1}{t'^2} - 4pq}} \qquad (i)$$

If we name u' the coefficient of t^x in the development of u, u' will be a function of t' and of x, generator of $y_{x,x'}$. If we name similarly T' the coefficient of t in the development of u, the product of $\frac{T'}{2^x p^{x-1}}$ by the function (i), will be the generating function of the second member of the preceding equation; this function is therefore equal to u'. Let us suppose $x = a + b$, then $y_{x,x'}$ becomes $y_{a+b,x'}$, and this quantity is equal to unity; because it is certain that A has won the game, when he has won all the tokens of B; u' is therefore then the generating function of unity; now x' is here zero or an even number, because the number of coups in which A can win the game, is equal to b plus an even number: indeed, A must for this win all the tokens of B, and moreover he must win again each token that he has lost, that which requires two coups. Next n expressing a number of coups in which A can win the game, it is equal to b plus an even number; x' being the number of coups which are lacking to player A in order to arrive to n, is therefore zero or an even number. Thence it follows in the case of $x = a + b$, u' becomes $\frac{1}{1-t'^2}$; we have therefore

$$\frac{T'}{2^{a+b}p^{a+b-1}}\frac{\left(\frac{1}{t'}+\sqrt{\frac{1}{t'^2}-4pq}\right)^{a+b}-\left(\frac{1}{t'}-\sqrt{\frac{1}{t'^2}-4pq}\right)^{a+b}}{\sqrt{\frac{1}{t'^2}-4pq}}=\frac{1}{1-t'^2};$$

that which gives the value of T'. By multiplying it by the function (i) divided by $2^a p^{a-1}$, and in which we make $x = a$, we will have the generating function of $y_{a,x'}$ equal to

$$\frac{2^b p^b t'^b[(1+\sqrt{1-4pqt'^2})^a-(1-\sqrt{1-4pqt'^2})^a]}{(1-t'^2)[(1+\sqrt{1-4pqt'^2})^{a+b}-(1-\sqrt{1-4pqt'^2})^{a+b}]}. \qquad (o)$$

In the case of $a = b$, it becomes [229]

$$\frac{2^a p^a t'^a}{(1-t'^2)[(1+\sqrt{1-4pqt'^2})^a+(1-\sqrt{1-4pqt'^2})^a]}.$$

By developing the function

$$(1+\sqrt{1-4pqt'^2})^a-(1-\sqrt{1-4pqt'^2})^a, \qquad (q)$$

according to the powers of t'^2, the radical disappears, and the highest exponent of t' in this development, is equal to or smaller than a. But if we develop $(1-\sqrt{1-4pqt'^2})^a$ according to the powers of t'^2, the least exponent of t' will be $2a$; the function (q) is therefore equal to the development of $(1+\sqrt{1-4pqt'^2})^a$, by rejecting the powers of t' superior to a.

Now we have, by §3 of the first Book,

$$z^a = 1 - a\alpha + \frac{a(a-3)}{1.2}\alpha^2 - \frac{a(a-4)(a-5)}{1.2.3}\alpha^3 + \text{etc.},$$

z being one of the roots of the equation

$$z = 1 - \frac{\alpha}{z},$$

which is reduced to unity, when α is null. This root is

$$\frac{1+\sqrt{1-4\alpha}}{2};$$

by supposing therefore $\alpha = pqt'^2$, we will have

$$\left(1+\sqrt{1-4pqt'^2}\right)^a = 2^a\left\{1 - apqt'^2 + \frac{a(a-3)}{1.2}p^2q^2t'^4 - \frac{a(a-4)(a-5)}{1.2.3}p^3q^3t'^6 + \text{etc.}\right\};$$

we will have thus,

$$\frac{2^ap^at'^a}{\left(1+\sqrt{1-4pqt'^2}\right)^a + \left(1-\sqrt{1-4pqt'^2}\right)^a} = \frac{p^at'^a}{1 - apqt'^2 + \frac{a(a-3)}{1.2}p^2q^2t'^4 - \frac{a(a-4)(a-5)}{1.2.3}p^3q^3t'^6 + \text{etc.}},$$

[230] the series of the denominator being continued exclusively until the powers of t' superior to a. This second member must be, by that which precedes, divided by $1 - t'^2$, in order to have the generating function of $y_{a,x'}$; the quantity $y_{a,x'}$ is therefore the sum of the coefficients of the powers of t', by considering in the development of this member, with respect to the powers of t', only the powers equal or inferior to x'. Each of these coefficients will express the probability that A will win the game at the coup indicated by the exponent of the power of t'.

If we name z_i the coefficient corresponding to t'^{a+2i}, we will have generally

$$0 = z_i - apqz_{i-1} + \frac{a(a-3)}{1.2}p^2q^2z_{i-2} - \text{etc.};$$

whence it is easy to conclude the values of z_1, z_2, etc., by observing that z_{-1}, z_{-2}, etc. are nulls, and that $z_0 = p^a$. The value of z_i being equal to $y_{a,a+2i} - y_{a,a+2i-2}$, we will have those of $y_{a,a}$, $y_{a,a+2}$, $y_{a,a+4}$, etc. The equation in the partial differences to which we are immediately led, is found thus restored to one equation in the ordinary differences which determines, by integrating it, the value of $y_{a,x'}$. But we can obtain this value by the following process which is applied in the general case where a and b are equal or different between them.

Let us resume the generating function of $y_{a,x'}$ found above; $y_{a,x'}$ is the coefficient of $t^{x'-b}$ in the development of the function

$$2^bp^b\frac{P}{Q(1-t'^2)},$$

by supposing

$$P = \frac{\left(1+\sqrt{1-4pqt'^2}\right)^a - \left(1-\sqrt{1-4pqt'^2}\right)^a}{\sqrt{1-4pqt'^2}}$$

$$Q = \frac{\left(1+\sqrt{1-4pqt'^2}\right)^{a+b} - \left(1-\sqrt{1-4pqt'^2}\right)^{a+b}}{\sqrt{1-4pqt'^2}}.$$

It results from §5 of the first Book, that if we consider the two terms

$$\frac{P}{2t'^{2i}Q}, \qquad -\frac{P}{(1-t'^2)t'^{2i+1}\,\frac{dQ}{dt'}};$$

that we make next successively $t'=1$ and $t'=-1$ in the first term, and t' equal [231]
successively to all the roots of the equation $Q=0$ in the second term; the sum of all the terms which we obtain in this manner, will be the coefficient of t'^{2i} in the development of the fraction

$$\frac{P}{Q(1-t'^2)}.$$

That which the first term produces in this sum is

$$\frac{p^a-q^a}{2^b(p^{a+b}-q^{a+b})}.$$

In order to have the roots of the equation $Q=0$, we make

$$t'=\frac{1}{2\sqrt{pq}\cos\varpi};$$

that which gives

$$Q=\frac{(\cos\varpi+\sqrt{-1}\sin\varpi)^{a+b}-(\cos\varpi-\sqrt{-1}\sin\varpi)^{a+b}}{\sqrt{-1}\sin\varpi\,(\cos\varpi)^{a+b-1}},$$

or

$$Q=\frac{2\sin(a+b)\varpi}{\sin\varpi\,(\cos\varpi)^{a+b-1}}.$$

The roots of the equation $Q=0$ are therefore represented by

$$\varpi=\frac{(r+1)\pi}{a+b},$$

r being a positive whole number which be extended from $r=0$ to $r=a+b-2$. When $a+b$ is an even number, $\frac{1}{2}\pi$ is one of the values of ϖ; it is necessary to exclude it, because, $\cos\varpi$ becoming null then, this value of ϖ does not render Q null. In this case, the equation $Q=0$ has only $a+b-2$ roots; but, as the term depending on the value $\varpi=\frac{1}{2}\pi$, is multiplied in the expression of $y_{a,x'}$, by a positive power of $\cos\frac{(r+1)\pi}{a+b}$, we can conserve the value of r which gives $\varpi=\frac{1}{2}\pi$, since the term which corresponds to it in the expression of $y_{a,x'}$ disappears.

Now we have

$$\frac{dQ}{dt'}=\left(\frac{dQ}{d\varpi}\right)\cdot\frac{d\varpi}{dt'};$$

whence we deduce, by virtue of the equation $\sin(a+b)\varpi=0$, [232]

$$\frac{dQ}{dt'}=\frac{4(a+b)\sqrt{pq}\cos(r+1)\pi}{\sin^2\varpi\,(\cos\varpi)^{a+b-3}}=\frac{4(a+b)\sqrt{pq}\,(-1)^{r+1}}{\sin^2\varpi\,(\cos\varpi)^{a+b-3}},$$

the term

$$\frac{-P}{(1-t'^2)t'^{2i+1}\,\frac{dQ}{dt'}}$$

becomes thus, by observing that

$$P = \frac{2\sin a\varpi}{\sin\varpi\,(\cos\varpi)^{a-1}},$$

$$\frac{(-1)^{r+1}2^{2i+2}(pq)^{i+1}\sin\frac{(r+1)\pi}{a+b}\sin\frac{(r+1)a\pi}{a+b}\left(\cos\frac{(r+1)\pi}{a+b}\right)^{b+2i+1}}{(a+b)\left(p^2-2pq\cos\frac{2(r+1)\pi}{a+b}+q^2\right)}; \qquad (h)$$

the sum of all the terms which we obtain, by giving to r all the whole and positive values, from $r=0$ to $r=a+b-2$, will be that which produces the function

$$\frac{-P}{(1-t'^2)t'^{2i+1}\,\frac{dQ}{dt'}}:$$

we will designate this sum by the characteristic S placed before the function (h).

If we make $r'+1=a+b-(r+1)$, we will have

$$\begin{aligned}
\sin\frac{(r'+1)\pi}{a+b} &= \sin\frac{(r+1)\pi}{a+b},\\
\cos\frac{(r'+1)\pi}{a+b} &= -\cos\frac{(r+1)\pi}{a+b},\\
\cos\frac{2(r'+1)\pi}{a+b} &= \cos\frac{2(r+1)\pi}{a+b},\\
\sin\frac{(r'+1)a\pi}{a+b} &= (-1)^{a+1}\sin\frac{(r+1)a\pi}{a+b}.
\end{aligned}$$

Thence it is easy to conclude that in the function (h), the term relative to $r+1$ is the same as the term relative to $r'+1$; we can therefore double this term, and extend then the characteristic S only to the values of r comprehended from $r=0$ to $r=\frac{a+b-2}{2}$, if
[233] $a+b$ is even, or $r=\frac{a+b-1}{2}$, if $a+b$ is odd. This premised, by observing that

$$\sin\frac{(r+1)a\pi}{a+b} = (-1)^r\sin\frac{(r+1)b\pi}{a+b},$$

we will have

$$\begin{aligned}
y_{a,b+2i} = \frac{p^b(p^a-q^a)}{p^{a+b}-q^{a+b}} &- \frac{2^{b+2i+2}p^b(pq)^{i+1}}{a+b}\\
\times S&\left\{\frac{\sin\frac{2(r+1)\pi}{a+b}\sin\frac{(r+1)b\pi}{a+b}\left(\cos\frac{(r+1)\pi}{a+b}\right)^{b+2i}}{p^2-2pq\cos\frac{2(r+1)\pi}{a+b}+q^2}\right\}.
\end{aligned} \qquad (H)$$

By changing a into b, p into q, and reciprocally, we will have the probability that player B will win the game before the coup $a+2i$, or at this coup.

Let us suppose $a=b$; $\sin\frac{(r+1)a\pi}{a+b}$ will become $\sin\frac{1}{2}(r+1)\pi$. This sine is null, when $r+1$ is even; therefore it suffices then to consider in the expression of $y_{a,a+2i}$, the odd values of $r+1$. By expressing them as $2s+1$, and observing that $\sin\frac{(2s+1)\pi}{2} = (-1)^s$, we will have

$$y_{a,a+2i} = \frac{p^a}{p^a+q^a} - \frac{2^{a+2i+2}p^a(pq)^{i+1}}{a} \times S\left\{\frac{(-1)^s \sin\frac{(2s+1)\pi}{a}\left(\cos\frac{(2s+1)\pi}{2a}\right)^{a+2i}}{p^2 - 2pq\cos\frac{(2s+1)\pi}{a} + q^2}\right\},$$

$2s+1$ needing to comprehend all the odd values contained in $a-1$.

If we change in this expression, p into q, and reciprocally, we will have the probability of player B to win the game in $a+2i$ coups. The sum of these two probabilities will be the probability that the game will end after this number of coups; this last probability is therefore

$$1 - \frac{2^{a+2i+1}}{a}(p^a+q^a)(pq)^{i+1} S\left\{\frac{(-1)^s \sin\frac{(2s+1)\pi}{a}\left(\cos\frac{(2s+1)\pi}{2a}\right)^{a+2i}}{p^2 - 2pq\cos\frac{(2s+1)\pi}{a} + q^2}\right\}.$$

If the skills p and q are equal, this expression becomes [234]

$$1 - \frac{2}{a} S\left\{\frac{(-1)^s \left(\cos\frac{(2s+1)\pi}{2a}\right)^{a+2i+1}}{\sin\frac{(2s+1)\pi}{2a}}\right\}.$$

When $a+2i$ is a large number, we can conclude from it in a manner quite near, the number of coups necessary in order that the probability that the game will end in this number of coups, be equal to a given fraction $\frac{1}{k}$. We will have then

$$\frac{2}{a} S\left\{\frac{(-1)^s \left(\cos\frac{(2s+1)\pi}{2a}\right)^{a+2i+1}}{\sin\frac{(2s+1)\pi}{2a}}\right\} = \frac{k-1}{k},$$

$a+2i$ being supposed a very great number quite superior to the number a, it suffices to consider the term of the first member which corresponds to s null, and then we have

$$a+2i+1 = \frac{\log\left(\frac{a(k-1)}{2k}\sin\frac{\pi}{2a}\right)}{\log(\cos\frac{\pi}{2a})},$$

these logarithms can be at will hyperbolic or tabular.

If in the preceding formulas, we suppose a infinite, b remaining a finite number; we will have the case in which player A plays against player B who has originally the number b of tokens, until he has won all the tokens of B, without that ever the latter is able to beat A, whatever be the number of tokens that he has won from him. In this case, the generating function (o) of $y_{a,x'}$ is reduced to

$$\frac{2^b p^b t'^b}{(1-t'^2)\left(1+\sqrt{1-4pqt'^2}\right)^b};$$

because then $\left(1-\sqrt{1-4pqt'^2}\right)^a$ and $\left(1-\sqrt{1-4pqt'^2}\right)^{a+b}$ developed, contain only infinite powers of t', powers which we must neglect, when we consider only a finite
[235] number of coups. We have by that which precedes

$$\left(1+\sqrt{1-4pqt'^2}\right)^{-b} = \frac{1}{2^b}\left\{\begin{array}{c} 1+bpqt'^2+\dfrac{b(b+3)}{1.2}p^2q^2t'^4+\dfrac{b(b+4)(b+5)}{1.2.3}p^3q^3t'^6 \\ \dots+\dfrac{b(b+i+1)(b+i+2)\dots(b+2i-1)p^iq^it'^{2i}}{1.2.3\dots i}+\text{etc.}\end{array}\right\}.$$

By multiplying this second member by $\frac{2^bp^bt'^b}{1-t'^2}$, the coefficient of t'^{b+2i} will be

$$p^b\left\{1+bpq+\frac{b(b+3)}{1.2}p^2q^2\dots+\frac{b(b+i+1)(b+i+2)\dots(b+2i-1)p^iq^i}{1.2.3\dots i}\right\};$$

this is the value of $y_{a,b+2i}$, or the probability that A will win the game before or at the coup $b+2i$.

This value will be very painful to reduce into numbers, if b and $2i$ were large numbers; it will be especially very difficult to obtain by its means, the number of coups in which A can wager one against one to win the game; but we can attain it easily in this manner.

Let us resume formula (H) found above. In the case of a infinite, and p being supposed equal or greater than q, if we suppose $\frac{(r+1)}{a}\pi=\phi$, and $\frac{\pi}{a}=d\phi$, it becomes

$$y_{a,b+2i}=1-\frac{2^{b+2i+2}p^b(pq)^{i+1}}{\pi}\int\frac{d\phi\sin 2\phi\sin b\phi\,(\cos\phi)^{b+2i}}{p^2-2pq\cos 2\phi\,+q^2},$$

the integral needing to be taken from $\phi=0$ to $\phi=\frac{1}{2}\pi$. In the case of p less than q, the same expression holds, provided that we change the first term 1, into $\frac{p^b}{q^b}$.

If $p=q$, this expression becomes

$$1-\frac{2}{\pi}\int\frac{d\phi\sin b\phi\,(\cos\phi)^{b+2i+1}}{\sin\phi},$$

the integral being taken from ϕ null to $\phi=\frac{1}{2}\pi$. Let us suppose now that b and i are great numbers. The *maximum* of the function

$$\frac{\phi\,(\cos\phi)^{b+2i+1}}{\sin\phi}$$

[236] corresponds to $\phi=0$, that which gives 1 for this *maximum*. The function decreases next with an extreme rapidity, and in the interval where it has a sensible value, we

can suppose

$$\begin{aligned}\log\sin\phi &= \log\phi + \log(1 - \tfrac{1}{6}\phi^2) = \log\phi - \tfrac{1}{6}\phi^2,\\ \log(\cos\phi)^{b+2i+1} &= (b+2i+1)\log(1 - \tfrac{1}{2}\phi^2 + \tfrac{1}{24}\phi^4)\\ &= -\frac{(b+2i+1)}{2}\phi^2 - \frac{(b+2i+1)}{12}\phi^4,\end{aligned}$$

that which gives, by neglecting the sixth powers of ϕ, and its fourth powers which are not multiplied by $b+2i+1$,

$$\log\left(\frac{(\cos\phi)^{b+2i+1}}{\sin\phi}\right) = -\log\phi - \frac{(b+2i+\frac{2}{3})}{2}\phi^2 - \frac{(b+2i+\frac{2}{3})}{12}\phi^4;$$

by making therefore

$$a^2 = \frac{b+2i+\frac{2}{3}}{2};$$

we will have

$$\frac{(\cos\phi)^{b+2i+1}}{\sin\phi} = \frac{(1-\frac{a^2}{6}\phi^4)}{\phi}c^{-a^2\phi^2};$$

hence,

$$\int\frac{d\phi\sin b\phi\,(\cos\phi)^{b+2i+1}}{\sin\phi} = \int\frac{d\phi\left(1-\frac{a^2}{6}\phi^4\right)}{\phi}\sin b\phi\, c^{-a^2\phi^2}.$$

This last integral can be taken from $\phi = 0$ to ϕ infinity; because it must be taken from $\phi = 0$ to $\phi = \frac{1}{2}\pi$; now a^2 being a considerable number, $c^{-a^2\phi^2}$ becomes excessively small, when we make $\phi = \frac{1}{2}\pi$, so that we can suppose it null, seeing the extreme rapidity with which this exponential diminishes, when ϕ increases. Now we have

$$\frac{d}{db}\int\frac{d\phi\left(1-\frac{a^2}{6}\phi^4\right)}{\phi}\sin b\phi\, c^{-a^2\phi^2} = \int d\phi\left(1-\frac{a^2}{6}\phi^4\right)\cos b\phi\, c^{-a^2\phi^2};$$

we have besides, by §25 of the first Book,

$$\begin{aligned}\int d\phi\cos b\phi\, c^{-a^2\phi^2} &= \frac{\sqrt{\pi}}{2a}c^{-\frac{b^2}{4a^2}},\\ \int\phi^4\, d\phi\cos b\phi\, c^{-a^2\phi^2} &= \frac{\sqrt{\pi}}{2a}\frac{d^4c^{-\frac{b^2}{4a^2}}}{db^4},\\ &= \frac{3\sqrt{\pi}}{8a^5}c^{-\frac{b^2}{4a^2}}\left(1-\frac{b^2}{a^2}+\frac{b^4}{12.a^4}\right);\end{aligned} \qquad [237]$$

whence we deduce, by supposing

$$\frac{b^2}{4a^2} = t^2,$$

$$\int\frac{d\phi\sin b\phi\,(\cos\phi)^{b+2i+1}}{\sin\phi} = \sqrt{\pi}\left\{\int dt\, c^{-t^2} - \frac{tc^{-t^2}}{8a^2}\left(1-\tfrac{2}{3}t^2\right)\right\}.$$

Thus the probability that A will win the game in the number $b+2i$ coups, is

$$1-\frac{2}{\sqrt{\pi}}\left[\int dt\, c^{-t^2}-\frac{T c^{-T^2}}{8a^2}\left(1-\tfrac{2}{3}T^2\right)\right];$$

the integral being taken from t null to $t=T$, T^2 being equal to $\frac{b^2}{4a^2}$.

If we seek the number of coups in which we can wager one against one that this will take place, we will make this probability equal to $\frac{1}{2}$, that which gives

$$\int dt\, c^{-t^2}=\frac{\sqrt{\pi}}{4}+\frac{T c^{-T^2}}{8a^2}\left(1-\tfrac{2}{3}T^2\right).$$

Let us name T' the value of t, which corresponds to

$$\int dt\, c^{-t^2}=\frac{\sqrt{\pi}}{4};$$

and let us suppose

$$T=T'+q,$$

[238] q being of order $\frac{1}{a^2}$. The integral $\int dt\, c^{-t^2}$ will be increased very nearly by $qc^{-T'^2}$; that which gives

$$qc^{-T'^2}=\frac{T' c^{-T'^2}}{8a^2}\left(1-\tfrac{2}{3}T'^2\right);$$

we will have therefore

$$T^2=T'^2+\frac{T'^2}{4a^2}\left(1-\tfrac{2}{3}T'^2\right).$$

Having therefore T^2 to the quantities near the order $\frac{1}{a^4}$, the equation

$$2a^2=b+2i+\tfrac{2}{3}=\frac{b^2}{2T^2}$$

will give, to the quantities near the order $\frac{1}{a^2}$,

$$b+2i=\frac{b^2}{2T'^2}-\frac{7}{6}+\frac{1}{3}T'^2.$$

In order to determine the value of T'^2, we will observe that here T' is smaller than $\frac{1}{2}$; thus the transcendent and integral equation

$$\int dt\, c^{-t^2}=\frac{\sqrt{\pi}}{4},$$

can be transformed into the following,

$$T'-\frac{1}{3}T'^3+\frac{1}{1.2}\cdot\frac{1}{5}T'^5-\frac{1}{1.2.3}\cdot\frac{1}{7}T'^7+\text{etc.}=\frac{\sqrt{\pi}}{4}.$$

By resolving this equation, we find

$$T'^2=0.2102497.$$

By supposing $b=100$, we will have

$$b+2i=23780,14.$$

There is therefore then disadvantage to wager one against one, that A will win the game in 23780 coups, but there is advantage to wager that he will win it in 23781 coups.

§11. A number $n+1$ of players play together with the following conditions. Two among them play first, and the one who loses is retired after having put a franc into the game, in order to return only after all the other players have played; that which [239] holds generally for all the players who lose, and who thence become the last. The one of the first two players who has won, plays with the third, and, if he wins it, he continues to play with the fourth, and so forth until he loses, or until he has beat successively all the players. In this last case, the game is ended. But if the player winning at the first coup, is vanquished by one of the other players, the vanquisher plays with the following player, and continues to play until he is vanquished, or until he has beat consecutively all the players; the game continues thus until there is one player who beats consecutively all the others, that which ends the game, and then the player who wins it, takes away all that which has been set into the game. This premised,

Let us determine first the probability that the game will end precisely at coup x; let us name z_x this probability. In order that the game finish at coup x, if is necessary that the player who enters into the game at coup $x-n+1$, wins this coup and the $n-1$ coups following; now he is able to enter against a player who has won only a single coup: by naming P the probability of this event, $\frac{P}{2^n}$ will be the corresponding probability that the game will end at coup x. But the probability z_{x-1} that the game will end at coup $x-1$, is evidently $\frac{P}{2^{n-1}}$. Because it is necessary for this that there is a player who has won a coup, at coup $x-n+1$, and who playing at this coup, wins it and the following $n-2$ coups; and the probability of each of these events being P and $\frac{1}{2^{n-1}}$, the probability of the composite event will be $\frac{P}{2^{n-1}}$; we will have therefore $z_{x-1} = \frac{P}{2^{n-1}}$, and consequently,

$$\frac{P}{2^n} = \frac{1}{2} z_{x-1};$$

$\frac{1}{2} z_{x-1}$ is therefore the probability that the game will end at coup x, relative to this case.

If the player who enters into the game at coup $x-n+1$, plays at this coup against a player who has already won two coups; by naming P' the probability of this case, $\frac{P'}{2^n}$ will be the probability relative to this case, that the game will end at coup x. But [240] we have

$$\frac{P'}{2^{n-2}} = z_{x-2};$$

because in order that the game end at coup $x-2$, it is necessary that at coup $x-n+1$, one of the players has already won two coups, and that he wins this coup and the $n-3$ following coups. We have therefore

$$\frac{P'}{2^n} = \frac{1}{2^2} z_{x-2};$$

$\frac{1}{2^2}z_{x-2}$ is therefore the probability that the game will end at coup x, relative to this case; and so forth.

By reassembling all these partial probabilities, we will have

$$z_x = \frac{1}{2}z_{x-1} + \frac{1}{2^2}z_{x-2} + \frac{1}{2^3}z_{x-3}\cdots + \frac{1}{2^{n-1}}z_{x-n+1}.$$

The generating function of z_x is, by the first Book,

$$\frac{\psi(t)}{1-\frac{1}{2}t-\frac{1}{2^2}t^2\cdots-\frac{1}{2^{n-1}}t^{n-1}}$$

or

$$\frac{\frac{1}{2}\psi(t)(2-t)}{1-t+\frac{1}{2^n}t^n}.$$

In order to determine $\psi(t)$, we will observe that the game can end no earlier than at coup n, and that the probability for this is $\frac{1}{2^{n-1}}$; because it is necessary that the vanquisher at the first coup, wins the $n-1$ following coups; $\psi(t)$ must therefore contain only the power n of t, and $\frac{1}{2^{n-1}}$ must be the coefficient of this power; that which gives $\psi(t) = \frac{t^n}{2^{n-1}}$: thus the generating function of z_x is

$$\frac{\frac{1}{2^n}t^n(2-t)}{1-t+\frac{1}{2^n}t^n}.$$

The sum of the coefficients of the powers of t to infinity, in the development of this
[241] function, is the probability that the game must end after an infinity of coups; now we have this sum by making $t=1$ in the function, that which reduces it to unity; it is therefore certain that the game must end.

We will have the probability that the game will be ended at coup x or before this coup, by determining the coefficient of t^x in the development of the preceding function, divided by $1-t$; the generating function of this probability is therefore

$$\frac{\frac{1}{2^n}t^n(2-t)}{(1-t)\left(1-t+\frac{1}{2^n}t^n\right)}.$$

Let us give to the generating function of z_x, this form

$$\frac{1}{2^n}\cdot\frac{t^n(2-t)}{1-t}\left(1-\frac{1}{2^n}\cdot\frac{t^n}{1-t}+\frac{1}{2^{2n}}\cdot\frac{t^{2n}}{(1-t)^2}-\text{etc.}\right);$$

the coefficient of t^x in $\frac{t^{rn}(2-t)}{2^{rn}(1-t)^r}$ is

$$\frac{1}{2^{rn}}\cdot\frac{(x-rn+1)(x-rn+2)\dots(x-rn+r-2)}{1.2.3\dots(r-1)}(x-rn+2r-2);$$

we have therefore

$$z_x = \frac{1}{2^n}-\frac{(x-2n+2)}{2^{2n}}+\frac{(x-3n+1)}{1.2.2^{3n}}(x-3n+4)$$
$$-\frac{(x-4n+1)(x-4n+2)}{1.2.3.2^{4n}}(x-4n+6)+\text{etc.},$$

an expression which is relative only to x greater than n, and in which it is necessary to take only as many terms as there are integral units in the quotient $\frac{x}{n}$: When $x = n$, we have $z_x = \frac{1}{2^{n-1}}$.

By developing in the same manner the generating function of the probability that the game will end before or at coup x, we will find for the expression of this probability,

$$\frac{x-n+2}{2^n} - \frac{(x-2n+1)}{1.2.2^{2n}}(x-2n+4) + \frac{(x-3n+1)(x-3n+2)}{1.2.3.2^{3n}}(x-3n+6) - \text{etc.},$$

this expression holding even in the case $x = n$.

Let us determine now the respective probabilities of the players in order to win the game at coup x. Let $y_{0,x,}$ be that of the player who has won the first coup. Let $y_{1,x}$, $y_{2,x}$, $\dots y_{n-1,x}$ be those of the following players, and $y_{n,x}$ that of the player who has lost at the first coup, and who thence became the last. Let us designate the players by (0), (1), (2), $\dots (n-1)$, (n). This premised, the probability $y_{r,x}$ of player (r) becomes $y_{r-1,x-1}$, if at the second coup player (0) is vanquished by player (1); because it is clear that (r) is found then, with respect to the vanquisher (1), in the same position where $(r-1)$ was with respect to the vanquisher (0); only, there is one coup less to play in order to arrive at coup x, that which changes x into $x-1$. Presently the probability that player (0) will be vanquished by (1) is $\frac{1}{2}$; thus $\frac{1}{2}y_{r-1,\,x-1}$ is the probability of player (r) to win the game at coup x, relative to the case where (0) is vanquished by (1). If (0) is vanquished only by (2), $y_{r,x}$ becomes $y_{r-2,\,x-2}$, and the probability of this event being $\frac{1}{4}$, we have $\frac{1}{4}y_{r-2,\,x-2}$ for the probability of player (r), to win the game at coup x, relative to this case. If player (0) is vanquished only by player (r), $y_{r,x}$ becomes $y_{0,x-r}$, and the probability of this event is $\frac{1}{2^r}$; thus $\frac{1}{2^r}y_{0,x-r}$ is the probability of player (r) to win the game at coup x, relative to this case. If player (0) is vanquished only by player $(r+1)$, $y_{r,x}$ is changed into $y_{n-1,\,x-r-1}$; because then player (r) is found, with respect to the vanquisher, in the original position of player $(n-1)$ with respect to player (0): only there remains only $x-r-1$ coups to play in order to arrive at coup x. Now the probability that (0) will be vanquished only by player $(r+1)$, is $\frac{1}{2^{r+1}}$; $\frac{1}{2^{r+1}}y_{n-1,x-r-1}$ is therefore the probability of (r) to win the game at coup x, relative to this case. By continuing thus, and reassembling all these partial probabilities, we will have the entire probability $y_{r,x}$ of player (r) to win the game; that which gives the following equation: [242]

$$y_{r,x} = \frac{1}{2}y_{r-1,x-1} + \frac{1}{2^2}y_{r-2,\,x-2}\cdots + \frac{1}{2^r}y_{0,x-r} + \frac{1}{2^{r+1}}y_{n-1,\,x-r-1} + \frac{1}{2^{r+2}}y_{n-2,\,x-r-2}\cdots + \frac{1}{2^{n-1}}y_{r+1,\,x-n+1}.$$

This expression holds from $r = 1$ to $r = n-2$. It gives [243]

$$\frac{1}{2}y_{r-1,x-1} = \frac{1}{2^2}y_{r-1,x-2} + \frac{1}{2^3}y_{r-3,\,x-3}\cdots + \frac{1}{2^n}y_{r,x-n}.$$

By subtracting this equation, from the preceding; we will have that here in the partial differences,

$$y_{r,x} - y_{r-1,x-1} + \frac{1}{2^n} y_{r,x-n} = 0; \tag{1}$$

this equation is extended from $r = 2$ to $r = n - 2$.

We have, by the preceding reasoning, the following equation,

$$y_{n-1,x} = \frac{1}{2} y_{n-2,\, x-1} + \frac{1}{2^2} y_{n-3,\, x-2} \cdots + \frac{1}{2^{n-1}} y_{0,x-n+1}.$$

But the preceding expression of $y_{r,x}$ gives

$$\frac{1}{2} y_{n-2,x-1} = \frac{1}{2^2} y_{n-3,x-2} \cdots + \frac{1}{2^{n-1}} y_{0,\, x-n+1} + \frac{1}{2^n} y_{n-1,x-n}.$$

By subtracting this equation from the preceding, we will have

$$y_{n-1,x} - y_{n-2,x-1} + \frac{1}{2^n} y_{n-1,x-n} = 0 :$$

thus equation (1) subsists in the case of $r = n - 1$.

The preceding reasoning leads further to this equation

$$y_{n,x} = \frac{1}{2} y_{n-1,x-1} + \frac{1}{2^2} y_{n-2,\, x-2} \cdots + \frac{1}{2^{n-1}} y_{1,\, x-n+1},$$

that which gives

$$\frac{1}{2} y_{n,\, x-1} = \frac{1}{2^2} y_{n-1,\, x-2} \cdots + \frac{1}{2^n} y_{1,x-n}.$$

By subtracting this equation, from that here which gives the general expression of $y_{r,x}$,

$$y_{1,x} = \frac{1}{2} y_{0,x-1} + \frac{1}{2^2} y_{n-1,\, x-2} \cdots + \frac{1}{2^{n+1}} y_{2,\, x-n+1};$$

[244] and making $\frac{1}{2}(y_{0,x} + y_{n,x}) = \bar{y}_{0,x}$; we will have

$$y_{1,x} - \bar{y}_{0,x-1} + \frac{1}{2^n} y_{1,x-n} = 0.$$

Equation (1) subsists therefore yet even in the case of $r = 1$, provided that we change $y_{0,x}$ into $\bar{y}_{0,x}$. We must observe that $\bar{y}_{0,x}$ is the probability to win the game at coup x, of each of the first two players, at the moment where the game commences; because this probability becomes, after the first coup, $y_{0,x}$ or $y_{n,x}$, according as the player wins or loses, and the probability of each of these events is $\frac{1}{2}$.

Now, the generating function of equation (1) is, by §20 of the first Book,

$$\frac{\phi(t)}{1 - tt' + \frac{1}{2^n} t^n}, \tag{a}$$

t being relative to the variable x, and t' being relative to the variable r, so that $y_{r,x}$ is the coefficient of $t'^r t^x$ in the development of this function; $\phi(t)$ is a function of t that there is concern to determine.

For this, we will make

$$T = \frac{1}{1 + \frac{1}{2^n} t^n};$$

the generating function of $y_{r,x}$ will be the coefficient of t'^r in the development of the function (a); it will be therefore

$$\phi(t) t^r T^{r+1};$$

the probability that the game will end precisely at coup x, is evidently the sum of the probabilities of each player to win at this coup; it is therefore

$$2\bar{y}_{0,x} + y_{1,x} + y_{2,x} \cdots + y_{n-1,x};$$

consequently the generating function of this probability is

$$T\phi(t)(2 + tT + t^2T^2 \cdots + t^{n-1}T^{n-1}),$$

or [245]

$$T\phi(t)\frac{(2 - tT - t^nT^n)}{1 - tT}.$$

By equating it to the generating function of this probability, that we have found above, and which is

$$\frac{\frac{1}{2^n} t^n (2 - t)}{1 - t + \frac{1}{2^n} t^n};$$

we will have

$$\phi(t) = \frac{\frac{1}{2^n} t^n (2 - t)(1 - tT)}{T(2 - tT - t^nT^n)\left(1 - t + \frac{1}{2^n} t^n\right)};$$

Thus the generating function of equation (1) in the partial differences, is

$$\frac{\frac{1}{2^n} t^n (2 - t)(1 - tT)}{T(2 - tT - t^nT^n)\left(1 - t + \frac{1}{2^n} t^n\right)\left(1 - tt' + \frac{1}{2^n} t^n\right)};$$

the generating function of $y_{r,x}$ is therefore

$$\frac{\frac{1}{2^n} t^{n+r} (2 - t)(1 - tT)T^r}{(2 - tT - t^nT^n)\left(1 - t + \frac{1}{2^n} t^n\right)}.$$

The coefficient of t^x in the development of this function, is the probability of player (r) to win the game at coup x. We will thus be able to determine this probability through this development. The sum of all these coefficients to x infinity, is the probability of player (r) to win the game; now we have this sum, by making $t = 1$ in the preceding function, that which gives $T = \frac{2^n}{1+2^n}$; let us name p this last quantity, and let us designate by y_r the probability of (r) to win the game, we will have

$$y_r = \frac{(1 - p)p^r}{2 - p - p^n}.$$

This expression is extended from $r = 0$ to $r = n - 1$, provided that we change y_0 [246]
into $\bar{y}_0$, $\bar{y}_0$ expressing the probability to win the game, of the first two players at the moment where they enter the game.

Now, each losing player depositing a franc into the game, let us determine the advantage of the different players. It is clear that after x coups, there were x tokens in the game; the advantage of player (r) relative to these x tokens, is the product of these tokens by the probability $y_{r,x}$ to win the game at coup x; this advantage is therefore $xy_{r,x}$. The value of $xy_{r,x}$ is the coefficient of $t^{x-1}dt$ in the differential of the generating function $y_{r,x}$; by dividing therefore this differential by dt, and by supposing next $t = 1$, we will have the sum of all the values of $xy_{r,x}$ to x infinity; this is the advantage of player (r). But it is necessary to subtract the tokens that he put into the game at each coup that he loses; now $y_{r,x}$ being his probability to win the game at coup x, $2^n y_{r,x-n+1}$ will be his probability to enter into the game, at coup $x - n + 1$, since this last probability, multiplied by the probability $\frac{1}{2^n}$, that he will win this coup, and the $n - 1$ following coups is his probability to win the game at coup x. By supposing therefore that he loses as many times as he enters into the game, the sum of all the values of $2^n y_{r,x-n+1}$ to x infinity, would be the disadvantage of player (r); and as the sum of all the values of $y_{r,x-n+1}$ is equal to the sum of all the values of $y_{r,x}$, or to y_r, we would have $2^n y_r$, or $\frac{2^n(1-p)p^r}{2-p-p^n}$ for the disadvantage of player (r). But he does not lose each time that he enters into the game, because he is able to enter into the game and win the game; it is necessary therefore to take off from $2^n y_r$, the sum of all the values of y_x or y_r, and then the disadvantage of (r) is $\frac{(2^n-1)(1-p)p^r}{2-p-p^n}$. In order to have the entire advantage of (r), it is necessary to subtract this last quantity, from the sum of the values of $xy_{r,x}$; by designating therefore by S this sum, the advantage of player (r) will be

$$S - \frac{(2^n - 1)(1 - p)p^r}{2 - p - p^n},$$

[247] S being, as we have just seen, the differential of the generating function of $y_{r,x}$, divided by dt, and in which we suppose next $t = 1$. Under this supposition, we have

$$T = p; \quad \frac{dT}{dt} = -np(1 - p).$$

Let us designate by Y_r the advantage of (r), we will find

$$Y_r = \frac{np + 1 - n}{2 - p - p^n} p^r \left\{ (1 - p)r + \frac{p^{n+1} + n(1 - p)p^n - p}{2 - p - p^n} \right\}.$$

This equation will serve from $r = 0$ to $r = n - 1$, provided that we change Y_0 into $\bar{Y}_0$, $\bar{Y}_0$ being the advantage of the first two players, at the moment where they enter into the game.

If at the commencement of the game, each of the players deposits into the game a sum a; the advantage of player (r) will be increased from it by $(n + 1)a$, multiplied by the probability y_r, that this player will win the game; but it is necessary to take off from it the stake a from this player; it is necessary therefore, in order to have then his advantage, to increase the preceding expression of Y_r, by the quantity

$$\frac{(n + 1)a(1 - p)p^r}{2 - p - p^n} - a.$$

When the advantage of (r) becomes negative, it is changed into disadvantage.

§12. Let q be the probability of a simple event, at each coup; we demand the probability to bring it forth i times consecutively, in the number x coups.

Let us name z_x the probability that this composite event will take place precisely at coup x. For this, it is necessary that the simple event not arrive at coup $x-i$, and that it arrives in the i coups following, the composite event being not at all arrived previously. Let then P be the probability that the simple event will not arrive at all at coup $x-i-1$. The corresponding probability that it will not arrive at all at coup $x-i$, will be $(1-q)P$; and the corresponding probability that the composed event will take place precisely at coup x, will be $(1-q)Pq^i$. This will be the part of z_x corresponding to this case. But the probability that the composed event will arrive [248] at coup $x-1$, is evidently Pq^i; we have therefore

$$P = \frac{z_{x-1}}{q^i};$$

thus the partial value of z_x, relative to this case, is $(1-q)z_{x-1}$.

Let us consider now the cases where the simple event will arrive at coup $x-i-1$. Let us name P' the probability that it will not arrive at coup $x-i-2$; the probability that it will arrive in this case at coup $x-i-1$, will be qP', and the probability that it will not arrive at coup $x-i$, will be $(1-q)qP'$; the partial value of z_x relative to this case, will be therefore $(1-q)qP'q^i$. But the probability that the composite event will arrive precisely at coup $x-2$, is $P'q^i$: this is the value of z_{x-2}; that which gives

$$P' = \frac{z_{x-2}}{q^i};$$

$(1-q)qz_{x-2}$ is therefore the partial value of z_x, relative to the case where the simple event will arrive at coup $x-i-1$, without arriving at coup $x-i-2$.

We will find in the same manner that $(1-q)q^2z_{x-3}$ is the partial value of z_x, relative to the case where the simple event will arrive at coups $x-i-1$ and $x-i-2$, without arriving at coup $x-i-3$; and so forth.

By uniting all these partial values of z_x, we will have

$$z_x = (1-q)(z_{x-1} + qz_{x-2} + q^2z_{x-3}\cdots + q^{i-1}z_{x-i}).$$

It is easy to conclude from it that the generating function of z_x is

$$\frac{q^i(1-qt)t^i}{1-t+(1-q)q^it^{i+1}};$$

because this generating function is

$$\frac{\phi(t)}{1-(1-q)(t+qt^2\cdots+q^{i-1}t^i)},$$

or

$$\frac{\phi(t)(1-qt)}{1-t+(1-q)q^it^{i+1}},$$

[249] The function $\phi(t)$ must be determined by the condition that it must contain only the power i of t, since the composed event is able to commence to be possible only at coup i; moreover, the coefficient of this power is the probability q^i, that this event will take place precisely at this coup.

By dividing the preceding generating function, by $1-t$, we will have

$$\frac{q^i(1-qt)t^i}{(1-t)^2\left(1+\frac{(1-q)q^it^{i+1}}{1-t}\right)}$$

for the generating function of the probability that the composite event will take place before or at coup x.

By developing this function, we will have for the coefficient of t^{x+i}, the series

$$\begin{aligned} q^i[(1-q)x+1] &- (1-q)q^{2i}\frac{(x-i)}{1.2}[(1-q)(x-i-1)+2] \\ &+ (1-q)^2q^{3i}\frac{(x-2i)(x-2i-1)}{1.2.3}[(1-q)(x-2i-2)+3] \\ &- (1-q)^3q^{4i}\frac{(x-3i)(x-3i-1)(x-3i-2)}{1.2.3.4}[(1-q)(x-3i-3)+4] \\ &+ \text{etc.}, \end{aligned}$$

the series being continued until we arrive to some negative factors. This is the expression of the probability that the composed event will take place at coup $x+i$ or before this coup.

Let us suppose further that two players A and B, of whom the respective skills to win a coup, are q and $1-q$, play with this condition, that the one of the two who will have first vanquished i times consecutively his adversary, will win the game; we demand the respective probabilities of the two players to win the game precisely at coup x.

Let y_x be the probability of A, and y'_x that of B. Player A is able to win the game at coup x, only as long as he commences or recommences to beat B at coup $x-i+1$, and that he continues to beat him the following $i-1$ coups. Now, before commencing coup $x-i+1$, B will have already beat A, either one time, or two times, ... or $i-1$
[250] times. In the first case, if we name P the probability of this case, $P(1-q)^{i-1}$ will be the probability y'_{x-1} of B to win the game at coup $x-1$, that which gives

$$P=\frac{y'_{x-1}}{(1-q)^{i-1}}.$$

But if B loses at coup $x-i+1$ and at the $i-1$ following coups, A will win the game at coup x, and the probability of this is Pq^i; $\frac{q^iy'_{x-1}}{(1-q)^{i-1}}$ is therefore the part of y_x, relative to the first case.

In the second case, if we name P' its probability, $P'(1-q)^{i-2}$ will be the probability y'_{x-2} of B to win the game at coup $x-2$. The probability of A to win the game at coup x, relative to this case, is $P'q^i$; we have therefore $\frac{q^iy'_{x-2}}{(1-q)^{i-2}}$ for this probability.

By continuing thus, we will have

$$y_x = \frac{q^i}{(1-q)^i}[(1-q)y'_{x-1} + (1-q)^2 y'_{x-2} \cdots + (1-q)^{i-1} y'_{x-i+1}].$$

If we change q into $1-q$, y_x into y'_x and reciprocally, we will have

$$y'_x = \frac{(1-q)^i}{q^i}(qy_{x-1} + q^2 y_{x-2} \cdots + q^{i-1} y_{x-i+1}).$$

Now, u being the generating function of y_x, that of y'_x will be, by all that which precedes,

$$kq.ut.(1 + qt + qt^2 \cdots + q^{i-2}t^{i-2}),$$

k being equal to $\frac{(1-q)^i}{q^i}$. But the preceding expression of y'_x commencing to hold only when $x = i+1$, because for the smaller values of x, y_{x-1}, y_{x-2}, etc. are nulls; it is necessary, in order to complete the preceding expression of the generating function of y'_x, to add to it a rational and integral function of t, of order i, and of which the coefficients of the powers of t are the values of y'_x, when x is equal or smaller than i. Now y'_x is null, when x is less than i; and when it is equal to i, y'_x is $(1-q)^i$, because it expresses then the probability of B to win the game after i coups; the function to [251]
add is therefore $(1-q)^i t^i$; thus the generating function of y'_x is

$$kq.ut.(1 + qt + qt^2 \cdots + q^{i-2}t^{i-2}) + (1-q)^i t^i.$$

If we name u' this function, the expression of y_x in y'_{x-1}, y'_{x-2}, etc., will give for the generating function of y_x, by changing in that of y'_x, k into $\frac{1}{k}$, q into $1-q$,

$$\frac{1}{k}(1-q).u'.t[1 + (1-q)t \cdots + (1-q)^{i-2}t^{i-2}] + q^i t^i.$$

This quantity is therefore equal to u; whence we deduce, by substituting in it for u its preceding value,

$$u = \frac{q^i t^i (1-qt)[1-(1-q)^i t^i]}{1 - t + q(1-q)^i t^{i+1} + (1-q)q^i t^{i+1} - q^i (1-q)^i t^{2i}}.$$

By changing q into $1-q$, we will have the generating function u' of y'_x. If we divide these functions by $1-t$, we will have the generating functions of the respective probabilities of A and of B, to win the game before or at coup x.

If we suppose $t = 1$ in u, we will have the probability that A will win the game; because it is clear that by developing u according to the powers of t, and by supposing next $t = 1$, the sum of all the terms of this development will be that of all the values of y_x. We find thus the probability of A to win the game equal to

$$\frac{[1-(1-q)^i]q^{i-1}}{(1-q)^{i-1} + q^{i-1} - q^{i-1}(1-q)^{i-1}};$$

the probability of B is therefore

$$\frac{(1-q)^{i-1}[1-q^i]}{(1-q)^{i-1} + q^{i-1} - q^{i-1}(1-q)^{i-1}}.$$

Let us suppose now that the players, at each coup that they lose, deposit a franc into the game, and let us determine their respective lot. It is clear that the gain
[252] of player A will be x, if he wins the game at coup x, since there will be x francs deposited into the game; thus the probability of this event being y_x by that which precedes. Sxy_x will be the expression of the advantage of A, the sign S extending to all the possible values of x. The generating function of y_x being u or $\frac{T'}{T}$, T' being the numerator of the preceding expression of u, and T being its denominator; it is easy to see that we will have Sxy_x by differentiating $\frac{T'}{T}$, and by supposing next $t = 1$ in this differential, that which gives with this condition,

$$Sxy_x = \frac{dT'}{Tdt} - \frac{T'dT}{T^2dt}.$$

In order to have the disadvantage of A, we will observe that at each coup that he plays, the probability that he will lose, and consequently that he will deposit a franc into the game, is $1-q$; his loss is therefore the product of $1-q$, by the probability that the coup will be played; now the probability that coup x will be played, is $1 - Sy_{x-1} - Sy'_{x-1}$; the generating function of unity, is here $\frac{t}{1-t}$, and that of $Sy_{x+1} + Sy'_{x+1}$ is $\frac{T't+T''t}{T(1-t)}$; T'' being that which T' becomes when we change q into $1 - q$ and reciprocally; thus the generating function of the disadvantage of A is

$$\frac{(1-q)t(T - T' - T'')}{(1-t)T}.$$

The numerator and the denominator of this function are divisible by $1-t$; moreover, we will have the sum of all the disadvantages of A, or his total disadvantage, by making $t = 1$ in this generating function; the total disadvantage is therefore by the known methods, and by observing that $T' + T'' = T$, when $t = 1$,

$$-\frac{(1-q)(dT - dT' - dT'')}{Tdt},$$

t being supposed equal to unity, after the differentiations. If we subtract this expression, from that of the total advantage of A, we will have, for the expression of the lot of this player,

$$\frac{qdT' + (1-q)(dT - dT'')}{Tdt} - \frac{T'dT}{T^2dt}.$$

[253] The lot of B will be

$$\frac{(1-q)dT'' + q(dT - dT')}{Tdt} - \frac{T''dT}{T^2dt},$$

t being supposed unity after the differentiations; that which gives

$$\begin{aligned}
T &= q(1-q)[q^{i-1} + (1-q)^{i-1} - q^{i-1}(1-q)^{i-1}];\\
\frac{dT}{dt} &= (i+1)q(1-q)[q^{i-1} + (1-q)^{i-1}] - 2^i q^i (1-q)^i - 1;\\
T' &= (1-q)q^i[1-(1-q)^i];\\
\frac{dT'}{dt} &= i(1-q)q^i[1-2(1-q)^i] - qq^i[1-(1-q)^i].
\end{aligned}$$

we will have T'' and $\frac{dT''}{dt}$ by changing in these last two expressions, q into $1-q$.

§13. An urn being supposed to contain $n+1$ balls, distinguished by the numbers 0, 1, 2, 3, ... n, we draw from it a ball which we replace into the urn after the drawing. We demand the probability that after i drawings, the sum of the numbers brought forth will be equal to s.

Let t_1, t_2, t_3, ... t_i be the numbers brought forth at the first drawing, at the second, at the third, etc.; we must have

$$t_1 + t_2 + t_3 \cdots + t_i = s. \tag{1}$$

t_2, t_3, ... t_i being supposed not to vary, this equation is susceptible only of one combination. But if we make vary at the same time t_1 and t_2, and if we suppose that these variables can be extended indefinitely from zero, then the number of combinations which give the preceding equation will be

$$s + 1 - t_3 - t_4 \cdots - t_i;$$

because t_1 can be extended from zero, that which gives

$$t_2 = s - t_3 - t_4 \cdots - t_i,$$

to $s - t_3 - t_4 \cdots - t_i$, that which gives $t_2 = 0$, the negative values of the variables t_1, t_2 needing to be excluded.

Now, the number $s + 1 - t_3 - t_4 \cdots - t_i$ is susceptible of many values, by virtue [254]
of the variations of t_3, t_4, etc. Let us suppose first t_4, t_5, etc. invariables, and that t_3 can be extended indefinitely from zero; then we make

$$s + 1 - t_3 - t_4 \cdots - t_i = x,$$

by integrating this variable of which the finite difference is unity, we will have $\frac{x(x-1)}{1.2}$ for its integral; but, in order to have the sum of all the values of x, it is necessary, as we know, to add x to this integral; this sum is therefore $\frac{x(x+1)}{1.2}$. It is necessary to make x equal to its greatest value, which we obtain by making t_3 null in the function $s+1-t_3-t_4 \cdots - t_i$: thus the total number of combinations relative to the variations of t_1, t_2 and t_3, is

$$\frac{(s + 2 - t_4 - t_5 \cdots - t_i)(s + 1 - t_4 - t_5 \cdots - t_i)}{1.2}.$$

By making next in this function

$$s + 2 - t_4 - t_5 \cdots - t_i = x,$$

it becomes $\frac{x(x-1)}{1.2}$; by integrating it from $x = 0$, and by adding the function itself, to this integral, we will have $\frac{(x+1)x(x-1)}{1.2.3}$; the value of x null corresponds to $t_4 = s + 2 - t_5 \cdots - t_i$, and its greatest value corresponds to t_4 null, and consequently it is equal to $s+2-t_5 \cdots - t_i$; by substituting therefore for x, this value into the preceding integral, we will have

$$\frac{(s+3-t_5-t_6-\cdots-t_i)(s+2-t_5-t_6-\cdots-t_i)(s+1-t_5-t_6-\cdots-t_i)}{1.2.3}$$

for the sum of all the combinations relative to the variations of t_1, t_2, t_3, t_4. By continuing thus, we will find generally that the total number of the combinations which give equation (1), under the supposition where the variables t_1, t_2, $\dots t_i$ can be extended indefinitely from zero, is

$$\frac{(s+i-1)(s+i-2)(s+i-3)\dots(s+1)}{1.2.3\dots(i-1)} \qquad (a)$$

[255] but in the present question, these variables can not be extended beyond n. In order to express this condition, we will observe that the urn containing $n+1$ balls, the probability to extract any one of them, is $\frac{1}{n+1}$; thus the probability of each of the values of t_1, from zero to n, is $\frac{1}{n+1}$. The probability of the values of t_1 equal or superior to $n+1$, is null; we can therefore represent it by $\frac{1-l^{n+1}}{n+1}$, provided that we make $l=1$ in the result of the calculation; then the probability of any value of t_1 can be generally expressed by $\frac{1-l^{n+1}}{n+1}$, provided that we make l to begin, only when t_1 will have attained $n+1$, and that we suppose it at the end, equal to unity: it is likewise of the probabilities of the other variables. Now, the probability of equation (1) is the product of the probabilities of the values of t_1, t_2, t_3, etc.; this probability is therefore $\left(\frac{1-l^{n+1}}{n+1}\right)^i$; the number of combinations which give this equation, multiplied by their respective probabilities, is thus the product of the fraction (a) by $\left(\frac{1-l^{n+1}}{n+1}\right)^i$, or

$$\frac{(s+1)(s+2)\dots(s+i-1)}{1.2.3\dots(i-1)}\left(\frac{1-l^{n+1}}{n+1}\right)^i; \qquad (b)$$

but it is necessary, in the development of this function, to apply l^{n+1} only to the combinations in which one of the variables begins to surpass n: it is necessary to apply l^{2n+2} only to the combinations in which two of the variables begin to surpass n, and thus of the rest. If in equation (1) we suppose that one of the variables, t_1, for example, surpasses n; by making $t_1 = n+1+t'$, this equation becomes

$$s-n-1 = t'_1 + t_2 + t_3 + \text{etc.},$$

the variable t'_1 being able to be extended indefinitely. If two of the variables such as t_1 and t_2 surpass n; by making

$$t_1 = n+1+t'_1, \qquad t_2 = n+1+t'_2;$$

[256] the equation becomes

$$s-2n-2 = t'_1 + t'_2 + t_3 + \text{etc.},$$

and so forth. We must therefore, in the function (a) which we have derived from equation (1), diminish s by $n+1$, relatively to the system of variables t'_1, t_2, t_3, etc. We must diminish it by $2n+2$, relatively to the variables t'_1, t'_2, t_3, etc.; and thus of the rest. It is necessary consequently, in the development of the function (b) with respect to the powers of l, to diminish in each term, s from the exponent of the power of l; by making next $l = 1$, this function becomes

$$\begin{aligned}\frac{(s+1)(s+2)\dots(s+i-1)}{1.2.3\dots(i-1)(n+1)^i} &- \frac{i(s-n)(s-n+1)\dots(s+i-n-2)}{1.2.3\dots(i-1)(n+1)^i} \\ &+ \frac{i(i-1)}{1.2}\cdot\frac{(s-2n-1)(s-2n)\dots(s+i-2n-3)}{1.2.3\dots(i-1)(n+1)^i} - \text{etc.};\end{aligned} \tag{c}$$

the series must be continued until one of the factors $s-n$, $s-2n-1$, $s-3n-2$, etc. becomes null or negative.

This formula gives the probability to bring forth a given number s, by projecting i dice with a number $n+1$ faces on each, the smallest number marked on the faces being 1. It is clear that this reverts to supposing in the preceding urn, all the numbers of the balls, increased by unity; and then the probability to bring forth the number $s+i$ in i drawings, is the same as that of bringing forth the number s in the case that we just considered; now, by making $s+i = s'$, we have $s = s' - i$; formula (c) will give therefore for the probability to bring forth the number s' by projecting the i dice,

$$\begin{aligned}\frac{(s'-1)(s'-2)\dots(s'-i+1)}{1.2.3\dots(i-1)(n+1)^i} &- \frac{i(s'-n-2)(s'-n-3)\dots(s'-i-n)}{1.2.3\dots(i-1)(n+1)^i} \\ &+ \frac{i(i-1)}{1.2}\cdot\frac{(s'-2n-3)(s'-2n-4)\dots(s'-i-2n-1)}{1.2.3\dots(i-1)(n+1)^i} - \text{etc.}\end{aligned}$$

Formula (c) applied to the case where s and n are infinite numbers, is transformed into the following

$$\frac{1}{1.2.3\dots(i-1)n}\left\{\left(\frac{s}{n}\right)^{i-1} - i\left(\frac{s}{n}-1\right)^{i-1} + \frac{i(i-1)}{1.2}\left(\frac{s}{n}-2\right)^{i-1} - \text{etc.}\right\}.$$

This expression can serve to determine the probability that the sum of the inclinations [257]
to the ecliptic, of a number i of orbits, will be comprehended within some given limits, by supposing that for each orbit, all the inclinations from zero to the right angle, are equally possible. In fact, if we imagine that the right angle $\frac{1}{2}\pi$, is divided into an infinite number n of equal parts, and if s contains an infinite number of these parts; by naming ϕ the sum of the inclinations of the orbits, we will have

$$\frac{s}{n} = \frac{\phi}{\frac{1}{2}\pi}.$$

By multiplying therefore the preceding expression by ds or by $\frac{n\,d\phi}{\frac{1}{2}\pi}$, and by integrating it from $\phi - \epsilon$ to $\phi + \epsilon$, we will have

$$\frac{1}{1.2.3\dots i}\left\{\begin{aligned}&\left(\frac{\phi+\epsilon}{\frac{1}{2}\pi}\right)^i - i\left(\frac{\phi+\epsilon}{\frac{1}{2}\pi}-1\right)^i + \frac{i(i-1)}{1.2}\left(\frac{\phi+\epsilon}{\frac{1}{2}\pi}-2\right)^i - \text{etc.} \\ &-\left(\frac{\phi-\epsilon}{\frac{1}{2}\pi}\right)^i + i\left(\frac{\phi-\epsilon}{\frac{1}{2}\pi}-1\right)^i - \frac{i(i-1)}{1.2}\left(\frac{\phi-\epsilon}{\frac{1}{2}\pi}-2\right)^i + \text{etc.}\end{aligned}\right\}; \tag{o}$$

this is the expression of the probability that the sum of the inclinations of the orbits will be comprehended within the limits $\phi - \epsilon$ to $\phi + \epsilon$.

Let us apply this formula to the orbits of the planets. The sum of the inclinations of the orbits of the planets to that of the Earth, was 91.4187° at the beginning of 1801: there are ten orbits, without including the ecliptic; we have therefore here $i = 10$. We make next

$$\phi - \epsilon = 0,$$
$$\phi + \epsilon = 91.4187^\circ.$$

The preceding formula becomes thus, by observing that $\frac{1}{2}\pi$, or the quarter of the circumference is 100°,[10]

$$\frac{1}{1.2.3\dots10}(0.914187)^{10}.$$

This is the expression of the probability that the sum of the inclinations of the orbits will be comprehended within the limits zero and 91, 4187°, if all the inclinations were equally possible. This probability is therefore 0, 00000011235. It is already very
[258] small; but it is necessary next to combine it with the probability of a very remarkable circumstance in the system of the world, and which consists in this that all the planets are moved in the same sense as the Earth. If the direct and retrograde movements are supposed equally possible, this last probability is $\left(\frac{1}{2}\right)^{10}$; it is necessary therefore to multiply 0, 00000011235 by $\left(\frac{1}{2}\right)^{10}$, in order to have the probability that all the movements of the planets and of the Earth will be directed in the same sense, and that the sum of their inclinations to the orbit of the earth, will be comprehended within the limits zero and 91, 4187°; we will have thus $\frac{1,0972}{(10)^{10}}$ for this probability; that which gives $1 - \frac{1,0972}{(10)^{10}}$ for the probability that this had not ought to take place; if all the inclinations, in the same way the direct and retrograde movements, have been equally facile. This probability approaches so to certainty, that the observed result becomes unlikely under this hypothesis; this result indicates therefore with a very great probability, the existence of an original cause which has determined the movements of the planets to bring themselves together to the plane of the ecliptic, or more naturally, to the plane of the solar equator, and to be moved in the sense of the rotation of the sun. If we consider next that the eighteen satellites observed until now, make their revolution in the same sense, and that the observed rotations in the number of thirteen in the planets, the satellites and the ring of Saturn, are yet directed in the same sense; finally, if we consider that the mean of the inclinations of the orbits of these stars, and of their equators to the solar equator, is quite removed from reaching a half right angle; we will see that the existence of a common cause, which has directed all these movements in the sense of the rotation of the sun, and onto some planes slightly inclined to the one of its equator, is indicated with a probability quite superior to the one of the greatest number of the historical facts on which we permit no doubt.

[10] *Translator's note*: These are decimal degrees. That is, $\pi/4 = 100^\circ$.

Let us see now if this cause has influence on the movement of the comets. The number of these which we have observed until the end of 1811, by counting for the same the diverse apparitions of the one of 1759, is raised to one hundred, of which fifty-three are direct, and forty-seven are retrograde. The sum of the inclinations of [259] the orbits of the first is $2657,993^\circ$, and that of the inclinations of the other orbits, is $2515,684^\circ$: the mean inclination of all these orbits is therefore $51,73677^\circ$; consequently the sum of all the inclinations is $\frac{i.\pi}{4} + i.1,73677^\circ$, i being here equal to 100. We see already that the mean inclination surpassing the half right angle, the comets, far from participating in the tendency of the bodies of planetary system, in order to be moved in some planes slightly inclined to the ecliptic, appear to have a contrary tendency. But the probability of this tendency is very small. In fact, if we suppose, in formula (o),

$$\phi = \frac{i.\pi}{4}, \qquad \epsilon = i.1,73677^\circ,$$

it becomes

$$\frac{1}{1.2.3\ldots i.2^i}\left\{\begin{aligned} &\left(i + \frac{4i.1,73677^\circ}{\pi}\right)^i - i\left(i + \frac{4i.1,73677^\circ}{\pi} - 2\right)^i \\ &\quad + \frac{i(i-1)}{1.2}\left(i + \frac{4i.1,73677^\circ}{\pi} - 4\right)^i - \text{etc.} \\ &\quad - \left(i - \frac{4i.1,73677^\circ}{\pi}\right)^i + i\left(i - \frac{4i.1,73677^\circ}{\pi} - 2\right)^i \\ &\quad - \frac{i(i-1)}{1.2}\left(i - \frac{4i.1,73677^\circ}{\pi} - 4\right)^i \text{etc.} \end{aligned}\right\}; \qquad (p)$$

π being 200°. This is the expression of the probability that the sum of the inclinations of the orbits of the i comets, must be comprehended within the limits $\pm i.1,73677^\circ$. The number of terms of this formula, and the precision with which it would be necessary to have each of them, renders the calculation of it impractical; it is necessary to recur to the methods of approximation developed in the second part of the first Book. We have, by §42 of the same Book,

$$\begin{aligned} &\frac{(i + r\sqrt{i})^i - i(i + r\sqrt{i} - 2)^i + \frac{i(i-1)}{1.2}(i + r\sqrt{i} - 4)^i - \text{etc.}}{1.2.3\ldots i.2^i} \\ &\qquad = \frac{1}{2} + \sqrt{\frac{3}{2\pi}} \int dr\, c^{-\frac{3}{2}r^2} - \frac{3}{20.i}\sqrt{\frac{3}{2\pi}}\, r(1 - r^2)c^{-\frac{3}{2}r^2}, \end{aligned}$$

the powers of the negative quantities being here excluded, as they are in the preceding [260] formula; by making therefore

$$r\sqrt{i} = \frac{4i.1,73677^\circ}{200^\circ},$$

formula (p) becomes

$$2\sqrt{\frac{3}{2\pi}}\int dr\, c^{-\frac{3}{2}r^2}-\frac{3}{10.i}\sqrt{\frac{3}{2\pi}}\,r(1-r^2)c^{-\frac{3}{2}r^2}$$

the integral being taken from r null. We find thus $0,474$ for the probability that the inclination of the 100 orbits must fall within the limits $50°\pm1,17377°$; the probability that the mean inclination must be inferior to the observed inclination, is therefore $0,737$. This probability is not great enough in order that the observed result makes rejection of the hypothesis of an equal facility of the inclinations of the orbits, and in order to indicate the existence of an original cause which has influence on these inclinations, a cause which we cannot forbid to admit in the inclinations of the orbits of the planetary system.

The same thing holds with respect to the sense of the movement. The probability that out of 100 comets, 47 moreover will be retrogrades, is the sum of the 48 first terms of the binomial $(p+q)^{100}$, by making in the result of the calculation $p=q=\frac{1}{2}$. But the sum of the 50 first terms, plus the half of the 51st or the middle term, is the half of the entire binomial, or of $\left(\frac{1}{2}+\frac{1}{2}\right)^{100}$, that is $\frac{1}{2}$; the sought probability is therefore

$$\frac{1}{2}-\frac{100.99\ldots51}{1.2.3\ldots50.2^{100}}\left(\frac{1}{2}+\frac{50}{51}+\frac{50.49}{51.52}\right)$$

or

$$=\frac{1}{2}-\frac{1.2.3\ldots100.1594}{(1.2.3\ldots50)^2.2^{100}.663}.$$

By virtue of the theorem

$$1.2.3\ldots s=s^{s+\frac{1}{2}}c^{-s}\left(1+\frac{1}{12s}+\text{etc.}\right)\sqrt{2\pi},$$

we have, very nearly,

$$1.2.3\ldots100=100^{100+\frac{1}{2}}c^{-100}\left(1+\frac{1}{1200}\right)\sqrt{2\pi},$$

$$2^{100}(1.2.3\ldots50)^2=100^{100+1}c^{-100}\left(1+\frac{1}{300}\right)\pi.$$

[261] The preceding probability becomes thus,

$$\frac{1}{2}-\frac{1}{\sqrt{50\pi}}\,\frac{1197.1594}{1200.663}=0.3046.$$

This probability is much too great to indicate a cause which has favored, at the origin, the direct movements. Thus the cause which has determined the sense of the movements of the revolution and of the rotation of the planets and of the satellites, seems to have no influence on the movement of the comets.

§14. The method of the preceding section has the advantage to be extended to the case where the number of balls of the urn, which bear the same label, is not equal to unity, but varies according to any law whatsoever. Let us imagine, for example, that there is only one ball bearing the n° 0, only one ball bearing the n° 1, and so forth until n° r inclusively. Let us suppose moreover that there are two balls bearing the n° $r+1$, two balls bearing the n° $r+2$, and so forth until n° n inclusively. The total number of balls in the urn will be $2n-r+1$, the probability to extract from it one of the labels inferior to $r+1$, will be therefore $\frac{1}{2n-r+1}$; and the probability to extract from it the n° $r+1$ or one of the superior labels, will be $\frac{2}{2n-r+1}$: we will represent it by $\frac{1+l^{r+1}}{2n-r+1}$; but we will make $l=1$ in the result of the calculation. Although there are no labels beyond n° n, we will be able however to consider in the urn some labels superior to n, to infinity, provided that we will give to their extraction, a null probability; we will be able therefore to represent this probability by $\frac{1+l^{r+1}-2l^{n+1}}{2n-r+1}$, by making $l=1$ in the result of the calculation. By this artifice, we will be able to represent generally the probability of any label whatsoever, by the preceding expression; provided that we will make l^{r+1} commence only when the labels will commence to surpass r, and that we will make l^{n+1} commence only when one of the labels will commence to surpass n. This premised, we will find, by applying here the reasonings of the previous section, that the probability to bring forth the number s, in i drawings, is equal to [262]

$$\frac{(s+i-1)(s+i-2)(s+i-3)\dots(s+1)}{1.2.3\dots(i-1)(2n-r+1)^i}(1+l^{r+1}-2l^{n+1})^i,$$

provided that in the development of this function, according to the powers of l, we diminish in each term, s from the exponent of the power of l, that we suppose next $l=1$, and that we arrest the series when we arrive to some negative factors.

§15. Let us apply now this method to the investigation of the mean result that any number of observations of which the laws of facility of the errors are known must give. For this, we will resolve the following problem:

Let i variable and positive quantities be t, t_1, t_2, $\dots t_{i-1}$, of which the sum is s, and of which the law of possibility is known; we propose to find the sum of the products of each value that a given function $\psi(t, t_1, t_2, \text{etc.})$ of these variables is able to receive, multiplied by the probability corresponding to this value.

Let us suppose for more generality, that the functions which express the possibilities of the variables t, t_1, etc. are discontinuous, and let us represent by $\phi(t)$ the possibility of t, from $t=0$ to $t=q$; by $\phi'(t)+\phi(t)$, its possibility from $t=q$ to $t=q'$; by $\phi''(t)+\phi'(t)+\phi(t)$, its possibility from $t=q'$ to $t=q''$, and so forth to infinity. Let us designate next the same quantities relative to the variables t_1, t_2, etc. by the same letters, by writing respectively at the base, the numbers 1, 2, 3, etc.; so that q_1, q_1', etc.; $\phi_1(t_1)$, $\phi_1'(t_1)$, etc. correspond, relatively to t_1, to that which q, q', etc., $\phi(t)$, $\phi'(t)$, etc. are respectively to t, and so forth. In this manner of representing the possibilities of the variables, it is clear that the function $\phi(t)$ holds from $t=0$ to t infinity; that the function $\phi'(t)$ holds from $t=q$ to t infinity, and so forth. In order to recognize the values of t, t_1, t_2, etc. when these diverse functions begin to hold, we will multiply conformably to the method exposed in the preceding sections, $\phi(t)$ by l^0 or unity, $\phi'(t)$ by l^q,$\phi''(t)$ by $l^{q'}$, etc.; we will multiply similarly $\phi_1(t_1)$ by unity, $\phi_1'(t_1)$

by l^{q_1}, and so forth: the exponents of the powers of l will indicate then these values.
[263] It will suffice next to make $l = 1$ in the last result of the calculation. By means of these very simple artifices, we can easily resolve the proposed problem.

The probability of the function $\psi(t, t_1, t_2, \text{etc.})$ is evidently equal to the product of the probabilities of t, t_1, t_2, etc., so that if we substitute for t its values $s - t_1 - t_2 - \text{etc.}$ that the equation gives

$$t + t_1 + t_2 \cdots + t_{i-1} = s,$$

the product of the proposed function by its probability, will be

$$\left.\begin{aligned}
&\psi(s - t_1 - t_2 - \text{etc.}, t_1, t_2, \text{etc.})\\
&\times [\phi(s - t_1 - t_2 - \text{etc.}) + l^q \phi'(s - t_1 - t_2 - \text{etc.})\\
&\qquad\qquad\qquad + l^{q'} \phi''(s - t_1 - t_2 - \text{etc.}) + \text{etc.}]\\
&\times [\phi_1(t_1) + l^{q_1} \phi_1'(t_1) + l^{q_1'} \phi_1''(t_1) + \text{etc.}]\\
&\times [\phi_2(t_2) + l^{q_2} \phi_2'(t_2) + l^{q_2'} \phi_2''(t_2) + \text{etc.}]\\
&\times \text{etc.}
\end{aligned}\right\} \quad (A)$$

we will have therefore the sum of all these products, 1° by multiplying the preceding quantity by dt_1, and by integrating for all the values of which t_1 is susceptible; 2° by multiplying this integral by dt_2, and by integrating for all the values of which t_2 is susceptible, and so forth to the last variable t_{i-1}; but these successive integrations require some particular attentions.

Let us consider any term whatsoever of the quantity (A), such as

$$\begin{aligned}
&l^{q+q_1+q_2+\text{etc.}} \psi(s - t_1 - t_2 - \text{etc.}, t_1, t_2, \text{etc.})\\
&\quad \times \phi'(s - t_1 - t_2 - \text{etc.}) \phi_1'(t_1) \phi_2''(t_2).\text{etc.};
\end{aligned}$$

by multiplying it by dt_1, it is necessary to integrate for all the possible values of t_1; now the function $\phi'(s - t_1 - t_2 - \text{etc.})$ holds only when t, of which the value is $s - t_1 - t_2 - \text{etc.}$, equals or surpasses q; the greatest value that t_1 is able to receive, is therefore $s - q - t_2 - t_3 - \text{etc.}$ Moreover, $\phi_1'(t_1)$ holding only when t_1 is equal or greater than q_1, this quantity is the smallest value that t_1 is able to receive; it is necessary therefore to take the integral of which there is concern, from $t_1 = q_1$ to

$$t_1 = s - q - q_1 - t_2 - t_3 - \text{etc.};$$

or, that which reverts to the same, from $t_1 - q_1 = 0$ to

$$t_1 - q_1 = s - q - q_1 - t_2 - t_3 - \text{etc.}$$

[264] We will find in the same manner that by multiplying this new integral by dt_2, it will be necessary to integrate it from $t_2 - q_2' = 0$ to

$$t_2 - q_2' = s - q - q_1 - q_2' - t_3 - \text{etc.}$$

By continuing to operate thus, we will arrive to a function of $s - q - q_1 - q_2' - \text{etc.}$, in which there will remain none of the variables t, t_1, t_2, etc. This function must be rejected, if $s - q - q_1 - q_2' - \text{etc.}$ is null or negative; because it is clear that in this case, the system of functions $\phi'(t)$, $\phi_1'(t_1)$, $\phi_2''(t_2)$, etc. can not be employed. In fact, the smallest values of t_1, t_2, etc. being by the nature of these functions, equals to q_1, q'_2, etc.; the greatest value that t can receive is $s - q_1 - q'_2 -$ etc.; thus the greatest value of $t - q$ is

$$s - q - q_1 - q'_2 - \text{etc.};$$

now the function $\phi'(t)$ can be employed only as long as $t - q$ is positive.

Thence results a very simple solution of the proposed problem. Let us substitute, 1° $q + t$ instead of t, into $\phi'(t)$; $q' + t$ instead of t, into $\phi''(t)$; $q'' + t$ instead of t, into $\phi'''(t)$, and so forth; 2° $q_1 + t_1$ instead of t_1, into $\phi'_1(t_1)$; $q'_1 + t_1$ instead of t_1, into $\phi''_1(t_1)$; etc.; 3° $q_2 + t_2$ instead of t_2, into $\phi'_2(t_2)$; $q'_2 + t_2$ instead of t_2, into $\phi''_2(t_2)$, etc.; and so forth; 4° finally, $k + t$ instead of t, $k_1 + t_1$ instead of t_1, and thus of the remainder, into $\psi(t, t_1, t_2, \text{etc.})$; the function (A) will become

$$\begin{aligned}\psi(k + s - t_1 - t_2 - t_3 - \text{etc.},\ & k_1 + t_1, k_2 + t_2, \text{etc.})\\ \times[\phi(s - t_1 - t_2 - t_3 - \text{etc.}) & + l^q\phi'(s + q - t_1 - t_2 - \text{etc.})\\ & + l^{q'}\phi''(s + q' - t_2 - t_3 - \text{etc.})]\\ \times[\phi_1(t_1) + l^{q_1}\phi'_1(q_1 + t_1) & + l^{q'_1}\phi''_1(q'_1 + t_1) + \text{etc.}]\\ \times[\phi_2(t_2) + l^{q_2}\phi'_2(q_2 + t_2) & + \text{etc.}].\end{aligned} \tag{A'}$$

by multiplying this function by dt_1, we will integrate it from t_1 null to $t_1 = s - t_2 - t_3 -$ etc. We will multiply next this first integral by dt_2, and we will integrate it from t_2 null to $t_2 = s - t_3 - t_4 -$ etc.. By continuing thus, we will arrive to a last integral which will be a function of s, and which we will designate by $\Pi(s)$; and this [265]
function will be the sought sum of all the values of $\psi(t, t_1, t_2, \text{etc.})$ multiplied by their respective probabilities. But for this, it is necessary to take care to change in any term whatsoever, multiplied by a power of l, such as $l^{q+q_1+q_2+\text{etc.}}$, k in the part of the exponent of the power relative to the variable t, and which in this case is q; and if this part is lacking, it is necessary to suppose k equal to zero. It is similarly necessary to change k_1 in the part of the exponent relative to the variable t_1, and so forth; it is necessary to diminish s from the entire exponent of the power of l, and to write thus, in the present case, $s - q - q_1 - q'_2 -$ etc., instead of s, and to reject the term, if s, thus diminished, becomes negative. Finally it is necessary to suppose $l = 1$.

If $\psi(t, t_1, t_2, \text{etc.})$, $\phi(t)$, $\phi'(t)$, etc.; $\phi_1(t_1)$, etc. are some rational and integral functions of the variables t, t_1, t_2, etc.; of their exponentials, and of sines and cosines; all the successive integrations will be possible, because it is of the nature of these functions, to reproduce themselves by the integrations. In the other cases, the integrations would not be able to be possible; but the preceding analysis reduces then the problem to quadratures. The case of the rational and integral functions, offer some simplifications that we will expose.

Let us suppose that we have

$$\begin{aligned}\phi(t) + l^q\phi'(q + t) + l^{q'}\phi''(q' + t) + \text{etc.} &= A + Bt + Ct^2 + \text{etc.},\\ \phi_1(t_1) + l^{q_1}\phi'_1(q_1 + t_1) + l^{q'_1}\phi''_1(q'_1 + t_1) + \text{etc.} &= A_1 + B_1t_1 + C_1t_1^2 + \text{etc.},\\ \phi_2(t_2) + l^{q_2}\phi'_2(q_2 + t_2) + l^{q'_2}\phi''_2(q'_2 + t_2) + \text{etc.} &= A_2 + B_2t_2 + C_2t_2^2 + \text{etc.},\\ &\text{etc.}\end{aligned}$$

and let us designate by $H.t^n.t_1^{n_1}.t_2^{n_2}$.etc. any term whatsoever of $\psi(k+t, k_1+t_1, k_2+t_2, \text{etc.})$; it is easy to be assured that the part of $\Pi(s)$ corresponding to this term, is

$$\begin{aligned} &1.2.3\ldots n.1.2.3\ldots n_1.1.2.3\ldots n_2.\text{etc.}.Hs^{i+n+n_1+n_2+\text{etc.}-1} \\ &\quad\times[A+(n+1)Bs+(n+1)(n+2)Cs^2+\text{etc.}] \\ &\quad\times[A_1+(n_1+1)B_1s+(n_1+1)(n_1+2)C_1s^2+\text{etc.}]; \\ &\quad\times[A_2+(n_2+1)B_2s+(n_2+1)(n_2+2)C_2s^2+\text{etc.}] \\ &\quad\times\text{etc.}, \end{aligned} \tag{B}$$

[266] provided that in the development of this quantity, instead of any power whatsoever a of s, we write $\frac{s^a}{1.2.3\ldots a}$. We will have next the corresponding part of the entire sum of the values of $\psi(t, t_1, t_2, \text{etc.})$, multiplied by their respective probabilities, by changing any term of this development, such as $H\lambda t^\mu s^a$ into $H\lambda(s-\mu)^a$, and by substituting into H, instead of k, the part of the exponent μ, which is relative to the variable t; instead of k_1, the part relative to t_1, and thus of the remainder.

If in formula (B) we suppose $H = 1$, and n, n_1, n_2, etc. nulls; we will have the sum of the values of unity, multiplied by their respective probability; now it is clear that this sum is nothing other than the sum of all the combinations in which the equation

$$t+t_1+t_2\cdots+t_{i-1}=s$$

holds, multiplied by their probability; it expresses consequently the probability of this equation. If under the preceding hypotheses, we suppose moreover that the law of probability is the same for the first r variables t, t_1, t_2,..., t_{r-1}, and if for the last $i-r$, it is again the same, but different than for the first; we will have

$$\begin{aligned} A&=A_1=A_2=\cdots=A_{r-1}, \\ B&=B_1=B_2=\cdots=B_{r-1}, \\ &\text{etc.} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ A_r&=A_{r+1}\ldots\cdots=A_{i-1}, \\ B_r&=B_{r-1}\ldots\cdots=B_{i-1}, \\ &\text{etc.}, \end{aligned}$$

and formula (B) will be changed into the following,

$$s^{i-1}(A+Bs+2Cs^2+\text{etc.})^r(A_r+B_rs+2C_rs^2+\text{etc.})^{i-r}. \tag{C}$$

This formula will serve to determine the probability that the sum of the errors of any number of observations whatsoever of which the law of facility of errors is known, will be comprehended within some given limits.

[267] Let us suppose, for example, that we have $i-1$ observations of which the errors for each observation are able to be extended from $-h$ to $+g$, and that by naming z the error of the first of these observations, the law of facility of this error is expressed by $a+bz+cz^2$. Let us suppose next that this law is the same for the errors z_1, z_2,

$\dots$, z_{i-2} of the other observations, and let us seek the probability that the sum of these errors, will be comprehended within the limits p and $p+e$.

If we make

$$z = t - h, \quad z_1 = t_1 - h, \quad z_2 = t_2 - h, \quad \text{etc.};$$

it is clear that t, t_1, t_2, etc. will be positive and will be able to be extended from zero, to $h+g$; moreover, we will have

$$z + z_1 + z_2 \cdots + z_{i-2} = t + t_1 + t_2 \cdots + t_{i-2} - (i-1)h;$$

therefore the greatest value of the sum $z+z_1+z_2 \cdots + z_{i-2}$ being by assumption, equal to $p+e$, and the smallest being equal to p; the greatest value of $t+t_1+t_2 \cdots + t_{i-2}$ will be $(i-1)h+p+e$, and the smallest will be $(i-1)h+p$; by making thus

$$(i-1)h + p + e = s$$

and

$$t + t_1 + t_2 \cdots + t_{i-2} = s - t_{i-1},$$

t_{i-1} will always be positive, and will be able to be extended from zero to e. This premised, if we apply in this case, formula (C); we will have $q = h+g$. Besides the law of facility of errors z being $a+bz+cz^2$, we will conclude from it the law of facility of t, by changing z into $t-h$; let

$$a' = a - bh + ch^2, \quad b' = b - 2ch;$$

we will have $a' + b't + ct^2$ for this law; this will be therefore the function $\phi(t)$. But as from $t = h+g$ to t infinity, the facility of the values of t is null by hypothesis; we will have

$$\phi'(t) + \phi(t) = 0,$$

that which gives

$$\phi'(t) = -(a' + b't + ct^2);$$

therefore if we make [268]

$$a'' = a' + b'(h+g) + c(h+g)^2,$$
$$b'' = b' + 2c(h+g),$$

we will have

$$\phi(t) + l^q \phi'(q+t) = a' + b't + ct^2 - l^{h+g}(a'' + b''t + ct^2);$$

and this equation will hold further, by changing t into t_1, t_2, etc.; since the law of facility of the errors is supposed the same for all the observations.

As for the variable t_{i-1}, we will observe that the probability of the equation

$$z + z_1 \cdots + z_{i-2} = \mu$$

being, whatever be μ, equal to the product of the probabilities of z, z_1, z_2, etc.; the probability of the equation

$$t + t_1 + t_2 \cdots + t_{i-2} = s - t_{i-1},$$

will be equal to the product of the probabilities of t, t_1, t_2, etc.; the law of probability of t_{i-1} is therefore constant and equal to unity; and, as this variable must be extended only from $t_{i-1} = 0$ to $t_{i-1} = e$; we will have

$$q_{i-1} = e, \quad \phi_{i-1}(t_{i-1}) = 1, \quad \phi'_{i-1}(t_{i-1}) + \phi_{i-1}(t_{i-1}) = 0;$$

and consequently

$$\phi'_{i-1}(t_{i-1}) = -1$$

that which gives

$$\phi_{i-1}(t_{i-1}) + l^{q_{i-1}}\phi'_{i-1}(q_{i-1} + t_{i-1}) = 1 - l^e;$$

formula (C) will become therefore

$$s^{i-1}[a' + b's + 2cs^2 - l^{h+g}(a'' + b''s + 2cs^2)]^{i-1}(1 - l^e). \tag{C$'$}$$

Let

$$\begin{aligned}
(a' + b's + 2cs^2)^{i-1} &= a^{(1)} + b^{(1)}s + c^{(1)}s^2 + f^{(1)}s^3 + \text{etc.},\\
(a' + b's + 2cs^2)^{i-2}(a'' + b''s + 2cs^2) &= a^{(2)} + b^{(2)}s^2 + c^{(2)}s + \text{etc.},\\
(a' + b's + 2cs^2)^{i-3}(a'' + b''s + 2cs^2) &= a^{(3)} + b^{(3)}s + c^{(3)}s^2 + \text{etc.},\\
&\text{etc.}
\end{aligned}$$

[269] The preceding formula (C′) will give, by changing any term whatsoever such as $\lambda l^\mu s^a$, into

$$\frac{\lambda(s-\mu)^a}{1.2.3\ldots a};$$

$$\frac{1}{1.2.3\dots(i-1)}\left\{\begin{array}{l}
\quad a^{(1)}\left[s^{i-1}-(s-e)^{i-1}\right]\\
+\dfrac{b^{(1)}}{i}\left[s^{i}-(s-e)^{i}\right]\\
+\dfrac{c^{(1)}}{i(i+1)}\left[s^{i+1}-(s-e)^{i+1}\right]\\
+\text{etc.}\\
-(i-1)\left\{\begin{array}{l}
\quad a^{(2)}\left[(s-h-g)^{i-1}-(s-h-g-e)^{i-1}\right]\\
+\dfrac{b^{(2)}}{i}\left[(s-h-g)^{i}-(s-h-g-e)^{i}\right]\\
+\dfrac{c^{(2)}}{i(i+1)}\left[(s-h-g)^{i+1}-(s-h-g-e)^{i+1}\right]\\
+\text{etc.}
\end{array}\right.\\
+\dfrac{(i-1)(i-2)}{1.2}\left\{\begin{array}{l}
\quad a^{(3)}\left[(s-2h-2g)^{i-1}-(s-2h-2g-e)^{i-1}\right]\\
+\dfrac{b^{(3)}}{i}\left[(s-2h-2g)^{i}-(s-2h-2g-e)^{i}\right]\\
+\dfrac{c^{(3)}}{i(i+1)}\left[(s-2h-2g)^{i+1}-(s-2h-2g-e)^{i+1}\right]\\
+\text{etc.}
\end{array}\right.\\
-\text{etc.}
\end{array}\right.$$

It is necessary to reject from this expression, the terms in which the quantity raised under the sign of the powers, is negative.

Let us suppose now that z, z_1, z_2, etc., representing always the errors of $i-1$ observations, the law of facility, so much of the error z as of the negative error $-z$, be $\beta(h-z)$, and that h and $-h$ are the limits of these errors. Let us suppose moreover that this law is the same for all the observations, and let us seek the probability that the sum of the errors will be comprehended within the limits p and $p+e$.

If we make $z=t-h$, $z_1=t_1-h$, etc.; it is clear that t, t_1, etc. will be always positive, and will be able to be extended from zero to $2h$; but here the law of facility is discontinuous at two points. From $t=0$ to $t=h$, it is expressed by βt. From $t=h$ to $t=2h$, it is expressed by $\beta(2h-t)$; finally, it is null from $t=2h$ to t infinity. We [270]
have therefore

$$q=h, \quad q'=2h;$$

we have next

$$\begin{aligned}\phi(t)&=\beta t,\\ \phi'(t)+\phi(t)&=(2h-t)\beta,\\ \phi''(t)+\phi'(t)+\phi(t)&=0,\end{aligned}$$

that which gives

$$\phi'(t)=(2h-2t)\beta, \quad \phi''(t)=(t-2h)\beta,$$

thus we have in this case,

$$\phi(t) + l^q\phi'(q+t) + l^{q'}\phi''(q'+t) = \beta t(1-l^h)^2;$$

an equation which holds further by changing t into t_1, t_2, etc. Presently we have

$$z + z_1 + z_2 \cdots + z_{i-2} = t + t_1 + t_2 \cdots + t_{i-2} - (i-1)h;$$

therefore the sum of the errors z, z_1, etc. needing to be by hypothesis, contained within the limits p and $p+e$, the sum of the values of t, t_1, $\ldots t_{i-2}$ will be comprehended within the limits $(i-1)h+p$ and $(i-1)h+p+e$; so that if we make

$$t + t_1 + t_2 \cdots + t_{i-2} = s - t_{i-1},$$

s being supposed equal to $(t-1)h+p+e$; t_{i-1} will be able to be extended from zero to e; and we will see, as in the preceding example, that its facility must be supposed equal to unity in this interval, and that it must be supposed null beyond this interval; thus we have $q_{i-1} = e$ and

$$\phi_{i-1}(t_{i-1}) + l^{q_{i-1}}\phi'_{i-1}(t_{i-1}) = 1 - l^e.$$

This premised, if we observe that $2\beta\int dz(h-z)$ being the probability that the error of an observation is comprehended within the limits $-h$ and $+h$, that which is certain, we have $\beta = \frac{1}{h^2}$; formula (C) will give for the expression of the sought probability,

[271]

$$\frac{1}{1.2.3\ldots(2i-2)h^{2i-2}}\begin{cases} s^{2i-2} - (s-e)^{2i-2} \\ \qquad - (2i-2)\left[(s-h)^{2i-2} - (s-h-e)^{2i-2}\right] \\ \qquad + \dfrac{(2i-2)(2i-3)}{1.2}\left[(s-2h)^{2i-2} - (s-2h-e)^{2i-2}\right] \\ \qquad - \text{etc.}\end{cases};$$

by taking care to reject all the terms in which the quantity elevated to the power $2i-2$, is negative.

We are going to apply next this analysis to the following problem. If we imagine a number i of points ranked in a straight line, and on these points, ordinates of which the first is at least equal to the second, the latter at least equal to the third, and so forth; and that the sum of these i ordinates are constantly equal to s. By supposing s partitioned into an infinity of parts, we can satisfy the preceding conditions, in an infinity of ways. We propose to determine the value of each of the ordinates, a mean among all the values that it can receive.

Let z be the smallest ordinate, or the i^{th} ordinate; let $z+z_1$ be the $(i-1)^{\text{st}}$ ordinate; let $z+z_1+z_2$, the $(i-2)^{\text{nd}}$ ordinate, and so forth to the first ordinate which will be $z+z_1\cdots+z_{i-1}$. The quantities z, z_1, z_2, etc. will be either nulls or positives, and their sum $iz+(i-1)z_1+(i-2)z_2\cdots+z_{i-1}$ will be, by the conditions of the problem, equal to s. Let

$$iz = t, \quad (i-1)z_1 = t_1, \quad (i-2)z_2 = t_2, \quad \ldots, \quad z_{i-1} = t_{i-1};$$

we will have

$$t + t_1 + t_2 \cdots + t_{i-1} = s;$$

the variables t, t_1, t_2, etc. will be able to be extended to s. The r^{th} ordinate will be

$$\frac{t}{i} + \frac{t_1}{i-1} \cdots + \frac{t_{i-r}}{r}.$$

It is necessary to determine the sum of all the variations that this quantity is able to receive, and to divide it by the total number of these variations, in order have the mean ordinate. Formula (B) gives very easily this sum, by observing that here

$$\psi(t, t_1, t_2, \text{etc.}) = \frac{t}{i} + \frac{t_1}{i-1} \cdots + \frac{t_{i-r}}{r};$$

and we find it equal to [272]

$$\frac{s^i}{1.2.3\dots i}\left(\frac{1}{i} + \frac{1}{i-1} \cdots + \frac{1}{r}\right).$$

By dividing this quantity by the total number of combinations, which can be only a function of i and of s, and which we will designate by N, we will have, for the mean value of the r^{th} ordinate,

$$\frac{s^i}{1.2.3\dots iN}\left(\frac{1}{i} + \frac{1}{i-1} \cdots + \frac{1}{r}\right).$$

In order to determine N, we will observe that all the mean values must together equal s; that which gives

$$N = \frac{s^{i-1}}{1.2.3\dots(i-1)};$$

the mean value of the r^{th} ordinate is therefore

$$\frac{s}{i}\left(\frac{1}{i} + \frac{1}{i-1} \cdots + \frac{1}{r}\right). \qquad (\epsilon)$$

Let us suppose that an observed effect has been able to be produced only by one of the i causes A, B, C, etc.; and that a person, after having estimated their respective probabilities, writes on a ballot, the letters which indicate these causes, in the order of the probabilities that he attributes to them, by writing first, the letter indicating the cause which seems to him most probable. It is clear that we will have by the preceding formula, the mean value of the probabilities that he is able to suppose to each of them, by observing that here the quantity s that we must apportion on each of the causes, is certitude or unity, since the person is assured that the effect must result from one of them. The mean value of the probability that he attributes to the cause that he has placed on his ballot at the r^{th} rank, is therefore

$$\frac{1}{i}\left(\frac{1}{i} + \frac{1}{i-1} \cdots + \frac{1}{r}\right).$$

Thence it follows that if a tribunal is summoned to decide on this object, and if each member expresses his opinion by a ballot similar to the preceding; then, by writing on each ballot, beside the letters which indicate the causes, the mean values which [273]
correspond to the rank that they have on the ballot; by making next a sum of all the

values which correspond to each cause, on the diverse ballots; the cause to which will correspond the greatest sum, will be that which the tribunal will judge most probable.

This rule is not at all applicable to the choice of the electoral assemblies, because the electors are not at all obliged, as the judges, to apportion one same sum taken for unity, on the diverse parts among which they must be determined: they can suppose to each candidate, all the nuances of merit comprehended between the null merit and the *maximum* of merit, which we will designate by a; the order of the names on each ballot, does only to indicate that the elector prefers the first to the second, the second to the third, etc. We will determine thus the numbers that it is necessary to write on the ballot, beside the names of the candidates.

Let t_1, t_2, t_3, $\dots t_i$ be the respective merits of the i candidates, in the opinion of the elector, t_1 being the merit that he supposes to the one of the candidates who he has set at the first rank, t_2 being the merit that he supposes at the second, and so forth. The integral $\int t_r dt_1 dt_2 \dots dt_i$ will express the sum of the merits that the elector can attribute to candidate r, provided that we integrate first with respect to t_i, from $t_i = 0$ to $t_i = t_{i-1}$; next with respect to t_{i-1}, from t_{i-1} to t_{i-2}, and so forth, to the integral relative to t_1, which we will take from t_1 null to $t_1 = a$. Because it is clear that then t_i never surpasses t_{i-1}, t_{i-1} never surpasses t_{i-2}, etc. By dividing the preceding integral by this here $\int dt_1 dt_2 \dots dt_i$ which expresses the total sum of the combinations in which the preceding condition is fulfilled, we will have the mean expression of the merit which the elector can attribute to the r^{th} candidate. In executing the integrations, we find $\frac{i-r+1}{i+1}a$ for this expression.

Thence it follows that we can write on the ballot of each elector i beside the first name, $i-1$ beside the second, $i-2$ beside the third, etc. By uniting next all the numbers relative to each candidate, on the diverse ballots; the one of the candidates who will have the greatest sum, must be presumed the candidate who, in the eyes of
[274] the electoral assembly, has the greatest merit, and must consequently be chosen.

This mode of election would be without doubt the better, if some strange considerations in the merit did not influence at all often with respect to the choice of the electors, even the most honest, and did not determine them at all to place in the last ranks the most formidable candidates to the one who they prefer; that which gives a great advantage to the candidates of a mediocre merit. Also experience has caused abandoning it in the establishments which have adopted it.

Let us suppose that the errors of an observation are able to be extended within the limits $+a$ and $-a$; but that ignoring the law of probability of these errors, we subject it only to the condition to give to them a probability so much smaller, as they are greater; the probability of the positive errors being supposed the same as that of the corresponding negative errors, all things that it is natural to admit. Formula (ϵ) will give again the mean law of the errors. For this we will imagine the interval a partitioned into an infinite number i of parts represented by dx, so that $i = \frac{a}{dx}$; we will make next $r = \frac{x}{dx}$; formula (ϵ) becomes thus

$$\frac{s\,dx}{a}\int\frac{dx}{x},$$

the integral being taken from $x = x$ to $x = a$. In the present question $s = \frac{1}{2}$; because the error must fall within the limits $-a$ and $+a$, the probability that it will fall within the limits zero and a is $\frac{1}{2}$; it is the quantity s that it is necessary to apportion on all the points of the interval a; formula (ϵ) becomes then

$$\frac{dx}{2a} \log \frac{a}{x}.$$

Thus the mean law of the probabilities of the positive errors x, or negatives $-x$, is

$$\frac{1}{2a} \log \frac{a}{x}.$$

CHAPTER 3

On the laws of probability, which result from the indefinite multiplication of events

§16. In measure as events are multiplied, their respective probabilities are developed more and more: their mean results and the profits or the losses which depend on them, converge toward some limits which they approach with probabilities always increasing. The determination of these increases and of these limits, is one of the most interesting and most delicate parts of the analysis of chances. [275]

Let us consider first the manner in which the possibilities of two simple events of which one alone must arrive at each trial,[1] is developed when we multiply the number of trials. It is clear that the event of which the facility is greatest, must probably arrive more often in a given number of trials; and we are carried naturally to think that by repeating the trials a very great number of times, each of these events will arrive proportionally to its facility, that we will be able thus to discover by experience. We are going to demonstrate analytically this important theorem.

We have seen in §6 that if p and $1-p$ are the respective probabilities of two events a and b; the probability that in $x+x'$ trials, the event a will arrive x times and the event b, x' times, is equal to

$$\frac{1.2.3\dots(x+x')}{1.2.3\dots x.1.2.3\dots x'}p^x(1-p)^{x'};$$

this is the $(x'+1)^{\text{st}}$ term of the binomial $[p+(1-p)]^{x+x'}$. Let us consider the greatest of these terms that we will designate by k. The anterior term will be $\frac{kp}{1-p}\cdot\frac{x'}{x+1}$, and [276] the following term will be $k\frac{1-p}{p}\cdot\frac{x}{x'+1}$. In order that k be the greatest term, it is necessary that we have at the same time

$$\frac{p}{1-p} < \frac{x+1}{x'} > \frac{x}{x'+1};$$

it is easy to conclude from it that if we make $x+x'=n$, we will have

$$x < (n+1)p > (n+1)p-1;$$

thus x is the greatest whole number comprehended within $(n+1)p$; by making therefore

$$x = (n+1)p - s,$$

[1] Herein trial translates *coup*.

that which gives

$$p=\frac{x+s}{n+1},\quad 1-p=\frac{x'+1-s}{n+1},\quad \frac{p}{1-p}=\frac{x+s}{x'+1-s},$$

s will be less than unity. If x and x' are very great numbers, we will have very nearly,

$$\frac{p}{1-p}=\frac{x}{x'},$$

that is that the exponents of p and of $1-p$, in the greatest term of the binomial, are quite nearly in the ratio of these quantities; so that of all the combinations which can take place in a very great number n of trials, the most probable is that in which each event is repeated proportionally to its probability.

The l^{th} term, after the greatest, is

$$\frac{1.2.3\dots n}{1.2.3\dots(x-l).1.2.3\dots(x'+l)}p^{x-l}(1-p)^{x'+l}.$$

We have, by §32 of the first Book,

$$1.2.3\dots n=n^{n+\frac{1}{2}}c^{-n}\sqrt{2\pi}\left\{1+\frac{1}{12n}+\text{etc.}\right\},$$

that which gives

[277]
$$\frac{1}{1.2.3\dots(x-l)}=(x-l)^{l-x-\frac{1}{2}}\frac{c^{x-l}}{\sqrt{2\pi}}\left\{1-\frac{1}{12(x-l)}-\text{etc.}\right\},$$
$$\frac{1}{1.2.3\dots(x'+l)}=(x'+l)^{-x'-l-\frac{1}{2}}\frac{c^{x'+l}}{\sqrt{2\pi}}\left\{1-\frac{1}{12(x'+l)}-\text{etc.}\right\}.$$

Let us develop the term $(x-l)^{l-x-\frac{1}{2}}$. Its hyperbolic logarithm is

$$\left(l-x-\tfrac{1}{2}\right)\left[\log x+\log\left(1-\tfrac{l}{x}\right)\right];$$

now we have

$$\log\left(1-\frac{l}{x}\right)=-\frac{l}{x}-\frac{l^2}{2x^2}-\frac{l^3}{3x^3}-\frac{l^4}{4x^4}-\text{etc.};$$

we will neglect the quantities of order $\frac{1}{n}$, and we will suppose that l^2 does not surpass at all the order n; then we will be able to neglect the terms of order $\frac{l^4}{x^3}$, because x and x' are of order n. We will have thus

$$\left(l-x-\tfrac{1}{2}\right)\left[\log x+\log\left(1-\tfrac{l}{x}\right)\right]$$
$$=\left(l-x-\frac{1}{2}\right).\log x+l+\frac{l}{2x}-\frac{l^2}{2x}-\frac{l^3}{6x^2},$$

that which gives, by passing again from the logarithms to the numbers,

$$(x-l)^{l-x-\frac{1}{2}}=c^{l-\frac{l^2}{2x}}x^{l-x-\frac{1}{2}}\left(1+\frac{l}{2x}-\frac{l^3}{6x^2}\right);$$

we will have similarly

$$(x'+l)^{-l-x'-\frac{1}{2}} = c^{-l-\frac{l^2}{2x'}} x'^{-l-x'-\frac{1}{2}} \left(1 - \frac{l}{2x'} + \frac{l^3}{6x'^2}\right).$$

We have next by that which precedes, $p = \frac{x+s}{n+1}$, s being less than unity; by making therefore $p = \frac{x-z}{n}$, z will be contained within the limits $\frac{x}{n+1}$ and $-\frac{n-x}{n+1}$, and consequently it will be, setting aside the sign, below unity. The value of p gives $1-p = \frac{x'+z}{n}$; we will have by the preceding analysis, [278]

$$p^{x-l}(1-p)^{x'+l} = \frac{x^{x-l} x'^{x'+l}}{n^n} \left(1 + \frac{nzl}{xx'}\right);$$

thence we deduce

$$\frac{1.2.3\ldots n}{1.2.3\ldots(x-l).1.2.3\ldots(x'+l)} p^{x-l}(1-p)^{x'+l}$$
$$= \frac{\sqrt{n} c^{-\frac{nl^2}{2xx'}}}{\sqrt{\pi}\sqrt{2xx'}} \left(1 + \frac{nzl}{xx'} + \frac{l(x'-x)}{2xx'} - \frac{l^3}{6x^2} + \frac{l^3}{6x'^2}\right).$$

We will have the term anterior to the greatest term, and which is extended from it at the distance l, by making l negative in this equation; by uniting next these two terms, their sum will be

$$\frac{2\sqrt{n}}{\sqrt{\pi}\sqrt{2xx'}} c^{-\frac{nl^2}{2xx'}}.$$

The finite integral

$$\sum \frac{2\sqrt{n}}{\sqrt{\pi}\sqrt{2xx'}} c^{-\frac{nl^2}{2xx'}},$$

taken from $l = 0$ inclusively, will express therefore the sum of all the terms of the binomial $[p + (1-p)]^n$, comprehended between the two terms, of which the one has p^{x+l} for factor, and the other has p^{x-l} for factor, and which are thus equidistant from the greatest term; but it is necessary to subtract from this sum, the greatest term which is evidently contained twice.

Now, in order to have this finite integral, we will observe that we have, by §10 of the first Book, y being function of l,

$$\sum y = \frac{1}{c^{\frac{dy}{dl}} - 1} = \left(\frac{dy}{dl}\right)^{-1} - \frac{1}{2}\left(\frac{dy}{dl}\right)^0 + \frac{1}{12}\frac{dy}{dl} + \text{etc.};$$

whence we deduce by the preceding section,

$$\sum y = \int y\, dl - \frac{1}{2} y + \frac{1}{12}\frac{dy}{dl} + \text{etc.} + \text{ constant.}$$

y being here equal to $\frac{2\sqrt{n}}{\sqrt{\pi}\sqrt{2xx'}} c^{-\frac{nl^2}{2xx'}}$, the successive differentials of y acquire for factor [279] $\frac{nl}{2xx'}$ and its powers; thus l being supposed to not be able to be more than order $\sqrt{n}$, this factor is of order $\frac{1}{\sqrt{n}}$, and consequently its differentials divided by the respective powers of dl, decrease more and more; by neglecting therefore, as we have done previously, the terms of order $\frac{1}{n}$, we will have, by starting with l the two finite and infinitely small integrals, and designating by Y the greatest term of the binomial,

$$\sum y = \int ydl - \frac{1}{2}y + \frac{1}{2}Y.$$

The sum of all the terms of the binomial $[p + (1 - p)]^n$ contained between the two terms equidistant from the greatest term by the number l, being equal to $\sum y - \frac{1}{2}Y$, it will be

$$\int ydl - \frac{1}{2}y;$$

and if we add there the sum of these extreme terms, we will have for the sum of all these terms,

$$\int ydl + \frac{1}{2}y.$$

If we make

$$t = \frac{l\sqrt{n}}{\sqrt{2xx'}},$$

this sum becomes

$$\frac{2}{\sqrt{\pi}} \int dt\, c^{-t^2} + \frac{\sqrt{n}}{\sqrt{\pi}\sqrt{2xx'}} c^{-t^2}. \qquad (o)$$

The terms that we have neglected being of the order $\frac{1}{n}$, this expression is so much more exact, as n is greater: it is rigorous, when n is infinity. It would be easy, by the preceding analysis, to have regard to the terms of order $\frac{1}{n}$, and of the superior orders.

[280] We have, by that which precedes, $x = np + z$, z being a number smaller than unity; we have therefore

$$\frac{x + l}{n} - p = \frac{l + z}{n} = \frac{t\sqrt{2xx'}}{n\sqrt{n}} + \frac{z}{n};$$

thus formula (o) expresses the probability that the difference between the ratio of the number of times that the event a must arrive, to the total number of trials, and the facility p of this event, is comprehended within the limits

$$\pm \frac{t\sqrt{2xx'}}{n\sqrt{n}} + \frac{z}{n}. \qquad (l)$$

$\sqrt{2xx'}$ being equal to

$$n\sqrt{2p(1 - p) + \frac{2z}{n}(1 - 2p) - \frac{2z^2}{n^2}};$$

we see that the interval comprehended between the preceding limits is of order $\frac{1}{\sqrt{n}}$.

If the limit of t, that we will designate by T, is supposed invariable, the probability determined by the function (o), remains very nearly the same; but the interval comprehended between the limits (l), diminishes without ceasing in measure as the trials are repeated, and it becomes null, when their number is infinite.

This interval being supposed invariable; when the events are multiplied, T increases without ceasing, and quite nearly as the square root of the number of trials. But when T is considerable, formula (o) becomes, by §27 of the first Book,

$$1-\frac{c^{-T^2}}{2T\sqrt{\pi}}\cfrac{1}{1+\cfrac{q}{1+\cfrac{2q}{1+\cfrac{3q}{1+\text{etc.}}}}}+\frac{c^{-T^2}}{\sqrt{2n\pi\left[p(1-p)+\frac{z}{n}(1-2p)-\frac{z^2}{n^2}\right]}},$$

q being equal to $\frac{1}{2T^2}$. When we make T increase, c^{-T^2} diminishes with an extreme rapidity, and the preceding probability approaches rapidly to unity to which it be- [281]
comes equal, when the number of trials is infinite.

There are here two sorts of approximations: one of them is relative to the limits taken on both sides of the facility of the event a; the other approximation is related to the probability that the ratio of the arrivals of this event, to the total number of trials, will be contained within these limits. The indefinite repetition of the trials increases more and more this probability, the limits remaining the same: it narrows more and more the interval of these limits, the probability remaining the same. Into infinity, this interval becomes null, and the probability is changed into certitude.

The preceding analysis unites to the advantage of demonstrating this theorem, the one to assign the probability that in a great number n of trials, the ratio of the arrivals of each event will be comprehended within some given limits. Let us suppose, for example, that the facilities of the births of boys and of girls are in the ratio of 18 to 17, and that there are born in one year, 14000 infants; we demand the probability that the number of boys will not surpass 7363, and will not be less than 7037.

In this case, we have

$$p=\frac{18}{35},\quad x=7200,\quad x'=6800,\quad n=14000,\quad l=163;$$

formula (o) gives quite nearly $0,994303$ for the sought probability.

If we know the number of times that out of n trials, the event a arrived; formula (o) will give the probability that its facility p supposed unknown, will be comprehended within the given limits. In fact, if we name i this number of times, we will have, by that which precedes, the probability that the difference $\frac{i}{n}-p$ will be comprehended within the limits $\pm\frac{T\sqrt{2xx'}}{n\sqrt{n}}+\frac{z}{n}$; consequently, we will have the probability that p will be comprehended within the limits

$$\frac{i}{n}\mp\frac{T\sqrt{2xx'}}{n\sqrt{n}}-\frac{z}{n}.$$

[282] The function $\frac{T\sqrt{2xx'}}{n\sqrt{n}}$ being of the order $\frac{1}{\sqrt{n}}$, we are able by neglecting the quantities of order $\frac{1}{n}$, to substitute there i instead of x, and $n-i$ instead of x'; the preceding limits become thus, by neglecting the terms of order $\frac{1}{n}$,

$$\frac{i}{n} \mp \frac{T\sqrt{2i(n-i)}}{n\sqrt{n}};$$

and the probability that the facility of the event a is comprehended within these limits, is equal to

$$\frac{2}{\sqrt{\pi}} \int dt\, c^{-t^2} + \frac{\sqrt{n} c^{-T^2}}{\sqrt{\pi}\sqrt{2i(n-i)}}. \tag{o'}$$

We see thus that in measure as the events are multiplied, the interval of the limits is narrowed more and more, and the probability that the value of p falls within these limits, approaches more and more unity or certitude. It is thus that the events, in being developed, make known their respective probabilities.

We arrive directly to these results, by considering p as a variable which can be extended from zero to unity, and by determining, after the observed events, the probability of its diverse values, as we will see it when we will treat the probability of causes, deduced from observed events.

If we have three or a greater number of events a, b, c, etc., of which one alone must arrive at each trial; we will have, by that which precedes, the probability that in a very great number n of trials, the ratio of the number x of times that one of these events, a for example, will arrive, to the number n, will be comprehended within the limits $p \pm \alpha$, α being a very small fraction; and we see that in the extreme case of the number n infinite, the interval 2α of these limits can be supposed null, and the probability can be supposed equal to certitude, so that the numbers of arrivals of each event will be proportional to their respective facilities.

Sometimes the events, instead of making known directly the limits of the value
[283] of p, give those of a function of this value; then we conclude from it the limits of p, by the resolution of equations. In order to give a quite simple example of it, let us consider two players A and B, of whom the respective skills are p and $1-p$, and playing together with this condition, that the game is won by the one of the two players who, out of three trials, will have vanquished twice his adversary, the third trial not being played, as useless, when one of the players is vanquished in the first two trials.

The probability of A to win the game, is the sum of the first two terms of the binomial $[p+(1-p)]^3$; it is consequently equal to $p^3+3p^2(1-p)$. Let P be this function; by raising the binomial $P+(1-P)$ to the power n, we will have, by the preceding analysis, the probability that, out of the number n of games, the number of games won by A will be comprehended within the given limits. It suffices for that to change p into P in formula (o).

If we name i the number of games won by A, formula (o') will give the probability that P will be comprehended within the limits

$$\frac{i}{n} \pm \frac{T\sqrt{2i(n-i)}}{n\sqrt{n}}.$$

Let therefore p' be the real and positive root of the equation

$$p^3 + 3p^2(1-p) = \frac{i}{n};$$

by designating by $p' \mp \delta p$ the limits of p, the corresponding limits of P will be very nearly $3p'^2 - 2p'^3 \mp 6p'(1-p')\delta p$; by equating these limits to the preceding, we will have

$$\delta p = \frac{T\sqrt{2i(n-i)}}{6p'(1-p')n\sqrt{n}};$$

thus formula (o') will give the probability that p will be comprehended within the limits

$$p' \mp \frac{T\sqrt{2i(n-i)}}{6p'(1-p')n\sqrt{n}}.$$

The number n of games does not determine the number of trials, since we are able to have some games of two trials, and others of three trials. We will have the [284]
probability that the number of games of two trials, will be comprehended within the given limits, by observing that the probability of a game with two trials, is $p^2+(1-p)^2$; Let us designate this function by P'. By elevating the binomial $P' + (1-P')$ to the power n, formula (o) will give the probability that the number of games of two trials will be comprehended within the limits $nP' \pm l$; now the number of games of two trials being $nP' \pm l$, the number of games with three trials will be $n(1-P') \mp l$; the total number of trials will be therefore $3n - nP' \mp l$; formula (o) will give therefore the probability that the number of trials will be comprehended within the limits

$$2n(1+p-p^2) \mp T\sqrt{2nP'(1-P')}.$$

§17. Let us consider an urn A containing a very great number n of white and black balls, and let us suppose that at each drawing, we draw one ball from the urn, and that we replace it with a black ball. We demand the probability that after r drawings, the number of white balls will be x.

Let us name $y_{x,r}$ this probability. After a new drawing, it becomes $y_{x,r+1}$. But in order that there are x white balls after $r+1$ drawings, it is necessary that there are either $x+1$ white balls after the drawing r, and that the following drawing makes a white ball exit, or x white balls after the drawing r, and that the following drawing makes a black ball exit. The probability that there will be $x+1$ white balls after r drawings, is $y_{x+1,r}$, and the probability that then the following drawing will make a white ball exit, is $\frac{x+1}{n}$; the probability of the composite event is therefore $\frac{x+1}{n}y_{x+1,r}$; this is the first part of $y_{x,r+1}$. The probability that there will be x white balls after the drawing r, is $y_{x,r}$; and the probability that then there will exit a black ball, is

$\frac{n-x}{n}$, because the number of black balls in the urn is $n-x$; the probability of the composite event is therefore $\frac{n-x}{n}y_{x,r}$; this is the second part of $y_{x,r+1}$. Thus we have

$$y_{x,r+1} = \frac{x+1}{n}y_{x+1,r} + \frac{n-x}{n}y_{x,r}.$$

If we make

$$x = nx', \quad r = nr', \quad y_{x,r} = y'_{x',r'},$$

[285] this equation becomes

$$y'_{x',r'+\frac{1}{n}} = \left(x' + \tfrac{1}{n}\right) y'_{x'+\frac{1}{n},r'} + (1-x')y'_{x',r'};$$

n being supposed a very great number, we are able to reduce into convergent series $y_{x,r'+\frac{1}{n}}$ and $y_{x'+\frac{1}{n},r'}$; we will have therefore, by neglecting the squares and the superior powers of $\frac{1}{n}$,

$$\frac{1}{n}\cdot\frac{dy'_{x',r'}}{dr'} = \frac{x'}{n}\cdot\frac{dy'_{x',r'}}{dx'} + \frac{1}{n}y'_{x',r'};$$

the integral of this equation in partial differences is

$$y'_{x',r'} = c^{r'}\phi(x'c^{r'}),$$

$\phi(x'c^{r'})$ being an arbitrary function of $x'c^{r'}$, that it is necessary to determine through the value of $y'_{x,0}$.

Let us suppose that urn A has been replenished in this manner. We project a right prism of which the base being a regular polygon of $p+q$ sides, is narrow enough so that the prism never falls on it. On the $p+q$ lateral faces, p are white and q are black, and we put into urn A, at each projection, a ball of the color of the face on which the prism falls. After n projections, the number of white balls will be quite nearly, by the preceding section, $\frac{np}{p+q}$, and the probability that it will be $\frac{np}{p+q}+l$, is, by the same section,

$$\frac{p+q}{\sqrt{2npq\pi}}c^{-\frac{(p+q)^2l^2}{2npq}}.$$

If we make

$$x = \frac{np}{p+q} + l, \quad \frac{(p+q)^2}{2pq} = i^2,$$

this function becomes

$$\frac{i}{\sqrt{\pi n}}c^{-\frac{i^2}{n}\left(x-\frac{np}{p+q}\right)^2};$$

[286] this is the value of $y_{x,0}$, or of $y'_{x',0}$; but the preceding value of $y'_{x',r'}$, gives

$$y_{x,0} = \phi\left(\frac{x}{n}\right);$$

we have therefore

$$\phi\left(\frac{x}{n}\right) = \frac{i}{\sqrt{n\pi}}c^{-i^2n\left(\frac{x}{n}-\frac{p}{p+q}\right)^2};$$

hence,

$$y'_{x',r'} = \frac{ic^{r'}}{\sqrt{n\pi}} c^{-i^2 n\left(\frac{xc^{r'}}{n} - \frac{p}{p+q}\right)^2}$$

whence we deduce

$$y_{x,r} = \frac{ic^{\frac{r}{n}}}{\sqrt{n\pi}} c^{-\frac{i^2}{n}\left(xc^{\frac{r}{n}} - \frac{np}{p+q}\right)^2}.$$

The most probable value of x is that which renders $xc^{\frac{r}{n}} - \frac{np}{p+q}$ null, and consequently it is equal to

$$\frac{np}{(p+q)c^{\frac{r}{n}}};$$

the probability that the value of x will be contained within the limits

$$\frac{np}{(p+q)c^{\frac{r}{n}}} \pm \frac{\mu\sqrt{n}}{c^{\frac{r}{n}}},$$

is

$$2\int \frac{i\,d\mu}{\sqrt{\pi}} c^{-i^2\mu^2},$$

the integral being taken from $\mu = 0$.

Let us seek now the mean value of the number of white balls contained within urn A, after r drawings. This value is the sum of all the possible numbers of white balls, multiplied by their respective probabilities; it is therefore equal to

$$\frac{2np}{(p+q)c^{\frac{r}{n}}} \int \frac{i\,d\mu}{\sqrt{\pi}} c^{-i^2\mu^2},$$

the integral being taken from $\mu = 0$ to $\mu = \infty$. This value is thus [287]

$$\frac{np}{(p+q)c^{\frac{r}{n}}};$$

consequently, it is the same as the most probable value of x.

Let us consider now two urns A and B containing each the number n of balls, and let us suppose that in the total number $2n$ of balls, there are as many white as black. Let us imagine that we draw at the same time, one ball, from each urn, and that next one puts into one urn, the ball extracted from the other. Let us suppose that we repeat this operation, any number r times, by agitating at each time the urns, in order to well mix the balls; and let us seek the probability that after this number r of operations, there will be x white balls in urn A.

Let $z_{x,r}$ be this probability. The number of possible combinations in r operations, is n^{2r}; because at each operation, the n balls of urn A are able to be combined with each of n balls from urn B, that which produces n^2 combinations; $n^{2r}z_{x,r}$ is therefore the number of combinations in which it is possible to have x white balls in urn A after these operations. Now, it can happen that the $(r+1)^{\text{st}}$ operation makes a white ball exit from urn A, and makes a white ball return; the number of cases in which this can arrive, is the product of $n^{2r}z_{x,r}$ by the number x of white balls of urn A, and by the number $n-x$ of white balls which must be then in urn B, since the total number of white balls of the two urns, is n. In all these cases, there remains x white balls in urn A; the product $x(n-x)n^{2r}z_{x,r}$ is therefore one of the parts of $n^{2r+2}z_{x,r+1}$.

It can happen further that the $(r+1)^{\text{st}}$ operation makes exit and return into urn A, a black ball, that which conserves in this urn x white balls. Thus $n-x$ being, after the r^{th} operation, the number of black balls of urn A, and x being the one of black balls of urn B, $(n-x)xn^{2r}x_{x,r}$ is further a part of $n^{2r+2}z_{x,r+1}$.

[288] If there are $x-1$ white balls in urn A after the r^{th} operation, and if the operation following makes a black ball exit from it, and makes a white ball return there; there will be x white balls in urn A after the $(r+1)^{\text{st}}$ operation; the number of cases in which that can happen, is the product of $n^{2r}z_{x-1,r}$ by the number $n-x+1$ of the black balls of urn A after the r^{th} drawing, and by the number $n-x+1$ of white balls of urn B, after the same operation; $(n-x+1)^2n^{2r}z_{x-1,r}$ is therefore again a part of $n^{2r+2}z_{x,r+1}$.

Finally, if there are $x+1$ white balls in urn A after the r^{th} operation, and if the operation following makes a white ball exit from it, and makes a black ball return there; there will be again, after this last operation, x white balls in the urn. The number of cases in which that can arrive, is the product of $n^{2r}z_{x+1,r}$ by the number $x+1$ of white balls of urn A, and by the number $x+1$ of black balls of urn B after the r^{th} operation; $(x+1)^2n^{2r}z_{x+1,r}$ is therefore further part of $n^{2r+2}z_{x,r+1}$.

By reuniting all these parts, and by equating their sum to $n^{2r+2}z_{x,r+1}$, we will have the equation in partial finite differences

$$z_{x,r+1} = \left(\frac{x+1}{n}\right)^2 z_{x+1,r} + \frac{2x}{n}\left(1-\frac{x}{n}\right)z_{x,r} + \left(1-\frac{x-1}{n}\right)^2 z_{x-1,r}.$$

Although this equation is in differences of the second order with respect to the variable x, however its integral contains only one arbitrary function which depends on the probability of the diverse values of x in the initial state of urn A. In fact, it is clear that if we knew the values of $z_{x,0}$ corresponding to all the values of x, from $x=0$ to $x=n$; the preceding equation will give all the values of $z_{x,1}$, $z_{x,2}$, etc., by observing that the negative values of x being impossible, $z_{x,r}$ is null when x is negative.

If n is a very great number, this equation is transformed into an equation in partial differences that we obtain thus. we have then, very nearly,

$$z_{x+1,r} = z_{x,r} + \left(\frac{dz_{x,r}}{dx}\right) + \frac{1}{2}\left(\frac{d^2z_{x,r}}{dx^2}\right),$$
$$z_{x-1,r} = z_{x,r} - \left(\frac{dz_{x,r}}{dx}\right) + \frac{1}{2}\left(\frac{d^2z_{x,r}}{dx^2}\right),$$
$$z_{x,r+1} = z_{x,r} + \left(\frac{dz_{x,r}}{dx}\right).$$

[289] Let

$$x = \frac{n+\mu\sqrt{n}}{2}, \quad r = nr', \quad z_{x,r} = U;$$

the preceding equation in the partial finite differences will become, by neglecting the terms of order $\frac{1}{n^2}$,

$$\left(\frac{dU}{dr'}\right) = 2U + 2\mu\left(\frac{dU}{d\mu}\right) + \left(\frac{d^2U}{d\mu^2}\right).$$

In order to integrate this equation which, as we are able to be assured by the method that I have given for this object, in the *Mémoires de l'Académie des Sciences*, of the year 1773,[2] is integrable in finite terms, only by means of definite integrals, let us make

$$U = \int \phi\, dtc^{-\mu t},$$

ϕ being a function of t and of r'. We will have

$$2\mu\left(\frac{dU}{d\mu}\right) = 2c^{-\mu t}t\phi - 2\int c^{-\mu t}(\phi\, dt + td\phi),$$
$$\left(\frac{ddU}{d\mu^2}\right) = \int c^{-\mu t}t^2\phi\, dt;$$

the equation in the partial differentials in U, become thus

$$\int c^{-\mu t}\left(\frac{dU}{dr'}\right)dt = 2c^{-\mu t}t\phi + \int c^{-\mu t}dt\left[t^2\phi - 2t\frac{d\phi}{dt}\right].$$

By equating between them the terms affected of the $\int$ sign, we will have the equation in the partial differentials

$$\left(\frac{d\phi}{dr'}\right) = t^2\phi - 2t\left(\frac{d\phi}{dt}\right).$$

The term outside the $\int$ sign, equated to zero, will give for the equation in the limits of the integral,

$$0 = t\phi c^{-\mu t}.$$

The integral of the preceding equation in the partial differentials of ϕ, is

$$\phi = c^{\frac{1}{4}t^2}\psi\left(\frac{t}{c^{2r'}}\right),$$

$\psi\left(\frac{t}{c^{2r'}}\right)$ being an arbitrary function of $\frac{t}{c^{2r'}}$; we have therefore [290]

$$U = \int dtc^{-\mu t + \frac{1}{4}t^2}\psi\left(\frac{t}{c^{2r'}}\right).$$

Let there be

$$t = 2\mu + 2s\sqrt{-1},$$

[2]This must refer to his "Recherches sur l'integration des équations différentielles aux différences finies, et sur leur usage dans la théorie des hasards." *Mémoires de l'Académie royale des Sciences de Paris (Savants étrangers)* [**5**].

the expression of U will take this form,

$$U = c^{-\mu^2} \int ds\, c^{-s^2} \Gamma\left(\frac{s-\mu\sqrt{-1}}{c^{2r'}}\right). \tag{A}$$

It is easy to see that the preceding equation, to the limits of the integral, requires that the limits of the integral relative to s, are taken from $s=-\infty$ to $s=\infty$. By taking the radical $\sqrt{-1}$, with the $-$ sign, we will have for U an expression of this form

$$U = c^{-\mu^2} \int ds\, c^{-s^2} \Pi\left(\frac{s+\mu\sqrt{-1}}{c^{2r'}}\right),$$

the arbitrary function $\Pi(s)$ being able to be different from $\Gamma(s)$. The sum of these two expressions of U will be its complete value. But it is easy to be assured that the integrals being taken from $s=-\infty$ to $s=\infty$, the addition of this new expression of U adds nothing to the generality of the first, in which it is comprehended.

Let us develop now the second member of equation (A), according to the powers of $\frac{1}{c^{2r'}}$, and let us consider one of the terms of this development, such as

$$\frac{H^{(i)}c^{-\mu^2}}{c^{4ir'}} \int ds\, c^{-s^2}(s-\mu\sqrt{-1})^{2i};$$

this term becoming, after the integrations,

$$\frac{1.3.5\ldots(2i-1)}{2^i}\sqrt{\pi}\frac{H^{(i)}c^{-\mu^2}}{c^{4ir'}} \times \left[1 - \frac{i(2\mu)^2}{1.2} + \frac{i(i-1)(2\mu)^4}{1.2.3.4} - \frac{i(i-1)(i-2)(2\mu)^6}{1.2.3.4.5.6} + \text{etc.}\right]$$

[291] Let us consider further one term of this development, relative to the odd powers of $\frac{1}{c^{2r'}}$, such as

$$\frac{L^{(i)}\sqrt{-1}c^{-\mu^2}}{c^{(4i+2)r'}} \int ds\, c^{-s^2}(s-\mu\sqrt{-1})^{2i+1},$$

This term becomes, after the integrations,

$$\frac{1.3.5\ldots(2i+1)L^{(i)}\sqrt{\pi}\mu c^{-\mu^2}}{2^i c^{(4i+2)r'}}\left[1 - \frac{i(2\mu)^2}{1.2.3} + \frac{i(i-1)(2\mu)^4}{1.2.3.4.5} - \text{etc.}\right].$$

Thus we will have therefore the general expression of the probability U, developed into a series ordered according to the powers of $\frac{1}{c^{2r'}}$, a series which becomes very convergent, when r' is a considerable number. This expression must be such, that $\int U dx$ or $\frac{1}{2}\int U d\mu\sqrt{n}$ be equal to unity, the integrals being extended to all the values of x and of μ, that is from x null to $x=n$, and from $\mu=-\sqrt{n}$ to $\mu=\sqrt{n}$; because it is certain that one of the values of x needing to take place, the sum of the probabilities of all these values must be equal to unity. By taking the integral $\int c^{-\mu^2}d\mu$ within the limits of μ, we have the same result to very nearly, as by taking it from $\mu=-\infty$ to $\mu=\infty$: the difference is only of the order $\frac{c^{-n}}{\sqrt{n}}$; and seeing the extreme rapidity with which c^{-n} diminishes in measure as n increases, we see that this difference is insensible when n is a great number. This premised, let us consider in the integral $\frac{1}{2}\int U d\mu\sqrt{n}$, the term

$$\frac{1.3.5\ldots(2i-1)H^{(i)}\sqrt{n\pi}}{2^i c^{4ir'}} \times \int d\mu c^{-\mu^2}\left[1-\frac{i(2\mu)^2}{1.2}+\frac{i(i-1)(2\mu)^4}{1.2.3.4}-\text{etc.}\right].$$

By extending the integral from $\mu=-\infty$ to $\mu=\infty$, this term becomes

$$\frac{1.3.5\ldots(2i-1)\frac{1}{2}H^(i)\pi\sqrt{n}}{2^i c^{4ir'}}\left[1-i+\frac{i(i-1)}{1.2}-\frac{i(i-1)(i-2)}{1.2.3}+\text{etc.}\right].$$

The factor $1-i+\frac{i(i-1)}{1.2}-\text{etc.}$ is equal to $(1-1)^i$; it is therefore null, except in the [292] case of $i=0$, where it is reduced to unity. It is clear that the terms of the expression of U which contain the odd powers of μ, give a null result in the integral $\frac{1}{2}\int U d\mu\sqrt{n}$, extended from $\mu=-\infty$ to $\mu=\infty$; because these terms have for factor $c^{-\mu^2}$, and we have generally within these limits

$$\int \mu^{2i+1}d\mu c^{-\mu^2}=0.$$

There is therefore only the first term of the expression of U, a term that we will represent by $Hc^{-\mu^2}$, which can give a result in the integral $\frac{1}{2}\int U d\mu\sqrt{n}$, and this result is $\frac{1}{2}H\sqrt{n\pi}$; we have therefore

$$\frac{1}{2}H\sqrt{n\pi}=1;$$

consequently,

$$H=\frac{2}{\sqrt{n\pi}}.$$

The general expression of U has thus the following form,

$$U=\frac{2c^{-\mu^2}}{\sqrt{n\pi}}\left\{\begin{array}{l}1+\dfrac{Q^{(1)}(1-2\mu^2)}{c^{4r'}}+\dfrac{Q^{(2)}(1-4\mu^2+\frac{4}{3}\mu^4)}{c^{8r'}}+\text{etc.}\\ +\dfrac{L^{(0)}\mu}{c^{2r'}}+\dfrac{L^{(1)}\mu(1-\frac{2}{3}\mu^2)}{c^{6r'}}+\dfrac{L^{(2)}\mu(1-\frac{4}{3}\mu^2+\frac{4}{15}\mu^4)}{c^{10r'}}+\text{etc.}\end{array}\right\}; \qquad (k)$$

$Q^{(1)}$, $Q^{(2)}$, etc., $L^{(0)}$, $L^{(1)}$, etc. being some indeterminate constants which depend on the initial value of U.

Let us suppose that U becomes X when r is null; X being a given function of μ. We have generally these two theorems,

$$0=Q^{(i)}\int \mu^{2q}d\mu\, U_i c^{-\mu^2},$$

$$0=L^{(i)}\int \mu^{2q+1}d\mu\, U_i' c^{-\mu^2},$$

when q is less than i; U_i and U_i' being functions of μ, by which $\frac{2Q^{(i)}c^{-\mu^2}}{\sqrt{n\pi}c^{4ir'}}$ and $\frac{2L^{(i)}c^{-\mu^2}}{\sqrt{n\pi}c^{(4i+2)r'}}$ are multiplied in the expression of U. In order to demonstrate these theorems, we
[293] will observe that, by that which precedes, $\frac{2Q^{(i)}c^{-\mu^2}U_i}{\sqrt{n\pi}}$ is equal to

$$(\sqrt{-1})^{2i}H^{(i)}c^{-\mu^2}\int ds\, c^{-s^2}(\mu+s\sqrt{-1})^{2i};$$

it is necessary therefore to show that we have

$$0=\iint \mu^{2q}ds\, d\mu\, c^{-\mu^2-s^2}(\mu+s\sqrt{-1})^{2i};$$

the integrals being taken from μ and s equal to $-\infty$ to μ and s equal to $+\infty$. By integrating first with respect to μ, this term becomes

$$\frac{2q-1}{2}\iint \mu^{2q-2}d\mu\, ds\, c^{-\mu^2-s^2}(\mu+s\sqrt{-1})^{2i}$$
$$+\,i\iint \mu^{2q-1}d\mu\, ds\, c^{-\mu^2-s^2}(\mu+s\sqrt{-1})^{2i-1}.$$

By continuing to integrate thus by parts relatively to μ, we arrive finally to some terms of the form

$$k\iint d\mu\, ds\, c^{-\mu^2-s^2}(\mu+s\sqrt{-1})^{2e},$$

e not being zero, and by that which precedes, these terms are null.

We will prove in the same manner, that we have

$$0=L^{(i)}\int \mu^{2q+1}\, d\mu\, U_i'\, c^{-\mu^2}.$$

Thence it follows that we have generally

$$0=\int U_i\, U_{i'}d\mu\, c^{-\mu^2},\quad 0=\int U_i'\, U_{i'}'\, d\mu\, c^{-\mu^2},$$

i and i' being different numbers. Because if, for example, i' is greater than i, all the powers of μ in U_i, are less than $2i'$; each of the terms of U_i will give therefore, by that which precedes, a result null in the integral $\int U_i\, U_{i'}d\mu\, c^{-\mu^2}$. The same reasoning holds for the integral $\int U_i'\, U_{i'}'\, d\mu c^{-\mu^2}$.

[294] But these integrals are not nulls, when $i=i'$. We will obtain them in this case, in this manner. We have, by that which precedes,

$$U_i=\frac{2^i(\sqrt{-1})^{2i}\int ds\, c^{-s^2}(\mu+s\sqrt{-1})^{2i}}{1.3.5\ldots(2i-1)\sqrt{\pi}}.$$

The term which has for factor μ^{2i} in this expression, is

$$\frac{2^i(\sqrt{-1})^{2i}\mu^{2i}}{1.3.5\ldots(2i-1)};$$

now, we are able to consider only this term in the first factor U_i of the integral $\int U_i U_i d\mu\, c^{-\mu^2}$; because the inferior powers of μ, in this factor, give a null result in the integral. we have therefore

$$\int U_i U_i d\mu\, c^{-\mu^2} = \frac{2^{2i}}{[1.3.5\ldots(2i-1)]^2\sqrt{\pi}} \iint \mu^{2i} d\mu\, ds\, c^{-\mu^2-s^2}(\mu+s\sqrt{-1})^{2i}.$$

We have, by integrating with respect to μ, from $\mu = -\infty$ to $\mu = \infty$,

$$\begin{aligned}&\iint \mu^{2i} d\mu\, ds\, c^{-\mu^2-s^2}(\mu+s\sqrt{-1})^{2i}\\ =\ &\frac{2i-1}{2}\iint \mu^{2i-2} d\mu\, ds\, c^{-\mu^2-s^2}(\mu+s\sqrt{-1})^{2i}\\ &+\frac{2i}{2}\iint \mu^{2i-1} d\mu\, ds\, c^{-\mu^2-s^2}(\mu+s\sqrt{-1})^{2i-1}\end{aligned}$$

The first term of the second member of this equation is null by that which precedes; this member is reduced therefore to its second term. We find in the same manner, that we have

$$\begin{aligned}&\iint \mu^{2i-1} d\mu\, ds\, c^{-\mu^2-s^2}(\mu+s\sqrt{-1})^{2i-1}\\ =\ &\frac{2i-1}{2}\iint \mu^{2i-2} d\mu\, ds\, c^{-\mu^2-s^2}(\mu+s\sqrt{-1})^{2i-2},\end{aligned}$$

and thus consecutively; we have therefore

$$\iint \mu^{2i} d\mu\, ds\, c^{-\mu^2-s^2}(\mu+s\sqrt{-1})^{2i} = \frac{1.2.3\ldots 2i\pi}{2^{2i}};$$

consequently,

$$\int U_i U_i d\mu\, c^{-\mu^2} = \frac{2.4.6\ldots 2i\sqrt{\pi}}{1.3.5\ldots(2i-1)}.$$

We will find in the same manner, [295]

$$\int U_i' U_i' d\mu\, c^{-\mu^2} = \frac{1}{2}\,\frac{2.4.6\ldots 2i\sqrt{\pi}}{1.3.5\ldots(2i+1)}.$$

We have evidently

$$\int U_i U_{i'}' d\mu\, c^{-\mu^2} = 0,$$

in the same case where i and i' are equal, because the product $U_i U_{i'}'$ contains only odd powers of μ. This premised.

The general expression of U gives for its initial value, that we have designated by X,

$$X = \frac{2c^{-\mu^2}}{\sqrt{n\pi}}\left\{\begin{matrix} 1 + Q^{(1)}\left(1-2\mu^2\right) + \text{etc.} \\ +L^{(0)}\mu + L^{(1)}\mu\left(1-\frac{3}{2}\mu^2\right) + \text{etc.}\end{matrix}\right\}.$$

If we multiply this equation by $U_i d\mu$, and if we take the integrals from $\mu = -\infty$ to $\mu = \infty$, we will have, by virtue of the preceding theorems,

$$\int XU_i d\mu = \frac{2}{\sqrt{n\pi}} Q^{(i)} \int U_i U_i . d\mu . c^{-\mu^2},$$

whence we deduce

$$Q^{(i)} = \frac{1.3.5\ldots(2i-1)\frac{1}{2}\sqrt{n}}{2.4.6\ldots 2i} \int XU_i . d\mu;$$

we will find in the same manner,

$$L^{(i)} = \frac{1.3.5\ldots(2i+1)\sqrt{n}}{2.4.6\ldots 2i} \int XU_i' . d\mu.$$

We will have therefore thus the successive values of $Q^{(1)}$, $Q^{(2)}$, etc.; $L^{(0)}$, $L^{(1)}$, etc., by means of definite integrals, when X or the initial value of U will be given.

In the case where X is equal to $\frac{2i}{\sqrt{n\pi}} c^{-i^2\mu^2}$, the general expression of U takes a very
[296] simple form. Then the arbitrary function $\Gamma\left(\frac{s-\mu\sqrt{-1}}{c^{2r'}}\right)$ of formula (A) is of the form $kc^{-\beta\left(\frac{s-\mu\sqrt{-1}}{c^{2r'}}\right)^2}$. In order to determine the constants β and k, we will observe that by supposing

$$\beta' = \frac{\beta}{c^{4r'}},$$

we will have

$$U = kc^{-\frac{\mu^2}{1+\beta'}} \int ds\, c^{-(1+\beta')\left(s - \frac{\beta'\mu\sqrt{-1}}{1+\beta'}\right)^2}.$$

By making next

$$\sqrt{1+\beta'}\left(s - \frac{\beta'\mu\sqrt{-1}}{1+\beta'}\right) = s',$$

and observing that the integral relative to s must be taken from $s = -\infty$ to $s = \infty$, the integral relative to s' must be taken within the same limits, we will have

$$U = \frac{k\sqrt{\pi}}{\sqrt{1+\beta'}} c^{-\frac{\mu^2}{1+\beta'}}.$$

By comparing this expression to the initial value of U, which is

$$U = \frac{2i}{\sqrt{n\pi}} c^{-i^2\mu^2};$$

and observing that β is the initial value of β', we will have

$$i^2 = \frac{1}{1+\beta};$$

whence we deduce

$$\beta = \frac{1-i^2}{i^2}, \quad \beta' = \frac{1-i^2}{i^2 c^{4r'}}.$$

We must have next

$$\frac{k\sqrt{\pi}}{\sqrt{1+\beta}} = \frac{2i}{\sqrt{n\pi}};$$

that which gives

$$k\sqrt{\pi} = \frac{2}{\sqrt{n\pi}},$$

a value that we obtain next, by the condition that $\frac{1}{2}\int U d\mu\sqrt{n} = 1$, the integral being [297]
taken from $\mu = -\infty$ to $\mu = \infty$; we will have, for the expression of U, whatever be r',

$$U = \frac{2}{\sqrt{n\pi(1+\beta')}} c^{-\frac{\mu^2}{1+\beta'}}.$$

We find, indeed, that this value of U, substituted into the equation in the partial differentials in U, satisfies it.

β' diminishing without ceasing when r' increases, the value of U varies without ceasing, and becomes in its limit, when r' is infinity,

$$U = \frac{2}{\sqrt{n\pi}} c^{-\mu^2}.$$

In order to give an application of these formulas, let us imagine, in an urn C, a very great number m of white balls, and a parallel number of black balls. These balls having been mixed, let us suppose that we draw from the urn, n balls that we put into urn A. Let us suppose next that we put into urn B, as many white balls, as there are black balls in urn A, and as many black balls, as there are white balls in the same urn. It is clear that the number of cases in which there will be x white balls, and consequently $n-x$ black balls in urn A, is equal to the product of the number of combinations of the m white balls of urn C, taken x by x, by the number of combinations of the m black balls of the same urn, taken $n-x$ by $n-x$. This product is, by §3, equal to

$$\frac{m(m-1)(m-2)\cdots(m-x+1)}{1.2.3\ldots x}\,\frac{m(m-1)(m-2)\cdots(m-n+x+1)}{1.2.3\ldots(n-x)}$$

or to

$$\frac{(1.2.3\ldots m)^2}{1.2.3\ldots x.1.2.3\ldots(n-x).1.2.3\ldots(m-x).1.2.3\ldots(m-n+x)}.$$

The number of all possible cases is the number of combinations of the $2m$ balls from urn C, taken n by n; this number is

$$\frac{1.2.3\ldots 2m}{1.2.3\ldots n.1.2.3\ldots(2m-n)};$$

by dividing the preceding fraction by that here, we will have, for the probability of x [298]
or for the initial value of U,

$$\frac{(1.2.3\ldots m)^2 1.2.3\ldots n.1.2.3\ldots(2m-n)}{1.2.3\ldots x.1.2.3\ldots(m-x).1.2.3\ldots(n-x).1.2.3\ldots(m-n+x).1.2.3\ldots 2m}:$$

Now, if we observe that we have very nearly, when s is a great number,

$$1.2.3\ldots s = s^{s+\frac{1}{2}}c^{-s}\sqrt{2\pi};$$

we will find easily after all the reductions, by making

$$x = \frac{n+\mu\sqrt{n}}{2},$$

and by neglecting the quantities of order $\frac{1}{n}$, which are not multiplied by μ^2,

$$U = \frac{2}{\sqrt{n\pi}}\sqrt{\frac{m}{2m-n}}c^{-\frac{m\mu^2}{2m-n}};$$

by making therefore

$$i^2 = \frac{m}{2m-n};$$

we will have

$$U = \frac{2i}{\sqrt{n\pi}}c^{-i^2\mu^2}.$$

If the number m is infinite, then $i^2 = \frac{1}{2}$, and the initial value of U is

$$U = \frac{\sqrt{2}}{\sqrt{n\pi}}c^{-\frac{1}{2}\mu^2}.$$

Its value, after any number of drawings, is

$$U = \frac{2}{\sqrt{n\pi\left(1+c^{-\frac{4r}{n}}\right)}}c^{-\frac{\mu^2}{1+c^{-\frac{4r}{n}}}}.$$

The case of m infinite returns to the one in which the urns A and B would be filled, by projecting n times a coin which would bring forth indifferently *heads* or *tails*, and
[299] putting into urn A, a white ball, each time that *heads* would arrive, and a black ball, each time that *tails* would arrive; and making the inverse for urn B. Because it is clear that the probability of drawing a white ball from urn C, is then $\frac{1}{2}$, as that to bring forth *heads* or *tails*.

By taking the integral $\int U dx$, or $\frac{1}{2}\int U d\mu\sqrt{n}$, from $\mu = -a$ to $\mu = a$, we will have the probability that the number of white balls of urn A, will be comprehended within the limits $\pm a\sqrt{n}$.

We are able to generalize the preceding result, by supposing the urn A filled as at the beginning of this section, by the projection of a prism of $p+q$ lateral faces, of which p are white and q are black. We have seen that then if we make

$$i^2 = \frac{(p+q)^2}{2pq},$$

we have at the origin, or when r is null,

$$U = \frac{i}{\sqrt{n\pi}}c^{-\frac{i^2}{n}\left(x-\frac{np}{p+q}\right)^2}.$$

Let us suppose p and q very little different, so that we have

$$p = \frac{p+q}{2}\left(1 + \frac{a}{\sqrt{n}}\right),$$
$$q = \frac{p+q}{2}\left(1 - \frac{a}{\sqrt{n}}\right),$$

we will have

$$i^2 = \frac{2}{1 - \frac{a^2}{n}},$$

or very nearly $i^2 = 2$; therefore

$$U = \frac{2}{\sqrt{2n\pi}} c^{-\frac{2}{n}\left(x - \frac{n}{2} - \frac{a\sqrt{n}}{2}\right)^2}.$$

By making therefore

$$x = \frac{n + \mu\sqrt{n}}{2};$$

we will have

$$U = \frac{2}{\sqrt{2n\pi}} c^{-\frac{1}{2}(\mu - a)^2}.$$

Let us suppose now that after any number whatsoever of drawings, we have [300]

$$U = \frac{2}{\sqrt{n\beta\pi}} c^{-\frac{(\mu-\alpha)^2}{\beta}},$$

β and α being some functions of r'. If we substitute this value into the equation in the partial differences in U, we will have

$$\begin{aligned} -\left(\frac{d\beta}{dr'}\right)\left[1 - \frac{2(\mu-\alpha)^2}{\beta}\right] + 4\left(\frac{d\alpha}{dr'}\right)(\mu - \alpha) \\ = 4(\beta - 1)\left[1 - \frac{2(\mu-\alpha)^2}{\beta}\right] - 8\alpha(\mu - \alpha), \end{aligned}$$

whence we deduce the two following equations,

$$\frac{\left(\frac{d\beta}{dr'}\right)}{\beta - 1} = -4, \quad \left(\frac{d\alpha}{dr'}\right) = -2\alpha.$$

By integrating them, and observing that at the origin of r', $\alpha = a$ and $\beta = 2$, we will have

$$\beta = 1 + c^{-4r'}, \quad \alpha = ac^{-2r'};$$

that which gives

$$U = \frac{2}{\sqrt{n\pi(1 + c^{-4r'})}} c^{-\frac{(\mu - ac^{-2r'})^2}{1 + c^{-4r'}}}.$$

Let us seek now the mean value of the number of white balls contained in urn A, after r drawings. This value is the sum of the products of the diverse numbers of white balls, multiplied by their respective probabilities; it is therefore equal to the integral

$$\int \frac{n+\mu\sqrt{n}}{2}.U.\frac{d\mu\sqrt{n}}{2},$$

taken from $\mu = -\infty$ to $\mu = \infty$. By substituting for U its value given by formula (k), we will have, by virtue of the preceding theorems, for this integral,

$$\frac{1}{2}n + \frac{\sqrt{n}}{4}L^{(0)}c^{-\frac{2r}{n}}.$$

[301] At the origin where r is null, this value is $\frac{1}{2}n + \frac{\sqrt{n}}{2}L^{(0)}$; thus we will have $L^{(0)}$ by means of the number of white balls that urn A contains at this origin.

We are able to obtain quite simply in the following manner, the mean value of the number of white balls, after r drawings. Let us imagine that each white ball has a value that we will represent by unity, the black balls being supposed to have no value. It is clear that the value of urn A will be the sum of the products of all the possible numbers of white balls which are able to exist in the urn, multiplied by their respective probabilities; this value is therefore that which we have named *mean value of the number of white balls*. Let us name it z, after the r^{th} drawing. At the following drawing, if there exits a white ball, this value diminishes by one unit; now if we suppose that x is the number of white balls contained in the urn after the r^{th} drawing, the probability of extracting a white ball from it will be $\frac{x}{n}$; by naming therefore U the probability of this supposition, the integral $\int \frac{Uxdx}{n}$, extended from $x = 0$ to $x = n$, will be the diminution of z, resulting from the probability to extract a white ball, from the urn. If we make, as above, $\frac{r}{n} = r'$, and if we designate the very small fraction $\frac{1}{n}$ by dr', this diminution will be equal to zdr'; because z is equal to $\int Ux\,dx$, a sum of the products of the numbers of white balls, by their respective probabilities. The value of urn A is increased, if we extract a white ball from urn B, in order to put it into urn A; now, x being supposed the number of white balls of urn A, $n - x$ will be the one of the white balls of urn B, and the probability to extract a white ball from this last urn, will be $\frac{n-x}{n}$; by multiplying this probability by the probability U of x, the integral $\int U\frac{n-x}{n}\,dx$, taken from x null to $x = n$, will be the increase of z. $\int U.(n-x)dx$ is the value of urn B; by naming therefore z' this value, $z'dr'$ will be the increase of z: we will have therefore

$$dz = z'dr' - zdr'.$$

[302] The sum of the values of the two urns is evidently equal to n, number of white balls that they contain, that which gives $z' = n - z$; substituting this value of z' into the preceding equation, it becomes

$$dz = (n - 2z)dr';$$

whence we deduce by integrating,

$$z = \frac{1}{2}n + \frac{L^{(0)}}{4c^{2r'}},$$

$L^{(0)}$ being an arbitrary constant; that which is conformed to that which precedes.

We can extend all this analysis, to the case of any number whatsoever of urns: we will limit ourselves here to seek the mean value of the number of white balls that each urn contains after r drawings.

Let us consider a number e of urns, disposed circularly, and each containing the number n of balls, some white, and the others black; n being supposed a very great number. Let us suppose that after r drawings, z_0, z_1, z_2, $\dots z_{e-1}$ are the respective values of the diverse urns. Each drawing consists in extracting at the same time, one ball from each urn, and to put it into the following, by departing from one of them in a determined sense. If we make $\frac{r}{n} = r'$ and $\frac{1}{n} = dr'$; we will have, by the reasoning that we have just made relatively to two urns,

$$dz_i = (z_{i-1} - z_i)dr';$$

this equation holds from $i = 1$ to $i = e - 1$. In the case of $i = e$, we have

$$dz_0 = (z_{e-1} - z_0)dr';$$

by integrating these equations, and supposing that at the origin the respective values of each urn, or the numbers of white balls that they contain, are

$$\lambda_0,\ \lambda_1,\ \dots,\ \lambda_{e-1}.$$

We arrive to this result which holds from $i = 0$ to $i = e - 1$,

$$z_i = \frac{1}{e}\mathrm{S}c^{-\left(1-\cos\frac{2s\pi}{e}\right)r'} \left\{ \begin{array}{l} \quad\lambda_0 \cos\left(\dfrac{2si\pi}{e} - ar'\right) \\ +\lambda_1 \cos\left(\dfrac{2s(i-1)\pi}{e} - ar'\right) \\ +\lambda_2 \cos\left(\dfrac{2s(i-2)\pi}{e} - ar'\right) \\ \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ +\lambda_{e-1} \cos\left(\dfrac{2s(i-e+1)\pi}{e} - ar'\right) \end{array} \right\} \qquad [303]$$

the sign S extending to all the values of s, from $s = 1$ to $s = e$, and a being equal to $\sin\frac{2s\pi}{e}$. The term of this expression, corresponding to $s = e$, is independent of r', and equal to $\frac{1}{e}(\lambda_0 + \lambda_1 + \cdots + \lambda_{e-1})$; that is, the entire sum of the white balls of the urns, divided by their number. This term is the limit of the expression of z_i; whence it follows that after an infinite number of drawings, the values of each urn are equal among them.

CHAPTER 4

On the probability of the errors of the mean results of a great number of observations, and on the most advantageous mean results

§18. Let us consider now the mean results of a great number of observations of which we know the law of the facility of errors. Let us suppose first that for each observation, the errors are able to be equally [304]

$$-n,\ -n+1,\ -n+2,\ \ldots,\ -1,\ 0,\ 1,\ 2,\ \ldots,\ n-2,\ n-1,\ n.$$

The probability of each error will be $\frac{1}{2n+1}$. If we name s the number of observations, the coefficient of $c^{l\varpi\sqrt{-1}}$ in the development of the polynomial

$$\left\{\begin{array}{r} c^{-n\varpi\sqrt{-1}} + c^{-(n-1)\varpi\sqrt{-1}} + c^{-(n-2)\varpi\sqrt{-1}} \cdots\cdots \\ \cdots + c^{-\varpi\sqrt{-1}} + 1 + c^{\varpi\sqrt{-1}} \cdots + c^{n\varpi\sqrt{-1}} \end{array}\right\}^s$$

will be the number of combinations in which the sum of the errors is l. This coefficient is the term independent of $c^{\varpi\sqrt{-1}}$ and of its powers, in the development of the same polynomial multiplied by $c^{-l\varpi\sqrt{-1}}$, and it is clearly equal to the term independent of ϖ in the same development multiplied by $\frac{c^{l\varpi\sqrt{-1}}+c^{-l\varpi\sqrt{-1}}}{2}$ or by $\cos l\varpi$, we will have therefore for the expression of this coefficient,

$$\frac{1}{\pi}\int d\varpi \cos l\varpi (1 + 2\cos\varpi + 2\cos 2\varpi \cdots + 2\cos n\varpi)^s,$$

the integral being taken from $\varpi = 0$ to $\varpi = \pi$.

We have seen, in §36 of the first book, that this integral is

$$\frac{(2n+1)^s\sqrt{3}}{\sqrt{n(n+1)2s\pi}} c^{-\frac{\frac{3}{2}l^2}{n(n+1)s}};$$

the total number of combinations of the errors is $(2n+1)^s$; by dividing the preceding quantity by that here, we will have [305]

$$\frac{\sqrt{3}}{\sqrt{n(n+1)2s\pi}} c^{-\frac{\frac{3}{2}l^2}{n(n+1)s}},$$

for the probability that the sum of the errors of the s observations will be l.

If we make

$$l = 2t\sqrt{\frac{n(n+1)s}{6}},$$

the probability that the sum of the errors will be comprehended within the limits $+2T\sqrt{\frac{n(n+1)s}{6}}$ and $-2T\sqrt{\frac{n(n+1)s}{6}}$ will be equal to

$$\frac{2}{\pi}\int dtc^{-t^2},$$

the integral being taken from $t=0$ to $t=T$. This expression holds further in the case of n infinite. Then by naming $2a$ the interval comprehended between the limits of the errors of each observation, we will have $n=a$, and the preceding limits would become $\pm\frac{2Ta\sqrt{s}}{\sqrt{6}}$: thus the probability that the sum of the errors will be comprehended within the limits $\pm ar\sqrt{s}$ is

$$2\sqrt{\frac{3}{2\pi}}\int drc^{-\frac{3}{2}r^2};$$

this is also the probability that the mean error will be comprehended within the limits $\pm\frac{ar}{\sqrt{s}}$; because we have the mean error, by dividing by s the sum of the errors.

The probability that the sum of the inclination of the orbits of s comets, will be comprehended within some given limits, by supposing all the inclinations equally possible, from zero to a right angle, is evidently the same as the preceding probability;
[306] the interval $2a$ of the limits of the errors of each observation is, in this case, the interval $\frac{\pi}{2}$ of the limits of the possible inclinations; then the probability that the sum of the inclinations must be comprehended within the limits $\pm\frac{\pi r\sqrt{s}}{4}$ is $2\sqrt{\frac{3}{2\pi}}\int drc^{-\frac{3}{2}r^2}$; that which accords with that which we have found in §13.

Let us suppose generally that the probability of each positive or negative error, is expressed by $\phi\left(\frac{x}{n}\right)$, x and n being some infinite numbers. Then, in the function

$$1+2\cos\varpi+2\cos 2\varpi+2\cos 3\varpi\cdots+2\cos n\varpi,$$

each term, such as $2\cos x\varpi$, must be multiplied by $\phi\left(\frac{x}{n}\right)$; now we have

$$2\phi\left(\frac{x}{n}\right)\cos x\varpi=2\phi\left(\frac{x}{n}\right)-\frac{x^2}{n^2}\phi\left(\frac{x}{n}\right)n^2\varpi^2+\text{etc.}$$

By making therefore

$$x'=\frac{x}{n},\qquad dx'=\frac{1}{n},$$

the function

$$\phi\left(\frac{0}{n}\right)+2\phi\left(\frac{1}{n}\right)\cos\varpi+2\phi\left(\frac{2}{n}\right)\cos 2\varpi\cdots+2\phi\left(\frac{n}{n}\right)\cos n\varpi,$$

becomes

$$2n\int dx'\phi(x')-n^3\varpi^2\int x'^2dx'\phi(x')+\text{etc.};$$

the integrals must be extended from $x'=0$ to $x'=1$. Let then

$$k=2\int dx'\phi(x'),\quad k''=\int x'^2dx'\phi(x'),\quad \text{etc.}$$

The preceding series becomes

$$nk\left(1 - \frac{k''}{k}n^2\varpi^2 + \text{etc.}\right).$$

Now the probability that the sum of the errors of the s observations will be comprehended within the limits $\pm l$, is, as it easy to be assured of it by the preceding reasonings,

$$\frac{2}{\pi}\iint d\varpi\, dl \cos l\varpi \left\{\begin{array}{r}\phi\left(\frac{0}{n}\right) + 2\phi\left(\frac{1}{n}\right)\cos\varpi + 2\phi\left(\frac{2}{n}\right)\cos 2\varpi + \cdots \\ \cdots + 2\phi\left(\frac{n}{n}\right)\cos n\varpi\end{array}\right\}^s,$$ [307]

the integral being taken from ϖ null to $\varpi = \pi$; this probability is therefore

$$2\frac{(nk)^s}{\pi}\iint d\varpi\, dl \cos l\varpi \left(1 - \frac{k''}{k}n^2\varpi^2 - \text{etc.}\right)^s. \qquad (u)$$

Let us suppose

$$\left(1 - \frac{k''}{k}n^2\varpi^2 - \text{etc.}\right)^s = c^{-t^2};$$

by taking the hyperbolic logarithms, we will have very nearly, when s is a great number,

$$s\frac{k''}{k}n^2\varpi^2 = t^2;$$

that which gives

$$\varpi = \frac{t}{n}\sqrt{\frac{k}{k''s}}.$$

If we observe next that nk or $2\int dx\,\phi\left(\frac{x}{n}\right)$ expressing the probability that the error of an observation is comprehended within the limits $\pm n$, this quantity must be equal to unity; the function (u) will become

$$\frac{2}{n\pi}\sqrt{\frac{k}{k''s}}\iint dl\, dt\, c^{-t^2}\cos\left(\frac{lt}{n}\sqrt{\frac{k}{k''s}}\right);$$

the integral relative to t needing to be taken from t null to $t = \pi n\sqrt{\frac{k''s}{k}}$, or to $t = \infty$, n being supposed infinite; now we have, by §25 of the first Book,

$$\int dt \cos\left(\frac{lt}{n}\sqrt{\frac{k}{k''s}}\right)c^{-t^2} = \frac{\sqrt{\pi}}{2}c^{-\frac{l^2}{4n^2}\frac{k}{k''s}};$$

by making therefore

$$\frac{l}{n} = 2t'\sqrt{\frac{k''s}{k}};$$

the function (u) becomes [308]

$$\frac{2}{\sqrt{\pi}}\int dt'\, c^{-t'^2}.$$

Thus by naming, as above, $2a$ the interval comprehended between the limits of the errors of each observation, the probability that the sum of the errors of the s observations, will be comprehended within the limits $\pm ar\sqrt{s}$, is

$$\sqrt{\frac{k}{k''s}}\int dr\, c^{-\frac{kr^2}{4k''}},$$

if $\phi\left(\frac{x}{n}\right)$ is constant, then $\frac{k}{k''}=6$, and this probability becomes

$$2\sqrt{\frac{3}{2\pi}}\int drc^{-\frac{3}{2}r^2},$$

that which is conformed to that which we have found above.

If $\phi\left(\frac{x}{n}\right)$ or $\phi(x')$ is a rational and entire function of x', we will have, by the method of §15, the probability that the sum of the errors will be comprehended within the limits $\pm ar\sqrt{s}$, expressed by a series of powers s, $2s$, etc. of quantities of the form $s-\mu\pm r\sqrt{s}$, in which μ increases in arithmetic progression, these quantities being continued until they become negatives. By comparing this series to the preceding expression of the same probability, we will obtain in a quite close manner, the value of the series; and we will arrive thus with respect to this kind of series, to some theorems analogous to those that we have given in §42 of the first Book, on the finite differences of the powers of a variable.

If the law of facility of the errors is expressed by a negative exponential which is able to be extended to infinity, and generally if the errors are able to be extended to infinity; then a becomes infinite, and the application of the preceding method can offer some difficulties. In all these cases, we will make

$$\frac{x}{h}=x', \qquad \frac{1}{h}=dx',$$

[309] h being any finite quantity whatsoever, and by following exactly the preceding analysis, we will find for the probability that the sum of the errors of the s observations is comprehended within the limits $\pm hr\sqrt{s}$,

$$\sqrt{\frac{k}{k''s}}\int dr\, c^{-\frac{kr^2}{4k''}},$$

an expression in which we must observe that $\phi\left(\frac{x}{h}\right)$ or $\phi(x')$ expresses the probability of the error $\pm x$, and that we have

$$k=2\int dx'\phi(x'), \quad k''=\int x'^2dx'\phi(x'),$$

the integrals being taken from $x'=0$ to $x'=\infty$.

§19. Let us determine presently the probability that the sum of the errors of a very great number of observations will be comprehended within some given limits, setting aside the sign of these errors, that is, by taking them all positively. For this, let us consider the series

$$\phi\left(\frac{n}{n}\right)c^{-n\varpi\sqrt{-1}}+\phi\left(\frac{n-1}{n}\right)c^{-(n-1)\varpi\sqrt{-1}}\cdots+\phi\left(\frac{0}{n}\right)\cdots$$
$$\cdots+\phi\left(\frac{n-1}{n}\right)c^{(n-1)\varpi\sqrt{-1}}+\phi\left(\frac{n}{n}\right)c^{n\varpi\sqrt{-1}},$$

$\phi\left(\frac{x}{n}\right)$ being the ordinate on the curve of probability of errors, corresponding to the error $\pm x$, and x being in the same way as n, considered as formed of an infinite number of units. If we raise this series to the power s, after having changed the sign of the negative exponentials; the coefficient of any one exponential, such as $c^{(l+\mu s)\varpi\sqrt{-1}}$, will be the probability that the sum of the errors taken setting aside the sign, is $l+\mu s$; this probability is therefore

$$\frac{1}{2\pi}\int d\varpi\, c^{-(l+\mu s)\varpi\sqrt{-1}}\left\{\begin{matrix}\phi\left(\frac{0}{n}\right)+2\phi\left(\frac{1}{n}\right)c^{\varpi\sqrt{-1}}+2\phi\left(\frac{2}{n}\right)c^{2\varpi\sqrt{-1}}\\ \cdots+2\phi\left(\frac{n}{n}\right)c^{n\varpi\sqrt{-1}}\end{matrix}\right\}^s,$$

the integral relative to ϖ being taken from $\varpi=-\pi$ to $\varpi=\pi$; because in this interval, [310]
the integral $\int d\varpi\, c^{-r\varpi\sqrt{-1}}$, or

$$\int d\varpi(\cos r\varpi-\sqrt{-1}\sin r\varpi)$$

disappears, whatever be r, provided that it is not null.

We have, by developing with respect to the powers of ϖ,

$$\log\left\{c^{-\mu s\varpi\sqrt{-1}}\left[\phi\left(\frac{0}{n}\right)+2\phi\left(\frac{1}{n}\right)c^{\varpi\sqrt{-1}}\cdots+2\phi\left(\frac{n}{n}\right)c^{n\varpi\sqrt{-1}}\right]^s\right\}$$
$$=s\log\left\{\begin{matrix}\phi\left(\frac{0}{n}\right)+2\phi\left(\frac{1}{n}\right)+2\phi\left(\frac{2}{n}\right)\cdots+2\phi\left(\frac{n}{n}\right)\\ +2\varpi\sqrt{-1}\left[\phi\left(\frac{1}{n}\right)+2\phi\left(\frac{2}{n}\right)\cdots+n\phi\left(\frac{n}{n}\right)\right]\\ -\varpi^2\left[\phi\left(\frac{1}{n}\right)+2^2\phi\left(\frac{2}{n}\right)\cdots+n^2\phi\left(\frac{n}{n}\right)\right]\\ -\text{etc.}\end{matrix}\right\}-\mu s\varpi\sqrt{-1} \quad (1)$$

By making therefore

$$\frac{x}{n}=x',\qquad \frac{1}{n}=dx',$$

we have

$$2\int dx'\phi(x')=k,\qquad \int x'dx'\phi(x')=k',\qquad \int x'^2dx'\phi(x')=k'',$$
$$\int x'^3dx'\phi(x')=k''',\qquad \int x'^4dx'\phi(x')=k^{\text{iv}},\qquad \text{etc.},$$

the integrals being taken from x' null to $x' = 1$; the second member of equation (1) becomes

$$s \log nk + s \log \left(1 + \frac{2k'}{k} n\varpi\sqrt{-1} - \frac{k''}{k} n^2\varpi^2 - \text{etc.}\right) - \mu s\varpi\sqrt{-1}.$$

the error of each observation needing to fall necessarily within the limits $\pm n$, we have $nk = 1$; the preceding quantity becomes thus

$$s\left(\frac{2k'}{k} - \frac{\mu}{n}\right) n\varpi\sqrt{-1} - \frac{(kk'' - 2k'^2)sn^2\varpi^2}{k^2} - \text{etc.};$$

by making therefore

$$\frac{\mu}{n} = \frac{2k'}{k},$$

and neglecting the powers of ϖ superior to the square, this quantity is reduced to its
[311] second term, and the preceding probability becomes

$$\frac{1}{2\pi} \int d\varpi c^{-lw\sqrt{-1} - \frac{(kk''-2k'^2)}{k^2} s.n^2.\varpi^2}.$$

Let

$$\beta = \frac{k}{\sqrt{kk'' - 2k'^2}}, \qquad \varpi = \frac{\beta t}{n\sqrt{s}}, \qquad \frac{l}{n} = r\sqrt{s},$$

the preceding integral becomes

$$\frac{1}{2\pi} \frac{c^{-\frac{\beta^2 r^2}{4}}}{n\sqrt{s}} \int \beta dt c^{-\left(t + \frac{l\beta\sqrt{-1}}{2n\sqrt{s}}\right)^2}.$$

This integral must be taken from $t = -\infty$ to $t = \infty$; and then the preceding quantity becomes

$$\frac{\beta}{2\sqrt{\pi}n\sqrt{s}} c^{-\frac{\beta^2 r^2}{4}}.$$

By multiplying it by dl or by $ndr\sqrt{s}$, the integral

$$\frac{1}{2\sqrt{\pi}} \int \beta dr c^{-\frac{\beta^2 r^2}{4}}$$

will be the half-probability that the value of l, and, consequently, the sum of the errors of the observations is comprehended within the limits $\frac{2k'}{k}as \pm ar\sqrt{s}$, $\pm a$ being the limits of the errors of each observation, limits that we designate by $\pm n$, when we imagine them partitioned into an infinite number of parts.

We see thus that the sum of the errors, the most probable, setting aside the sign, is that which corresponds to $r = 0$. This sum is $\frac{2k'}{k}as$. In the case where $\phi(x)$ is constant, $\frac{2k'}{k} = \frac{1}{2}$, the sum of the errors, the most probable, is therefore then the half of the greatest sum possible, a sum which is equal to sa. But if $\phi(x)$ is not constant and diminishes in measure as the error x increases, then $\frac{2k'}{k}$ is less than $\frac{1}{2}$, and the
[312] sum of the errors, setting aside the sign, is below the half of the greatest sum possible.

We can, by the same analysis, determine the probability that the sum of the squares of the errors, will be $l+\mu s$; it is easy to see that this probability has for expression the integral

$$\frac{1}{2\pi}\int d\varpi\, c^{-(l+\mu s)\varpi\sqrt{-1}}\left\{\begin{array}{r}\phi\left(\frac{0}{n}\right)+2\phi\left(\frac{1}{n}\right)\,c^{\varpi\sqrt{-1}}+2\phi\left(\frac{2}{n}\right)\,c^{2^2\varpi\sqrt{-1}}\\ \cdots+2\phi\left(\frac{n}{n}\right)\,c^{n^2\varpi\sqrt{-1}}\end{array}\right\}^s,$$

taken from $\varpi=-\pi$ to $\varpi=\pi$. By following exactly the preceding analysis, we will have

$$\mu=\frac{2n^2k''}{k},$$

and by making

$$\beta'=\frac{k}{\sqrt{kk^{\text{iv}}-2k''^2}},$$

the probability that the sum of the squares of the errors of the s observations will be comprehended within the limits $\frac{2k''}{k}a^2s\pm a^2r\sqrt{s}$, will be

$$\frac{1}{\sqrt{\pi}}\int\beta'dr\,c^{-\frac{\beta'^2r^2}{4}}.$$

The most probable sum is that which corresponds to r null; it is therefore $\frac{2k''}{k}a^2s$. If s is a very great number, the result of the observations will deviate very little from this value, and consequently it will make known very nearly the factor $\frac{a^2k''}{k}$.

§20. When we wish to correct an element already known quite nearly, by the collection of a great number of observations, we form equations of condition in the following manner. Let z be the correction of the element, and β the observation; the analytic expression of the latter will be a function of the element. By substituting, instead of the element, its approximate value, plus the correction z; by reducing into series with respect to z, and neglecting the square of z, this function will take the [313]
form $h+pz$; by equating it to the observed quantity β, we will have

$$\beta=h+pz;$$

z would be therefore determined, if the observation was rigorous; but as it is susceptible of error, by naming ϵ this error, we have exactly, to the quantities near of order z^2,

$$\beta+\epsilon=h+pz;$$

and by making $\beta-h=\alpha$, we have

$$\epsilon=pz-\alpha.$$

Each observation furnishes a similar equation, that we can represent for the $(i+1)^{\text{st}}$ observation, by this one

$$\epsilon^{(i)}=p^{(i)}z-\alpha^{(i)}.$$

By uniting all these equations, we have

$$S\epsilon^{(i)} = zSp^{(i)} - S\alpha^{(i)}, \tag{1}$$

the sign S being related to all the values of i, from $i = 0$ to $i = s - 1$, s being the total number of observations. By supposing null the sum of the errors, this equation gives

$$z = \frac{S\alpha^{(i)}}{Sp^{(i)}};$$

it is that which we name ordinarily *mean result of the observations.*

We have seen, in §18, that the probability that the sum of the errors of s observations will be comprehended within the limits $\pm ar\sqrt{s}$, is

$$\sqrt{\frac{k}{k''\pi}} \int dr\, c^{-\frac{kr^2}{4k''}}.$$

Let us name $\pm u$ the error of the result z; by substituting into equation (1), $\pm ar\sqrt{s}$ instead of S$\epsilon^{(i)}$, and $\frac{S\alpha^{(i)}}{Sp^{(i)}} \pm u$ instead of z, it gives

$$r = \frac{uSp^{(i)}}{a\sqrt{s}};$$

[314] the probability that the error of the result z, will be comprehended within the limits $\pm u$ is therefore,

$$\sqrt{\frac{k}{k''s\pi}} Sp^{(i)} \int \frac{du}{a}\, c^{-\frac{ku^2(Sp^{(i)})^2}{4k''a^2s}}.$$

Instead of supposing null the sum of the errors, we are able to suppose null any linear function of these errors, that we will represent thus,

$$m\epsilon + m^{(1)}\epsilon^{(1)} + m^{(2)}\epsilon^{(2)} \cdots + m^{(s-1)}\epsilon^{(s-1)}, \tag{m}$$

m, $m^{(1)}$, $m^{(2)}$, etc. being positive or negative whole numbers. By substituting into this function (m), instead of ϵ, $\epsilon^{(1)}$, $\epsilon^{(2)}$, etc., their values given by the equations of condition, it becomes

$$zSm^{(i)}p^{(i)} - Sm^{(i)}\alpha^{(i)};$$

by equating therefore to zero, the function (m), we have

$$z = \frac{Sm^{(i)}\alpha^{(i)}}{Sm^{(i)}p^{(i)}}.$$

Let u be the error of this result, so that we have

$$z = \frac{Sm^{(i)}\alpha^{(i)}}{Sm^{(i)}p^{(i)}} + u;$$

the function (m) becomes

$$uSm^{(i)}p^{(i)}.$$

Let us determine the probability of the error u, when the observations are in great number.

For this, let us consider the product

$$\int \phi\left(\frac{x}{a}\right) c^{mx\varpi\sqrt{-1}} \times \int \phi\left(\frac{x}{a}\right) c^{m^{(1)}x\varpi\sqrt{-1}} \cdots \times \int \phi\left(\frac{x}{a}\right) c^{m^{(s-1)}xn\varpi\sqrt{-1}},$$

the $\int$ sign extending to all the values of x, from the extreme negative value of x, to its positive extreme value. $\phi\left(\frac{x}{a}\right)$ is, as in the preceding sections, the probability of an error x, in each observation; x being supposed, in the same way as a, formed from [315] an infinity of parts taken for unity. It is clear that the coefficient of any exponential $c^{l\varpi\sqrt{-1}}$, in the development of this product, will be the probability that the sum of the errors of the observations, multiplied respectively by m, $m^{(1)}$, etc., that is, the function (m), will be equal to l; by multiplying therefore the preceding product by $c^{-l\varpi\sqrt{-1}}$, the term independent of $c^{\varpi\sqrt{-1}}$ and of its powers, in this new product, will express this probability. If we suppose, as we will do here, the probability of the positive errors, the same as that of the negative errors; we will be able, in the sum $\int \phi\left(\frac{x}{a}\right) c^{mx\varpi\sqrt{-1}}$, to reunite the terms multiplied, one by $c^{mx\varpi\sqrt{-1}}$, and the other by $c^{-mx\varpi\sqrt{-1}}$; then this sum takes the form $2\int \phi\left(\frac{x}{a}\right) \cos\, mx\varpi$. It is likewise of it of all the similar sums. Thence it follows that the probability that the function (m) will be equal to l, is equal to

$$\frac{1}{2\pi}\int d\varpi \left\{ \begin{array}{l} \mathrm{c}^{-l\varpi\sqrt{-1}} \times 2\int \phi\left(\frac{x}{a}\right)\cos mx\varpi \\ \quad \times 2\int \phi\left(\frac{x}{a}\right)\cos m^{(i)}x\varpi \cdots \times 2\int \phi\left(\frac{x}{a}\right)\cos m^{(s-1)}x\varpi \end{array} \right\}; \qquad (i)$$

the integral being taken from $\varpi = -\pi$ to $\varpi = \pi$. We have by reducing the cosines into series,

$$\int \phi\left(\frac{x}{a}\right) \cos\ mx\varpi = \int \phi\left(\frac{x}{a}\right) - \frac{1}{2}m^2a^2\varpi^2 \int \frac{x^2}{a^2}\phi\left(\frac{x}{a}\right) + \text{etc.}$$

If we make $\frac{x}{a} = x'$ and if we observe that the variation of x being unity, we have $dx' = \frac{1}{a}$; we will have

$$\int \phi\left(\frac{x}{a}\right) = a\int dx'\phi(x').$$

Let us name, as in the preceding sections, k the integral $2\int dx'\phi(x')$, taken from x' null to its extreme positive value; let us name similarly k'' the integral $\int x'^2dx'\phi(x')$, taken within the same limits, and so forth; we will have

$$2\int \phi\left(\frac{x}{a}\right)\cos mx\varpi = ak\left(1 - \frac{k''}{k}m^2a^2\varpi^2 + \frac{k^{\text{iv}}}{12k}m^4a^4\varpi^4 - \text{etc.}\right).$$

The logarithm of the second member of this equation is [316]

$$-\frac{k''}{k}m^2a^2\varpi^2 + \frac{kk^{\text{iv}} - 6k''^2}{12k^2}m^4a^4\varpi^4 - \text{etc.} + \log ak;$$

ak or $2a\int dx'\phi(x')$ expresses the probability that the error of each observation, will be comprehended within its limits, that which is certain; we have therefore $ak = 1$; that which reduces the preceding logarithm to

$$-\frac{k''}{k}m^2a^2\varpi^2 + \frac{kk^{\text{iv}} - 6k''^2}{12k^2}m^4a^4\varpi^4 - \text{etc.}$$

Thence it is easy to conclude that the product

$$2\int\phi\left(\frac{x}{a}\right)\cos mx\varpi \times 2\int\phi\left(\frac{x}{a}\right)\cos m^{(i)}x\varpi \cdots \times 2\int\phi\left(\frac{x}{a}\right)\cos m^{(s-1)}x\varpi,$$

is

$$\left(1 + \frac{kk^{\text{iv}} - 6k''^2}{12k^2}a^4\varpi^4 Sm^{(i)4} + \text{etc.}\right)c^{-\frac{k''}{k}a^2\varpi^2 Sm^{(i)2}};$$

the preceding integral (i) is reduced therefore to

$$\frac{1}{2\pi}\int d\varpi\left\{1 + \frac{kk^{\text{iv}} - 6k''^2}{12k^2}a^4\varpi^4 Sm^{(i)4} + \text{etc.}\right\} \times c^{-lw\sqrt{-1}-\frac{k''}{k}a^2\varpi^2 Sm^{(i)2}}.$$

By making $sa^2\varpi^2 = t^2$, this integral becomes

$$\frac{1}{2a\pi\sqrt{s}}\int dt\left\{1 + \frac{kk^{\text{iv}} - 6k''^2}{12k^2}\cdot\frac{Sm^{(i)4}}{s^2}t^4 + \text{etc.}\right\} \times c^{-\frac{lw\sqrt{-1}}{a\sqrt{s}}-\frac{k''}{k}\cdot\frac{Sm^{(i)2}}{s}t^2};$$

$Sm^{(i)2}, \mathrm{S}m^{(i)4}$, etc. are evidently quantities of order s; thus $\frac{Sm^{(i)4}}{s^2}$ is of order $\frac{1}{s}$; by neglecting therefore the terms of this last order, vis-à-vis of unity, the last integral is reduced to

$$\frac{1}{2a\pi\sqrt{s}}\int dtc^{-\frac{lt\sqrt{-1}}{a\sqrt{s}}-\frac{k''}{k}\cdot\frac{Sm^{(i)2}}{s}t^2}.$$

[317] The integral relative to ϖ needing to be taken from $\varpi = -\pi$ to $\varpi = \pi$, the integral relative to t must be taken from $t = -a\pi\sqrt{s}$ to $t = a\pi\sqrt{s}$; and in these cases, the exponential under the $\int$ sign is insensible at these two limits, either because s is a great number, or because a is here supposed divided into an infinity of parts taken for unity; we are able therefore to take the integral from $t = -\infty$ to $t = \infty$. Let us make

$$t' = \sqrt{\frac{k''Sm^{(i)2}}{ks}}\left\{t + \frac{l\sqrt{-1}k\sqrt{s}}{2ak''Sm^{(i)2}}\right\},$$

the preceding integral function becomes

$$\frac{c^{-\frac{kl^2}{4k''a^2Sm^{(i)2}}}}{2a\pi\sqrt{\frac{k''}{k}Sm^{(i)2}}}\int dt'c^{-t'^2}.$$

The integral relative to t' must be taken, as the integral relative to t, from $t' = -\infty$ to $t' = \infty$; that which reduces the preceding quantity to this one,

$$\frac{c^{-\frac{kl^2}{4k''a^2 Sm^{(i)2}}}}{2a\sqrt{\pi}\sqrt{\frac{k''}{k}Sm^{(i)2}}}.$$

If we make $l = ar\sqrt{s}$, and if we observe that the variation of l being unity, we have $adr = 1$, we will have

$$\frac{\sqrt{s}}{2\sqrt{\frac{k''\pi}{k}Sm^{(i)2}}}\int dr\, c^{-\frac{kr^2 s}{4k'' Sm^{(i)2}}},$$

for the probability that the function (m) will be comprehended within the limits zero and $ar\sqrt{s}$, the integral being taken from r null.

We have need here to know the probability of the error u, of the element determined by making null the function (m). This function being supposed equal to l or to $ar\sqrt{s}$; we will have, by that which precedes,

$$uSm^{(i)}p^{(i)} = ar\sqrt{s};$$

by substituting this value into the preceding integral function, it becomes [318]

$$\frac{Sm^{(i)}p^{(i)}}{2a\sqrt{\frac{k''\pi}{k}Sm^{(i)2}}}\int du\, c^{-\frac{ku^2(Sm^{(i)}p^{(i)})^2}{4k''a^2 Sm^{(i)2}}};$$

this is the expression of the probability that the value of u will be comprehended within the limits zero and u: it is also the expression of the probability that u will be comprehended within the limits zero and $-u$. If we make

$$u = 2at\sqrt{\frac{k''}{k}}\cdot\frac{\sqrt{Sm^{(i)2}}}{Sm^{(i)}p^{(i)}},$$

the preceding probability becomes

$$\frac{1}{\sqrt{\pi}}\int dt\, c^{-t^2}.$$

Now the probability remaining the same, t remains the same, and the interval of the two limits of u, are tightened so much more as $a\sqrt{\frac{k''}{k}}\cdot\frac{\sqrt{Sm^{(i)2}}}{Sm^{(i)}p^{(i)}}$ is smaller. This interval remaining the same, the value of t, and consequently the probability that the error of the element falls within this interval, is so much greater, as the same quantity $a\sqrt{\frac{k''}{k}}\cdot\frac{\sqrt{Sm^{(i)2}}}{Sm^{(i)}p^{(i)}}$ is smaller; it is necessary therefore to choose the system of factors $m^{(i)}$, which renders this quantity a *minimum*; and as a, k, k'' are the same in all these systems, it is necessary to choose the system which renders $\frac{\sqrt{Sm^{(i)2}}}{Sm^{(i)}p^{(i)}}$ a *minimum.*

We are able to arrive to the same result, in this manner. Let us resume the expression of the probability that u will be comprehended within the limits zero and u. The coefficient of du in the differential of this expression, is the ordinate of the curve of probabilities of the errors u of the element, errors represented by the

abscissa u of this curve, that we can extend to infinity, on each side of the ordinate which corresponds to u null. This premised, each error, either positive, or negative, must be considered as a disadvantage or a real loss, in any game whatsoever; now, by
[319] the principles of the theory of probabilities, exposed at the beginning of this Book, we evaluate this disadvantage, by taking the sum of all the products of each disadvantage by its probability; the mean value of the positive error to fear, is therefore the sum of the products of each error by its probability; it is consequently equal to the integral

$$\frac{\int u\,du\,Sm^{(i)}p^{(i)}c^{-\frac{ku^2(Sm^{(i)}p^{(i)})^2}{4k''a^2Sm^{(i)2}}}}{2a\sqrt{\frac{k''\pi}{k}Sm^{(i)2}}},$$

taken from u null to u infinity; thus this error is

$$a\sqrt{\frac{k''}{k\pi}}\cdot\frac{\sqrt{Sm^{(i)2}}}{Sm^{(i)}p^{(i)}}.$$

This quantity taken with the $-$ sign, gives the mean error to fear less. It is clear that the system of factors $m^{(i)}$ that it is necessary to choose, must be such that these errors are some *minima*, and consequently such that $\frac{\sqrt{Sm^{(i)2}}}{Sm^{(i)}p^{(i)}}$ be a *minimum*.

If we differentiate this function with respect to $m^{(i)}$, we will have by equating its differential to zero, by the condition of the *minimum*,

$$\frac{m^{(i)}}{Sm^{(i)2}}=\frac{p^{(i)}}{Sm^{(i)}p^{(i)}}.$$

This equation holds whatever be i; and as the variation of i does not change the fraction $\frac{Sm^{(i)2}}{Sm^{(i)}p^{(i)}}$ at all; by naming μ this fraction, we will have

$$m=\mu p,\quad m^{(1)}=\mu p^{(1)},\quad \dots\quad m^{(s-1)}=\mu p^{(s-1)};$$

and we can, whatever be p, $p^{(1)}$, etc., take μ such that the numbers m, $m^{(1)}$, etc. are whole numbers, as the preceding analysis supposes them. Then we have

$$z=\frac{Sp^{(i)}\alpha^{(i)}}{Sp^{(i)2}},$$

[320] and the mean error to fear becomes

$$\pm\frac{a\sqrt{\frac{k''}{k\pi}}}{Sp^{(i)2}}:$$

it is under all the hypotheses that we can make on the factors m, $m^{(1)}$, etc., the smallest possible mean error.

If we make the values of m, $m^{(1)}$, etc. equal to ± 1, the mean error to fear will be smaller when the sign $\pm$ will be determined in a manner that $m^{(i)}p^{(i)}$ is positive, that which returns to supposing $1=m=m^{(1)}=$ etc., and to prepare the equations of condition, so that the coefficient of z in each of them, is positive; this is that which we do in the ordinary method. Then the mean result of the observations is

$$z = \frac{S\alpha^{(i)}}{Sp^{(i)}},$$

and the mean error to fear positive or negative,[1] is

$$\pm\frac{a\sqrt{\frac{k''s}{k\pi}}}{Sp^{(i)}};$$

but this error surpasses the preceding which, as we have seen, is the smallest possible. We are able to be convinced of it besides in this manner. It suffices to show that we have the inequality

$$\frac{\sqrt{s}}{Sp^{(i)}} > \frac{1}{\sqrt{Sp^{(i)2}}},$$

or

$$sSp^{(i)2} > (Sp^{(i)})^2.$$

In fact, $2pp^{(1)}$ is less than $p^2 + p^{(1)2}$, since $(p^{(1)} - p)^2$ is a positive quantity; we can therefore, in the second member of the preceding inequality, substitute, for $2pp^{(1)}$, $p^2 + p^{(1)2} - f$, f being a positive quantity. By making some similar substitutions for all the similar products, this second member will be equal to the first, less a positive quantity.

The result [321]

$$z = \frac{Sp^{(i)}\alpha^{(i)}}{Sp^{(i)2}},$$

to which corresponds the *minimum* of mean error to fear, is the one which the method of least squares of the errors of the observations gives; because, the sum of these squares being

$$(pz - \alpha)^2 + (p^{(1)}z - \alpha^{(1)})^2 \cdots + (p^{(s-1)}z - \alpha^{(s-1)})^2,$$

the condition of the *minimum* of this function, by making z vary, gives for this variable, the preceding expression; this method must therefore be employed in preference, whatever be the law of facility of the errors, a law on which the ratio $\frac{k''}{k}$ depends.

This ratio is $\frac{1}{6}$, if $\phi(x)$ is a constant; it is less than $\frac{1}{6}$, if $\phi(x)$ is variable, and such that it diminishes in measure as x increases, as it is natural to suppose. By adopting the mean law of errors, that we have given in §15, and according to which $\phi(x)$ is equal to $\frac{1}{2a}\log\frac{a}{x}$, we have $\frac{k''}{k} = \frac{1}{18}$. As for the limits $\pm a$, we are able to take for these limits, the deviations from the mean result, which would cause to reject an observation.

But we can, by the same observations, determine the factor $a\sqrt{\frac{k''}{k}}$ of the expression of the mean error. In fact, we have seen, in the preceding section, that the sum of the squares of the errors of the observations, is very nearly $2s\frac{a^2k''}{k}$, and that if they

[1] *en plus ou en moins*: more or less, i.e. positive or negative.

are in great number, it becomes extremely probable that the observed sum will not deviate from this value, by a sensible quantity; we are able therefore to equate them; now the observed sum is equal to $S\epsilon^{(i)2}$, or to $S(p^{(i)}z - \alpha^{(i)})^2$, by substituting for z its value $\frac{Sp^{(i)}\alpha^{(i)}}{Sp^{(i)2}}$; we find thus,

$$2s\frac{a^2k''}{k} = \frac{Sp^{(i)2}.S\alpha^{(i)2} - (Sp^{(i)}\alpha^{(i)})^2}{Sp^{(i)2}}.$$

[322] The preceding expression of the mean error to fear respecting the result z, becomes then

$$\pm\frac{\sqrt{Sp^{(i)2}.S\alpha^{(i)2} - (Sp^{(i)}\alpha^{(i)})^2}}{Sp^{(i)2}\sqrt{2s\pi}},$$

an expression in which there is nothing which is not given by the observations and by the coefficients of the equations of condition.

§21. Let us suppose now that we have two elements to correct by the collection of a great number of observations. By naming z and z' the respective corrections of these elements, we will form, as in the preceding section, some equations of condition, which will be comprehended under this general form

$$\epsilon^{(i)} = p^{(i)}z + q^{(i)}z' - \alpha^{(i)},$$

$\epsilon^{(i)}$ being, as in that section, the error of the $(i+1)^{\text{st}}$ observation. If we multiply respectively by m, $m^{(1)}$, ..., $m^{(s-1)}$ these equations, and if we add together these products, we will have a first final equation

$$Sm^{(i)}\epsilon^{(i)} = z.Sm^{(i)}p^{(i)} + z'.Sm^{(i)}q^{(i)} - Sm^{(i)}\alpha^{(i)}.$$

By multiplying further the same equations respectively by n, $n^{(1)}$, ..., $n^{(s-1)}$ and adding these products, we will have a second final equation

$$Sn^{(i)}\epsilon^{(i)} = z.Sn^{(i)}p^{(i)} + z'.Sn^{(i)}q^{(i)} - Sn^{(i)}\alpha^{(i)},$$

the sign S extending here, as in the preceding section, to all the values of i, from $i = 0$ to $i = s - 1$.

If we suppose null the two functions $Sm^{(i)}\epsilon^{(i)}$, $Sn^{(i)}\epsilon^{(i)}$, functions which we will designate respectively by (m) and (n); the two preceding final equations will give the corrections z and z' of the two elements. But these corrections are susceptible of errors relative to that of which the supposition that we have just made, is itself susceptible. Let us imagine therefore that the functions (m) and (n), instead of being nulls, are respectively l and l', and let us name u and u' the errors corresponding to
[323] the corrections z and z', determined by that which precedes; the two final equations will become

$$l = u.Sm^{(i)}p^{(i)} + u'.Sm^{(i)}q^{(i)},$$
$$l' = u.Sn^{(i)}p^{(i)} + u'.Sn^{(i)}q^{(i)}.$$

It is necessary now to determine the factors m, $m^{(1)}$, etc.; n, $n^{(1)}$, etc., in a manner that the mean error to fear respecting each element, is a *minimum.* For this, let us consider the product

$$\int \phi\left(\frac{x}{a}\right) c^{-(m\varpi+n\varpi')x\sqrt{-1}} \times \int \phi\left(\frac{x}{a}\right) c^{-(m^{(1)}\varpi+n^{(1)}\varpi')x\sqrt{-1}} \cdots$$
$$\cdots \times \int \phi\left(\frac{x}{a}\right) c^{-(m^{(s-1)}\varpi+n^{(s-1)}\varpi')x\sqrt{-1}},$$

the sign $\int$ referring to all the values of x, from $x = -a$ to $x = a$; $\phi\left(\frac{x}{a}\right)$ being, as in the preceding section, the probability of the error x, in the same way as of the error $-x$. The preceding function becomes, by reuniting the two exponentials relative to x and to $-x$,

$$2\int \phi\left(\frac{x}{a}\right) \cos(mx\varpi + nx\varpi') \times 2\int \phi\left(\frac{x}{a}\right) \cos(m^{(1)}x\varpi + n^{(1)}x\varpi')$$
$$\cdots \times 2\int \phi\left(\frac{x}{a}\right) \cos(m^{(s-1)}x\varpi + n^{(s-1)}x\varpi'),$$

the sign $\int$ extending here to all the values of x, from $x = 0$ to $x = a$; x being supposed, in the same way as a, divided into an infinity of parts taken for unity. Presently, it is clear that the term independent of the exponentials, in the product of the preceding function, by $c^{-l\varpi\sqrt{-1}-l'\varpi'\sqrt{-1}}$, is the probability that the sum of the errors of each observation, multiplied respectively by m, $m^{(1)}$, etc., or the function (m), will be equal to l, at the same time as the function (n), sum of the errors of each observation, multiplied respectively by n, $n^{(1)}$, etc., will be equal to l'; this probability is therefore

$$\frac{1}{4\pi^2}\iint d\varpi d\varpi' c^{-l\varpi\sqrt{-1}-l'\varpi'\sqrt{-1}} \left\{ \begin{aligned} 2\int \phi\left(\frac{x}{a}\right) \cos(m\varpi + n\varpi')x \cdots\cdots \\ \ldots \times 2\int \phi\left(\frac{x}{a}\right) \cos(m^{(s-1)}\varpi + n^{(s-1)}\varpi')x \end{aligned} \right\}$$

the integrals being taken from ϖ and ϖ' equal to $-\pi$, to ϖ and ϖ' equal to π. This [324]
premised;

By following exactly the analysis of the preceding section, we find that the preceding function is reduced to very nearly

$$\frac{1}{4\pi^2}\iint d\varpi d\varpi' c^{-l\varpi\sqrt{-1}-l'\varpi'\sqrt{-1}-\frac{k''}{k}a^2[\varpi^2 Sm^{(i)2}+2\varpi\varpi'.Sm^{(i)}n^{(i)}+\varpi'^2.Sn^{(i)2}]},$$

k and k'' having here the same signification as in the section cited. We see further, by the same section, that the integrals are able to be extended from $a\varpi = -\infty$, $a\varpi' = -\infty$, to $a\varpi = \infty$ and $a\varpi' = \infty$. If we make

$$t = a\varpi + \frac{a\varpi'.Sm^{(i)}n^{(i)}}{Sm^{(i)2}} + \frac{kl\sqrt{-1}}{2k''a.Sm^{(i)2}}$$
$$t' = a\varpi' - \frac{k}{2k''a} \cdot \frac{(lSm^{(i)}n^{(i)} - l'Sm^{(i)2})\sqrt{-1}}{Sm^{(i)2}.Sn^{(i)2} - (Sm^{(i)}n^{(i)})^2};$$

if we make next

$$E = Sm^{(i)2}.Sn^{(i)2} - (Sm^{(i)}n^{(i)})^2;$$

the preceding double integral becomes

$$c^{-\frac{k}{4k''a^2E}[l^2Sn^{(i)2} - 2ll'Sm^{(i)}n^{(i)} + l'^2Sm^{(i)2}]} \times \iint \frac{dt\,dt'}{4\pi^2a^2} c^{-\frac{k''l^2}{k}Sm^{(i)2} - \frac{k''t'^2E}{kSm^{(i)2}}}.$$

By taking the integrals within the positive and negative infinite limits, as those relative to $a\varpi$ and $a\varpi'$, we will have

$$\frac{1}{\frac{4k''\pi}{k}a^2\sqrt{E}} c^{-\frac{k}{4k''a^2}\cdot\frac{l^2Sn^{(i)2} - 2ll'Sm^{(i)}n^{(i)} + l'^2Sm^{(i)2}}{E}}. \qquad (o)$$

It is necessary now, in order to have the probability that the values of l and of l' will be comprehended within the given limits, to multiply this quantity by $dl\,dl'$, and to integrate next within these limits. By naming X this quantity, the probability of
[325] which there is concern will be therefore $\iint X\,dl\,dl'$. But in order to have the probability that the errors u and u' of the corrections of the elements will be comprehended within the given limits, it is necessary to substitute into this integral, instead of l and l', their values in u and u'. Now if we differentiate the expressions of l and of l', by supposing l' constant, we have

$$dl = duSm^{(i)}p^{(i)} + du'Sm^{(i)}q^{(i)},$$
$$0 = duSn^{(i)}p^{(i)} + du'Sn^{(i)}q^{(i)};$$

that which gives

$$dl = \frac{du[Sm^{(i)}p^{(i)}.Sn^{(i)}q^{(i)} - Sn^{(i)}p^{(i)}.Sm^{(i)}q^{(i)}]}{Sn^{(i)}q^{(i)}}.$$

If we differentiate next the expression of l', by supposing u constant, we have

$$dl' = du'Sn^{(i)}q^{(i)};$$

we will have therefore

$$dl\,dl' = [Sm^{(i)}p^{(i)}.Sn^{(i)}q^{(i)} - Sn^{(i)}p^{(i)}.Sm^{(i)}q^{(i)}]du\,du'.$$

By making next

$$F = Sn^{(i)2}(Sm^{(i)}p^{(i)})^2 - 2Sm^{(i)}n^{(i)}.Sm^{(i)}p^{(i)}.Sn^{(i)}p^{(i)} + Sm^{(i)2}.(Sn^{(i)}p^{(i)})^2,$$
$$G = Sn^{(i)2}.Sm^{(i)}p^{(i)}.Sm^{(i)}q^{(i)} + Sm^{(i)2}.Sn^{(i)}p^{(i)}.Sn^{(i)}q^{(i)}$$
$$- Sm^{(i)}n^{(i)}.[Sn^{(i)}p^{(i)}.Sm^{(i)}q^{(i)} + Sm^{(i)}p^{(i)}.Sn^{(i)}q^{(i)}],$$
$$H = Sn^{(i)2}(Sm^{(i)}q^{(i)})^2 - 2Sm^{(i)}n^{(i)}.Sm^{(i)}q^{(i)}.Sn^{(i)}q^{(i)} + Sm^{(i)2}.(Sn^{(i)}q^{(i)})^2,$$
$$I = Sm^{(i)}p^{(i)}.Sn^{(i)}q^{(i)} - Sn^{(i)}p^{(i)}.Sm^{(i)}q^{(i)},$$

the function (o) becomes

$$\iint \frac{k}{4k''\pi} \cdot \frac{1}{\sqrt{E}} \cdot \frac{du\,du'}{a^2} c^{-\frac{k(Fu^2+2Guu'+Hu'^2)}{4k''a^2E}}.$$

Let us integrate first this function from $u' = -\infty$ to $u' = \infty$. If we make

$$t = \frac{\sqrt{\frac{kH}{4k''}}\left(u' + \frac{Gu}{H}\right)}{a\sqrt{E}},$$

and if we take the integral from $t = -\infty$ to $t = \infty$, we will have by considering in it [326]
only the variation of u',

$$\int \sqrt{\frac{k}{4k''\pi}} \cdot \frac{du}{a} \cdot \frac{1}{\sqrt{H}} \cdot c^{-\frac{ku^2}{4k''a^2}\frac{FH-G^2}{EH}}.$$

Now we have

$$\frac{FH - G^2}{E} = I^2;$$

the preceding integral becomes therefore

$$\int \frac{I}{\sqrt{H}} \frac{du}{a} \sqrt{\frac{k}{4k''\pi}} c^{-\frac{k}{4k''}\frac{I^2u^2}{a^2H}}.$$

We will have, by the preceding section, the mean error to fear positive or negative, respecting the correction of the first element, by multiplying the quantity under the sign $\int$ by $\pm u$, and taking the integral from $u = 0$ to $u = \infty$, that which gives, for this error,

$$\pm\frac{a\sqrt{H}}{I\sqrt{\frac{k\pi}{k''}}},$$

the $+$ sign indicating the mean positive error to fear, and the $-$ sign the mean negative error to fear.

Let us determine presently the factors $m^{(i)}$ and $n^{(i)}$, in a manner that this error is a *minimum.* By making $m^{(i)}$ alone vary, we have

$$d\log\frac{\sqrt{H}}{I} = dm^{(i)}\frac{[-p^{(i)}Sn^{(i)}q^{(i)} + q^{(i)}Sn^{(i)}p^{(i)}]}{I} + dm^{(i)}\frac{\left\{\begin{aligned} &q^{(i)}Sn^{(i)2}.Sm^{(i)}q^{(i)} - n^{(i)}.Sm^{(i)}q^{(i)}.Sn^{(i)}q^{(i)} \\ &\qquad - q^{(i)}.Sm^{(i)}n^{(i)}.Sn^{(i)}q^{(i)} + m^{(i)}(Sn^{(i)}q^{(i)})^2\end{aligned}\right\}}{H}$$

It is easy to see that this differential disappears, if we suppose in the coefficients of $dm^{(i)}$,

$$m^{(i)} = \mu p^{(i)}, \quad n^{(i)} = \mu q^{(i)},$$

μ being an arbitrary coefficient independent of i, and by means of which we can render $m^{(i)}$ and $n^{(i)}$ whole numbers; the preceding supposition renders therefore null [327]
the differential of $\frac{\sqrt{H}}{I}$, taken with respect to $m^{(i)}$. We will see in the same manner,

that this supposition renders null the differential of the same quantity, taken with respect to $n^{(i)}$. Thus this supposition renders a *minimum* the mean error to fear respecting the correction of the first element; and we will see in the same manner, that it renders further a *minimum*, the mean error to fear respecting the correction of the second element, an error that we obtain by changing in the expression of the preceding, H into F. Under this supposition, the corrections of the two elements are

$$z = \frac{Sq^{(i)2}.Sp^{(i)}\alpha^{(i)} - Sp^{(i)}q^{(i)}.Sq^{(i)}\alpha^{(i)}}{Sp^{(i)2}.Sq^{(i)2} - (Sp^{(i)}q^{(i)})^2},$$

$$z' = \frac{Sp^{(i)2}.Sq^{(i)}\alpha^{(i)} - Sp^{(i)}q^{(i)}.Sp^{(i)}\alpha^{(i)}}{Sp^{(i)2}.Sq^{(i)2} - (Sp^{(i)}q^{(i)})^2}.$$

It is easy to see that these corrections are those that the method of least squares of the errors of the observations gives, or of the *minimum* of the function

$$S(p^{(i)}z + q^{(i)}z' - \alpha^{(i)})^2;$$

whence it follows that this method holds generally, whatever be the number of elements to determine; because it is clear that the previous analysis can be extended to any number of elements.

By substituting for $a\sqrt{\frac{k''}{k\pi}}$, the quantity $\sqrt{\frac{S\epsilon^{(i)2}}{2s\pi}}$, to which we can, by §20, suppose it equal, ϵ, $\epsilon^{(1)}$, etc. being that which remains in the equations of condition, after having substituted there the corrections given by the method of least squares of the errors; the mean error to fear respecting the first element, is

$$\pm\frac{\sqrt{\frac{S\epsilon^{(i)2}}{2s\pi}}\sqrt{Sq^{(i)2}}}{\sqrt{Sp^{(i)2}.Sq^{(i)2} - (Sp^{(i)}q^{(i)})^2}}.$$

The mean positive or negative error to fear respecting the second element, is

$$\pm\frac{\sqrt{\frac{S\epsilon^{(i)2}}{2s\pi}}\sqrt{Sp^{(i)2}}}{\sqrt{Sp^{(i)2}.Sq^{(i)2} - (Sp^{(i)}q^{(i)})^2}}.$$

[328] whence we see that the first element is more or less well determined as the second, according as $Sq^{(i)2}$ is smaller or greater than $Sp^{(i)2}$.

If the r first equations of condition do not contain q at all, and if the $s - r$ last do not contain p at all; then $Sp^{(i)}q^{(i)} = 0$, and the preceding formulas coincide with that of the preceding section.

We are able to obtain thus the mean error to fear respecting each element determined by the method of least squares of the errors, whatever be the number of elements, provided that we consider a great number of observations. Let z, z', z'', z''', etc., be the corrections of each element, and let us represent generally the equations of condition, by the following:

$$\epsilon^{(i)} = p^{(i)}z + q^{(i)}z' + r^{(i)}z'' + t^{(i)}z''' + \text{etc.} - \alpha^{(i)}.$$

In the case of a single element, the mean error to fear is, as we have seen,

$$\pm\sqrt{\frac{S\epsilon^{(i)2}}{2s\pi}}\cdot\frac{1}{\sqrt{Sp^{(i)2}}}. \tag{a}$$

When there are two elements, we will have the mean error to fear respecting the first element, by changing, in the function (a), $Sp^{(i)2}$ into $Sp^{(i)2}-\frac{(Sp^{(i)}q^{(i)})^2}{Sq^{(i)2}}$, that which gives for this error,

$$\pm\frac{\sqrt{\frac{S\epsilon^{(i)2}}{2s\pi}}\sqrt{Sq^{(i)2}}}{\sqrt{Sp^{(i)2}.Sq^{(i)2}-(Sp^{(i)}q^{(i)})^2}}. \tag{a'}$$

When there are three elements, we will have the error to fear respecting the first element, by changing in this expression (a'), $Sp^{(i)2}$ into $Sp^{(i)2}-\frac{(Sp^{(i)}r^{(i)})^2}{Sr^{(i)2}}$, $Sp^{(i)}q^{(i)}$ into $Sp^{(i)}q^{(i)}-\frac{Sp^{(i)}r^{(i)}.Sq^{(i)}r^{(i)}}{Sr^{(i)2}}$, and $Sq^{(i)2}$ into $Sq^{(i)2}-\frac{(Sq^{(i)}r^{(i)})^2}{Sr^{(i)2}}$; this which gives for this error,

$$\pm\frac{\sqrt{\frac{S\epsilon^{(i)2}}{2s\pi}}\sqrt{Sq^{(i)2}.Sr^{(i)2}-(Sq^{(i)}r^{(i)})^2}}{\sqrt{\begin{array}{l}Sp^{(i)2}.Sq^{(i)2}.Sr^{(i)2}-Sp^{(i)2}(Sq^{(i)}r^{(i)})^2-Sq^{(i)2}(Sp^{(i)}r^{(i)})^2\\ \quad -Sr^{(i)2}(Sp^{(i)}q^{(i)})^2+2Sp^{(i)}q^{(i)}.Sp^{(i)}r^{(i)}.Sq^{(i)}r^{(i)}\end{array}}}. \tag{a''}$$

In the case of four elements, we will have the mean error to fear respecting the first [329] element, by changing in this expression (a''), $Sp^{(i)2}$ into $Sp^{(i)2}-\frac{(Sp^{(i)}t^{(i)})^2}{St^{(i)2}}$, $Sp^{(i)}q^{(i)}$ into $Sp^{(i)}q^{(i)}-\frac{Sp^{(i)}t^{(i)}.Sq^{(i)}t^{(i)}}{St^{(i)2}}$, etc. By continuing thus, we will have the mean error to fear respecting the first element, whatever be the number of elements. By changing in the expression of this error, that which is relative to the first element, into that which is relative to the second, and reciprocally; we will have the mean error to fear respecting the second element, and thus of the others.

Thence results a simple way to compare among them diverse astronomical tables, on the side of precision. These tables can always be supposed reduced to the same form, and then they differ only by the epochs, the mean movements, and the coefficients of their arguments; because, if one of them, for example, contains an argument which is not found at all in the others, it is clear that that returns to supposing in the latter, this coefficient null. Now, if we compared these tables to the totality of the good observations, by rectifying them through this comparison, these tables thus rectified, would satisfy, by that which precedes, the condition that the sum of the squares of the errors that they would permit to yet subsist, be a *minimum.* The tables which would approach most to fulfill this condition, would merit therefore preference; whence it follows that by comparing these diverse tables, to a considerable number of observations, the presumption of exactitude must be in favor of that in which the sum of the squares of the errors is smaller than in the others.

§22. To here we have supposed the facilities of the positive errors, the same as those of the negative errors. Let us consider now the general case in which these

facilities are able to be different. Let us name a the interval in which the errors of each observation are able to be extended, and let us suppose it divided into an infinite number $n + n'$ of equal parts and taken for unity, n being the number of the parts which correspond to the negative errors, and n' being the number of the parts
[330] which correspond to the positive errors. On each point of the interval a, let us raise an ordinate which expresses the probability of the corresponding error, and let us designate by $\phi\left(\frac{x}{n+n'}\right)$, the ordinate corresponding to the error x. This premised, let us consider the series

$$\phi\left(\frac{-n}{n+n'}\right)c^{-qn\varpi\sqrt{-1}} + \phi\left[\frac{-(n-1)}{n+n'}\right]c^{-q(n-1)\varpi\sqrt{-1}}\cdots$$
$$\cdots+\phi\left(\frac{-1}{n+n'}\right)c^{-q\varpi\sqrt{-1}} + \phi\left(\frac{0}{n+n'}\right) + \phi\left(\frac{1}{n+n'}\right)c^{q\varpi\sqrt{-1}}\cdots$$
$$\cdots+\phi\left(\frac{n'-1}{n+n'}\right)c^{q(n'-1)\varpi\sqrt{-1}} + \phi\left(\frac{n'}{n+n'}\right)c^{qn'\varpi\sqrt{-1}}.$$

Let us represent this series by $\int\phi\left(\frac{x}{n+n'}\right)c^{qx\varpi\sqrt{-1}}$, the $\int$ sign extending to all the values of x, from $x = -n$ to $x = n'$. The term independent of $c^{\varpi\sqrt{-1}}$ and of its powers, in the development of the function

$$c^{-(l+\mu)\varpi\sqrt{-1}} \times \int\phi\left(\frac{x}{n+n'}\right)c^{qx\varpi\sqrt{-1}} \times \int\phi\left(\frac{x}{n+n'}\right)c^{q^{(1)}x\varpi\sqrt{-1}}\cdots$$
$$\cdots\times\int\phi\left(\frac{x}{n+n'}\right)c^{q^{(s-1)}x\varpi\sqrt{-1}},$$

will be, by §21, the probability that the function

$$q\epsilon + q^{(1)}\epsilon^{(1)}\cdots + q^{(s-1)}\epsilon^{(s-1)} \qquad (m)$$

will be equal to $l + \mu$; this probability is therefore

$$\frac{1}{2\pi}\int d\varpi\, c^{-l\varpi\sqrt{-1}}c^{-\mu\varpi\sqrt{-1}} \times \int\phi\left(\frac{x}{n+n'}\right)c^{qx\varpi\sqrt{-1}} \times \text{etc.}, \qquad (1)$$

the integral being taken from $\varpi = -\pi$ to $\varpi = \pi$. The logarithm of the function

$$c^{-\mu\varpi\sqrt{-1}}\int\phi\left(\frac{x}{n+n'}\right)c^{qx\varpi\sqrt{-1}} \times \int\phi\left(\frac{x}{n+n'}\right)c^{q^{(1)}x\varpi\sqrt{-1}} \times \text{etc.}, \qquad (2)$$

is

$$-\mu\varpi\sqrt{-1} + \log\left[\int\phi\left(\frac{x}{n+n'}\right)c^{qx\varpi\sqrt{-1}}\right] + \text{etc.},$$

[331] n and n' being supposed infinite numbers, if we make

$$\frac{x}{n+n'} = x', \quad \frac{1}{n+n'} = dx';$$

if moreover we suppose

$$k = \int dx'\phi(x'), \quad k' = \int x'dx'\phi(x'), \quad k'' = \int x'^2dx'\phi(x'), \quad \text{etc.},$$

the integrals being taken from $x' = -\frac{n}{n+n'}$ to $x' = \frac{n'}{n+n'}$, we will have

$$\int \phi\left(\frac{x}{n+n'}\right) c^{qx\varpi\sqrt{-1}} = (n+n')k \begin{Bmatrix} 1 + \dfrac{k'}{k}\cdot q.(n+n').\varpi\sqrt{-1} \\ -\dfrac{k''}{2k}\cdot q^2.(n+n')^2.\varpi^2 + \text{etc.} \end{Bmatrix}.$$

The error of each observation needing to fall within the limits $-n$ and $+n'$, and the probability that this will hold being $\int \phi\left(\frac{x}{n+n'}\right)$, or $(n+n')k$, this quantity must be equal to unity. Thence it is easy to conclude that the logarithm of the function (2) is, by making $\mu' = \frac{\mu}{n+n'}$,

$$\left(\frac{k'}{k} Sq^{(i)} - \mu'\right)(n+n')\varpi\sqrt{-1} - \frac{kk'' - k'^2}{2k^2}\cdot Sq^{(i)2}(n+n')^2.\varpi^2 + \text{etc.},$$

the sign S embracing all the values of i, from i null to $i = s-1$. We will make the first power of ϖ vanish, by making

$$\mu' = \frac{k'}{k}\cdot Sq^{(i)},$$

and if we consider only the second power, that which we can do by that which precedes, when s is a very great number, we will have, for the logarithm of the function (2),

$$-\frac{kk''-k'^2}{2k^2} Sq^{(i)2}(n+n')^2\varpi^2.$$

By passing again from the logarithms to the numbers, the function (2) is transformed into the following

$$c^{-\frac{kk''-k'^2}{2k^2}(n+n')^2\varpi^2 Sq^{(i)2}};$$

the integral (1) becomes thus [332]

$$\frac{1}{2\pi}\int d\varpi\, c^{-lw\sqrt{-1}}\, c^{-\frac{kk''-k'^2}{2k^2}(n+n')^2\varpi^2 \mathrm{S}q^{(i)2}}.$$

Let us suppose

$$l = (n+n')r\sqrt{Sq^{(i)2}},$$
$$t = \sqrt{\frac{(kk''-k'^2)Sq^{(i)2}}{2k^2}}(n+n')\varpi - \frac{r\sqrt{-1}}{2}\sqrt{\frac{2k^2}{kk''-k'^2}}.$$

The variation of l being unity, we will have

$$1 = (n+n')dr\sqrt{Sq^{(i)2}};$$

the preceding integral becomes thus, after having integrated it from $t = -\infty$ to $t = \infty$,

$$\frac{kdr}{\sqrt{2(kk''-k'^2)\pi}} c^{-\frac{k^2r^2}{2(kk''-k'^2)}}$$

Thus the probability that the function (m) will be comprehended within the limits

$$\frac{ak'}{k}Sq^{(i)} \pm ar\sqrt{Sq^{(i)2}},$$

is equal to

$$\frac{2}{\sqrt{\pi}}\int \frac{kdr}{\sqrt{2(kk''-k'^2)}}c^{-\frac{k^2r^2}{2(kk''-k'^2)}},$$

the integral being taken from r null.

$\frac{ak'}{k}$ is the abscissa of the ordinate which passes through the center of gravity of the area of the curve of the probabilities of the errors of each observation; the product of this abscissa by $Sq^{(i)}$ is therefore the mean result toward which the function (m) converges without ceasing. If we suppose $1 = q = q^{(1)} =$ etc.; the function (m) becomes the sum of the errors, and then $Sq^{(i)}$ becomes s; therefore by dividing the sum of the errors by s, in order to have the mean error; this error converges without
[333] ceasing toward the abscissa of the center of gravity, in a manner that by taking on both sides any interval whatsoever as small as we will wish, the probability that the mean error will fall within that interval, will finish, by multiplying indefinitely the observations, by differing from certainty, only by a quantity less than every given magnitude.

§23. We just investigated the mean result that observations numerous and not yet made, must indicate with most advantage, and the law of probability of the errors of this result. Let us consider presently the mean result of observations already made, and of which we know the respective deviations. For this, let us imagine a number s of observations of the same kind, that is, such that the law of errors is the same for all. Let us name A the result of the first; $A+q$, the one of the second; $A+q^{(1)}$ the one of the third, and so forth; q, $q^{(1)}$, $q^{(2)}$, etc. being positive and increasing quantities, that which we can always obtain by a convenient disposition of the observations. Let us designate further by $\phi(z)$, the probability of the error z for each observation, and let us suppose that $A+x$ be the true result. The error of the first observation is then $-x$; $q-x$, $q^{(1)}-x$, etc. are the errors of the second, of the third, etc. The probability of the simultaneous existence of all these errors, is the product of their respective probabilities; it is therefore

$$\phi(-x)\phi(q-x)\phi(q^{(1)}-x).\text{etc.}$$

Now, x being susceptible of an infinity of values; by considering them as so many causes of the observed event, the probability of each of them will be, by §1,

$$\frac{dx\,\phi(-x)\phi(q-x)\phi(q^{(1)}-x).\text{etc.}}{\int dx\,\phi(-x)\phi(q-x)\phi(q^{(1)}-x).\text{etc.}},$$

the integral of the denominator being taken for all the values of which x is susceptible. Let us name $\frac{1}{H}$ this denominator. This premised, let us imagine a curve of which x is the abscissa, and of which the ordinate y is

$$H\phi(-x)\phi(q-x)\phi(q^{(1)}-x).\text{etc.};$$

this curve will be that of the probabilities of the values of x. The value that it is [334] necessary to choose for the mean result, is that which renders the mean error to fear, a *minimum*. Each error, either positive, or negative, needing to be considered as a disadvantage, or a real loss in the game; we have the mean disadvantage, by taking the sum of the products of each disadvantage, by its probability; the mean value of the error to fear, is therefore the sum of the products of each error, setting aside the sign, by its probability. Let us determine the abscissa that it is necessary to choose in order that this sum be a *minimum*. For this, let us give to the abscissas, for origin, the first extremity of the preceding curve, and let us name x' and y' the coordinates of the curve, by departing from this origin. Let l be the value that it is necessary to choose. It is clear that if the true result were x', the error of the result l would be, setting aside the sign, $l-x'$, as much as x' would be less than l; now y' is the probability that x' is the true result; the sum of the errors to fear, setting aside the sign, multiplied by their probability, is therefore for all the values of x', less than l, $\int(l-x')y'dx'$, the integral being taken from $x'=0$ to $x'=l$. We will see in the same manner, that for the values of x' superior to l, the sum of the errors to fear, multiplied by their probability, is $\int(x'-l)y'dx'$, the integral being taken from $x'=l$ to the abscissa x' corresponding to the last extremity of the curve; the entire sum of the errors to fear, setting aside the sign, multiplied by their respective probabilities, is therefore

$$\int(l-x')y'dx'+\int(x'-l)y'dx'.$$

The differential of this function, taken with respect to l, is

$$dl\int y'dx'-dl\int y'dx';$$

because we have the differential of $\int(l-x')y'dx'$, by differentiating first the value of l under the $\int$ sign, and by adding to this differential, the increase which results from the variation of the limit of the integral, a limit which is changed into $l+dl$. This increase is equal to the element $(l-x')y'dx'$, to the limit where $x'=l$; it is therefore null, and $dl\int y'dx'$ is the differential of the integral $\int(l-x')y'dx'$. We will see in the [335] same manner, that $-dl\int y'dx'$ is the differential of the integral $\int(x'-l)y'dx'$. The sum of these differentials is null relatively to the abscissa l, for which the mean error to fear is a *minimum*; we have therefore, relatively to this abscissa,

$$\int y'dx'=\int y'dx',$$

the first integral being taken from $x'=0$ to $x'=l$, and the second being taken from $x'=l$ to the extreme value of x'.

It follows thence that the abscissa which renders the mean error to fear, a *minimum*, is that of which the ordinate divides the area of the curve into two equal parts. This point enjoys further the property to be the one on the side of which it is as probable that the true result falls, as the other side; and by this reason, it is able further to be named *middle of probability.* Some celebrated geometers have taken for the middle that it is necessary to choose, the one which renders the observed result, the most probable, and consequently the abscissa which corresponds to the greatest ordinate of the curve; but the middle that we adopt, is evidently indicated by the theory of probabilities.

If we put $\phi(x)$ under the form of an exponential, and if we designate it by $c^{-\psi(x^2)}$, so that it is able equally to agree to the positive and negative errors; we will have

$$y = Hc^{-\psi(x^2)-\psi(x-q)^2-\psi(x-q^{(1)})^2-\text{etc.}} \tag{1}$$

If we make $x = a + z$, and if we develop the exponent of c with respect to the powers of z, y will take this form

$$y = Hc^{-M-2Nz-Pz^2-Qz^3-\text{etc.}},$$

an expression in which we have

$$\begin{aligned} M =&\psi(a^2) + \psi(a-q)^2 + \psi(a-q^{(1)})^2 + \text{etc.},\\ N =&a\psi'(a^2) + (a-q)\psi'(a-q)^2 + (a-q^{(1)})\psi'(a-q^{(1)})^2 + \text{etc.},\\ P =&\psi'(a^2) + \psi'(a-q)^2 + \psi'(a-q^{(1)})^2 + \text{etc.} + 2a^2\psi''(a^2)\\ &+ 2(a-q)^2\psi''(a-q)^2 + a(a-q^{(1)})^2\psi''(a-q^{(1)})^2 + \text{etc.},\\ &\text{etc.}, \end{aligned}$$

[336] $\psi'(t)$ being the coefficient of dt in the differential of $\psi(t)$, $\psi''(t)$ being the coefficient of dt in the differential of $\psi'(t)$, and so forth.

Let us suppose the number s of observations, very great, and let us determine a by the equation $N = 0$ which gives the condition of the *maximum* of y; then we have

$$y = Hc^{-M-Pz^2-Qz^3-\text{etc.}}.$$

M, P, Q, etc. are of order s; now, if z is very small of order $\frac{1}{\sqrt{s}}$, Qz^3 becomes of order $\frac{1}{\sqrt{s}}$, and the exponential $c^{-Qz^3-\text{etc.}}$ is able to be reduced to unity. Thus, in the interval from $z = 0$ to $z = \frac{r}{\sqrt{s}}$, we are able to suppose

$$r = Hc^{-M-Pz^2}.$$

Farther on, and when z is of order $s^{-\frac{m}{2}}$, m being smaller than unity, Pz^2 becomes of order s^{1-m}; consequently c^{-Pz^2} becomes in the same way as y, insensible; so that we can, in all extent of the curve, suppose

$$y = Hc^{-M-Pz^2}.$$

The value of a given by the equation $N = 0$, or

$$0 = a\psi'(a^2) + (a-q)\psi'(a-q)^2 + (a-q^{(1)})\psi'(a-q^{(1)})^2 + \text{etc.},$$

is then the abscissa x corresponding to the ordinate which divides the area of the curve into equal parts. The condition that the entire area of the curve must represent certitude or unity, gives

$$\frac{1}{H} = \int dz\, c^{-M-Pz^2},$$

the integral being taken from $z = -\infty$ to $z = \infty$, that which gives

$$H = \frac{c^M \sqrt{P}}{\sqrt{\pi}}.$$

The mean positive or negative error to fear, by taking a for the mean result of the observations, is $\pm \int zydz$, the integral being taken from z null to z infinity, that which gives for this error [337]

$$\pm \frac{1}{2\sqrt{\pi P}}.$$

But the entire ignorance which we have of the law $c^{-\psi(x^2)}$ of the errors of each observation, does not permit forming the equation

$$0 = a\psi'(a^2) + (a-q)\psi'(a-q)^2 + \text{etc.}$$

Thus knowledge of the values of q, $q^{(1)}$, etc., shedding *a posteriori*, no light on the mean result a of the observations; it is necessary to be held to the most advantageous result determined *a priori*, and that we have seen to be the one which the method of least squares of the errors furnishes.

Let us seek the function $\psi(x^2)$ which gives constantly the rule of the arithmetic means, admitted by the observers. For this, let us imagine that out of the s observations, the first i coincide, in the same way as the $s - i$ last. The equation $N = 0$ becomes then

$$0 = ia\psi'(a^2) + (s-i)(a-q)\psi'(a-q)^2.$$

The rule of the arithmetic mean gives

$$a = \frac{s-i}{s} q;$$

the preceding equation becomes thus

$$\psi'\left[\left(\frac{s-i}{s}\right)^2 q^2\right] = \psi'\left(\frac{i^2}{s^2} q^2\right).$$

This equation must hold whatever be $\frac{i}{s}$ and q, it is necessary that $\psi'(t)$ be independent of t, that which gives

$$\psi'(t) = k,$$

k being a constant. By integrating, we have

$$\psi(t) = kt - L,$$

L being an arbitrary constant; hence, [338]

$$c^{-\psi(x^2)} = c^{L-kx^2}.$$

Such is therefore the function which can alone, give generally the rule of the arithmetic means. The constant L must be determined in a manner that the integral $\int dx\, c^{L-kx^2}$, taken from $x=-\infty$ to $x=\infty$, be equal to unity; because it is certain that the error x of an observation must fall within these limits; we have therefore

$$c^L = \sqrt{\frac{k}{\pi}};$$

consequently the probability of the error x is $\sqrt{\frac{k}{\pi}}c^{-kx^2}$.

In truth, this expression gives infinity for the limit of the errors, that which is not admissible; but, seeing the rapidity with which this kind of exponential diminishes in measure as x increases, we are able to take k rather great, for which beyond the admissible limit of the errors, their probabilities are insensible, and can be supposed null.

The preceding law of errors gives for the general expression (1) of y,

$$y = \sqrt{\frac{sk}{\pi}}e^{-ksu^2};$$

by determining H in a manner that the entire integral $\int ydx$ be unity, and making

$$x = \frac{Sq^{(i)}}{s} + u.$$

The ordinate which divides the area of the curve into two equal parts, is that which corresponds to $u=0$, and consequently to

$$x = \frac{Sq^{(i)}}{s};$$

this is therefore the value of x that it is necessary to choose for the mean result of the observations; now, this value is that which the rule of the arithmetic means gives;
[339] the preceding law of errors of each observation, gives therefore constantly the same results as this rule, and we have seen that it is the only law which enjoys this property.

By adopting this law, the probability of the error $\epsilon^{(i)}$ of the $(i+1)^{\text{st}}$ observation, is

$$\sqrt{\frac{k}{\pi}}e^{-k\epsilon^{(i)2}};$$

now we have seen in §20, that z being the correction of an element, this observation furnishes the equation of condition

$$\epsilon^{(i)} = p^{(i)}z - \alpha^{(i)}.$$

The probability of the value of $p^{(i)}z - \alpha^{(i)}$ is therefore

$$\sqrt{\frac{k}{\pi}}e^{-k(p^{(i)}z-\alpha^{(i)})^2};$$

the probability of the simultaneous existence of the s values $p.z - \alpha$, $p^{(1)}.z - \alpha^{(1)}$, $\dots p^{(s-1)}.z - \alpha^{(s-1)}$, will be therefore

$$\left(\sqrt{\frac{k}{\pi}}\right)^{s-1} e^{-kS(p^{(i)}z-\alpha^{(i)})^2}.$$

This probability varies with z; we will have the probability of any value whatsoever of z, by multiplying this quantity by dz, and dividing the product by the integral of this product, taken from $z = -\infty$ to $z = \infty$. Let

$$z = \frac{Sp^{(i)}q^{(i)}}{Sp^{(i)2}} + u,$$

this probability becomes

$$du\sqrt{\frac{kSp^{(i)2}}{\pi}}e^{-ku^2Sp^{(i)2}};$$

so that if we describe a curve of which the coefficient of du is the ordinate, and of which u is the abscissa, this curve extended from $u = -\infty$ to $u = \infty$, can be considered as the curve of the probabilities of the errors u, of which the result

$$z = \frac{Sp^{(i)}\alpha^{(i)}}{Sp^{(i)2}}$$

is susceptible. The ordinate which divides the area of the curve into two equal parts, [340] is that which corresponds to $u = 0$, and consequently to z equal to $\frac{Sp^{(i)}\alpha^{(i)}}{Sp^{(i)2}}$; this result is therefore the one that it is necessary to choose; now, it is the same as the one which the method of least squares of the errors of observations gives; the preceding law of errors of each observation, leads therefore to the same results as this method.

The method of least squares of the errors becomes necessary, when there is concern to take a mean among many given results, each, by the collection of a great number of observations of diverse kinds. Let us suppose that one same element is given, 1° by the mean result of s observations of a first kind, and that it is by these observations, equal to A; 2° by the mean result of s' observations of a second kind, and that it is equal to $A+q$; 3° by the mean result of s'' observations of a third kind, and that it is equal to $A+q'$, and thus of the remaining. If we represent by $A+x$, the true element; the error of the result of the observations s will be $-x$; by supposing therefore β equal to

$$\sqrt{\frac{k}{k''}} \cdot \frac{\sqrt{Sp^{(i)2}}}{2a},$$

if we make use of the method of least squares of the errors in order, to determine the mean result; or to

$$\sqrt{\frac{k}{k''}}\frac{Sp^{(i)2}}{2a\sqrt{s}},$$

if we make use of the ordinary method; the probability of this error will be, by §20,

$$\frac{\beta}{\sqrt{\pi}}c^{-\beta^2x^2}.$$

The error of the result of the s' observations will be $q - x$, and by designating by β' for these observations, that which we have named β for the s observations, the probability of this error will be

$$\frac{\beta'}{\sqrt{\pi}} c^{-\beta'^2(x-q)^2}.$$

[341] Similarly the error of the result of the s'' observations will be $q' - x$; and by naming for them, β'', that which we have named β for the s observations; the probability of this error will be

$$\frac{\beta''}{\sqrt{\pi}} c^{-\beta''^2(x-q')^2},$$

and so forth. The product of all these probabilities will be the probability that $-x$, $q - x$, $q' - x$, etc. will be the errors of the mean results of the observations s, s', s'', etc. By multiplying it by dx, and taking the integral from $x = -\infty$ to $x = \infty$, we will have the probability that the mean results of the observations s', s'', etc., will surpass respectively by q, q', etc., the mean result of the s observations.

If we take the integral within the determined limits, we will have the probability that the preceding condition being fulfilled, the error of the first result will be comprehended within these limits; by dividing this probability by that of the condition itself, we will have the probability that the error of the first result will be comprehended within some given limits, when we are certain that the condition effectively holds; this probability is therefore

$$\frac{\int dx c^{-\beta^2 x^2 - \beta'^2(x-q)^2 - \beta''^2(x-q')^2 - \text{etc.}}}{\int dx c^{-\beta^2 x^2 - \beta'^2(x-q)^2 - \beta''^2(x-q')^2 - \text{etc.}}},$$

the integral of the numerator being taken within the given limits, and that of the denominator being taken from $x = -\infty$ to $x = \infty$. We have

$$\begin{aligned} &\beta^2 x^2 + \beta'^2(x-q)^2 + \beta''^2(x-q')^2 + \text{etc.} \\ =&(\beta^2 + \beta'^2 + \beta''^2 + \text{etc.})x^2 - 2x(\beta'^2 q + \beta''^2 q' + \text{etc.}) \\ &+\beta'^2 q^2 + \beta''^2 q'^2 + \text{etc.} \end{aligned}$$

Let

$$x = \frac{\beta'^2 q + \beta''^2 q' + \text{etc.}}{\beta^2 + \beta'^2 + \beta''^2 + \text{etc.}} + t;$$

the preceding probability will become

$$\frac{\int dt\, c^{-(\beta^2+\beta'^2+\beta''^2+\text{etc.})t^2}}{\int dt\, c^{-(\beta^2+\beta'^2+\beta''^2+\text{etc.})t^2}},$$

[342] the integral of the numerator being taken within some given limits, and that of the denominator being taken from $t = -\infty$ to $t = \infty$. This last integral is

$$\frac{\sqrt{\pi}}{\sqrt{\beta^2 + \beta'^2 + \beta''^2 + \text{etc.}}}.$$

By making therefore

$$t' = t\sqrt{\beta^2 + \beta'^2 + \beta''^2 + \text{etc.}};$$

the preceding probability becomes

$$\frac{1}{\sqrt{\pi}} \int dt' c^{-t'^2}.$$

The most probable value of t', is that which corresponds to t' null; whence it follows that the most probable value of x, is that which corresponds to $t = 0$, thus the correction of the first result, that the collection of all the observations s, s', s'', etc. give with most probability, is

$$\frac{\beta'^2 q + \beta''^2 q' + \text{etc.}}{\beta^2 + \beta'^2 + \beta''^2 + \text{etc.}}.$$

This correction added to the result A, gives for the result that it is necessary to choose,

$$\frac{A\beta^2 + (A+q)\beta'^2 + (A+q')\beta''^2 + \text{etc.}}{\beta^2 + \beta'^2 + \beta''^2 + \text{etc.}}.$$

The preceding correction is that which renders a *minimum*, the function

$$(\beta x)^2 + (\beta'.\overline{x-q})^2 + (\beta''.\overline{x-q'})^2 + \text{etc.}$$

Now the greatest ordinate of the curve of probabilities of the first result is, as we have just seen, $\frac{\beta}{\sqrt{\pi}}$; that of the curve of probabilities of the second result, is $\frac{\beta'}{\sqrt{\pi}}$, and so forth; the mean that it is necessary to choose among the diverse results, is therefore the one which renders a *minimum*, the sum of the squares of the error of each result, multiplied by the greatest ordinate of its curve of probability. Thus the law of the *minimum* of the squares of the errors, becomes necessary, when we must take a mean [343]
among some given results, each, by a great number of observations.

§24. We have seen previously, that in all the manners to combine the equations of condition, in order to form the linear final equations, necessary to the determination of the elements; the most advantageous is that which results from the method of least squares of the errors of the observations, at least when the observations are in great number. If instead of considering the *minimum* of the squares of the errors, we considered the *minimum* of other powers of the errors, or even of every other function of the errors; the final equations would cease to be linear, and their resolution would become impractical, if the observations were in great number. However there is a case which merits a particular attention, in this that it gives the system in which the greatest error, setting aside the sign, is less than in every other system. This case is the one of the *minimum* of the infinite and even powers of the errors. Let us consider here only the correction of a single element; and z expressing this correction, let us represent, as previously, the equations of condition, by the following,

$$\epsilon^{(i)} = p^{(i)} z - \alpha^{(i)},$$

i being able to vary from zero to $s-1$, s being the number of observations. The sum of the powers $2n$ of the errors will be $\mathrm{S}(\alpha^{(i)} - p^{(i)} z)^{2n}$, the sign S extending

to all the values of i. We can suppose, in this sum, all the values of $p^{(i)}$ positive; because if one of them was negative, it would become positive by changing, as we are able to do, the signs of the two terms of the binomial raised to the power $2n$, to which it corresponds. We will suppose therefore the quantities $\alpha - pz$, $\alpha^{(1)} - p^{(1)}z$, $\alpha^{(2)} - p^{(2)}z$, etc., disposed in a manner that the quantities p, $p^{(1)}$, $p^{(2)}$, etc. are positive and increasing. This premised, if $2n$ is infinite, it is clear that the greatest term of the sum $\mathrm{S}(\alpha^{(i)} - p^{(i)}z)^{2n}$, will be the entire sum, unless there was one or many other terms which were equal to it, and this is that which must hold in the case of the *minimum*
[344] of the sum. In fact, if there was only a single greatest quantity, setting aside the sign, such as $\alpha^{(i)} - p^{(i)}z$, we would be able to diminish it by making z vary conveniently, and then the sum $S(\alpha^{(i)} - p^{(i)}z)^{2n}$ would diminish and would not be a *minimum*. It is necessary moreover that if $\alpha^{(i)} - p^{(i)}z$ and $\alpha^{(i')} - p^{(i')}z$ are, setting aside the sign, the two greatest quantities and equal between them, they be of contrary sign. In fact, the sum $(\alpha^{(i)} - p^{(i)}z)^{2n} + (\alpha^{(i')} - p^{(i')}z)^{2n}$ needing to be then a *minimum*, its differential $-2ndz[p^{(i)}(\alpha^{(i)} - p^{(i)}z)^{2n-1} + p^{(i')}(\alpha^{(i')} - p^{(i')}z)^{2n-1}]$ must be null, that which can be when n is infinite, only in the case where $\alpha^{(i)} - p^{(i)}z$ and $\alpha^{(i')} - p^{(i')}z$ are infinitely little different, and of contrary sign. If there are three greatest quantities, and equals among them, setting aside the sign, we will see in the same manner that their signs are not able to be the same.

Now, let us consider the sequence

$$\begin{gathered}\alpha^{(s-1)} - p^{(s-1)}z,\ \alpha^{(s-2)} - p^{(s-2)}z,\ \alpha^{(s-3)} - p^{(s-3)}z,\ \ldots\ \alpha - pz,\\ -\alpha + pz,\ \ldots\ -\alpha^{(s-3)} + p^{(s-3)}z,\ -\alpha^{(s-2)} + p^{(s-2)}z, -\alpha^{(s-1)} + p^{(s-1)}z.\end{gathered} \tag{o}$$

If we suppose $x = -\infty$, the first term of the sequence surpasses the following, and continues to surpass them by making z increase, to the moment where it becomes equal to one of them. Then this one, by the increase of z, becomes greatest of all; and in measure as we make z increase, it continues always to surpass those which precede it. In order to determine this term, we will form the sequence of quotients

$$\frac{\alpha^{(s-1)} - \alpha^{(s-2)}}{p^{(s-1)} - p^{(s-2)}},\ \frac{\alpha^{(s-1)} - \alpha^{(s-3)}}{p^{(s-1)} - p^{(s-3)}},\ \ldots,\ \frac{\alpha^{(s-1)} - \alpha}{p^{(s-1)} - p},\ \frac{\alpha^{(s-1)} + \alpha}{p^{(s-1)} + p},\ \ldots,\ \frac{\alpha^{(s-1)} + \alpha^{(s-1)}}{p^{(s-1)} + p^{(s-1)}}.$$

Let us suppose that $\frac{\alpha^{(s-1)} - \alpha^{(r)}}{p^{(s-1)} - p^{(r)}}$ is the smallest of these quotients by having regard to the sign, that is by regarding a greater negative quantity, as smaller than another lesser negative quantity. If there are many least and equal quotients, we will consider the one which is related to the most distant term of the first in the sequence (o); this term will be the greatest of all, to the moment where, by the increase of z, it
[345] becomes equal to one of the following, which begins then to be the greatest. In order to determine this new term, we will form a new sequence of quotients

$$\frac{\alpha^{(r)} - \alpha^{(r-1)}}{p^{(r)} - p^{(r-1)}},\ \frac{\alpha^{(r)} - \alpha^{(r-2)}}{p^{(r)} - p^{(r-2)}},\ \ldots,\ \frac{\alpha^{(r)} - \alpha}{p^{(r)} - p},\ \frac{p^{(r)} + \alpha}{p^{(r)} + p},\ \text{etc.};$$

the term of the sequence (o), to which the least of these quotients correspond, will be the new term. We will continue thus until one of the two terms which become equal and the greatest, is in the first half of the sequence (o), and the other in the second half. Let $\alpha^{(i)} - p^{(i)}z$ and $-\alpha^{(i')} + p^{(i')}z$ be these two terms; then the value of z which corresponds to the system of the *minimum* of the greatest of the errors, setting aside the sign, is

$$z = \frac{\alpha^{(i)} + \alpha^{(i')}}{p^{(i)} - p^{(i')}}.$$

If there are many elements to correct, the equations of condition which determine their corrections, contain many unknowns, and the investigation of the system of correction, in which the greatest error is, setting aside the sign, smaller than in every other system, becomes more complicated. I have considered this case in a general manner, in the third Book of the *Mécanique Céleste*. I will observe only here, that then the sum of the $2n$ powers of the errors of the observations is, as in the case of a single unknown, a *minimum*, when $2n$ is infinite; whence it is easy to conclude that in the system of which there is concern, it must have as many errors plus one, equals, and greatest setting aside the sign, as there are elements to correct. We imagine that the results corresponding to $2n$ equal to a great number must differ little from those which $2n$ infinite gives. It is not even necessary for this, that the $2n$ power be quite elevated, and I have recognized through many examples, that in the same case where this power does not surpass the square, the results differ little from those that the system of the *minimum* of the greatest errors gives, that which is a new advantage of the method of least squares of the errors of observations.

For a long time, geometers take an arithmetic mean among their observations; and, in order to determine the elements that they wish to know, they choose the [346]
most favorable circumstances for this object, namely, those in which the errors of the observations alter the least that it is possible, the value of these elements. But Cotes is, if I do not deceive myself, the first who has given a general rule in order to make agree in the determination of an element, many observations, proportionally to their influence. By considering each observation as a function of the element, and regarding the error of the observation as an infinitely small differential; it will be equal to the differential of the function, taken with respect to that element. The more the coefficient of the differential of the element will be considerable, the less it will be necessary to make the element vary, in order that the product of its variation, by this coefficient, be equal to the error of the observation; this coefficient will express therefore the influence of the observation on the value of the element. This premised, Cotes represents all the values of the element, given by each observation, by the parts of an indefinite straight line, all these parts having a common origin. He imagines next, at their other extremities, weights proportional to the respective influences of the observations. The distance from the common origin of the parts, to the common center of gravity of all these weights, is the value that he chose for the element.

Let us take the equation of condition of §20,

$$\epsilon^{(i)} = p^{(i)}z - \alpha^{(i)},$$

$\epsilon^{(i)}$ being the error of the $(i+1)^{\text{st}}$ observation, and z being the correction of the element already known quite nearly; $p^{(i)}$ that we are always able to suppose positive, will express the influence of the corresponding observation. $\frac{\alpha^{(i)}}{p^{(i)}}$ being the value of z resulting from the observation, the rule of Cotes reverts to multiplying this value by $p^{(i)}$, to make a sum of all the products relative to the diverse values, and to divide it by the sum of all the $p^{(i)}$, that which gives

$$z = \frac{S\alpha^{(i)}}{Sp^{(i)}}.$$

[347] This was indeed the correction adopted by the observers, before the usage of the method of least squares of the errors of the observations.

However, we do not see that since this excellent geometer, we have employed his rule, until Euler who in his first piece on Jupiter and Saturn, appears to me the first to have served himself, of the equations of the condition, in order to determine the elements of the elliptic movement of these two planets. Near the same time, Tobie Mayer made use of it in his beautiful researches on the libration of the moon, and next in order to form his lunar Tables. Since, the best astronomers have followed this method, and the success of the Tables which they have constructed by its means, has verified the advantage of it.

When we have only one element to determine, this method leaves no difficulty; but when we must correct at the same time many elements, it is necessary to have as many final equations formed by the reunion of many equations of condition, and by means of which we determine by elimination, the corrections of the elements. But what is the most advantageous manner to combine the equations of condition, in order to form the final equations? It is here that the observers abandoned themselves to some arbitrary gropings which must have led them to some different results, although deduced from the same observations. In order to avoid these gropings, Mr. Legendre had the simple idea to consider the sum of the squares of the errors of the observations, and to render it a *minimum*; that which furnishes directly as many final equations, as there are elements to correct. This scholarly geometer is the first who has published this method; but we owe to Mr. Gauss the justice to observe that he had had, many years before this publication, the same idea of which he made a habitual usage, and that he had communicated to many astronomers. Mr. Gauss, in his Theory of elliptic movement, has sought to connect this method to the Theory of Probabilities, by showing that the same law of errors of the observations, which give generally the rule of the arithmetic mean among many observations, admitted by the observers, gives similarly the rule of the least squares of the errors of the observations; and it
[348] is that which we have seen in §23. But, as nothing proves that the first of these rules gives the most advantageous result, the same uncertainty exists with respect to the second. The investigation of the most advantageous manner to form the final equations, is without doubt one of the most useful of the Theory of Probabilities: its importance in physics and astronomy, carries me to occupy myself with it. For this, I will consider that all the ways to combine the equations of condition, in order to form a linear final equation, reverted to multiplying them respectively by some factors which were null relatively to the equations that we employed not at all, and to make a sum of all these products; that which gives a first final equation. A second system

of factors gives a second final equation, and so forth, until we have as many final equations, as elements to correct. Now, it is clear that it is necessary to choose the system of factors, such that the mean positive or negative error to fear respecting each element, be a *minimum*; the mean error being the sum of the products of each error by its probability. When the observations are in small number, the choice of these systems depends on the law of errors of each observation. But if we consider a great number of observations, that which holds most often in the astronomical researches, this choice becomes independent of this law; and we have seen in that which precedes, that analysis leads then directly to the results of the method of least squares of the errors of the observations. Thus this method which offered first only the advantage to furnish, without groping, the final equations necessary to the correction of the elements, gives at the same time the most precise corrections, at least when we wish to employ only final equations which are linear, an indispensable condition, when we consider at the same time a great number of observations; otherwise, the elimination of the unknowns and their determination would be impractical.

CHAPTER 5

Application of the Calculus of Probabilities, to the research of phenomena and of their causes

§25. The phenomena of nature present themselves most often accompanied with so many strange circumstances; so great a number of perturbing causes mix their influence, that it is very difficult, when they are very small, to recognize them. We can then arrive there, only by multiplying the observations, so that the strange effects coming to destroy themselves, the mean result of the observations no longer permit perceiving but these phenomena. Let us imagine by that which precedes, that that holds rigorously, only in the case of an infinite number of observations. In every other case, the phenomena are indicated by the mean results, only in a probable manner, but which is so much more, as the observations are in greater number. The investigation of this probability is therefore very important for physics, astronomy, and generally for all the natural sciences. We will see that it returns to the methods that we have just exposed. In the preceding chapter, the existence of a phenomenon was certain; its extent alone has been the object of the Calculus of Probabilities: here the existence of a phenomenon and its extent, are the object of this calculus. [349]

Let us take for example, the diurnal variation of the barometer, that we observe between the tropics, and which becomes sensible even in our climates, when we choose and when we multiply the observations conveniently. We have recognized that in general, toward nine hours of the morning, the barometer is more elevated than toward four hours of the evening; next it rises again toward eleven hours of the evening, and it descends again toward four hours of the morning, in order to arrive again to its *maximum* height, toward nine hours. Let us suppose that we have observed the [350] height of the barometer toward nine hours of the morning and toward four hours of the evening, during the number s days; and, in order to avoid the too great influence of the perturbing causes, let us choose these days in a manner that, in the interval from nine hours to four hours, the barometer has not varied beyond four millimeters. Let us suppose next that by making the sum of the s heights of the morning, and the sum of the s heights of the evening, the first of these sums surpasses the second by the quantity q; this difference will indicate a constant cause which tends to raise the barometer toward nine hours of the morning, and to lower it toward four hours of the evening. In order to determine with what probability this cause is indicated, let us imagine that this cause not exist at all, and that the observed difference q, results from accidental perturbing causes, and from the errors of the observations. The probability that then the observed difference between the sums of the heights of the morning and of the evening, must be below q, is, by §18, equal to

$$\sqrt{\frac{k}{4k''\pi}}\int dr\, c^{-\frac{kr^2}{4k''}},$$

the integral being taken from $r = -\infty$ to $r = \frac{q}{a\sqrt{s}}$, k and k'' being constants dependent on the law of probability of the differences between the heights of the morning and of the evening, and $\pm a$ being the limits of these differences, a being here equal to four millimeters, $\frac{k}{k''}$ being at least equal to 6, as we have seen in §20, $\frac{k}{4k''}$ is not able to be supposed less than $\frac{3}{2}$; by making therefore $s = 400$, and supposing the extent of the diurnal variation, of one millimeter, that which is very nearly that which Mr. Ramond has found in our climates, by the comparison of a very great number of observations, we will have $q = 400^{\text{mm}}$. Thus $r = 5$, and $\frac{kr^2}{4k''}$ is at least equal to 37.5; by making therefore

$$t^2 = \frac{kr^2}{4k''},$$

[351] the preceding probability becomes at least

$$1 - \frac{\int dt\, c^{-t^2}}{\sqrt{\pi}},$$

the integral being taken from $t = \sqrt{37,5}$ to $t = \infty$. This integral is quite nearly, by §27 of the first Book,

$$1 - \frac{c^{-37.5}}{2\sqrt{37,5\pi}},$$

and it approaches so to unity or to certitude, that it is extremely probable that if there existed no constant cause at all of the observed excess of the sum of the barometric heights of the morning, over those of the heights of the evening, this excess would be smaller than 400^{mm}; it indicates therefore with an extreme probability, the existence of a constant cause which has produced it.

The phenomenon of a diurnal variation being thus well established, let us determine the most probable value of its extent, and the error that we are able to commit with respect to its evaluation. Let us suppose for this, that this value is $\frac{q}{s} \pm \frac{ar}{\sqrt{s}}$; the probability that the extent of the diurnal variation from the morning to the evening, will be comprehended within these limits is, by §18,

$$2\sqrt{\frac{k}{4k''\pi}}\int dr\, c^{-\frac{kr^2}{4k''}},$$

the integral being taken from $r = 0$.

We can eliminate $\frac{k''}{k}$, by observing that by §20, this fraction is nearly equal to $\frac{S\epsilon^{(i)2}}{2as^2}$; $\pm\epsilon^{(i)2}$ being the difference from $\frac{q}{s}$ to the observed extent the $(i+1)^{\text{st}}$ day, and the sign S extending to all the values of i, from $i = 0$ to $i = s-1$; by making therefore

$$ar = t\sqrt{\frac{2S\epsilon^{(i)2}}{s}},$$

the probability that the extent of the diurnal variation from the morning to the [352]
evening, is comprehended within the limits $\frac{q}{s} \pm \frac{t}{\sqrt{s}}\sqrt{\frac{2S\epsilon^{(i)2}}{s}}$, will be $\frac{2}{\sqrt{\pi}}\int dt\, c^{-t^2}$, the integral being taken from t null.

The diurnal variation of the heights of the barometer, depends uniquely on the sun; but these heights are still affected by the aerial tides that the attraction of the sun and of the moon produce on our atmosphere, and of which I have given the theory in fourth Book of the *Mécanique Céleste*. It is therefore necessary to consider at the same time these two variations, and to determine their magnitudes and their respective epochs, by forming equations of condition analogous to those of which the astronomers make use, in order to correct the elements of the celestial movements. These variations being principally sensible at the equator, and the perturbing causes being extremely small; we will be able, by means of excellent barometers, to determine them with a great precision; and I do not doubt at all that we recognize then, in the collection of a very great number of observations, the laws that the theory of the gravity indicate in the atmospheric tides, and that manifests itself in a manner so remarkable in the observations of the tides of the Ocean, that I have discussed extensively, in the Book cited of the *Mécanique Céleste*.

We see, by that which precedes, that we are able to recognize the very small effect of a constant cause, by a long sequence of observations of which the errors are able to exceed this effect itself. But then, it is necessary to take care to vary the circumstances of each observation, in a manner that the mean result of their collection, is not at all altered sensibly by it, and is nearly entirely the effect of the cause of which there is concern: it is necessary to multiply the observations, until the analysis indicates a very great probability that the error of this result will be comprehended within some very narrowed limits.

Let us suppose, for example, that we wish to recognize by observation, the small deviation to the east, produced by the rotation of the earth, in the fall of a body. I have shown, in the tenth Book of the *Mécanique Céleste*, that if from the summit of a
quite high tower, we abandon a body to its weight, it will fall onto a horizontal plane [353]
passing through the foot of the tower, at a small distance to the east from the point of contact of this plane with a ball suspended by a wire of which the point of suspension is the one of the departure of the body. I have given in the Book cited, the expression of this deviation, and there results from it that by setting aside the resistance of the air, it is uniquely toward the east; that it is proportional to the cosine of the latitude, and to the square root of the cube of the height, and that at the latitude of Paris, it is raised to 5.1 millimeters, when the height of the tower is 50 meters. The resistance of the air changes this last result; I have given similarly the expression of it in this case, in the Book cited.

We have already made a great number of experiments in order to confirm, by this means, the movement of the rotation of the Earth, which besides is demonstrated by so many other phenomena, that this confirmation becomes useless. The small errors of these very delicate experiments, have often exceeded the effect that we would wish to determine; and it is only by multiplying considerably the experiments, that we can thus establish its existence and fix its value. We will submit this object to the analysis of probabilities.

If we take for origin of the coordinates, the point of contact of the plane and of the ball suspended by a wire of which the summit of the suspension is the one of the departure of a ball that we make fall; if we next mark on this plane, the diverse points where the ball will touch the plane in each experiment; by determining the common center of gravity of these points, the line drawn from the origin of the coordinates to this center, will determine the sense and the mean quantity by which the ball is deviated from this origin; and both will be determined with so much more exactitude, as the experiments will be more numerous and more precise.

Let us consider now, as axis of the abscissas, the line drawn from the origin of the coordinates, to the east; and let us designate by x, $x^{(1)}$, $x^{(2)}$, ..., $x^{(s-1)}$, y, $y^{(1)}$, $y^{(2)}$, ..., $y^{(s-1)}$ the respective coordinates of the points determined by the experiments of which the number is s. In expressing by X and Y the coordinates of the center of
[354] gravity of all these points; we will have

$$X = S\frac{x^{(i)}}{s}, \qquad Y = S\frac{y^{(i)}}{s},$$

the sign S extending to all the values of i, from $i = 0$ to $i = s - 1$. This premised, by designating by $\pm a$ the limits of the errors of each experiment, in the sense of x; the probability that the mean deviation of the ball, from the point of origin of the coordinates, is comprehended within the limits $X \pm \frac{ar}{\sqrt{s}}$, will be, by §18,

$$2\sqrt{\frac{k}{4k''\pi}}\int dr\, c^{-\frac{kr^2}{4k''}},$$

k and k'' being constants which depend on the law of facility of the errors of each experiment in the sense of x.

Similarly, $\pm a'$ being the limits of the errors of each experiment in the sense of y; the probability that the mean value of the deviation in the sense of y, is comprehended within the limits $Y \pm \frac{a'r}{\sqrt{s}}$, will be

$$2\sqrt{\frac{\overline{k}}{4\overline{k}''\pi}}\int dr\, c^{-\frac{\overline{k}r^2}{4\overline{k}''}},$$

$\overline{k}$ and $\overline{k}''$ being constants depending on the law of errors of the experiments in the sense of y. The fractions $\frac{k}{4k''}$ and $\frac{\overline{k}}{4\overline{k}''}$ being, by that which precedes, greater than $\frac{3}{2}$; we will be able to judge the degree of approximation and of probability of the values of X and of Y, and to determine the probability of the deviation to the south and to the north, indicated by the observations.

The preceding analysis is able further to be applied to the investigation of the small inequalities of the celestial movements, of which the extent is comprehended within the limits, either of the errors of observations, or of the perturbations produced by accidental causes. It is nearly thus that Tycho Brahe recognized that the equation of the times, relative to the sun and to the planets, was not at all applicable to the moon, and that it was necessary to subtract the part depending on the anomaly of [355] the sun, and even a much greater quantity, that which led Flamsteed to the discovery of the lunar inequality that we name *annual equation*. It is further in the results of a great number of observations, that Mayer recognized that the equation of precession, relative to the planets and to the stars, was not at all applicable to the moon; he evaluated to around 12″ decimals, the quantity by which it was necessary then to diminish it, a quantity that Mason raised next to nearly 24″, by the comparison of all the observations of Bradley, and that Mr. Bürg has reduced to 21″, by means of a much greater number of observations of Maskelyne. This inequality, although indicated by the observations, was neglected by the greatest number of astronomers; because it did not appear to result from the theory of universal gravitation. But having submitted its existence to the calculus of probabilities, it appeared to me indicated with a probability so strong, that I believed a duty to seek the cause of it. I saw well that it was able to result only from the ellipticity of the terrestrial spheroid, that we had neglected until then in the theory of the lunar movement, as needing to produce only insensible terms; and I concluded from it that it was extremely probable that these terms became sensible by the successive integrations of the differential equations. Having determined these terms by a particular analysis, that I have exposed in the seventh Book of the *Mécanique Céleste*, I discovered first the inequality of the movement of the moon in latitude, and which is proportional to the sine of its longitude: by its means, I recognized that the theory of gravity gives effectively the diminution observed by the astronomers cited, in the inequality of the precession, applicable to the lunar movement in longitude. The quantity of this diminution, and the coefficient of the inequality in latitude of which I just spoke, are therefore very proper to determine the flatness of the earth. Having made part of my researches to Mr. Bürg who occupied himself then with his *Tables of the Moon*, I prayed him to determine with a particular care, the coefficients of the two inequalities. By a remarkable concurrence, the coefficients that he has determined, accord to give to the earth, the flatness $\frac{1}{305}$, a flatness which differs little from the mean concluded from the measures of the degrees of the meridian and from the pendulum, but which [356] seeing the influence of the errors of the observations and of the perturbing causes, on these measures, appears to me more exactly determined by the lunar inequalities. Mr. Burckhardt who has just formed new Tables of the Moon, very precise, on the collection of observations of Bradley and of Maskelyne, has found the same coefficient as Mr. Bürg, for the lunar inequality in latitude: he finds a thirty-fourth to add to the coefficient of the inequality in longitude, that which reduces the flatness to $\frac{1}{301}$, by this inequality. The very slight difference of these results, proves that by fixing at $\frac{1}{304}$, this flatness, the error is insensible.

The analysis of probabilities has led me similarly to the cause of the great irregularities of Jupiter and of Saturn. The difficulty to recognize the law of it, and to restore them to the theory of universal attraction, had made conjecture that they were due to the momentary actions of comets; but a theorem to which I was arrived on the mutual attraction of the planets, made me reject this hypothesis, indicating to me the mutual attraction of the two planets, as the true cause of these irregularities.

According to this theorem, if the movement of Jupiter is accelerated by virtue of some great inequality with very long period; the one of Saturn must be decelerated in the same manner, and this deceleration is to the acceleration of Jupiter, as the product of the mass of this last planet, by the square root of the great axis of its orbit, is to the similar product relative to Saturn. Thus by taking for unity, the deceleration of Saturn, the corresponding acceleration of Jupiter must be $0,40884$; now Halley had found, by the comparison of the modern observations to the old, that the acceleration of Jupiter corresponded to the deceleration of Saturn, and that it was $0,44823$ of this deceleration. These results, so well in accord with the theory, led me to think that there exists in the movement of these planets, two great inequalities corresponding and of contrary sign, which produced these phenomena. I have recognized that the mutual action of the planets was not able to occasion in their mean movements, some
[357] variations always increasing, or periodic, but of a period independent of their mutual configuration; it was therefore in the relation of the mean movements of Jupiter and of Saturn, that I needed to seek that of which there is concern; now by examining this relation, it is easy to recognize that twice the mean movement of Jupiter surpasses only by a very small quantity, five times the one of Saturn; thus the inequalities which depend on this difference, and of which the period is around nine centuries, can become very great by the successive integrations which give to them for divisor, the square of the very small coefficient of the time, in the argument of these inequalities. By fixing toward the epoch of Tycho Brahe, the origin of this argument; I saw that Halley had had to find by the comparison of the modern observations to the old, the alterations that he had observed; while the comparison of the modern observations among them, must have presented alterations contrary and parallel to those that Lambert had noted. The existence of the inequalities of which I just spoke, appeared to me therefore extremely probable, and I hesitated not at all to undertake the long and painful calculation, necessary in order to assure myself of it completely. The result of this calculation, not only confirmed them, but it made known to me many other inequalities of which the collection has carried the Tables of Jupiter and of Saturn, to the degree of precision of the same observations.

We see thence how it is necessary to be attentive to the indications of nature, when they are the result of a great number of observations, although besides they be inexplicable by known means. I engage thus the astronomers to follow with a particular attention, the lunar inequality with long period, which depends principally on the movement of the perigee of the moon, added to the double of the mean movement of its nodes; an inequality of which I have spoken in the seventh Book of the *Mécanique Céleste*, and that already the observations indicate with much likelihood. The preceding cases are not the only ones in which the observations have redressed the analysts. The movement of the lunar perigee and the acceleration of the movement of the moon, which was not given at all at first by the approximations, has made
[358] felt the necessity to rectify these approximations. Thus we can say that nature itself has concurred with the analytic perfection of the theories based on the principle of universal gravitation; and it is in my sense, one of the strongest proofs of the truth of this admirable principle.

We can further, by the analysis of probabilities, verify the existence or the influence of certain causes of which we have believed to notice the action on organized beings. Of all the instruments that we are able to use in order to understand the imperceptible agents of nature, the most sensible are the nerves, especially when their sensibility is enhanced by some particular circumstances. It is by their means, that we have discovered the weak electricity that the contact of two heterogeneous metals develop; that which has opened a vast field to the investigations of the physicians and of the chemists. The singular phenomena which result from the extreme sensitivity of the nerves in some individuals, have given birth to diverse opinions on the existence of a new agent that we have named *animal magnetism*, on the action of ordinary magnetism and the influence of the sun and of the moon, in some nervous affections; finally with respect to the impressions that the proximity of metals or of running water can give birth. It is natural to think that the action of these causes is very weak, and can be troubled easily by a great number of accidental circumstances; thus of that which, in some cases, it is not at all manifested, we must not conclude that it never exists. We are so far from knowing all the agents of nature, that it would be less philosophical to deny the existence of the phenomena, uniquely because they are inexplicable in the actual state of our knowledge. Alone we must examine them with an attention so much more scrupulous, as it appears more difficult to admit them; and it is here that the analysis of the probabilities becomes indispensable in order to determine to what point it is necessary to multiply the observations or the experiments, in order to have in favor of the existence of the agents that they seem to indicate, a probability superior to all the reasons that we are able to have besides to reject it.

The same analysis is able to be extended to the diverse results of medicine and of political economy, and even to the influence of moral causes; because the action [359] of these causes, when it is repeated a great number of times, offers in its results as much regularity, as physical causes.

We are able to determine further by the analysis of probabilities, compared to a great number of experiences, the advantage and the disadvantage of the players, in the cases of which complication renders impossible their direct investigation. Such is the advantage to the hand, in the game of piquet: such are further the respective probabilities to bring forth the different faces of a right rectangular prism, of which the length, the width and the height are unequal; when the prism projected into the air, falls again on a horizontal plane.

Finally, we would be able to make use of the calculus of probabilities, in order to rectify curves or to square their surfaces. Without doubt, the geometers will not employ this means; but, as it gives me place to speak of a particular kind of combinations of chance, I will expose it in a few words.

Let us imagine a plane divided by parallel lines, equidistant by the quantity a; let us imagine moreover a very narrow cylinder, of which $2r$ is the length, supposed equal or less than a. We require the probability that by casting it on it, it will encounter one of the divisions of the plane.

Let us erect on any point of one of these divisions, a perpendicular extended to the following division. Let us suppose that the center of the cylinder be on this perpendicular, and at the height y above the first of these two divisions. In making the cylinder rotate about its center, and naming ϕ the angle that the cylinder makes with the perpendicular, at the moment where it encounters this division; 2ϕ will be the part of the circumference described by each extremity of the cylinder, in which it encounters the division; the sum of all these parts will be therefore $4\int \phi dy$, or $4\phi y - 4\int y d\phi$; now we have $y = r\cos\phi$; this sum is therefore

$$4\phi y - 4r\sin\ \phi + \text{constant}$$

In order to determine this constant, we will observe that the integral must be extended
[360] from y null to $y = r$, and consequently from $\phi = \frac{\pi}{2}$ to $\phi = 0$, that which gives

$$\text{constant} = 4r;$$

thus the sum of which there is concern is $4r$. From $y = a - r$ to $y = a$, the cylinder is able to encounter the following division, and it is clear that the sum of all the parts relative to this encounter, is again $4r$; $8r$ is therefore the sum of all the parts relative to the encounter of one or of the other of the divisions by the cylinder, in the movement of its center the length of the perpendicular. But the number of all the arcs that it describes in rotating in entirety with respect to itself, at each point of this perpendicular, is $2a\pi$; this is the number of all the possible combinations; the probability of the encounter of one of the divisions of the plane by the cylinder, is therefore $\frac{4r}{a\pi}$. If we cast this cylinder a great number of times, the ratio of the number of times where the cylinder will encounter one of the divisions of the plane, to the total number of casts, will be, by §16, very nearly, the value of $\frac{4r}{a\pi}$, that which will make known the value of the circumference 2π. We will have, by the same section, the probability that the error of this value will be comprehended within some given limits; and it is easy to see that the ratio $\frac{8r}{a\pi}$ which, for a given number of projections, renders the error to fear least, is unity; that which gives the length of the cylinder equal to the interval of the divisions, multiplied by the ratio of the circumference to four diameters.

Let us imagine now the preceding plane divided again by some lines perpendicular to the preceding, and equidistant by a quantity b equal or greater than the length $2r$ of the cylinder. All these lines will form with the first ones, a sequence of rectangles of which b will be the length and a the height. Let us consider one of these rectangles; let us suppose that in its interior, we draw at the distance r from each side, lines which are parallel to them. They will form first an interior rectangle, of which $b - 2r$ will be the length, and $a - 2r$ the height; next two small rectangles, of which r will
[361] be the height, and $b - 2r$ the length; then two other small rectangles of which r will be the length, and $a - 2r$ the height; finally, four small squares of which the sides will be equal to r.

As long as the center of the cylinder will be placed in the interior rectangle, the cylinder in rotating on its center, will never encounter the sides of the large rectangle.

When the center of the cylinder will be placed in the interior of one of the rectangles of which r is the height and $b-2r$ the length; it is easy to see by that which precedes, that the product of $8r$, by the length $b-2r$, will be the number of corresponding combinations, in which the cylinder will encounter one or the other of the sides b of the great rectangle. Thus $8r(b-2r)$ will be the total number of combinations corresponding to the cases in which, the center of the cylinder being placed in one or the other of these small rectangles, the cylinder encounters the outline of the great rectangle. By the same reason, $8r(a-2r)$ will be the total number of combinations in which the center of the cylinder being placed in the interior of the small rectangles of which r and $a-2r$ are the dimensions, the cylinder encounters the outline of the great rectangle.

There remains for us to consider the four small squares. Let $ABCD$ be one of them. From the angle A common to this square and to the great rectangle, as center, and from the radius r, let us describe a quarter circumference, terminating itself at the points B and D. As long as the center of the cylinder will be comprehended within the quarter circle formed by this arc, the cylinder in turning, will encounter in all its positions, the outline of the rectangle; the number of combinations in which this will take place, is therefore equal to the product of 2π by the area of the quarter circle, and consequently it is equal to $\frac{\pi^2 r^2}{2}$. If the center of the cylinder is in the part of the square which is outside of the quarter circle; the cylinder, in turning around its center, will be able to encounter one or the other of the two sides AB and AD extended, without ever encountering both at the same time. In order to determine the number of combinations relative to this encounter, I conceive on any point of side AB, distant by x from point A, a perpendicular y of which the extremity is beyond the quarter circle. I place the center of the cylinder on this extremity from which I let down four straight lines equal to r, and of which two descend onto the side AB [362]
extended, if that is necessary, and two others onto the side AD similarly prolonged. I name 2ϕ the angle comprehended between the first two lines, and $2\phi'$ the angle comprehended between the second two. It is clear that the cylinder in turning on its center, will encounter the side AB extended, as often as one of its halves will be within the angle 2ϕ, and that it will encounter the side AD extended, as often as one of its halves will be within the angle $2\phi'$; the total number of all combinations in which the cylinder will encounter one or the other of these sides, is therefore $4(\phi+\phi')$; thus this number, relatively to the part of the square, exterior to the quarter circle, is

$$4\int(\phi+\phi')dx\ dy;$$

now we have evidently

$$x = r\cos\ \phi', \qquad y = r\cos\ \phi;$$

the preceding integral becomes thus,

$$4r^2\iint(\phi+\phi')d\phi\, d\phi' \sin\phi\sin\phi',$$

and it is easy to see that the integral relative to ϕ', must be taken from $\phi' = 0$ to $\phi' = \frac{\pi}{2} - \phi$, and that the integral relative to ϕ must be taken from $\phi = 0$ to $\phi = \frac{\pi}{2}$, that which gives $\frac{1}{2}r^2(12 - \pi^2)$ for this integral. By adding to it $\frac{\pi^2 r^2}{2}$, we will have the number of combinations relative to the square; and in quadrupling this number, and joining it to the preceding numbers of combinations relative to the encounter of the outline of the great rectangle, by the cylinder; we will have, for the total number of combinations,

$$8(a+b)r - 8r^2.$$

But the total number of possible combinations, is evidently equal to 2π multiplied by the area ab of the great rectangle; the probability of the encounter of the divisions of the plane by the cylinder, is therefore

$$\frac{4(a+b)r - 4r^2}{ab\pi}.$$

CHAPTER 6

On the probability of causes and of future events, deduced from observed events

§26. The probability of the greater part of simple events, is unknown: by considering it *a priori*, it appears to us susceptible of all the values comprehended between zero and unity; but, if we have observed a result composed of many of these events, the manner by which they enter there, renders some of these values more probable than the others. Thus in measure as the observed result is composed by the development of the simple events, their true possibility is made more and more known, and it becomes more and more probable that it falls within some limits which being tightened without ceasing, would end by coinciding, if the number of simple events became infinite. In order to determine the laws according to which this possibility is discovered, we will name it x. The theory exposed in the preceding chapters, will give the probability of the observed result, as a function of x. Let y be this function; if we consider the different values of x as so many causes of this result, the probability of x will be, by the third principal of §1, equal to a fraction of which the numerator is y, and of which the denominator is the sum of all the values of y; by multiplying therefore the numerator and the denominator of this fraction by dx, this probability will be [363]

$$\frac{y\ dx}{\int y\ dx},$$

the integral of the denominator being taken from $x = 0$ to $x = 1$. The probability that the value of x is comprehended within the limits $x = \theta$, and $x = \theta'$ is consequently equal to

$$\frac{\int y\,dx}{\int y\,dx}, \tag{1}$$

the integral of the numerator being taken from $x = \theta$ to $x = \theta'$, and that of the denominator being taken from $x = 0$ to $x = 1$. [364]

The most probable value of x, is that which renders y a *maximum*. We will designate it by a. If at the limits of x, y is null, then each value of y has a corresponding equal value on the other side of the *maximum*.

When the values of x, considered independently of the observed result, are not equally possible; by naming z the function of x which expresses their probability; it is easy to see, by that which has been said in first chapter of this Book, that by changing in formula (1), y into yz, we will have the probability that the value of x is comprehended within the limits $x = \theta$ and $x = \theta'$. This reverts to supposing all the values of x equally possible *a priori*, and by considering the observed result, as being formed of two independent results, of which the probabilities are y and z. We are able to restore thus all the cases to the one where we suppose *a priori*, before the event, an equal possibility to the different values of x, and, by this reason, we will adopt this hypothesis in that which will follow.

We have given in §22 and the following of the first Book, the formulas necessary in order to determine by some convergent approximations, the integrals of the numerator and of the denominator of formula (1), when the simple events of which the observed event is composed, are repeated a very great number of times; because then y has for factors, functions of x raised to very great powers. We will, by means of these formulas, determine the law of probability of the values of x, in measure as they deviate from the value a, the most probable, or which renders y a *maximum*. For that, let us resume formula (c) of §27 of the first Book,

$$\begin{aligned}\int y\,dx =& Y\left\{U + \frac{1}{2}\cdot\frac{d^2U^3}{1.2\,dx^2} + \frac{1.3}{2^2}\cdot\frac{d^4U^5}{1.2.3.4\,dx^4} + \text{etc.}\right\}\int dt\,c^{-t^2}\\ &+ \frac{Y}{2}c^{-T^2}\left\{\frac{dU^2}{dx} - T\cdot\frac{d^2U^3}{1.2\,dx^2} + (T^2+1)\cdot\frac{d^3U^4}{1.2.3\,dx^3} - \text{etc.}\right\} \qquad (2)\\ &- \frac{Y}{2}c^{-T'^2}\left\{\frac{dU^2}{dx} + T'\cdot\frac{d^2U^3}{1.2\,dx^2} + (T'^2+1)\cdot\frac{d^3U^4}{1.2.3\,dx^3} + \text{etc.}\right\};\end{aligned}$$

[365] ν is equal to $\frac{x-a}{\sqrt{\log Y - \log y}}$, and U, $\frac{dU^2}{dx}$, $\frac{d^2U^3}{dx^2}$, etc. are that which ν, $\frac{d\nu^2}{dx}$, $\frac{d^2\nu^3}{dx^2}$, etc. become, when we change after the differentiations, x into a, a being the value of x which renders y a *maximum*: T is equal to that which the function $\sqrt{\log Y - \log y}$ becomes, when we change x into $a-\theta$ in y, and T' is that which the same function becomes, when we change x into $a+\theta'$. The preceding expression of $\int y\,dx$ gives the value of this integral, within the limits $x = a-\theta$ and $x = a+\theta'$, the integral $\int dt\,c^{-t^2}$ being taken from $t = -T$ to $t = T'$.

Most often, at the limits of the integral $\int y\,dx$, extended from $x = 0$ to $x = 1$, y is null; now when y is not null, it becomes so small at these limits, that we are able to suppose it null. Then, we can make at these limits T and T' infinite, that which gives for the integral $\int y\,dx$, extended from $x = 0$ to $x = 1$,

$$\int y\,dx = Y\left\{U + \frac{1}{2}\cdot\frac{d^2U^3}{1.2\,dx^2} + \frac{1.3}{2^2}\cdot\frac{d^4U^5}{1.2.3.4\,dx^4} + \text{etc.}\right\}\sqrt{\pi};$$

thus the probability that the value of x is comprehended within the limits $x = a-\theta$ and $x = a+\theta'$ is equal to

$$\frac{\int dt\,c^{-t^2}}{\sqrt{\pi}} + \frac{\left\{\begin{array}{l}\frac{1}{2}c^{-T^2}\left\{\frac{dU^2}{dx} - T\cdot\frac{d^2U^3}{1.2\,dx^2} + (T^2+1)\cdot\frac{d^3U^4}{1.2.3\,dx^3} - \text{etc.}\right\}\\ -\frac{1}{2}c^{-T'^2}\left\{\frac{dU^2}{dx} + T'\cdot\frac{d^2U^3}{1.2\,dx^2} + (T'^2+1)\cdot\frac{d^3U^4}{1.2.3\,dx^3} + \text{etc.}\right\}\end{array}\right\}}{\left\{U + \frac{1}{2}\cdot\frac{d^2U^3}{1.2\,dx^2} + \frac{1.3}{2^2}\cdot\frac{d^4U^5}{1.2.3.4\,dx^4} + \text{etc.}\right\}\sqrt{\pi}}; \qquad (3)$$

We see, by §23 of the first Book, that in the case where y has for factors, some functions of x raised to great powers of order $\frac{1}{\alpha}$, α being an extremely small fraction, then U is most often of order $\sqrt{\alpha}$, so that its successive differences; U, $\frac{dU^2}{dx}$, $\frac{d^2U^3}{dx^2}$, etc. are respectively of the orders $\sqrt{\alpha}$, α, $\alpha^{\frac{3}{2}}$, etc.; whence it follows that the convergence of the series of formula (3), requires that T and T' are not of an order superior to $\frac{1}{\sqrt{\alpha}}$.

If we suppose $\theta = \theta'$, then we have very nearly $T = T'$, and formula (3) is reduced, [366] by neglecting the terms of order α, to the integral $\frac{\int dt\, c^{-t^2}}{\sqrt{\pi}}$, taken from $t = -T$ to $t = T'$; that which reverts in neglecting the square of the difference $T'^2 - T^2$, to doubling the preceding integral, and to taking it from t null to

$$t = \sqrt{\frac{T^2 + T'^2}{2}}:$$

now we have

$$T^2 = \log Y - \log y,$$

and we can suppose

$$\log y = \frac{1}{\alpha}\log\phi,$$

ϕ being a function of x or of $a - \theta$, which no longer contains factors raised to great powers; by naming therefore Φ, $\frac{d\Phi}{dx}$, $\frac{d^2\Phi}{dx^2}$, etc., that which ϕ, $\frac{d\phi}{dx}$, $\frac{d^2\phi}{dx^2}$, etc. become, when θ is null; by observing next that the condition of Y or Φ, a *maximum*, gives $\frac{d\Phi}{dx} = 0$, we will have

$$\alpha T^2 = -\theta^2\frac{dd\Phi}{2\Phi\, dx^2} + \theta^3\frac{d^3\Phi}{6\Phi\, dx^3} - \frac{\theta^4}{8}\left[\frac{d^4\Phi}{3\Phi\, dx^4} - \left(\frac{dd\Phi}{\Phi\, dx^2}\right)^4\right] + \text{etc.}$$

By changing θ into $-\theta$, we will have the value of $\alpha T'^2$; we will have therefore, by neglecting the terms of order α^2,

$$\frac{\alpha(T^2 + T'^2)}{2} = -\theta^2\frac{dd\Phi}{2\Phi\, dx^2};$$

hence,

$$\sqrt{\frac{T^2 + T'^2}{2}} = \frac{\theta}{\sqrt{\alpha}}\sqrt{-\frac{dd\Phi}{2\Phi\, dx^2}}.$$

Let us make

$$k = \sqrt{-\frac{dd\Phi}{2\Phi\, dx^2}} = \sqrt{-\frac{\alpha ddY}{2Y dx^2}},$$

$$\theta = \frac{t\sqrt{\alpha}}{k};$$

the probability that the value of x is comprehended within the limits $a \pm \frac{t\sqrt{\alpha}}{k}$ will be [367]

$$\frac{2\int dt\, c^{-t^2}}{\sqrt{\pi}},$$

the integral being taken from $t = 0$, and being able to be obtained in a very close manner, from the formulas of §27 from the first Book.

There results from this expression, that the most probable value of x is a, or that which renders the observed event, the most probable; and that by multiplying to infinity the simple events of which the observed event is composed, we are able at the same time to narrow the limits $a \pm \frac{t\sqrt{\alpha}}{k}$, and to increase the probability that the value of x will fall between these limits; so that at infinity, this interval becomes null, and the probability is confounded with certitude.

If the observed event depends on simple events of two different kinds, by naming x and x' the possibilities of these two kinds of events, we will see by the preceding reasonings, that y being then the probability of the composite event, the fraction

$$\frac{y\,dx\,dx'}{\iint y\,dx\,dx'}, \tag{4}$$

will be the probability of the simultaneous values of x and of x', the integrals of the denominator being taken from $x = 0$ to $x = 1$, and from $x' = 0$ to $x' = 1$. By naming a and a' the values of x and x' which render y a *maximum*, and making $x = a + \theta$, $x' = a' + \theta'$, we will find, by the analysis of §27 from the first Book, that if we suppose

$$\frac{\theta}{\sqrt{2Y}}\sqrt{-\left(\frac{ddY}{dx^2}\right)} - \theta'\frac{\left(\frac{ddY}{dx\,dx'}\right)}{2Y}\sqrt{\frac{2Y}{-\left(\frac{ddY}{dx^2}\right)}} = t,$$

$$\frac{\theta'}{\sqrt{-2Y\left(\frac{ddY}{dx^2}\right)}}\sqrt{\left(\frac{ddY}{dx^2}\right)\left(\frac{ddY}{dx'^2}\right) - \left(\frac{ddY}{dxdx'}\right)^2} = t',$$

[368] the fraction (4) will take form

$$\frac{dt\,dt'c^{-t^2-t'^2}}{\iint dt\,dt'c^{-t^2-t'^2}}.$$

The integrals of the denominator must be taken from $t = -\infty$ to $t = \infty$, and from $t' = -\infty$ to $t' = \infty$; because the integrals relative to x and x' of the fraction (4) being taken from $x = 0$ and $x' = 0$ to x and x' equal to unity, and at these limits, the values of θ and of θ' being $-a$ and $1 - a$, $-a'$ and $1 - a'$, the limits of t and of t' are equal to these last limits multiplied by some quantities of order $\frac{1}{\sqrt{\alpha}}$: thus the exponential $c^{-t^2-t'^2}$ is excessively small at these limits, and we can without sensible error, extend the integrals of the denominator of the preceding fraction, to the positive and negative infinite values of the variables t and t'. This denominator becomes thus equal to π; and the probability that the values of θ' and of θ are comprehended within the limits

$$\theta' = 0, \qquad \theta' = \frac{t'\sqrt{-2Y\left(\frac{ddY}{dx^2}\right)}}{\sqrt{\left(\frac{ddY}{dx^2}\right)\left(\frac{ddY}{dx'^2}\right) - \left(\frac{ddY}{dxdx'}\right)^2}},$$

$$\theta = 0, \qquad \theta = \frac{t\sqrt{2Y}}{\sqrt{-\left(\frac{ddY}{dx^2}\right)}} + \frac{t'\left(\frac{ddY}{dxdx'}\right)}{\sqrt{\left(\frac{ddY}{dx^2}\right)\left(\frac{ddY}{dx'^2}\right) - \left(\frac{ddY}{dxdx'}\right)^2}}\sqrt{\frac{2Y}{-\left(\frac{ddY}{dx^2}\right)}},$$

is equal to

$$\frac{1}{\pi}\iint dt\,dt' c^{-t^2-t'^2},$$

the integrals being taken from t and t' nulls.

We see by this formula, that in the case of two different kinds of simple events, the probability that their respective possibilities are those which render the composite event, most probable, becomes more and more great, and ends by being confounded with certitude; that which holds generally for any number whatsoever of different kinds of simple events, which enter into the observed event.

If we imagine an urn containing an infinity of balls of many different colors, and [369]
if after having drawn a great number n from it, p out of this number, had been of the first color, q of the second, r of the third, etc.; by designating by x, x', x'', etc. the respective probabilities to bring forth in a single drawing, one of these colors, the probability of the observed event will be the term which has for factor $x^p x'^q x''^r$.etc., in the development of the polynomial

$$(x + x' + x'' + \cdots)^n,$$

where we have

$$x + x' + x'' + \text{etc.} = 1,$$
$$p + q + r + \text{etc.} = n;$$

we will be able therefore to suppose here $y = x^p x'^q x''^r$.etc.; and then we have for the values of x, x', x'', etc. which render the observed event the most probable

$$x = \frac{p}{n}, \quad x' = \frac{q}{n}, \quad x'' = \frac{r}{n}, \quad \text{etc.}$$

Thus the most probable values are proportional to the numbers of the arrivals of the colors; and when the number n is a great number, the respective probabilities of the colors, are very nearly equal to the numbers of times that they have arrived, divided by the number of drawings.

§27. In order to give an application of the preceding formula, let us consider the case where two players A and B play together with this condition, that the one who out of three coups will have won two of them, wins the game; and let us suppose that out of a very great number n of games, A has won a number i of them. By naming x the probability of A to win a coup, and consequently $1 - x$, the corresponding probability of B; the probability of A to win a game, will be the sum of the first two terms of the binomial $(x + 1 - x)^3$, and the corresponding probability of B, will be the sum of the last two terms. These probabilities are therefore $x^2(3 - 2x)$ and $(1-x)^2(1+2x)$; thus the probability that out of n games, A will win i of them, and B, $n-i$, will be proportional to $x^{2i}(3-2x)^i(1-x)^{2n-2i}(1+2x)^{n-i}$. By naming therefore y this function, and a the value of x which renders it a *maximum*, the probability [370]
that the value of x is comprehended within the limits $a - \theta$ and $a + \theta$ will be

$$\frac{\int y\ dx}{\int y\ dx},$$

the integral of the numerator being taken from $x = a - \theta$ to $x = a + \theta$, and that of the denominator being taken from $x = 0$ to $x = 1$. If we make

$$\frac{1}{n} = \alpha, \quad \frac{i}{n} = i',$$

we will have by the preceding section,

$$\phi = x^{2i'}(3-2x)^{i'}(1-x)^{2-2i'}(1+2x)^{1-i'}.$$

The condition of the *maximum* of y or of ϕ, gives $d\phi = 0$; consequently a being the value of x corresponding to this *maximum*, we will have

$$0 = \frac{2i'}{a} - \frac{2i'}{3-2a} - \frac{2(1-i')}{1-a} + \frac{2(1-i')}{1+2a};$$

whence we deduce

$$i' = a^2(3-2a), \quad 1 - i' = (1-a)^2(1+2a);$$

next we have

$$\frac{-dd\Phi}{2\Phi dx^2} = \frac{18}{(3-2a)(1+2a)} = k^2.$$

The probability that the value of x is comprehended within the limits $a \pm \frac{r}{\sqrt{n}}$, will be therefore, by the preceding section, equal to

$$\frac{6\sqrt{2}}{\sqrt{\pi(3-2a)(1+2a)}} \int dr\, c^{\frac{-18r^2}{(3-2a)(1+2a)}}.$$

We will see easily that this result agrees with the one that we have found in §16, by an analysis less direct than this one.

The game ends in two coups, if A or B wins the first two coups, the third coup
[371] not being played, because it becomes useless. Thus the numbers of games won by one and the other of the players, does not indicate the number of games played; but they indicate that this last number is contained within some given limits, with a probability that increases without ceasing, in measure as the games are multiplied. The investigation of this number and of this probability being very proper to clarify the preceding analysis; we will occupy ourselves with it.

The probability that A will win a game in two coups, is x^2, x expressing, as above, his probability to win at each coup. The probability that he will win the game in three coups, is $2x^2(1-x)$. The sum $x^2(3-2x)$ of these two probabilities, is the probability that A will win the game. Thus in order to have the probability that out of i games won by player A, s will be of two coups, it is necessary to raise to the power i, the binomial

$$\frac{x^2}{x^2(3-2x)} + \frac{2x^2(1-x)}{x^2(3-2x)}$$

or

$$\frac{1}{3-2x} + \frac{2(1-x)}{3-2x},$$

and the term $i-s+1$ of the development of this power, will be that probability which is thus equal to

$$\frac{1.2.3\dots i.2^{i-s}(1-x)^{i-s}}{1.2.3\dots s.1.2.3\dots(i-s)(3-2x)^i}.$$

The greatest term of this development is, by §16, the one in which the exponents s and $i-s$ of the first and of the second term of the binomial are very nearly in the ratio of these terms, that which gives

$$s=\frac{i}{3-2x}.$$

We will name s' this quantity, and we will make

$$s=s'+l.$$

We will have, by §16,

$$\sqrt{\frac{i}{2s'\pi(i-s')}}dl\,c^{\frac{-il^2}{2s'(i-s')}}$$

for the probability of s, corresponding to the skill x of player A. [372]

We will find similarly that, if we name z the number of the games of two coups, won by player B, out of the number $n-i$ of games that he has won; the most probable value of z will be $\frac{n-i}{1+2x}$; and that by designating by z' this quantity, and making

$$z=z'+l',$$

the probability of z corresponding to x will be

$$\sqrt{\frac{n-i}{2z'(n-i-z')\pi}}dl'\,c^{\frac{-(n-i)l'^2}{2z'(n-i-z')}}.$$

The product of these two probabilities is therefore the probability corresponding to x, that the number of games of two coups, won by player A, will be $s'+l$, while the number of games of two coups, won by player B, will be $z'+l'$. Let

$$q=\frac{i}{2s'(i-s')},\quad q'=\frac{n-i}{2z'(n-i-z')};$$

we will have, for this composite probability,

$$\frac{\sqrt{qq'}}{\pi}dl\,dl'c^{-ql^2-q'l'^2}.$$

It is necessary to multiply this probability by that of x, which, as we have seen in the preceding section, is $\frac{y\,dx}{\int y\,dx}$; the product is

$$\frac{\sqrt{qq'}}{\pi}\,\frac{y\,dx}{\int y\,dx}dl\,dl'\,c^{-ql^2-q'l'^2}; \qquad (\epsilon)$$

the integral of the denominator must be taken from $x = 0$ to $x = 1$; and by §27 of the first Book, this integral is very nearly,

$$Y\sqrt{\pi}\sqrt{-\frac{2Y\,dx^2}{ddY}}.$$

If we name X the function

$$\sqrt{qq'}\,c^{-ql^2-q'l'^2},$$

[373] and if we designate by a' the value of x, which renders Xy a *maximum*, and by X' and Y', that which X and y become, when we change x into a' there; we will have, by the preceding section, by making $x = a' + \theta$,

$$ydx\sqrt{qq'}\,c^{-ql^2-q'l'^2} = Y'X'd\theta\,c^{\frac{\theta^2 d^2(X'Y')}{2X'Y'dx^2}}.$$

It is easy to see that a' differs from the value a of x, which renders y a *maximum*, only by a quantity of order α, which we will designate by $f\alpha$; by substituting into Y, $a+f\alpha$ instead of a', in order to form Y', and developing with respect to the powers of α, we will see that $\frac{dY}{da}$ being null, because Y is the *maximum* of y, Y' differs from Y, only by quantities of order α; thus we have, to the quantities near of an order inferior to the one that we conserve, and by observing that $\frac{dX'}{X'dx}$ and $\frac{d^2X'}{X'dx^2}$ can be neglected with respect to $\frac{dY'}{Y'dx}$,

$$\frac{d^2X'Y'}{2X'Y'dx^2} = \frac{d^2Y}{2Y\,dx^2};$$

the function (ϵ) becomes thence

$$\frac{\sqrt{qq'}}{\pi\sqrt{\pi}}\sqrt{-\frac{ddY}{2Y\,dx^2}}\,dl\,dl'd\theta\,c^{-ql^2-q'l'^2+\frac{\theta^2 ddY}{2Y\,dx^2}}. \qquad (\epsilon')$$

We must in this function, suppose $x = a$, that which gives, by substituting for i, its value $na^2(3-2a)$,

$$q = \frac{3-2a}{4na^2(1-a)}, \quad q' = \frac{1+2a}{4na(1-a)^2}.$$

Next, x being equal to $a' + \theta$, it is equal to $a + f\alpha + \theta$; by neglecting therefore the quantities of order α, we will have

$$x = a + \theta.$$

Now the number of games of two coups, being

$$\frac{i}{3-2x} + \frac{n-i}{1+2x} + l + l',$$

[374] this number will be

$$\frac{i}{3-2a} + \frac{n-i}{1+2a} + \left[\frac{2i}{(3-2a)^2} - \frac{2(n-i)}{(1+2a)^2}\right]\theta + l + l'.$$

Let us make

$$t = \left[\frac{2i}{(3-2a)^2} - \frac{2(n-i)}{(1+2a)^2}\right]\theta + l + l';$$

and let us designate by q'' the quantity

$$-\frac{ddY}{2Y\,dx^2\left[\frac{2i}{(3-2a)^2}-\frac{2(n-i)}{(1+2a)^2}\right]^2},$$

which, after all the reductions, is reduced to

$$\frac{9(3-2a)(1+2a)}{2n(1-2a)^2(3-2a+2a^2)^2};$$

the function (ϵ') will become

$$\frac{\sqrt{qq'q''}}{\pi\sqrt{\pi}}\,dt\,dl\,dl'\,c^{-ql^2-q'l'^2-q''(t-l-l')^2}. \qquad (\epsilon'')$$

By integrating it from $l=-\infty$ to $l=\infty$, and from $l'=-\infty$ to $l'=\infty$, we will have the probability that the number of games of two coups, will be equal to

$$\frac{i}{3-2a}+\frac{n-i}{1+2a}+t;$$

now we have

$$\int dl\,c^{-ql^2-q'l'^2-q''(t-l-l')^2}$$
$$=\int dl\,c^{-\frac{qq''}{q+q''}(t-l')^2-q'l'^2-(q+q'')\left[l-\frac{q''}{q+q''}(t-l')\right]^2}.$$

This last integral, taken from $l=-\infty$ to $l=\infty$, is, by that which precedes,

$$\frac{\sqrt{\pi}}{\sqrt{q+q''}}c^{-\frac{qq''}{q+q''}(t-l')^2-q'l'^2}.$$

By multiplying it by dl', and by putting it under this form, [375]

$$\frac{\sqrt{\pi}\;dl'}{\sqrt{q+q''}}c^{-\frac{qq'q''t^2}{qq'+qq''+q'q''}-\frac{qq'+qq''+q'q''}{q+q''}\left(l'-\frac{qq''t}{qq'+qq''+q'q''}\right)^2},$$

and integrating from $l'=-\infty$ to $l'=\infty$; we will have

$$\frac{\pi}{\sqrt{qq'+qq''+q'q''}}\,c^{-\frac{qq'q''t^2}{qq'+qq''+q'q''}}$$

The function (ϵ'') integrated with respect to l and l', within the positive and negative infinite limits of these variables, becomes thus

$$\frac{1}{\sqrt{\pi}}\sqrt{\frac{qq'q''}{qq'+qq''+q'q''}}\,c^{-\frac{qq'q''t^2}{qq'+qq''+q'q''}}.$$

Thus the probability that the number of games of two coups, will be comprehended within the limits

$$\frac{i}{3-2a}+\frac{n-i}{1+2a}\pm t=n(a^2+\overline{1-a}^2)\pm t,$$

is equal to the double of the integral of the preceding differential, taken from t null. We must observe that q, q', q'' are of order $\frac{1}{n}$, so that the quantity $\frac{qq'q''}{qq'+qq''+q'q''}$ is of the same order. Let us represent it by $\frac{k'^2}{n}$, and let us make $t = r\sqrt{n}$; we will have

$$\frac{2}{\sqrt{\pi}} \int k' dr\, c^{-k'^2 r^2}, \qquad (\epsilon''')$$

for the expression of the probability that the number of games of two coups, will be comprehended within the limits

$$n(a^2 + \overline{1-a}^2) \pm r\sqrt{n},$$

the integral being taken from r null. The interval of these two limits is $2r\sqrt{n}$, and the
[376] ratio of this interval to the number n of games, is $\frac{2r}{\sqrt{n}}$; this ratio diminishes without ceasing, in measure as n increases, and r can increase at the same time indefinitely; so that the preceding integral approaches indefinitely unity.

The total number of coups, is the triple of the number of games of three coups, plus the double of the number of games of two coups, or the triple of the total number n of games, less the number of games of two coups; it is therefore

$$2n(1 + a - a^2) \mp r\sqrt{n},$$

The integral (ϵ''') is therefore the expression of the probability that the number of coups will be comprehended within these limits.

If instead of knowing the number i of games won by player A, and the total number n of games, we knew the number i and the total number of coups; the same analysis will be able to serve to determine the unknown number n of games. For this, let us designate by h, the total number of coups; we will have, by that which precedes, the two equations

$$3n - \frac{i}{3-2a} - \frac{n-i}{1+2a} = h \pm r\sqrt{n},$$
$$\frac{i}{a} - \frac{i}{3-2a} = \frac{n-i}{1-a} - \frac{n-i}{1+2a}.$$

These equations give a and n as functions of $h \pm r\sqrt{n}$. Let us suppose

$$n = i\psi\left(\frac{h \pm r\sqrt{n}}{i}\right), \quad a = \Gamma\left(\frac{h \pm r\sqrt{n}}{i}\right);$$

we will have, by reducing into series,

$$n = i\psi\left(\frac{h}{i}\right) \pm ir\sqrt{n}\frac{d\psi\left(\frac{h}{i}\right)}{dh} + \text{etc.};$$

we will substitute into k', instead of n and of a, $i\psi\left(\frac{h}{i}\right)$ and $\Gamma\left(\frac{h}{i}\right)$: the integral (ϵ''') is then the probability that the number n of games, is comprehended within the limits

$$i\psi\left(\frac{h}{i}\right) \pm ir\sqrt{i\psi\left(\frac{h}{i}\right)}\frac{d\psi\left(\frac{h}{i}\right)}{dh}.$$

§28. It is principally in the births, that the preceding analysis is applicable, and we are able to deduce from it, not only for the human race, but for all the kinds of organized beings, some interesting results. Until here the observations of this kind have been made in great number, only on the human race: we will submit the principals to the calculus. [377]

Let us consider first the births observed at Paris, at London and in the realm of Naples. In the space of 40 years elapsed from the commencement of 1745, an epoch where we have begun to distinguish at Paris, out of the registers, the births of two sexes, to the end of 1784, we have baptized in this capital, 393386 boys, and 377555 girls, the found infants being comprehended in this number: this gives nearly $\frac{25}{24}$ for the ratio of the baptisms of the boys to those of the girls.

In the space of 95 years elapsed from the commencement of 1664 to the end of 1758, there was born at London, 737629 boys, and 698958 girls; that which gives $\frac{19}{18}$ nearly, for the ratio of the births of boys to those of girls.

Finally, in the space of nine years elapsed, from the commencement of 1774 to the end of 1782, there was born in the realm of Naples, Sicily not included, 782352 boys, and 746821 girls; that which gives $\frac{22}{21}$ for the ratio of the births of the boys to those of the girls.

The smallest of these numbers of births, are relative to Paris; besides, it is in this city that the births of the boys and of the girls, more approach equality. For these two reasons, the probability that the possibility of the birth of a boy surpasses $\frac{1}{2}$, must be less than at London and in the realm of Naples. Let us determine numerically this probability.

Let us name p the number of masculine births observed at Paris, q the one of the feminine births, and x the possibility of a masculine birth, that is the probability that an infant who must be born, will be a boy; $1 - x$ will be the possibility of a feminine birth, and we will have the probability that out of $p + q$ births, p will be masculine, [378]
and q will be feminine, equal to

$$\frac{1.2.3\dots(p+q)}{1.2.3\dots p.1.2.3\dots q}x^p(1-x)^q;$$

by making therefore

$$y = x^p(1-x)^q,$$

the probability that the value of x is comprehended within some given limits, will be by §26, equal to

$$\frac{\int y\,dx}{\int y\,dx},$$

the integral of the denominator being taken from $x = 0$ to $x = 1$, and that of the numerator being taken within the given limits. If we take zero and $\frac{1}{2}$ for these limits, we will have the probability that the value of x not surpass $\frac{1}{2}$. The value which corresponds to the *maximum* of y is $\frac{p}{p+q}$; and seeing the magnitude of the numbers p and q, the excess of $\frac{p}{p+q}$ over $\frac{1}{2}$, is too considerable in order to employ here formula (c) from §27 of the first Book, in the approximation of the integral $\int y\,dx$, taken from

$x = 0$ to $x = \frac{1}{2}$; it is necessary therefore, in this case, to make use of formula (A) from §22 of the same Book. Here we have

$$\nu = -\frac{y\,dx}{dy} = -\frac{x(1-x)}{p-(p-q)x};$$

the formula cited (A) gives thus for the integral $\int y\,dx$, taken from $x = 0$ to $x = \frac{1}{2}$,

$$\frac{1}{2^{p+q+1}(p-q)}\left[1 - \frac{p+q}{(p-q)^2} + \text{etc.}\right].$$

As for the integral $\int y\,dx$, taken from $x = 0$ to $x = 1$, we have, by §26,

$$\int y\,dx = Y\left[U + \frac{1}{2}\cdot\frac{d^2U^3}{1.2\,dx^2} + \text{etc.}\right]\sqrt{\pi},$$

Y being that which y becomes at its *maximum*, or when we substitute $\frac{p}{p+q}$ for x. ν is
[379] here equal to $\frac{x-\frac{p}{p+q}}{\sqrt{\log Y - \log y}}$; and U, $\frac{d^2U^3}{dx^2}$, etc. are that which ν, $\frac{d^2\nu^3}{dx^2}$, etc. become, when we make, after the differentiations, $x = \frac{p}{p+q}$. We find thus for the integral $\int y\,dx$ taken from x null to $x = 1$,

$$\int y\ dx = \frac{p^{p+\frac{1}{2}}q^{q+\frac{1}{2}}\sqrt{2\pi}}{(p+q)^{p+q+\frac{3}{2}}}\left[1 + \frac{(p+q)^2 - 13pq}{12pq(p+q)} + \text{etc.}\right];$$

the probability that the value of x does not surpass $\frac{1}{2}$, is therefore equal to

$$\begin{aligned}&\frac{(p+q)^{p+q+\frac{3}{2}}}{(p-q)\sqrt{\pi}\,2^{p+q+\frac{3}{2}}p^{p+\frac{1}{2}}q^{q+\frac{1}{2}}}\\ &\times\left[1 - \frac{p+q}{(p-q)^2} - \frac{\overline{p+q}^2 - 13pq}{12pq(p+q)} - \text{etc.}\right].\end{aligned} \qquad (o)$$

In order to apply large numbers to this formula, it would be necessary to have the logarithms of p, q and $p-q$, with twelve decimals at least: we are able to supply it in this manner. We have

$$\log\left[\frac{\left(\frac{p+q}{2}\right)^{p+q}}{p^pq^q}\right] = -p\ \log\left(1 + \frac{p-q}{p+q}\right) - q\ \log\left(1 - \frac{p-q}{p+q}\right).$$

When the logarithms are hyperbolic, the second member of this equation, reduced to series, becomes

$$-(p+q)\left[\frac{\left(\frac{p-q}{p+q}\right)^2}{1.2} + \frac{\left(\frac{p-q}{p+q}\right)^4}{3.4} + \frac{\left(\frac{p-q}{p+q}\right)^6}{5.6} + \frac{\left(\frac{p-q}{p+q}\right)^8}{7.8} + \text{etc.}\right];$$

we will have therefore by this very convergent series, the hyperbolic logarithm of $\frac{(p+q)^{p+q}}{2^{p+q}p^pq^q}$. In multiplying it by $0,43429448$, we will convert it into tabular logarithm, and by adding to it the tabular logarithm of $\frac{(p+q)^{\frac{3}{2}}}{2(p-q)\sqrt{2pq\pi}}$, we will have the tabular logarithm of the factor which multiplies series (o). If we name $\frac{1}{\mu}$ this factor, and if we make

$$p = 393386, \quad q = 377555;$$

we find by tabular logarithm [380]

$$\log \mu = 72,2511780,$$

the series (o) becomes

$$\frac{1}{\mu}(1 - 0,0030761 + \text{etc.}).$$

This quantity of an excessive smallness, subtracted from unity, will give the probability that at Paris, the possibility of the births of the boys, surpasses that of girls; whence we see that we must regard this probability as being equal, at least, to that of the most authenticated historical facts.

If we apply formula (o) to the births observed in the principal cities of Europe, we find that the superiority of the births of boys over the births of girls, observed everywhere from Naples to Petersburg, indicates a greater possibility of the births of boys, with a probability extremely near to certitude; this result appears therefore to be a general law, at least in Europe; and if, in some small cities, where we have observed only a not very considerable number of births, nature seems to deviate from it; there is everywhere to believe that this deviation was only apparent, and that at length, the observed births in these cities would offer, in being multiplied, a result similar to the one of the great cities. Many philosophers, deceived by these anomalies, have sought the cause of phenomena which are only the effect of chance; that which proves the necessity to make precede parallel investigations, by that of the probability with which the observations indicate the phenomena of which we wish to determine the cause. I take for example, the small city of Vitteaux, in which, out of 415 births observed during five years, there were born 203 boys and 212 girls. p being here less than q, the natural order appears reversed. Let us see what is according to these observations, the probability that the facilities of the births of boys surpasses in this city, those of the births of girls. This probability is $\frac{\int y\,dx}{\int y\,dx}$, the integral of the numerator being taken from $x = \frac{1}{2}$ to $x = 1$, and that of the denominator being taken from $x = 0$ to $x = 1$. Formula (o) which, subtracted from unity, gives this fraction, becomes here [381]
divergent; we will employ then formula (3) from §26, which is reduced very nearly to its first term $\frac{\int dt\,c^{-t^2}}{\sqrt{\pi}}$, the integral being taken from the value of t which corresponds to $x = \frac{1}{2}$ to the value of t which corresponds to $x = 1$. Now we have, by the section cited,

$$t^2 = \log Y - \log y,$$

y being $x^p(1-x)^q$, and Y being the value of y corresponding to the *maximum* of y, which holds when $x = \frac{p}{p+q}$; the value of t^2 which corresponds to $x = \frac{1}{2}$ is $-\log\left[\frac{\left(\frac{p+q}{2}\right)^{p+q}}{p^p q^q}\right]$, this logarithm being hyperbolic, and being given, by that which precedes, by a very convergent series. The value of t^2 which corresponds to $x = 1$, is $t^2 = \infty$; thus we have therefore the two limits of the integral $\int dt\, c^{-t^2}$, an integral which it will be easy to obtain by the formulas which we have given for this object. We find thus the probability that at Vitteaux, the facilities of the births of boys surpasses over those of girls, equal to $0,33$; the superiority of the facility of the births of girls, is therefore indicated by these observations, with a probability equal to $0,67$, a probability much too weak to balance the analogy which carried us to think that at Vitteaux, as in all the cities where we have observed a considerable number of births, the possibility of the births of boys surpasses that of the births of girls.

§29. We have seen at London, the observed ratio of the births of boys to those of girls, is equal to $\frac{19}{18}$, while at Paris, the one of the baptisms of boys to those of girls, is only $\frac{25}{24}$. This seems to indicate a constant cause of this difference. Let us determine the probability of this cause.

Let p and q be the numbers of baptisms of boys and girls, made at Paris in the interval from the beginning of 1745 to the end of 1784; by designating by x, the
[382] possibility of the baptism of a boy, and making, as in the preceding section,

$$y = x^p(1-x)^q,$$

the most probable value of x, will be that which renders y a *maximum*; it is therefore $\frac{p}{p+q}$; by supposing next

$$x = \frac{p}{p+q} + \theta;$$

the probability of the value of θ will be, by §26, equal to

$$\frac{d\theta}{\sqrt{\pi}}\sqrt{\frac{(p+q)^3}{2pq}}\,c^{-\frac{(p+q)^3}{2pq}\theta^2}.$$

By designating by p', q' and θ' that which p, q and θ become for London, we will have

$$\frac{d\theta'}{\sqrt{\pi}}\sqrt{\frac{(p'+q')^3}{2p'q'}}\,c^{-\frac{(p'+q')^3}{2p'q'}\theta'^2}$$

for the probability of θ'; the product

$$\frac{d\theta\ d\theta'}{\pi}\sqrt{\frac{(p+q)^3(p'+q')^3}{4pqp'q'}}\,c^{-\frac{(p+q)^3}{2pq}\theta^2-\frac{(p'+q')^3}{2p'q'}\theta'^2}$$

of these two probabilities, will be therefore the probability of the simultaneous existence of θ and θ'. Let us make

$$\frac{p'}{p'+q'} + \theta' = \frac{p}{p+q} + \theta + t;$$

the preceding differential function becomes

$$\frac{d\theta\ dt}{\pi}\sqrt{\frac{(p+q)^3(p'+q')^3}{4pqp'q'}}\,c^{-\frac{(p+q)^3}{2pq}\theta^2-\frac{(p'+q')^3}{2p'q'}\left(\theta+t-\frac{p'q-pq'}{(p+q)(p'+q')}\right)^2}.$$

By integrating it for all the possible values of θ, and next for all the positive values of t; we will have the probability that the possibility of the baptisms of boys is greater at London than at Paris. The values of θ are able to be extended from θ equal to $-\frac{p}{p+q}$ to θ equal to $1-\frac{p}{p+q}$; but when p and q are very great numbers, the factor [383] $c^{-\frac{(p+q)^3}{2pq}\theta^2}$ is so small at these two limits, that we are able to regard it as null; we are able therefore to extend the integral relative to θ, from $\theta=-\infty$ to $\theta=\infty$. We see for the same reason, that the integral relative to t, is able to be extended from $t=0$ to $t=\infty$. By following the process from §27 for these multiple integrations, we will find easily that if we make

$$k^2 = \frac{(p+q)^3(p'+q')^3}{2p'q'(p+q)^3+2pq(p'+q')^3},$$

$$h = \frac{p'q-pq'}{(p+q)(p'+q')},$$

$$\theta + \frac{2pqk^2}{(p+q)^3}(t-h) = t',$$

that which gives $d\theta = dt'$; the preceding differential integrated first with respect to t' from $t'=-\infty$ to $t'=\infty$, and next from $t=0$ to t infinity, will give

$$\int \frac{k\,dt}{\sqrt{\pi}} c^{-k^2(t-h)^2}$$

for the probability that at London, the possibility of the baptisms of boys is greater than at Paris. If we make

$$k(t-h) = t'',$$

this integral becomes

$$\int \frac{dt''}{\sqrt{\pi}} c^{-t''^2},$$

the integral being taken from $t''=-kh$ to $t''=\infty$; and it is clear that it is equal to

$$1 - \int \frac{dt''}{\sqrt{\pi}} c^{-t''^2},$$

the integral being taken from $t''=kh$ to $t''=\infty$. Thence it follows, by §27 of the first Book, that if we suppose

$$i^2 = \frac{p'q'(p+q)^3+pq(p'+q')^3}{(p+q)(p'+q')(p'q-pq')^2};$$

[384] the probability that the possibility of the baptisms of boys is greater at London than at Paris, has for expression

$$1 - \frac{ic^{-\frac{1}{2i^2}}}{\sqrt{2\pi}} \cfrac{1}{1 + \cfrac{i^2}{1 + \cfrac{2i^2}{1 + \cfrac{3i^2}{1 + \cfrac{4i^2}{1 + \text{etc.}}}}}} \qquad (\mu)$$

By making in this formula

$$p = 393386, \qquad q = 377555,$$
$$p' = 737629, \qquad q' = 698958,$$

it becomes

$$1 - \frac{1}{328269}.$$

There is therefore odds of 328268 against one, that at London, the possibility of the baptisms of boys was greater than at Paris. This probability approaches so much to certitude, that there is place to investigate the cause of this superiority.

Among the causes which can produce it, it has appeared to me that the baptisms of the found infants, who are part of the annual list of the baptisms at Paris, must have a sensible influence on the ratio of the baptisms of the boys to those of the girls; and that they should diminish this ratio, if, as it is natural to believe, the parents in the surrounding country, finding advantage to retain near to them the male infants, have sent them to the hospice of the Enfants-trouvés[1] of Paris, in a ratio less than the one of the births of the two sexes. This is that which the summary from the registers of this hospice has made me see with a very great probability. From the commencement of 1745 to the end of 1809, we have baptized 163499 boys and 159405 girls, a number of which the ratio is $\frac{39}{38}$, and differs too much from the ratio $\frac{25}{24}$ of the baptisms of the boys and the girls at Paris, in order to be attributed to simple chance.

§30. Let us determine, according to the preceding principles, the probabilities
[385] of the results founded on the tables of mortality or of assurance, constructed on a great number of observations. Let us suppose first that with respect to a number p of individuals of a given age A, we have observed that there exists yet the number q, at the age $A + a$; we demand the probability that out of p' individuals of age A, there will exist $q' + z$ of them at the age $A + a$, the ratio of p' and q' being the same as that of p to q.

Let x be the probability of an individual of age A, to survive to age $A + a$; the probability of the observed event is then the term of the binomial $(x + \overline{1 - x})^p$ which

[1] *Translator's note*: Foundling Hospital of Paris.

has x^q for factor; this probability is therefore

$$\frac{1.2.3\dots p}{1.2.3\dots\overline{p-q}1.2.3\dots q}\cdot x^q(1-x)^{p-q};$$

thus the probability of the value of x, taken from the observed event, is

$$\frac{x^q dx\,(1-x)^{p-q}}{\int x^q dx(1-x)^{p-q}},$$

the integral of the denominator being taken from $x=0$ to $x=1$.

The probability that out of the p' individuals of age A, $q'+z$ will live to age $A+a$, is

$$\frac{1.2.3\dots p'}{1.2.3\dots(q'+z)1.2.3\dots(p'-q'-z)}x^{q'+z}(1-x)^{p'-q'-z}.$$

By multiplying this probability by the preceding probability of the value of x; the product integrated from $x=0$ to $x=1$, will be the probability of the existence of $q'+z$ persons at age $A+a$; by naming therefore P this probability, we will have

$$P=\frac{1.2.3\dots p'\int x^{q+q'+z}dx(1-x)^{p+p'-q-q'-z}}{1.2.3\dots(q'+z)1.2.3\dots(p'-q'-z)\int x^q dx(1-x)^{p-q}},$$

the integrals of the numerator and of the denominator being taken from $x=0$ to $x=1$. We have by §28, very nearly,

$$\int x^{q+q'+z}dx(1-x)^{p+p'-q-q'-z}$$ [386]

$$=\sqrt{2\pi}\left[(q+q')\left(1+\frac{z}{q+q'}\right)\right]^{q+q'+z+\frac{1}{2}}$$

$$\times\frac{\left[(p+p'-q-q')\left(1-\frac{z}{p+p'-q-q'}\right)\right]^{p+p'-q-q'-z+\frac{1}{2}}}{(p+p')^{p+p'+\frac{3}{2}}},$$

$$\int x^q dx(1-x)^{p-q}=\sqrt{2\pi}\,\frac{q^{q+\frac{1}{2}}(p-q)^{p-q+\frac{1}{2}}}{p^{p+\frac{3}{2}}}.$$

Next, by §33 of the first Book, we have

$$1.2.3\dots\dots\dots p'=p'^{p'+\frac{1}{2}}c^{-p'}\sqrt{2\pi},$$

$$1.2.3\dots\dots(q'+z)=q'^{q'+z+\frac{1}{2}}\left(1+\frac{z}{q'}\right)^{q'+z+\frac{1}{2}}c^{-q'-z}\sqrt{2\pi},$$

$$1.2.3\dots(p'-q'-z)=(p'-q')^{p'-q'-z+\frac{1}{2}}\left(1-\frac{z}{p'-q'}\right)^{p'-q'-z+\frac{1}{2}}c^{-p'+q'+z}\sqrt{2\pi};$$

finally we have $q'=\frac{qp'}{p}$. This premised, we find after all the reductions,

$$P = \sqrt{\frac{p^3}{qp'(p-q)(p+p')2\pi}}$$
$$\times \frac{\left(1+\frac{z}{q+q'}\right)^{q+q'+z+\frac{1}{2}}\left(1-\frac{z}{p+p'-q-q'}\right)^{p+p'-q-q'-z+\frac{1}{2}}}{\left(1+\frac{z}{q'}\right)^{q'+z+\frac{1}{2}}\left(1-\frac{z}{p'-q'}\right)^{p'-q'-z+\frac{1}{2}}}.$$

If we take the hyperbolic logarithm of the second member of this equation, if we reduce this logarithm into series ordered with respect to the powers of z, and if we neglect the powers superior to the square, we will have by passing again from the logarithm to the function,

$$P = \sqrt{\frac{p^3}{qp'(p-q)(p+p')2\pi}}\left[1+\frac{(2q-p)p^2z}{2qp'(p-q)(p+p')}\right]c^{\frac{-p^3z^2}{2qp'(p-q)(p+p')}}.$$

[387] p, q, p' being supposed very great numbers of order $\frac{1}{\alpha}$, the coefficient of z is very small of order α; the one of $-z^2$ is very small and of the same order. But if we suppose $\frac{z}{p}$ of the order $\sqrt{\alpha}$, we will be able to neglect in the preceding expression, the term depending on the first power of z, as very small of order $\sqrt{\alpha}$. Moreover, this term is itself destroyed, when we have regard at the same time to the positive and negative values of z. By neglecting it therefore, we will have

$$2\sqrt{\frac{p^3}{qp'(p-q)(p+p')2\pi}}\int dz\, c^{-\frac{p^3z^2}{2qp'(p-q)(p+p')}}$$

for the expression of the probability that out of p' individuals of age A, the number of those who will arrive to age $A+a$ will be comprehended within the limits $q \pm z$, the integral being taken from z null.

Let us suppose now that we have found by observation, that out of p individuals of age A, q lived yet to age $A+a$, and r to age $A+a+a'$; we demand the probability that out of p' individuals of the same age A, $\frac{qp'}{p}+z$ will live to age $A+a$, and $\frac{rp'}{p}+z'$ will live to age $A+a+a'$.

The probability that out of p' individuals of age A, $\frac{qp'}{p}+z$ will live to age $A+a$ is, by that which precedes,

$$\sqrt{\frac{p^3}{2qp'(p-q)(p+p')\pi}}\,c^{-\frac{p^3z^2}{2qp'(p-q)(p+p')}}.$$

We will have the probability that out of $\frac{qp'}{p}+z$ individuals of age $A+a$, $\left(\frac{qp'}{p}+z\right)\frac{r}{q}+u$ will live to age $A+a+a'$, by changing in the preceding function, p' into $\frac{qp'}{p}+z$, p into q, q into r, and z into u; that which gives, by neglecting z with respect to $\frac{qp'}{p}$,

$$\sqrt{\frac{qp^2}{2rp'(q-r)(p+p')\pi}}\; c^{-\frac{qp^2u^2}{2rp'(p-r)(p+p')}}.$$

The product of these two probabilities, is the probability of the simultaneous existence of z and of u; now we have [388]

$$\left(\frac{qp'}{p}+z\right)\frac{r}{q}+u=\frac{rp'}{p}+z';$$

that which gives

$$u=z'-\frac{rz}{q};$$

by making therefore

$$\beta^2=\frac{p^3}{2qp'(p-q)(p+p')},$$

$$\beta'^2=\frac{qp^2}{2rp'(q-r)(p+p')}.$$

The probability P of the simultaneous existence of the values of z and of z' will be

$$P=\int\frac{\beta\,dz}{\sqrt{\pi}}\cdot\frac{\beta'\,dz'}{\sqrt{\pi}}\,c^{-\beta^2z^2-\beta'^2\left(z'-\frac{rz}{q}\right)^2}.$$

By following this analysis, we find generally that, if we make

$$\beta''^2=\frac{rp^2}{2sp'(r-s)(p+p')},$$

$$\beta'''^2=\frac{sp^2}{2tp'(s-t)(p+p')},$$

etc.;

the probability P that out of p' individuals of age A, the numbers of those who will live to ages $A+a$, $A+a+a'$, $A+a+a'+a''$, etc. will be comprehended within the respective limits

$$\frac{qp'}{p},\frac{qp'}{p}+z;\quad\frac{rp'}{p},\frac{rp'}{p}+z';\quad\frac{sp'}{p},\frac{sp'}{p}+z'';\quad\frac{tp'}{p},\frac{tp'}{p}+z''';\quad\text{etc.}$$

is

$$P=\int\frac{\beta\,dz}{\sqrt{\pi}}\cdot\frac{\beta'\,dz'}{\sqrt{\pi}}\cdot\frac{\beta''\,dz''}{\sqrt{\pi}}.\text{etc.}\,c^{-\beta^2z^2-\beta'^2\left(z'-\frac{rz}{q}\right)^2-\beta'''^2\left(z''-\frac{sz'}{r}\right)^2-\text{etc.}}.$$

We can estimate by this formula, the respective probabilities of the numbers of a table of mortality, constructed on a great number of observations. The manner to form these tables, is very simple. We take out of the registers of births and of deaths, a great number of infants who we follow during the course of their life, by determining [389] how many there remain of them at the end of each year of their age; and we write this number vis-à-vis dying each year. But as in the first two or three years of life, mortality is very rapid; it is necessary, for more exactitude, to indicate in this first age, the number of the surviving at the end of each half-year. If the number p of infants were infinite, we would have thus exact tables which would represent the true law of mortality in the place and at the epoch of their formation. But the number of infants that we choose being finite; however great it be, the numbers of the table are susceptible of errors. Let us represent by p', q', r', s', t', etc., these diverse numbers. The true numbers, for a number p' of births, are $\frac{qp'}{p}$, $\frac{rp'}{p}$, $\frac{sp'}{p}$, $\frac{tp'}{p}$, etc. If we make $q' = \frac{qp'}{p} + z$, z will be the error of q'; similarly, if we suppose $r' = \frac{rp'}{p} + z'$, z' will be the error of r', and so forth. The preceding expression of P is therefore the probability that the errors of q', r', s', etc. are comprehended within the limits zero and z, zero and z', zero and z'', etc. The values of β, β', etc. depend on p, q, r, etc. which are unknowns; but the supposition of p infinite gives

$$\beta^2 = \frac{p^2}{2qp'(p-q)}.$$

We are able to substitute without sensible error, $\frac{q'}{p'}$ instead of $\frac{q}{p}$, that which gives

$$\beta^2 = \frac{p'}{2q'(p'-q')}.$$

We will have in the same manner,

$$\beta'^2 = \frac{q'}{2r'(q'-r')},$$
$$\beta''^2 = \frac{r'}{2s'(r'-s')},$$
etc.

If we wish to consider only the error of one of the numbers of the table, such as s',
[390] then we will integrate the expression of P, relatively to z''', z^{iv}, etc., from the infinite negative values of these variables to their infinite positive values; and then we have

$$P = \int \frac{\beta\, dz}{\sqrt{\pi}} \cdot \frac{\beta'\, dz'}{\sqrt{\pi}} \cdot \frac{\beta''\, dz''}{\sqrt{\pi}}\, c^{-\beta^2 z^2 - \beta'^2\left(z' - \frac{r'z}{q'}\right)^2 - \beta'''^2\left(z'' - \frac{s'z'}{r'}\right)^2}.$$

The integrals relative to z and z' must be taken from their negative infinite values, to their positive infinite values; we will find thus, by the process of which we have often made use for this kind of integration, that if we suppose

$$\gamma^2 = \frac{p'}{2s'(p'-s')},$$

we will have

$$P = \int \frac{\gamma\, dz''}{\sqrt{\pi}}\, c^{-\gamma^2 z''^2}.$$

The probability that the error of any number from the table, will be comprehended within the limits zero and any quantity, is therefore independent, either of the intermediate numbers, or of the subsequent numbers.

If we make $\gamma z'' = t$, we will have

$$\frac{z''}{s'} = t\sqrt{\frac{2(p'-s')}{p's'}},$$

and the probability P that the ratio of the error of the number s' from the table, to this number itself, will be comprehended within the limits $\pm t\sqrt{\frac{2(p'-s')}{p's'}}$ is

$$P = 2\int \frac{dt}{\sqrt{\pi}}\, c^{-t^2},$$

the integral being taken from t null. We see thus that the value of t, and consequently the probability P remaining the same, this ratio increases when s' diminishes; thus the numbers from the table are so much less certain, as they are more extended from the first p'. We see further that this ratio diminishes in measure as p' increases, or in measure as we multiply the observations; in a manner that we are able by this multiplication, to diminish at the same time this ratio and to increase t; this ratio [391]
becoming null when p' is infinite, and P becoming then equal to unity.

§31. Let us apply the preceding analysis to the research on the population of a great empire. One of the simplest and most proper ways to determine this population, is the observation of the annual births of which we are obliged to take account in order to determine the civil state of the infants. But this way supposes that we know very nearly the ratio of the population to the annual births, a ratio that we obtain by making at many points of the empire, the exact denumeration of the inhabitants, and by comparing it to the corresponding births observed during some consecutive years: we conclude from it next, by a simple proportion, the population of all the empire. The government has well wished, at my prayer, to give orders to have with precision, these data. In thirty departments distributed over the area of France, in a manner to outweigh the effects of the variety of climates, we have made a choice of the townships of which the mayors, by their zeal and their intelligence, would be able to furnish the most precise information. The exact denumeration of the inhabitants of these townships, for 22 September 1802, is totaled to 2037615 individuals. The summary of the births, of the marriages and of the deaths, from 22 September 1799 to 22 September 1802, has given for these three years,

Births	*Marriages*	*Deaths*
110312 boys,	46037,	103659 males,
105287 girls,		99443 females.

The ratio of the births of boys to those of girls, that this summary presents, is the one of 22 to 21; and the marriages are to the births, as 3 to 14; the ratio of the population to the annual births is 28,352845. In supposing therefore the number of annual births in France, equal to one million, that which deviates little from the truth; we will have, by multiplying by the preceding ratio, this last number, the population of France equal to 28352845 individuals. Let us see the error that we are able to fear in this evaluation.

[392] For this, let us imagine an urn which contains an infinity of white and black balls in an unknown ratio. Let us suppose next that having drawn at random a great number p of these balls, q have been white, and that in a second drawing, out of an unknown number of extracted balls, there are q' of them white. In order to deduce from it this unknown number, we suppose its ratio to q', the same as the one of p to q; that which gives $\frac{pq'}{q}$ for this number. Let us seek the probability that the number of balls extracted in the second drawing, is comprehended within the limits $\frac{pq'}{q} \pm z$. Let us name x the unknown ratio of the number of white balls, to the total number of balls in the urn. The probability of the observed event in the first drawing, will be expressed by the term which has for factor $x^q(1-x)^{p-q}$ in the development of the binomial $(x+\overline{1-x})^p$, whence it is easy to conclude, as in the preceding section, that the probability of x is

$$\frac{x^q dx\,(1-x)^{p-q}}{\int x^q dx\,(1-x)^{p-q}};$$

the integral of the denominator being taken from $x = 0$ to $x = 1$. Let us imagine now that in the second drawing, the total number of balls extracted is $\frac{pq'}{q} + z$; the probability of the observed number q' of white balls, will be the term of the binomial $(x+\overline{1-x})^{\frac{pq'}{q}+z}$, which has for factor $x^{q'}(1-x)^{\frac{pq'}{q}+z-q'}$; this probability is therefore

$$\frac{1.2.3\ldots\left(\frac{pq'}{q}+z\right)}{1.2.3\ldots q'.1.2.3\ldots\left(\frac{pq'}{q}+z-q'\right)} x^{q'}(1-x)^{\frac{pq'}{q}+z-q'}.$$

By multiplying it by the preceding probability of x, by integrating the product from $x = 0$ to $x = 1$, and by dividing it by this same product multiplied by dz, and integrated for all the positive and negative values of z, we will have the probability
[393] that the total number of balls extracted, is $\frac{pq'}{q} + z$. We will find thus, by the analysis of the preceding section, this probability equal to

$$\sqrt{\frac{q^3}{2pq'(p-q)(q+q')\pi}}\; c^{-\frac{q^3z^2}{2pq'(p-q)(q+q')}};$$

by naming therefore P the probability that the number of balls extracted in the second drawing, is comprehended within the limits $\frac{pq'}{q} \pm z$, we will have

$$P = 1 - 2\int dz\sqrt{\frac{q^3}{2pq'(p-q)(q+q')\pi}}\; c^{-\frac{q^3z^2}{2pq'(p-q)(q+q')}},$$

the integral being taken from $z = z$ to z infinity.

Now, the number p of balls extracted in the first drawing, can represent a denumeration; and the number q of white balls which are comprehended, can express the number of women who, in this denumeration, must become mothers in the year, or the number of annual births, corresponding to the denumeration. Then q' expresses the number of annual births observed in all the empire, and whence we conclude the population $\frac{pq'}{q}$. In this case, the preceding value of P expresses the probability that this population is comprehended within the limits $\frac{pq'}{q} \pm z$.

We will suppose, conformably to the preceding data,

$$p = 2037615, \qquad q = \frac{110313 + 105287}{3};$$

we will suppose next

$$q' = 1500000, \qquad z = 500000;$$

the preceding formula gives then

$$P = 1 - \frac{1}{1162}.$$

There is odds therefore around 1161 against one, that in fixing at 42529267, the [394]
population corresponding to fifteen hundred thousand births, we will not be deceived by a half-million.

The difference between certitude and the probability P diminishes with a very great rapidity, when z increases: it would be insensible, if we suppose $z = 700000$.

§32. Let us consider now the probability of future events, deduced from observed events; and let us suppose that having observed an event composed of any number of simple events, we seek the probability of a future result, composed of similar events.

Let us name x the probability of each simple event, y the corresponding probability of the observed result, and z the one of the future result; the probability of x will be, as we have seen,

$$\frac{y\,dx}{\int y\,dx},$$

the integral being taken from $x = 0$ to $x = 1$; $\frac{yz\,dx}{\int y\,dx}$ is therefore the probability of the future result, taken from the value of x, considered as cause of the simple event; thus, by naming P the entire probability of the future event, we will have

$$P = \frac{\int yz\,dx}{\int y\,dx},$$

the integrals of the numerator and of the denominator being taken from $x = 0$ to $x = 1$.

Let us suppose, for example, that an event having arrived m times consecutively, we demand the probability that it will arrive the following n times. In this case, x being supposed to represent the possibility of the simple event, x^m will be that of the observed event, and x^n that of the future event; that which gives

$$y = x^m, \quad z = x^n;$$

whence we deduce

$$P = \frac{m+1}{m+n+1}.$$

Let us suppose the observed event, composed of a very great number of simple events; [395]
let a be the value of x which renders y a *maximum*, and Y that *maximum*; let a' be

the value of x which renders yz a *maximum*, and Y' and Z' that which y and z become at this *maximum*. We will have by §27 of the first Book, very nearly

$$\int y\,dx = \frac{Y^{\frac{3}{2}}\sqrt{2\pi}}{\sqrt{-\frac{ddY}{dx^2}}},$$

$$\int yz\,dx = \frac{(Y'Z')^{\frac{3}{2}}\sqrt{2\pi}}{\sqrt{-\frac{dd(Y'Z')}{dx^2}}}.$$

The observed result being composed of a very great number of simple events, let us suppose that the future event is much less composite. The equation which gives the value of a' of x, corresponding to the *maximum* of yz, is

$$0 = \frac{dy}{y\,dx} + \frac{dz}{z\,dx}.$$

$\frac{dy}{y\,dx}$ is a very great quantity, of order $\frac{1}{\alpha}$; and since the future result is much less composite with respect to the observed result, $\frac{dz}{z\,dx}$ will be of a lesser order, which we will designate by $\frac{1}{\alpha^{1-\lambda}}$; thus a being the value of x which satisfies the equation $0 = \frac{dy}{y\,dx}$; the difference between a and a' will be very small of order α^λ, and we will be able to suppose

$$a' = a + \alpha^\lambda \mu.$$

This supposition gives

$$Y' = Y + \alpha^\lambda \mu \cdot \frac{dY}{dx} + \frac{\alpha^{2\lambda}\mu^2}{1.2} \cdot \frac{d^2Y}{dx^2} + \text{etc.}$$

But we have $\frac{dY}{dx} = 0$, and it is easy to conclude from it that $\frac{d^nY}{Y\,dx^n}$ is of an order equal
[396] or less than $\frac{1}{\alpha^{\frac{n}{2}}}$; the term $\frac{\alpha^{\lambda n}\mu^n}{1.2.3\ldots n} \cdot \frac{d^nY}{Y\,dx^n}$ will be consequently more than order $\alpha^{n\left(\lambda - \frac{1}{2}\right)}$. Thus the convergence of the expression of Y' in series, requires that λ surpass $\frac{1}{2}$; and in this case, Y' differs from Y, only by quantities of order $\alpha^{2\lambda-1}$.

If we name Z that which z becomes when we make $x = a$; we will be assured in the same manner that Z' can be reduced to Z. Finally, we will prove by a similar reasoning, that $\frac{d^2(Y'Z')}{dx^2}$ is reduced to very nearly $Z\frac{d^2Y}{dx^2}$. By substituting these values into the expression of P, we will have

$$P = Z;$$

that is that we can then determine the probability of the future result, by supposing x equal to the value which renders the observed result most probable. But if it is necessary for that that the future result rather be not very composite, so that the exponents of the factors of z are of an order of magnitude smaller than the square root of the factors of y; otherwise, the preceding supposition would expose some sensible errors.

If the future result is a function of the observed result, z will be a function of y, which we will represent by $\phi(y)$. The value of x, which renders zy a *maximum* is, in this case, the same which renders y a *maximum*; thus we have $a' = a$; and if we designate $\frac{d\phi(y)}{dy}$ by $\phi'(y)$, the expression of P will become, by observing that $\frac{dY}{dx} = 0$,

$$P = \frac{\phi(Y)}{\sqrt{1 + \frac{Y\phi'(y)}{\phi(Y)}}}.$$

If $\phi(Y) = y^n$, so that the future event is n times the repetition of the observed event, we will have

$$P = \frac{Y^n}{\sqrt{n+1}}.$$

The probability P calculated under the supposition that the possibility of the simple events is equal to that which renders the observed result most probable, is Y^n: we see thus that the small errors which result from this supposition, are accumulated at the [397] rate of the simple events which enter into the future result, and become very sensible when these events are in great number.

§33. Since 1745, an epoch where we have commenced to distinguish at Paris upon the registers, the baptisms of boys from those of girls, we have constantly observed that the number of the first has been superior to the one of the second. Let us determine the probability that this superiority will be maintained during a given time, for example, in the space of a century.

Let p be the observed number of baptisms of boys; q the one of girls; $2n$ the number of annual baptisms; x the probability that the infant who will be born and be baptized will be a boy. By raising $x + (1 - x)$ to the power $2n$, and developing this power, we will have

$$x^{2n} + 2nx^{2n-1}(1-x) + \frac{2n(n-1)}{1.2}x^{2n-2}(1-x)^2 + \text{etc.}$$

The sum of the n first terms of this development, will be the probability that each year, the number of baptisms of boys will surpass the one of the baptisms of girls. Let us name z this sum; z^i will be the probability that this superiority will be maintained during the number i of consecutive years; therefore, if we designate by P the entire probability that this will take place; we will have, by the preceding section,

$$P = \frac{\int x^p dx\, z^i(1-x)^q}{\int x^p dx\,(1-x)^q},$$

the integrals of the numerator and of the denominator being taken from $x = 0$ to $x = 1$.

If we name a the value of x which renders $x^p z^i (1-x)^q$ a *maximum*, and if we designate by Z, $\frac{dZ}{dx}$, $\frac{d^2Z}{dx^2}$ that which z, $\frac{dz}{dx}$, $\frac{d^2z}{dx^2}$ become, when we change x into a; we will have, by §26,

$$\int x^p dx\, z^i(1-x)^q = \frac{a^{p+1}(1-a)^{q+1}Z^i\sqrt{2\pi}}{\sqrt{p(1-a)^2 + qa^2 + ia^2(1-a)^2\left(\frac{dZ^2 - ZddZ}{Z^2dx^2}\right)}}.$$

[398] z being the sum of the first n terms of the function

$$x^{2n}\left[1+2n\frac{(1-x)}{x}+\frac{2n(2n-1)}{1.2}\cdot\frac{(1-x)^2}{x^2}+\text{etc.}\right],$$

we have by §37 of the first Book,

$$z=\frac{\int\frac{u^{n-1}du}{(1+u)^{2n+1}}}{\int\frac{u^{n-1}du}{(1+u)^{2n+1}}},$$

the integral of the numerator being taken from $u=\frac{1-x}{x}$ to $u=\infty$, and that of the denominator being taken from $u=0$ to $u=\infty$. Let there be $u=\frac{1-s}{s}$; this value of z will become

$$z=\frac{\int s^n ds\,(1-s)^{n-1}}{\int s^n ds\,(1-s)^{n-1}},$$

the integral of the numerator being taken from $s=0$ to $s=x$, and that of the denominator taken from $s=0$ to $s=1$. Thence we deduce

$$\frac{dz}{z\,dx}=\frac{x^n(1-x)^{n-1}}{\int s^n ds\,(1-s)^{n-1}},$$

the integral of the denominator being taken from $x=0$ to $s=x$. We will have next

$$\frac{ddz}{z\,dx^2}=\frac{dz}{z\,dx}=\frac{n-(2n-1)x}{x(1-x)}.$$

By changing x into a in these expressions, we will have those of Z, $\frac{dZ}{Z\,dx}$, $\frac{ddZ}{Z\,dx^2}$.

In order to determine a, we will observe that the condition of the *maximum* of $x^p z^i(1-x)^q$ gives

$$0=\frac{p}{a}-\frac{q}{1-a}+i\frac{dZ}{Z\,dx};$$

whence we deduce, by substituting for $\frac{dZ}{Z\,dx}$, its preceding value,

$$a=\frac{p}{p+q}+\frac{ia^{n+1}(1-a)^n}{(p+q)\int s^n ds\,(1-s)^{n-1}},$$

[399] the integral of the denominator being taken from $s=0$ to $s=a$. In order to conclude a from this equation, we will observe that the value of s which renders $s^n(1-s)^{n-1}$ a *maximum*, is very nearly $\frac{1}{2}$, and consequently, less than $\frac{p}{p+q}$ which itself is smaller than a. Thus n being supposed a large number, we can, without sensible error, extend the integral of this expression of a, from $s=0$ to $s=1$, the term which depends on it being very small. This gives, by §28,

$$\int s^n ds\,(1-s)^{n-1}=\frac{n^{n+\frac{1}{2}}(n-1)^{n-\frac{1}{2}}\sqrt{2\pi}}{(2n-1)^{2n+\frac{1}{2}}}=\frac{\sqrt{\pi}}{2^{2n}\sqrt{n}};$$

the equation which determines a becomes thus quite nearly,

$$a=\frac{p}{p+q}+\frac{ia^{n+1}(1-a)^n 2^{2n}\sqrt{n}}{(p+q)\sqrt{\pi}}.$$

In order to resolve it, we will observe that a differs very little from $\frac{p}{p+q}$: so that if we make

$$a = \frac{p}{p+q} + \mu,$$

μ will be quite small, and we will have in a very close manner,

$$\mu = i\sqrt{n}\frac{p\left[1-\left(\frac{p-q}{p+q}\right)^2\right]^n}{(p+q)^2\sqrt{\pi}}c^{-\frac{n\mu(p+q)(p-q)}{pq}-\frac{(p+q)^2n\mu^2}{pq}}; \tag{1}$$

we will have next very nearly,

$$a^p(1-a)^q = \left(\frac{p}{p+q}\right)^p\left(\frac{q}{p+q}\right)^q c^{-\frac{(p+q)^3}{2pq}\mu^2}.$$

By substituting into the radical

$$\sqrt{p(1-a)^2 + qa^2 + ia^2(1-a)^2\left(\frac{dZ^2 - ZddZ}{Z^2dx^2}\right)},$$

for a, its value $\frac{p}{p+q}+\mu$; for $\frac{dZ}{Z\,dx}$, its value $\frac{(p+q)a-p}{ia(1-a)}$ or $\frac{(p+q)\mu}{ia(1-a)}$; and for $\frac{ddZ}{Z\,dx^2}$, its value [400]
$\frac{dZ}{Z\,dx}\ \frac{n-(2n-1)a}{a(1-a)}$; this radical becomes very nearly

$$\sqrt{\frac{pq}{p+q}}\sqrt{1+\frac{(p+q)\mu}{pq}[n(p-q)-p]+\frac{(p+q)^2}{pq}\mu^2\left(2n+\frac{p+q}{i}\right)}.$$

Finally, we have by §28,

$$\int x^p dx\,(1-x)^q = \left(\frac{p}{p+q}\right)^p\left(\frac{q}{p+q}\right)^q\sqrt{\frac{pq}{p+q}}\cdot\frac{\sqrt{2\pi}}{p+q}.$$

This premised, the expression of P will become very nearly,

$$P = \frac{Z^i c^{-\frac{(p+q)^3}{2pq}\mu^2}}{\sqrt{1+\frac{(p+q)\mu}{pq}[n(p-q)-p]+\frac{(p+q)^2\mu^2}{pq}\left(2n+\frac{(p+q)}{i}\right)}}. \tag{2}$$

The concern is therefore no longer but to determine Z. We have

$$Z = \frac{\int s^n ds\,(1-s)^{n-1}}{\int s^n ds\,(1-s)^{n-1}},$$

the integral of the numerator being taken from $s=0$ to $s=a$, and that of the denominator being taken from $s=0$ to $s=1$. It is easy to conclude from it that we have

$$Z = 1 - \frac{\int s^n ds\,(1-s)^{n-1}}{\int s^n ds\,(1-s)^{n-1}},$$

the integral of the numerator being taken from $s = a$ to $s = 1$ and that of the denominator being taken from $s = 0$ to $s = 1$; we will have thus quite nearly, by §29,

$$Z = 1 - \frac{\int dt\, c^{-t^2}}{\sqrt{\pi}}, \tag{3}$$

the integral relative to t being taken from

$$t^2 = \frac{2n-1}{2n(n-1)} \left(\frac{n(p-q)}{p+q} - \frac{p}{p+q} + (2n-1)\mu \right)^2,$$

to $t^2 = \infty$.

[401] In order to apply numbers to these formulas, we will observe that, by that which precedes, in the interval from the commencement of 1745 to the end of 1784, we have by §28, relatively to Paris,

$$p = 393386, \quad q = 377555.$$

By dividing by 40 the sum of these two numbers, we will have 19273,5 for the mean number of annual baptisms; that which gives $n = 9636,75$; we will suppose moreover $i = 100$. By means of these values, we will determine that of μ, by equation (1); we will determine next the value of Z by equation (3); finally equation (2) will give the value of P. We will find thus

$$P = 0,782.$$

There was therefore at the end of 1784, according to these data, odds nearly four against one that in the space of a century, the baptisms of boys at Paris, will surpass, each year, over those of the girls.

CHAPTER 7

On the influence of the unknown inequalities which are able to exist among the chances that we suppose perfectly equal

§34. I have already considered this influence in §1, where we have seen that these inequalities increase the probability of the events composed of the repetition of simple events. I will resume here this important object in the applications of the analysis of probabilities. [402]

There results from the section cited, that if in the game of *heads* and *tails*, there exists an unknown difference between the possibilities to bring forth one or the other; by naming α this difference, so that $\frac{1\pm\alpha}{2}$ is the possibility to bring forth *heads*, and consequently $\frac{1\mp\alpha}{2}$ that to bring forth *tails*, the one of the two signs $+$ and $-$ that we must adopt being unknown; the probability to bring forth *heads* n times consecutively, will be

$$\frac{(1+\alpha)^n+(1-\alpha)^n}{2^{n+1}}$$

or

$$\frac{1}{2^n}\left(1+\frac{n.\overline{n-1}}{1.2}\alpha^2+\frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{1.2.3.4}\alpha^4+\text{etc.}\right). \tag{1}$$

The game of *heads* and *tails* consists, as we know, in casting into the air a very thin coin, which falls again necessarily on one of its two opposite faces that we name *heads* and *tails*. We are able to diminish the value of α, by rendering these two faces the most equal as it is possible. But it is physically impossible to obtain a perfect equality; and then, the one who wagers to bring forth *heads* twice consecutively, or *tails* twice consecutively, has the advantage over the one who wagers that in two trials, *heads* and *tails* will alternate; its probability being $\frac{1+\alpha^2}{2}$.

We are able to diminish the influence of the inequality of the two faces of the coin, by submitting them themselves to the chances of hazard. Let us designate by A this coin, and let us imagine a second coin B similar to the first. Let us suppose that after having projected this second coin, we project the coin A in order to form a first trial, and let us determine the probability that in n consecutive similar trials, the coin A will present the same faces as the coin B. If we name p the probability to bring forth *heads* with the coin A, and q the probability to bring forth *tails*; if we designate next by p' and q' the same probabilities for the coin B, $pp'+qq'$ will be the probability that in one trial, the coin A will present the same faces as the coin B; thus $(pp'+qq')^n$ will be the probability that that will take place constantly in n [403]

trials. Let

$$p = \frac{1+\alpha}{2}, \qquad q = \frac{1-\alpha}{2},$$
$$p' = \frac{1+\alpha'}{2}, \qquad q' = \frac{1-\alpha'}{2};$$

we will have

$$(pp' + qq')^n = \frac{1}{2^n}(1+\alpha\alpha')^n.$$

But as we are ignorant of what the faces are that the inequalities α and α' favor, the preceding probability is able to be equally either $\frac{1}{2^n}(1+\alpha\alpha')^n$, or $\frac{1}{2^n}(1-\alpha\alpha')^n$, according as α or α' are of like sign or of contrary signs; the true value of this probability is therefore, α and α' being supposed positives,

$$\frac{1}{2^{n+1}}\left[(1+\alpha\alpha')^n + (1-\alpha\alpha')^n\right]$$

or

$$\frac{1}{2^n}\left(1 + \frac{n.\overline{n-1}}{1.2}\alpha^2\alpha'^2 + \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{1.2.3.4}\alpha^4\alpha'^4 + \text{etc.}\right).$$

If we compare this formula to formula (1), we see that it is approached more than it, to $\frac{1}{2^n}$, or to the probability which would hold, if the faces of the coins were perfectly equal. Thus the inequality of these faces, is thence corrected in great part: it would
[404] even be it in totality, if α' were null, or if the two faces of the coin B were perfectly equal.

p representing the probability of *heads*, with the coin A, and q that of *tails*; the probability to bring forth *heads* an odd number of times in n trials will be

$$\frac{1}{2}\left[(p+q)^n \mp (p-q)^n\right],$$

the $-$ sign holding if n is even, and the $+$ sign holding if n is odd. Making $p = \frac{1+\alpha}{2}$, $q = \frac{1-\alpha}{2}$, the preceding function becomes

$$\frac{1}{2}(1 \mp \alpha^n).$$

If n is odd and equal to $2i+1$, this function is

$$\frac{1}{2}(1+\alpha^{2i+1});$$

but as we are able to suppose equally α positive or negative, it is necessary to take the half of the sum of its two values relative to these suppositions; that which gives $\frac{1}{2}$ for its true value; the inequality of the faces of the coin changes therefore not at all the probability $\frac{1}{2}$ to bring forth *heads* an odd number of times. But if n is even and equal to $2i$, this probability becomes

$$\frac{1}{2}(1-\alpha^{2i}), \tag{2}$$

$\pm\alpha$ being the unknown inequality of the probability between *heads* and *tails*; there is therefore disadvantage to wager to bring forth *heads* or *tails* an odd number of times in $2i$ trials, and consequently, there is advantage to wager to bring forth one or the other, an even number of times.

We are able to diminish this advantage, by changing the wager to bring forth *heads* an odd number of times in $2i$ trials, into the wager to bring forth in the same number of trials, an odd number of resemblances between the faces of the two coins A and B, projected as we have said above. In fact, the probability of a resemblance at each trial is, as we have seen, $pp' + qq'$, and the probability of a dissemblance is $pq' + p'q$. Let us name P the first of these two quantities, and Q the second; the probability to bring forth an odd number of resemblances in $2i$ trials, will be [405]

$$\frac{1}{2}\left[(P+Q)^{2i} - (P-Q)^{2i}\right].$$

If we make, as previously,

$$p = \frac{1+\alpha}{2}, \quad q = \frac{1-\alpha}{2}, \quad p' = \frac{1+\alpha'}{2}, \quad q' = \frac{1-\alpha'}{2};$$

we will have

$$P = \frac{1+\alpha\alpha'}{2}, \quad Q = \frac{1-\alpha\alpha'}{2};$$

the preceding function becomes thus,

$$\frac{1}{2}(1 - \alpha^{2i}\alpha'^{2i}).$$

This function remains the same, whatever change that we make in the signs of α and of α'; it is therefore the true probability to bring forth an odd number of resemblances; but α and α' being small fractions, we see that it is nearer $\frac{1}{2}$, more than formula (2); the disadvantage of an odd number is therefore thence diminished.

We see by that which precedes, that we are able to diminish the influence of the unknown inequalities among the chances that we suppose equals, by submitting them themselves to chance. For example, if we put into an urn the tickets 1, 2, 3, $\dots n$ following this order, and if next after having agitated the urn in order to mix well the tickets, we draw one from it; if there is among the probabilities to exit of the tickets a small difference depending on the order according to which they have been placed in the urn; we will diminish it considerably, by putting into a second urn, these tickets, according to their order of exit from the first urn, and by agitating next this second urn, in order to well mix the tickets. Then the order according to which we have placed the tickets in the first urn, will have extremely little influence on the extraction of the first ticket which will exit from the second urn. We would diminish further this influence, by considering in the same manner a third urn, a fourth, etc.

Let us consider two players A and B playing together, in a manner that at each trial, the one who loses, gives a token to his adversary, and that the game endures until one of them has won all the tokens of the other. Let p and q be their respective [406]
skills; a and b their numbers of tokens at commencement. There results from formula

(H) of §10, by making i infinity, that the probability of A, to win the game, is

$$\frac{p^b(p^a - q^a)}{p^{a+b} - q^{a+b}}.$$

If we make in this expression,

$$p = \frac{1 \pm \alpha}{2}, \quad q = \frac{1 \mp \alpha}{2},$$

we will have, by taking the superior sign, the probability relative to the case where A is stronger than B; and by taking the inferior sign, we will have the probability relative to the case where A is less strong than B. If we are ignorant of who is the strongest of the players, the half-sum of these two probabilities will be the probability of A, that we find thus equal to

$$\frac{\frac{1}{2}\left[(1+\alpha)^a - (1-\alpha)^a\right]\left[(1+\alpha)^b + (1-\alpha)^b\right]}{(1+\alpha)^{a+b} - (1-\alpha)^{a+b}}; \tag{3}$$

by changing a into b, and reciprocally, we will have the probability of B. If we suppose α infinitely small or null; these probabilities become $\frac{a}{a+b}$ and $\frac{b}{a+b}$; they are therefore proportionals to the numbers of tokens of the players; thus for equality of the game, their stakes must be in this ratio. But then the inequality which is able to exist between them, is favorable to the player who has the smallest number of tokens; because if we suppose a less than b, it is easy to see that expression (3) is greater than $\frac{a}{a+b}$. If the players agree to double, to triple, etc. their tokens; the advantage of A increases without ceasing, and in the case of a and b infinite, its probability becomes $\frac{1}{2}$ or the same as that of B.

P being the probability of an event composed of two simple events of which p and $1-p$ are the respective probabilities; if we suppose that the value of p is susceptible of an unknown inequality z which is able to be extended from $-\alpha$ to $+\alpha$; by naming
[407] ϕ the probability of $p+z$, ϕ being a function of z; we will have for the true probability of the composite event,

$$\frac{\int P'\phi\, dz}{\int \phi\, dz},$$

P' being that which P becomes when we change p into $p+z$, and the integrals being taken from $z = -\alpha$ to $z = \alpha$.

If we have no other data in order to determine z, but one observed event, formed from the same simple events; by naming Q the probability of this event, $p+z$ and $1-p-z$ being the probabilities of the simple events; the preceding expression gives, by changing ϕ into Q, for the probability of the composite event,

$$\frac{\int P'Q\, dz}{\int Q\, dz},$$

the integrals being taken here from $z = -p$ to $z = 1-p$; that which is conformed to that which we have found in the preceding chapter.

CHAPTER 8

On the mean duration of life, of marriages and of any associations whatsoever

§35. Let us suppose that we have followed with respect to a very great number n of infants, the law of mortality, from their birth to their total extinction; we will have their mean life, by making a sum of the durations of all their lives, and by dividing it by the number n. If this number were infinite, we would have exactly the duration of mean life. Let us seek the probability that the mean life of n infants, will deviate from this one, only within some given limits. [408]

Let us designate by $\phi\left(\frac{x}{a}\right)$, the probability to die at age x, a being the limit of x; a and x being supposed to contain an infinite number of parts taken for unity. We will consider the power

$$\left\{\begin{matrix}\phi\left(\dfrac{0}{a}\right)+\phi\left(\dfrac{1}{a}\right)c^{-\varpi\sqrt{-1}}+\phi\left(\dfrac{2}{a}\right)c^{-2\varpi\sqrt{-1}}\cdots+\phi\left(\dfrac{x}{a}\right)c^{-x\varpi\sqrt{-1}}\\ \cdots+\phi\left(\dfrac{a}{a}\right)c^{-a\varpi\sqrt{-1}}\end{matrix}\right\}^n,$$

it is clear that the coefficient of $c^{-(l+n\mu)\varpi\sqrt{-1}}$, in the development of this power, is the probability that the sum of the ages to which the n infants will arrive, will be $l+n\mu$; by multiplying therefore by $c^{(l+n\mu)\varpi\sqrt{-1}}$ the preceding power, the term independent of the powers of $c^{\pm\varpi\sqrt{-1}}$ in the product, will be this probability which consequently, is equal to

$$\frac{1}{2\pi}\int d\varpi c^{l\varpi\sqrt{-1}}\left\{c^{\mu\varpi\sqrt{-1}}\left[\begin{matrix}\phi\left(\dfrac{0}{a}\right)+\phi\left(\dfrac{1}{a}\right)c^{-\varpi\sqrt{-1}}\cdots+\phi\left(\dfrac{x}{a}\right)c^{-x\varpi\sqrt{-1}}\\ \cdots+\phi\left(\dfrac{a}{a}\right)c^{-a\varpi\sqrt{-1}}\end{matrix}\right]\right\}^n, \quad (1)$$

the integral being taken from $\varpi=-\pi$ to $\varpi=\pi$.

If we take in this integral, the hyperbolic logarithm of the quantity under the $\int$ sign, raised to the power n, we will have, by developing the exponentials into series, this logarithm equal to [409]

$$n\mu\varpi\sqrt{-1}+n\log\left\{\int\phi\left(\frac{x}{a}\right)-\varpi\sqrt{-1}\int x\phi\left(\frac{x}{a}\right)-\frac{\varpi^2}{2}\int x^2\phi\left(\frac{x}{a}\right)+\text{etc.}\right\}; \quad (2)$$

the $\int$ sign referring here to all the values of x, from $x=0$ to $x=a$. If we make $\frac{x}{a}=x'$, and if we observe that the variation of x being unity, we have $adx'=1$; we will have

$$\int \phi\left(\frac{x}{a}\right) = a \int dx'\phi(x'),$$
$$\int x\phi\left(\frac{x}{a}\right) = a^2 \int x'dx'\phi(x'),$$
$$\int x^2\phi\left(\frac{x}{a}\right) = a^3 \int x'^2dx'\phi(x'),$$
$$\text{etc.},$$

the integrals relative to x' being taken from $x' = 0$ to $x' = 1$. Let us name k, k', k'', etc. these successive integrals; the probability that the duration of life of an infant, will be comprehended within the limits zero and a, is $\int \phi\left(\frac{x}{a}\right)$ or $a \int dx'\phi(x')$; now this probability is certitude itself; we have therefore $ak = 1$. This premised, the function (2) becomes

$$n\mu\varpi\sqrt{-1} + n\log\left(1 - \frac{k'}{k}a\varpi\sqrt{-1} - \frac{k''}{k}\cdot\frac{a^2\varpi^2}{2} + \text{etc.}\right)$$

or

$$\left(\frac{n\mu}{a} - \frac{nk'}{k}\right)a\varpi\sqrt{-1} - n\frac{(kk'' - k'^2)}{2k^2}\cdot a^2\varpi^2 - \text{etc.}$$

If we make

$$\mu = \frac{ak'}{k} = \frac{a^2k'}{ak} = a^2k',$$

the first power of ϖ disappears; and moreover, n being supposed a very great number, we can be arrested at the second power of ϖ; the function (1) becomes thus, by passing again from the logarithms to the numbers,

$$\frac{1}{2\pi}\int d\varpi\, c^{l\varpi\sqrt{-1} - n\frac{(kk''-k'^2)}{2k^2}\cdot a^2\varpi^2}.$$

[410] If we make

$$\beta^2 = \frac{k^2}{2(kk'' - k'^2)}, \quad t = \frac{a\varpi\sqrt{n}}{2\beta} - \frac{\beta l\sqrt{-1}}{a\sqrt{n}},$$

this integral becomes, by taking it from $t = -\infty$ to $t = \infty$,

$$\frac{\beta}{a\sqrt{n\pi}}c^{-\frac{\beta^2l^2}{a^2n}}.$$

By multiplying it by dl, and making $l = ar\sqrt{n}$, we will have

$$\frac{2}{\sqrt{\pi}}\int \beta dr\, c^{-\beta^2r^2}$$

for the probability that the sum of the ages to which the n infants will arrive, will be comprehended within the limits $na^2k' \pm ar\sqrt{n}$.

The quantity a^2k' or $\int x\phi\left(\frac{x}{a}\right)$ is the sum of the products of each age, by the probability of arriving there; it is therefore the true duration of mean life; thus the probability that the sum of the ages to which the n infants cease to live, divided by their number, is comprehended within these limits

True duration of mean life, more or less $\frac{ar}{\sqrt{n}}$, has for expression

$$\frac{2}{\sqrt{\pi}}\int \beta dr\, c^{-\beta^2 r^2}.$$

The mean value of r, positive or negative, is by §20,

$$\pm\frac{1}{\sqrt{\pi}}\int \beta r\, dr\, c^{-\beta^2 r^2},$$

the integral being taken from $r = 0$ to r infinity. By multiplying it by $\frac{a}{\sqrt{n}}$, we will have the mean error to fear positive or negative, when we take for mean duration of life, the sum of the ages that the n infants considered above have lived, divided by n, a quotient that we will designate by G; this error is therefore [411]

$$\pm\frac{a}{2\beta\sqrt{n\pi}}.$$

We have very nearly,

$$a^2 k' = G;$$

and as $ak = 1$, we will have

$$\frac{k'}{k} = \frac{G}{a}.$$

If we name next H, the sum of the squares of the ages that the n infants have lived, divided by n; we will find, by the analysis of §19,

$$\frac{k''}{k}a^2 = H;$$

these values give

$$\beta^2 = \frac{a^2}{2(H - G^2)}:$$

the mean error to fear positive or negative, with respect to the duration of life, becomes thus

$$\pm\frac{\sqrt{H - G^2}}{\sqrt{2n\pi}}.$$

It is clear that these results hold equally relatively to the mean duration of that which remains to live, when instead of departing from the epoch of birth, we depart from any epoch of life.

We are able to determine easily, by means of the tables of mortality, formed from year to year, the mean duration of that which remains to live to a person of whom the age is an entire number of years A. For that, we will add all the numbers of the table which follow the one which corresponds to age A; we will divide the sum by this last number, and we will add $\frac{1}{2}$ to the quotient. In fact, if we designate by (1), (2), (3), etc., the numbers of the table, corresponding to the year A and to the following years; the number of individuals who die in the first year, in departing from year A, will be $(1)-(2)$; but in this short interval, the mortality is able to be supposed constant; $\frac{1}{2}[(1) - (2)]$ is therefore the sum of the durations of their life, in departing from age A. Similarly $\frac{3}{2}[(2) - (3)]$, $\frac{5}{2}[(3) - (4)]$, etc. are the sums of the durations

[412] of life, by departing from the same age, of those who die in the second, third, etc. years counted from year A. The reunion of all these sums is $\frac{(1)}{2}+(2)+(3)+(4)+$etc.; and by dividing it by (1), we will have the mean duration of that which remains to live to the person of age A. We will form thus a table of the mean durations of that which remains to live at the different ages. We will be able likewise to conclude these durations from one another, by observing that if F designates this duration for age A, and F' the corresponding duration at age $A+1$, we have

$$F = \frac{(2)}{(1)}\left(F' + \frac{1}{2}\right) + \frac{1}{2}.$$

§36. Let us determine now the mean duration of life, which would hold, if one of the causes of mortality were to be extinguished. Let U be the number of infants who out of the number n of births, would survive yet to the age x under this hypothesis, u being the one of the infants living to this age out of the same number of births, in the case where that cause of mortality subsists. Let us name $z\triangle x$, the probability that one individual of age x, will perish of this malady in the very short interval of time $\triangle x$; $uz\triangle x$ will be very nearly by §25, the number of individuals u, who will perish of this malady in the interval of time $\triangle x$, if this number is considerable. Similarly if we designate by $\phi\triangle x$ the probability that one individual of age x will perish by the other causes of mortality in the interval $\triangle x$; $u\phi\triangle x$ will be the number of individuals who will perish by these causes, in the interval of time $\triangle x$; this will be therefore the value of $-\triangle u$; I affect $\triangle u$ with the $-$ sign, because u diminishes in measure as x increases; we have therefore

$$-\triangle u = u\triangle x(\phi + z).$$

We will have similarly

$$-\triangle U = U\phi\triangle x.$$

By eliminating ϕ from these two equations, we will have

$$\frac{\triangle U}{U} = \frac{\triangle u}{u} + z\triangle x.$$

[413] $\triangle x$ being a very small quantity, we can transform the characteristic $\triangle$ into the differential characteristic d, and then the preceding equation becomes

$$\frac{dU}{U} = \frac{du}{u} + z\,dx;$$

whence we deduce by integrating, and observing that at age zero $U = u = n$,

$$U = uc^{\int z\,dx}, \tag{3}$$

the integral being taken from x null. We can obtain this integral, by means of the registers of mortality, in which we take account of the age of the dead individuals, and of the causes of their death. In fact, $uz\triangle x$ being by that which precedes, the number of those who, arrived to age x, have perished in the interval of time $\triangle x$, by the malady of which there is concern; we will have very nearly the integral $\int z\,dx$, by supposing $\triangle x$ equal to one year, and by taking from the birth of the n infants that we have considered, until the year x, the sum of the fractions which have for numerator the number of individuals who the malady has made perish each year, and for denominator, the number of the n infants who survive yet to the middle of the same year. Thus we will be able to transform by means of equation (3), a table of ordinary mortality, into that which would hold, if the malady of which there is concern, did not exist.

Smallpox has this in particular, namely, that the same individual is never twice attacked, or at least this case is so rare, that, if it exists, we are able to set it aside. Let us imagine that out of a very great number n of infants, u arrive to age x, and that in the number u, y have not had smallpox at all. Let us imagine further that out of this number y, $iy\,dx$ take this malady in the instant dx, and that out of this number, $iry\,dx$ perish from this malady. By designating, as above, by ϕ the probability to perish at age x, by some other causes; we will have evidently

$$du = -u\phi\,dx - iry\,dx.$$

We will have next

$$dy = -y\phi\,dx - iy\,dx.$$

In fact, y diminishes by the number of those who, in the instant dx, take smallpox, and this number is by the supposition, $iy\,dx$. y diminishes further by the number of [414]
individuals comprehended in y, who perish by some other causes, and this number is $y\phi\,dx$.

Now, if from the first of the two preceding equations, multiplied by y, we subtract the second multiplied by u, and if we divide the difference by y^2, we will have

$$d\frac{u}{y} = i\frac{u}{y}\,dx - ir\,dx;$$

that which gives, by integrating from x null, and observing that at this origin, $u = y = n$,

$$\frac{u}{y} = \left(1 - \int ir\,dx\,c^{-\int i\,dx}\right) c^{\int i\,dx}; \tag{4}$$

this equation will make known the number of individuals of age x, who have not at all yet had smallpox. We have next

$$z\,dx = \frac{iry\,dx}{u},$$

$nz\,dx$ being, as above, those who perish in the time dx, of the malady that we consider. By substituting instead of $\frac{y}{n}$, its preceding value; we will have, after having integrated,

$$c^{\int z\,dx} = \frac{1}{1 - \int ir\,dx\,c^{-\int i\,dx}};$$

equation (3) will give therefore

$$U = \frac{u}{1 - \int ir\,dx\,c^{-\int i\,dx}}. \tag{5}$$

This value of U supposes that we knew by observation i and r. If these numbers were constants, it would be easy to determine them; but as they are able to vary from

age to age, the elements of formula (3) are easier to know, and this formula seems to me more proper to determine the law of mortality which would hold, if smallpox was extinct. By applying to it the data that we have been able to procure with
[415] respect to the mortality caused by this malady, at the diverse ages of life; we find that its extinction by means of the vaccine, would increase more than three years, the duration of mean life, if besides this duration was not at all restrained by the diminution related to the subsistances, due to a greater increase of population.

§37. Let us consider presently the mean duration of marriages. For that let us imagine a great number n of marriages among n young men of age a, and n young women of age a'; and let us determine the number of these marriages subsisting after x years elapsed from their origin. Let us name ϕ the probability that a young man who is married at age a, will arrive to age $a+x$; and ψ the probability that a young woman who is married at age a', will arrive to age $a'+x$. The probability that their marriage will subsist after its x^{th} year, will be $\phi\psi$; therefore if we develop the binomial $(\phi\psi + \overline{1-\phi\psi})^n$, the term $H(\phi\psi)^i(1-\phi\psi)^{n-i}$ of this development, will express the probability that out of n marriages, i will subsist after x years. The greatest term of the development is, by §16, the one in which i is equal to the greatest whole number contained in $\overline{n+1}.\phi\psi$; and, by the same section, it is extremely probable that the number of the marriages subsisting will deviate only very little positive or negative from this number. Thus, by designating by i, the number of subsisting marriages, we will be able to suppose very nearly,

$$i = n\phi\psi.$$

$n\phi$ is quite near the number of the n husbands surviving to the age $a+x$. The tables of mortality will make it known in a quite close manner, if they have been formed out of the numerous lists of mortality; because if we designate by p' the number of men surviving to age a, out of the collection of these lists, and by q' the number of the surviving to age $a+x$, we will have quite nearly, by §29,

$$n\phi = \frac{nq'}{p'}.$$

If we name similarly p'' the number of women surviving to age a' and by q'' the number
[416] of the survivors to age $a'+x$, we will have very nearly,

$$n\psi = \frac{nq''}{p''};$$

therefore

$$i = \frac{nq'q''}{p'p''}.$$

We will form thus from year to year, a table of values of i. By making next a sum of all the numbers of this table, and by dividing it by n; we will have the mean duration of the marriages made at age a for the young men, and at the age a' for the young women.

Let us seek now the probability that the error of the preceding value of i, will be comprehended within some given limits. Let us suppose in order to simplify the calculation, that the two spouses are of the same age, and that the probability of the life of the men is the same as that of the women; then we have

$$a' = a, \quad q'' = q', \quad p'' = p', \quad \phi = \psi;$$

and the preceding expression of i becomes

$$i = \frac{nq'^2}{p'^2}.$$

Let us imagine that the value of i is $\frac{nq'^2}{p'^2} + s$; s will be the error of this expression of i. We have seen in §30, that if we have observed that out of a very great number p of individuals of age a, q are arrived to the age $a + x$; the probability that out of p' other individuals of the age a, $\frac{p'q}{p} + z$ will arrive to the age $a + x$, is

$$\sqrt{\frac{p^3}{2qp'(p-q)(p+p')\pi}}\, c^{-\frac{p^3z^2}{2qp'(p-q)(p+p')}}.$$

If we suppose p and q infinite, we will have evidently

$$\phi = \frac{q}{p},$$

and if we make

$$\frac{p'q}{p} + z = q';$$

we will have [417]

$$\phi = \frac{q'}{p'} - \frac{z}{p'};$$

that which gives very nearly, by neglecting the square $\frac{nz^2}{p'^2}$,

$$n\phi^2 = \frac{nq'^2}{p'^2} - \frac{2nq'z}{p'^2};$$

thus the preceding probability of z, is at the same time the probability of this expression of $n\phi^2$. Let us suppose now $i = n\phi^2 + l$; by considering the binomial $(\phi^2 + \overline{1-\phi^2})^n$, the probability of this expression of i is by §16,

$$\frac{1}{\sqrt{\pi.2n\phi^2(1-\phi^2)}}\, c^{-\frac{l^2}{2n\phi^2(1-\phi^2)}}.$$

But the preceding value of i becomes, by substituting for $n\phi^2$ its value,

$$i = \frac{nq'^2}{p'^2} - \frac{2nq'z}{p'^2} + l;$$

the probability of this last expression of i is equal to the product of those of i and of z, found above; it is therefore equal to

$$\frac{c^{-\frac{z^2}{2p'\phi(1-\phi)}-\frac{l^2}{2n\phi^2(1-\phi^2)}}}{2\pi\sqrt{np'\phi^3(1-\phi)^2(1+\phi)}}.$$

Having supposed previously $i = \frac{nq'^2}{p'^2} + s$, we will have $s = l - \frac{2nq'z}{p'^2}$; by substituting therefore for l its value deduced from this equation, and observing that we have very nearly $\frac{q'}{p'} = \phi$; we will have for the probability that the value of s will be comprehended within some given limits, the integral expression

$$\frac{\iint dz\, ds\, c^{-\frac{z^2}{2\phi(1-\phi)p'}-\frac{\left(s+\frac{2nq'z}{p'^2}\right)^2}{2n\phi^2(1-\phi^2)}}}{2\pi\sqrt{np'\phi^3(1-\phi)^2(1+\phi)}},$$

[418] the integral relative to z being able to be taken from $z = -\infty$ to $z = \infty$. Thence it is easy to conclude by the methods exposed previously, that if we make

$$k^2 = \frac{p'}{2n\phi^2(1-\phi)[p' + (p' + 4n)\phi]};$$

the preceding integral becomes

$$\int \frac{k\, ds}{\sqrt{\pi}} c^{-k^2s^2};$$

thus the probability that the error of the expression $i = \frac{nq'^2}{p'^2}$ will be $\pm s$, is

$$\frac{2}{\sqrt{\pi}} \int k\, ds\, c^{-k^2s^2},$$

the integral being taken from s null.

The preceding analysis is applied equally to the mean duration of a great number of associations formed of three individuals, or of four individuals, etc. Let n be this number, and let us suppose that all the associates are of the same age a at the moment of association; let us designate by p the number of individuals from the table of mortality, of the age a, and by q the number of individuals of the age $a + x$; the number i of the associations existing after x years elapsed from the origin of the associations, will be quite nearly

$$i = \frac{nq^r}{p^r},$$

r being the number of individuals of each association. We will find by the same analysis, the probability that this number will be contained within the given limits. The sum of the values of i corresponding to all the values of x, divided by n, will be the mean duration of this kind of associations.

CHAPTER 9

On the benefits depending on the probability of future events.

§38. Let us imagine that the arrival of an event procures the benefit ν, and that its non-arrival causes the loss μ. A person A awaits the arrival of a number s of similar events, all equally probable, but independent of one another; we demand what is his advantage. [419]

Let q be the probability of the arrival of each event, and consequently $1-q$ that of its non-arrival; if we develop the binomial $(q+\overline{1-q})^s$, the term

$$\frac{1.2.3\ldots s}{1.2.3\ldots i.1.2.3\ldots(s-i)}q^i(1-q)^{s-i}$$

of this development, will be the probability that out of s events, i will arrive. In this case, the benefit of A is $i\nu$, and his loss is $(s-i)\mu$; the difference is $i(\nu+\mu)-s\mu$; by multiplying it by its probability expressed by the preceding term, and taking the sum of these products for all the values of i, we will have the advantage of A, which, consequently is equal to

$$-s\mu(q+\overline{1-q})^s+(\nu+\mu)S\frac{1.2.3\ldots s}{1.2.3\ldots i.1.2.3\ldots\overline{\overline{s-i}}}q^i(1-q)^{s-i},$$

the sign S extending to all the values of i. We have

$$S\frac{i.1.2.3\ldots s}{1.2.3\ldots i.1.2.3\ldots\overline{\overline{s-i}}}q^i(1-q)^{s-i}$$
$$=\frac{d}{dt}S\frac{1.2.3\ldots s}{1.2.3\ldots i.1.2.3\ldots\overline{\overline{s-i}}}q^it^i(1-q)^{s-i}=\frac{d}{dt}(qt+\overline{1-q})^s,$$

provided that we suppose $t=1$, after the differentiation, that which reduces this last member to qs; the advantage of A is therefore $s(q\nu-\overline{1-q}.\mu)$. This advantage is null, if $\nu q=\mu(1-q)$; that is, if the benefit of the arrival of the event, multiplied by its probability, is equal to the loss caused by its non-arrival, multiplied by its probability. The advantage becomes negative, and is changed into disadvantage, if the second product surpasses the first. In all cases, the advantage or the disadvantage of A is proportional to the number s of the events. [420]

We will determine by the analysis of §16, the probability that the real benefit of A will be comprehended within the given limits, if s is a large number. Following this analysis, the sum of the diverse terms of the binomial $(q+\overline{1-q})^s$ comprehended between the two terms distant by $l+1$, on both sides of the greatest, is

$$\frac{2}{\sqrt{\pi}}\int dt\,c^{-t^2}\frac{1}{\sqrt{2s\pi q(1-q)}}c^{-\frac{l^2}{2sq(1-q)}},$$

the integral being taken from $t = 0$ to $t = \frac{l}{\sqrt{2sq(1-q)}}$. The exponent of q in the greatest term is very nearly by the same section, equal to sq; and the exponents of q, corresponding to the extreme terms comprehended within the preceding interval, are respectively $sq - l$ and $sq + l$. The benefits corresponding to these three terms are

$$\begin{aligned} &s(q\nu - \overline{1-q}.\mu) - l(\nu+\mu),\\ &s(q\nu - \overline{1-q}.\mu),\\ &s(q\nu - \overline{1-q}.\mu) + l(\nu+\mu); \end{aligned}$$

by making therefore $l = r\sqrt{s}$, the probability that the real benefit of A will not exceed the limits $s(q\nu - \overline{1-q}.\mu) \pm r\sqrt{s}(\nu+\mu)$ is equal to

$$\frac{2}{\sqrt{\pi}}\frac{\int dr\, c^{-\frac{r^2}{2q(1-q)}}}{\sqrt{2q(1-q)}} + \frac{1}{\sqrt{2s\pi q(1-q)}}c^{-\frac{r^2}{2q(1-q)}},$$

the integral being taken from $r = 0$, and the last term being able to be neglected. We
[421] see by this formula that if $q\nu - \overline{1-q}.\mu$ is not null, the real benefit increases without ceasing, and becomes infinitely great and certain, in the case of an infinite number of events.

We are able to extend by the following analysis, this result, to the case where the probability of the s events are different, in the same way as the benefits and the losses which are attached. Let q, $q^{(1)}$, $q^{(2)}$, ..., $q^{(s-1)}$ be the respective probabilities of these events; ν, $\nu^{(1)}$, $\nu^{(2)}$,..., $\nu^{(s-1)}$ the benefits which their arrivals procure. We can, in order to simplify, set aside the losses which their non-arrivals cause, by comprehending in the benefit which the arrival of each event procures, the quantity that A would lose by its non-arrival, and by subtracting next from the total advantage of A, the sum of these last quantities; because it is easy to see that that changes not at all the position of A.

This premised, let us consider the product

$$\left(1-q+qc^{\nu\varpi\sqrt{-1}}\right)\left(1-q^{(1)}+q^{(1)}c^{\nu^{(1)}\varpi\sqrt{-1}}\right)\cdots \\ \cdots\left(1-q^{(s-1)}+q^{(s-1)}c^{\nu^{(s-1)}\varpi\sqrt{-1}}\right).$$

It is clear that the probability that the sum of the benefits will be $f + l'$, is equal to the coefficient of $c^{(f+l')\varpi\sqrt{-1}}$ in the development of this product; it is therefore equal to

$$\frac{1}{2\pi}\int d\varpi c^{-(f+l')\varpi\sqrt{-1}}\left(1-q+qc^{\nu\varpi\sqrt{-1}}\right)\cdots \\ \cdots\left(1-q^{(s-1)}+q^{(s-1)}c^{\nu^{(s-1)}\varpi\sqrt{-1}}\right), \qquad (a)$$

the integral being taken from $\varpi = -\pi$ to $\varpi = \pi$, and the numbers ν, $\nu^{(1)}$, etc. being supposed, as we can make it, whole numbers. Let us take the logarithm of the product

$$c^{-f\varpi\sqrt{-1}}\left(1-q+qc^{\nu\varpi\sqrt{-1}}\right)\cdots\left(1-q^{(s-1)}+q^{(s-1)}c^{\nu^{(s-1)}\varpi\sqrt{-1}}\right); \qquad (b)$$

by developing it according to the powers of ϖ, it becomes

$$(Sq^{(i)}\nu^{(i)} - f)\varpi\sqrt{-1} - \frac{\varpi^2}{2} Sq^{(i)}(1 - q^{(i)})\nu^{(i)2} - \text{etc.}$$

the sign S corresponding to all the values of i, from $i = 0$ to $i = s-1$. The supposition of f equal to $Sq^{(i)}\nu^{(i)}$ makes the first power of ϖ disappear; and the size of s, a very [422] great number, renders insensible the terms depending on the powers of ϖ, superior to the square; by passing again therefore from the logarithms to the numbers in the preceding development, the product (b) becomes very nearly

$$c^{-\frac{\varpi^2}{2} Sq^{(i)}(1-q^{(i)})\nu^{(i)2}};$$

that which changes the integral (a) into this one

$$\frac{1}{2\pi} \int d\varpi\, c^{-l'\varpi\sqrt{-1} - \frac{\varpi^2}{2} Sq^{(i)}(1-q^{(i)})\nu^{(i)2}}.$$

The integral must be taken from $\varpi = -\pi$ to $\varpi = \pi$, and $Sq^{(i)}(1 - q^{(i)})\nu^{(i)2}$ being a great number of order s; it is clear that this integral is able to be extended without sensible error, to the positive and negative infinite values of ϖ. By making therefore

$$\varpi\sqrt{\frac{Sq^{(i)}(1 - q^{(i)})\nu^{(i)2}}{2}} + \frac{l'\sqrt{-1}}{\sqrt{2Sq^{(i)}(1 - q^{(i)})\nu^{(i)2}}} = t,$$

and integrating from $t = -\infty$ to $t = \infty$, the integral (a) becomes

$$\frac{1}{\sqrt{2\pi Sq^{(i)}(1 - q^{(i)})\nu^{(i)2}}} c^{-\frac{l'^2}{2Sq^{(i)}(1-q^{(i)})\nu^{(i)2}}}.$$

If we multiply this quantity by $2dl'$, and if next we integrate it from $l' = 0$, this integral will be the expression of the probability that the benefit of A will be comprehended within the limits $f \pm l'$, or $Sq^{(i)}\nu^{(i)} \pm l'$; by making thus

$$l' = r\sqrt{2Sq^{(i)}(1 - q^{(i)})\nu^{(i)2}},$$

the probability that the benefit of A will be comprehended within the limits

$$Sq^{(i)}\nu^{(i)} \pm r\sqrt{2Sq^{(i)}(1 - q^{(i)})\nu^{(i)2}},$$

is

$$\frac{2}{\sqrt{\pi}} \int dr\, c^{-r^2},$$

the integral being taken from $r = 0$.

Now it is necessary, by that which precedes, to change within the preceding limits, [423] $\nu^{(i)}$ into $\nu^{(i)} + \mu^{(i)}$, and to subtract $S\mu^{(i)}$ from it; the probability that the real benefit of A will be comprehended within the limits

$$S[q^{(i)}\nu^{(i)} - (1 - q^{(i)})\mu^{(i)}] \pm r\sqrt{2Sq^{(i)}(1 - q^{(i)})(\nu^{(i)} + \mu^{(i)})^2}$$

is therefore

$$\frac{2}{\sqrt{\pi}}\int dr\, c^{-r^2}.$$

We see by this formula, that as little as the mathematical expectation of each event, surpasses zero; by multiplying the events to infinity, the first term of the expression of the limits being of order s, while the second is only of order $\sqrt{s}$, the real benefit is increased without ceasing, and becomes at the same time infinitely great and certain, in the case of an infinite number of events.

§39. Let us consider now the case where, at each event, the person A has any number of chances to hope or to fear. Let us suppose, for example, that an urn contains some balls of diverse colors; that we draw a ball from this urn, by replacing it into the urn after the drawing, and that the benefit of A is ν, if the extracted ball is of the first color; that it is ν', if the extracted ball is of the second color, that it is ν'' if the extracted ball is of the third color, and so forth; the benefits becoming negative, when A is forced to give instead of receive. Let us name a, a', a'', etc. the probabilities that the ball extracted at each drawing, will be of the first, or of the second, or of the third, etc. color, and let us suppose that we have thus s drawings; we will have first

$$a + a' + a'' + \text{etc.} = 1.$$

By multiplying next the terms of the first member of this equation, respectively by $c^{\nu\varpi\sqrt{-1}}$, $c^{\nu'\varpi\sqrt{-1}}$, $c^{\nu''\varpi\sqrt{-1}}$, etc., the term independent of the powers of $c^{\varpi\sqrt{-1}}$, in the development of the function

$$c^{-(l+s\mu)\varpi\sqrt{-1}}\left[ac^{\nu\varpi\sqrt{-1}} + a'c^{\nu'\varpi\sqrt{-1}} + a''c^{\nu''\varpi\sqrt{-1}} + \text{etc.}\right]^s,$$

[424] will be, by that which precedes, the probability that in s drawings, the benefit of A will be $s\mu + l$; this probability is therefore equal to

$$\frac{1}{2\pi}\int d\varpi\, c^{-l\varpi\sqrt{-1}}\left[c^{-\mu\varpi\sqrt{-1}}\left(ac^{\nu\varpi\sqrt{-1}} + a'c^{\nu'\varpi\sqrt{-1}} + \text{etc.}\right)\right]^s; \qquad (c)$$

the integral relative to ϖ being taken from $\varpi = -\pi$ to $\varpi = \pi$. If we develop with respect to the powers of ϖ, the hyperbolic logarithm of the quantity raised to the power s, under the $\int$ sign, and if we observe that $a + a' + a'' + \text{etc.} = 1$, we will have for this logarithm,

$$\begin{aligned}&\varpi\sqrt{-1}[a\nu + a'\nu' + a''\nu'' + \text{etc.} - \mu]\\ &-\frac{\varpi^2}{2}\begin{bmatrix} a\nu^2 + a'\nu'^2 + a''\nu''^2 + \text{etc.}\\ -(a\nu + a'\nu' + a''\nu'' + \text{etc.})^2\end{bmatrix} - \text{etc.}\end{aligned}$$

We will make the first power of ϖ disappear, by making

$$\mu = a\nu + a'\nu' + a''\nu'' + \text{etc.};$$

If we suppose next

$$2k^2 = a\nu^2 + a'\nu'^2 + a''\nu''^2 + \text{etc.} - (a\nu + a'\nu' + a''\nu'' + \text{etc.})^2,$$

and if we observe that s being supposed a great number, we can neglect the powers of ϖ superior to the square; we will have, by passing again from the logarithms to the numbers,

$$\left[c^{-\mu\varpi\sqrt{-1}}\left(ac^{\nu\varpi\sqrt{-1}} + a'c^{\nu'\varpi\sqrt{-1}} + a''c^{\nu''\varpi\sqrt{-1}} + \text{etc.}\right)\right]^s = c^{-sk^2\varpi^2};$$

that which changes the integral (c) into this one,

$$\frac{1}{2\pi}\int d\varpi\, c^{-l\varpi\sqrt{-1}-sk^2\varpi^2},$$

which becomes by integrating as in the preceding section,

$$\frac{1}{2k\sqrt{s\pi}}c^{-\frac{l^2}{4sk^2}}.$$

By multiplying it by $2dl$, and integrating the product from $l = 0$, we will have the probability that the real benefit of A, will be comprehended within the limits [425]

$$s(a\nu + a'\nu' + a''\nu'' + \cdots) \pm l;$$

by making therefore

$$l = 2kr'\sqrt{s},$$

this probability will be, by taking the integral from $r' = 0$,

$$\frac{2}{\sqrt{\pi}}\int dr'\, c^{-r'^2}.$$

We have supposed in that which precedes, the probabilities of the events, known; let us examine the case where they are unknown. Let us suppose that out of m similar events awaited, n have arrived, and that A awaits s similar events of which each procures to him by its arrival, the benefit ν, the non-arrival causing to him the loss μ. If we represent by $\frac{n}{m}s + z$, the number of events which will arrive out of the s events awaited, the probability that z will be contained within the limits $\pm kt$ will by §30,

$$\frac{2}{\sqrt{\pi}}\int dt\, c^{-t^2},$$

the integral being taken from $t = 0$, k^2 being equal to

$$\frac{2ns(m-n)(m+s)}{m^3}.$$

But $\frac{n}{m}s + z$ being the number of arrived events, the real benefit of A is

$$\left(\frac{n\nu}{m} - \frac{(m-n)\mu}{m}\right)s + z(\nu + \mu);$$

the preceding integral is therefore the probability that the real benefit of A will be comprehended within the limits

$$\left(\frac{n\nu}{m} - \frac{(m-n)\mu}{m}\right)s \pm kt(\nu + \mu).$$

k is of order $\sqrt{s}$, if m and n are of order equal or greater than s; thus howsoever small that the mathematical expectation be, relative to each event, the real benefit becomes at infinity, certain and infinitely great, when the number of past events, is supposed infinite, as the one of future events.

[426] §40. We will now determine the benefits of the establishments founded on the probabilities of human life. The most simple way to calculate these benefits, is to reduce them to actual capitals. Let us take for example, life pensions. A person of age A wishes to constitute on his head, a life pension h; we demand the capital that he must for it, give to the funds of the establishment which makes this pension for him.

If we name y_0 the number of individuals of age A, in the table of mortality of which we make use, and y_x the number of individuals of age $A+x$; the probability to pay the pension at the end of the year $A+x$, will be $\frac{y_x}{y_0}$; consequently, the value of the payment will be $\frac{hy_x}{y_0}$. But if we designate by r the annual interest on unity, so that the capital 1 becomes $1+r$ after one year; it will become $(1+r)^x$ after x years; thus, the payment $(1+r)^x$ made at the end of the x^{th} year, reduced to actual capital, becomes unity, or this same payment divided by $(1+r)^x$; the payment $\frac{hy_x}{y_0}$ reduced to actual capital, is therefore $\frac{hy_x}{y_0(1+r)^x}$. The sum of all the payments made during the duration of life of the person who constitutes the pension, and multiplied by their probability, is equivalent therefore to an actual capital represented by the finite integral

$$\sum \frac{hy_x}{y_0(1+r)^x},$$

the characteristic Σ needing to embrace all the values of the function that it affects.

We are able to determine this integral by forming all these values according to the table of mortality, and by adding them together: we will deduce next the capitals from one another, by observing that if we name F the capital relative to the age A, and F' the capital relative to age $A+1$, we have

$$F = \frac{y_1}{y_0} \cdot \frac{F'+h}{1+r}.$$

But this process is simplified, when the law of mortality is known, and especially when it is given by a rational and entire function of x, that which is always possible,
[427] by considering the numbers of the table of mortality, as ordinates of which the corresponding ages are the abscissas, and by making a parabolic curve pass through the extremities of the two extreme ordinates, and of many intermediate ordinates. The differences which exist among the diverse tables of mortality, permit regarding this mean, as exact also as these tables, and even to be satisfied with a small number of ordinates.

Let us make

$$\frac{1}{1+r} = p, \quad \frac{y_x}{y_0} = u;$$

let us resume formula (16) of §11 of the first Book, which gives

$$\sum p^x u = \frac{p^x}{pc^{\frac{du}{dx}} - 1} + f,$$

f being an arbitrary constant. It is necessary in the development of the first term of the second member of this equation, with respect to the ratio to the powers of $\frac{du}{dx}$, to change any power $\left(\frac{du}{dx}\right)^i$ into $\frac{d^i u}{dx^i}$, and to multiply by u, the first term, which is independent of $\frac{du}{dx}$. We have thus

$$\sum p^x u = f - \frac{p^x u}{1-p} - \frac{p^{x+1}\frac{du}{dx}}{(1-p)^2} - \frac{(p+1)p^{x+1}}{1.2.(1-p)^3} \cdot \frac{d^2u}{dx^2} + \text{etc.}$$

In order to determine f, we will observe that the integral $\sum p^x u$ is null, when $x = 1$, and that it is terminated, when $x = n+1$, $A+n$ being the limit of life; because then it embraces the terms corresponding to all the numbers 1, 2, 3, ... n. Let us designate therefore by (u), $\left(\frac{du}{dx}\right)$, etc.; u', $\left(\frac{du'}{dx}\right)$, etc., the values of u, $\frac{du}{dx}$, etc. corresponding to $x = 1$, and to $x = n+1$; we will have

$$\sum \frac{hp^x y_x}{y_0} = h \left\{ \begin{array}{l} \frac{p}{1-p}[(u) - p^n(u')] \\ + \frac{p^2}{(1-p)^2}\left[\left(\frac{du}{dx}\right) - p^n\left(\frac{du'}{dx}\right)\right] \\ + \frac{(p+1)p^2}{1.2.(1-p)^3}\left[\left(\frac{d^2u}{dx^2}\right) - p^n\left(\frac{d^2u'}{dx^2}\right)\right] \\ +\text{etc.} \end{array} \right\}. \qquad (o)$$

If u or $\frac{y_x}{y_0}$ is constant and equal to unity, from $x = 1$ to $x = n$; then the life pension [428]
must be paid certainly during the number n of years, and it becomes an annuity. In this case, $\frac{du}{dx}$ is null, and the preceding formula gives $\frac{hp(1-p^n)}{1-p}$ for the capital equivalent to the annuity h.

If $u = 1 - \frac{x}{n}$; then the probability of life decreases in arithmetic progression, and the preceding formula gives

$$\frac{hp}{1-p}\left(1 - \frac{1-p^n}{n(1-p)}\right)$$

for the capital equivalent to the life annuity h, and so forth.

Let us suppose now that we wish to constitute a life pension h, on many individuals of ages A, $A+a$, $A+a+a'$, etc., so that the pension remains to the survivors. Let us designate by y_x, y_{x+a}, $y_{x+a+a'}$, etc., the numbers of the table of mortality, corresponding to the ages A, $A+a$, $A+a+a'$, etc.; the probability that the first individual has, to live to the age $A+x$, being $\frac{y_x}{y_0}$; the probability that at this age, he will have ceased to live, is $1 - \frac{y_x}{y_0}$. Similarly, the probability that the second individual has, to live to age $A+a+x$, or to the end of the x^{th} year of the constitution of the pension, being $\frac{y_{x+a}}{y_a}$; the probability that he will have ceased to live then, is $1 - \frac{y_{x+a}}{y_a}$; the probability that the third individual will have ceased to live, at the same epoch of the constitution of the pension, is $1 - \frac{y_{x+a+a'}}{y_{a+a'}}$, and so forth. The probability that none of these individuals will exist at this epoch, is

$$\left(1 - \frac{y_x}{y_0}\right)\left(1 - \frac{y_{x+a}}{y_a}\right)\left(1 - \frac{y_{x+a+a'}}{y_{a+a'}}\right).\text{etc.}$$

By subtracting this product, from unity; the difference will be the probability that one of these individuals at least, will be living at the end of the x^{th} year of the constitution of the pension. Let us name u this probability; $\sum hp^x u$ will be the
[429] actual capital equivalent to the life annuity h. But we must observe by taking this integral, that the quantities y_x, y_{x+a}, etc. are nulls, when their indices x, $x+a$, etc. surpass the number n, $A+n$ being the limit of the life.

If y_x is a rational and entire function of x, and of exponentials, such as q^x, r^x, etc.; we will have easily, by the formulas of the first Book, the integral $h\sum p^x u$. But we are able in all cases, to form by means of a table of mortality, all the terms of this integral, by taking the sum, and constructing thus tables of life annuities, on one or many heads.

The preceding analysis similarly serves to determine the life annuity that one must make at an establishment, in order to assure to his heirs a capital after his death. The capital equivalent to the life annuity h, made on a person of age A, is by that which precedes, $hS\frac{p^x y_x}{y_0}$, the sign S comprehending all the terms inclusively, from $x = 1$ to the limit of the life of the person. Let us name hq this integral, and let us imagine that the establishment receives from this person the rent h, and gives to him in exchange, the capital hq. Let us imagine next that the same person places this capital at perpetual interest with the establishment itself; the annual interest of unity being r or $\frac{1-p}{p}$. It is clear that the establishment must render the capital hq, to the heirs of the person. But the person has made during his life, the rent h to the establishment, and the person has received from it the pension $\frac{hq(1-p)}{p}$; the rent that the person has made really is therefore $h\left[1 - \frac{q(1-p)}{p}\right]$; this is therefore that which the person must give annually to the establishment, in order to assure to his heirs the capital hq.

I will not insist further on these objects, in the same way with respect to those which are relative to the establishments of assurance of each kind, because they present no difficulties. I will observe only that all these establishments must, in order to prosper, reserve themselves a benefit, and multiply considerably their affairs; so that, their real benefit becoming near certain, they are exposed the least as it is possible, to some great losses which would be able to destroy them. In fact, if the
[430] number of the affairs is s, and if the advantage of the establishment in each of them, is b; then it becomes extremely probable that the real benefit of the establishment will be sb, s being supposed a very great number.

In order to show it, let us suppose that s persons of age A constitute, each on his head, a life annuity h; and let us consider one of these persons who we will designate by C. If C dies in the interval from the end of the year x elapsed from the constitution of his pension, to the end of the year $x+1$; the establishment will have paid to him the pension h during x years, and the sum of these payments, reduced to actual capital, will be $h(p+p^2+\cdots+p^x)$ or $\sum hp^{x+1}$; now the probability that C will die within this interval, is $\frac{y_x-y_{x+1}}{y_0}$, or $-\frac{\triangle y_x}{y_0}$; the value of the loss that the establishment must then support, is therefore $-\frac{\triangle y_x}{y_0}\sum hp^{x+1}$. The sum of all these losses is

$$-\sum\left(\frac{\triangle y_x}{y_0}\sum hp^{x+1}\right); \qquad (r)$$

this is the capital that C must deposit into the funds of the establishment, in order to receive from it the life pension h. We are able to observe here that we have

$$-\triangle y_x\sum p^{x+1} = -y_{x+1}\sum p^{x+1} + y_x\sum p^x + y_x p^x:$$

by integrating the second member of this equation, the function (r) is reduced to

$$-\frac{y_x}{y_0}\sum hp^x + \frac{\sum hy_xp^x}{y_0}\text{constant};$$

now $\sum p^x$ is reduced to zero, when $x=1$, and when $x=n+1$, y_x is null by that which precedes; the function (r), or the capital that C must pay to the establishment, is therefore $\frac{\sum hy_xp^x}{y_0}$; that which is conformed to that which precedes. But under the form of the function (r), we are able to apply to the benefit of the establishment, the analysis of §39. In fact, we have in this case, by the section cited,

$$a\nu + a'\nu' + a''\nu'' + \text{etc.} = -\sum\left(\frac{\triangle y_x}{y_0}\sum hp^{x+1}\right);$$

next a, a', etc. being the successive values of $-\frac{\triangle y_x}{y_0}$, we will have [431]

$$a\nu^2 + a'\nu'^2 + \text{etc.} = \sum\left[-\frac{\triangle y_x}{y_0}\left(\sum hp^{x+1}\right)^2\right],$$

so that

$$2k^2 = \sum\left[-\frac{\triangle y_x}{y_0}\left(\sum hp^{x+1}\right)^2\right] - \left[\sum\frac{\triangle y_x}{y_0}\left(\sum hp^{x+1}\right)\right]^2.$$

In supposing that each of s persons who constitute the pension h on his head, deposit to the funds of the establishment, beyond the capital corresponding to this pension, a sum b, in order to defray the expense of the establishment; we will have by §39,

$$\frac{2}{\sqrt{\pi}}\int dr'c^{-r'^2},$$

for the probability that the real benefit of the establishment will be comprehended within the limits

$$sb \pm 2kr'\sqrt{s}.$$

Thus in the case of an infinite number of affairs, the real benefit of the establishment, becomes certain and infinite. But then those who treat with it, have a mathematical disadvantage which must be compensated by a moral advantage, of which the estimation will be the object of the following chapter.

CHAPTER 10

On moral expectation

§41. We have seen in §2, the difference which exists between mathematical expectation and moral expectation. Mathematical expectation resulting from the probable awaiting of one or many goods, being the product of these goods, by the probability to obtain them, it can be evaluated by the analysis exposed in that which precedes. Moral expectation is ruled on a thousand circumstances which it is nearly impossible to evaluate well. But we have given in the section cited, a principle, which being applied to the most common cases, leads to some often useful results, and of which we are going to develop the principals. [432]

According to this principle, x being the physical fortune of an individual, the increase dx that he receives, produces in the individual a moral good reciprocal to this fortune; the increase of his moral fortune can therefore be expressed by $\frac{k\,dx}{x}$, k being a constant. Thus by designating by y the moral fortune corresponding to the physical fortune x, we will have

$$y = k \log x + \log h,$$

h being an arbitrary constant that we will determine by means of a value of y corresponding to a given value of x. With respect to that, we will observe that we can never suppose x and y nulls or negatives, in the natural order of things; because a man who possesses nothing regards his existence, as a moral good which can be compared to the advantage that a physical fortune of which it is quite difficult to assign the value would procure to him, but that we can not fix below that which it would be for him rigorously necessary in order to exist; because we imagine that he [433]
would not agree at all to receive a moderate sum, such as one hundred francs, with the condition to claim nothing, when he would have spent it.

Let us suppose now that the physical fortune of an individual is a, and that the expectancy of one of the increases α, β, γ, etc., occurs to him, these quantities being able to be nulls or even negatives, that which changes the increases to diminutions. Let us represent by p, q, r, etc., the respective probabilities of these increases, the sum of these probabilities being supposed equal to unity. The corresponding moral fortunes of the individual, will be able to be

$$k\log(a+\alpha)+\log h, \quad k\log(a+\beta)+\log h, \quad k\log(a+\gamma)+\log h, \quad \text{etc.}$$

By multiplying these fortunes respectively by their probabilities p, q, r, etc.; the sum of their products will be the moral fortune of the individual, by virtue of his expectancy; by naming therefore Y this fortune, we will have

$$Y = kp\log(a+\alpha) + kq\log(a+\beta) + kr\log(a+\gamma) + \text{etc.} + \log h.$$

Let X be the physical fortune which corresponds to this moral fortune, we will have

$$Y = k \log X + \log h.$$

The comparison of these two values of Y gives

$$X = (a+\alpha)^p (a+\beta)^q (a+\gamma)^r.\text{etc.}$$

If we subtract the original fortune a, from this value from X; the difference will be the increase of the physical fortune which would procure to the individual, the same moral advantage which results for him, from his expectancy. This difference is therefore the expression of this advantage, instead that the mathematical advantage has for expression

$$p\alpha + q\beta + r\gamma + \text{etc.}$$

Thence result many important consequences. One of them is that the mathematically most equal game, is always disadvantageous. In fact, if we designate by a the physical fortune of the player before commencing the game; by p, his probability to win, and
[434] by μ his stake; that of his adversary must be, for equality of the game, $\frac{(1-p)\mu}{p}$; thus the player winning the game, his physical fortune becomes $a + \frac{1-p}{p}\mu$, and the probability of that is p. If he loses the game, his physical fortune becomes $a - \mu$, and the probability of that is $1-p$; by naming therefore X his physical fortune, by virtue of his expectation, we will have by that which precedes,

$$X = \left(a + \frac{1-p}{p}\mu\right)^p (a-\mu)^{1-p};$$

now this quantity is smaller than a, that is that we have

$$\left(1 + \frac{1-p}{p}\cdot\frac{\mu}{a}\right)^p \left(1 - \frac{\mu}{a}\right)^{1-p} < 1$$

or by taking the hyperbolic logarithms,

$$p \log\left(1 + \frac{1-p}{p}\cdot\frac{\mu}{a}\right) + (1-p)\log\left(1 - \frac{\mu}{a}\right) < 0.$$

The first member of this equation can be put under the form

$$\int (1-p)\cdot\frac{d\mu}{a}\left(\frac{1}{1+\frac{1-p}{p}\cdot\frac{\mu}{a}} - \frac{1}{1-\frac{\mu}{a}}\right),$$

a quantity which is evidently negative.

There results further from the preceding analysis, that it is worth more to expose his fortune, by parts, to some dangers independent from one another, than to expose all entire to the same danger. In order to show it, let us suppose that one merchant having to make come by sea, a sum ϵ, exposes it on a single vessel, and that observation has made known the probability p of the arrival of a vessel of the same kind, in the port; the mathematical advantage of the merchant, resulting from his expectation, will be $p\epsilon$. But if we represent by unity his physical fortune, independently of his expectancy; his moral fortune will be by that which precedes,

$$kp\log(1+\epsilon)+\log h,$$

and his moral advantage will be, by virtue of his expectancy, [435]

$$(1+\epsilon)^p-1,$$

a quantity smaller than $p\epsilon$: because we have

$$(1+\epsilon)^p<1+p\epsilon,$$

since $\log(1+\epsilon)^p$ or $p\log(1+\epsilon)$ is less than $\log(1+p\epsilon)$, that which is evident, when we put these two logarithms under the form $\int\frac{p\,d\epsilon}{1+\epsilon}$ and $\int\frac{p\,d\epsilon}{1+p\epsilon}$.

Let us suppose now, that the merchant exposes the sum ϵ by equal parts, on r vessels. His physical fortune will become $1+\epsilon$, if all the vessels arrive, and the probability of this event is p^r. If $r-1$ vessels arrive, the physical fortune of the merchant becomes $1+\frac{(r-1)\epsilon}{r}$, and the probability of this event is $rp^{r-1}(1-p)$. If $r-2$ vessels arrive, the physical fortune of the merchant becomes $1+\frac{r-2}{r}\epsilon$, and the probability of this event is $\frac{r.\overline{r-1}}{2}p^{r-2}(1-p)^2$, and so forth; the moral fortune of the merchant is therefore by that which precedes,

$$k\left\{\begin{aligned}&p^r\log(1+\epsilon)+rp^{r-1}(1-p)\log\left(1+\frac{r-1}{r}\epsilon\right)\\&\quad+\frac{r.\overline{r-1}}{2}p^{r-2}(1-p)^2\log\left(1+\frac{r-2}{r}\epsilon\right)+\text{etc.}\end{aligned}\right\}+\log h,$$

an expression that we are able to put under this form,

$$kp\int d\epsilon\left[\frac{p^{r-1}}{1+\epsilon}+\frac{\overline{r-1}.p^{r-2}(1-p)}{1+\frac{r-1}{r}\epsilon}+\frac{\overline{r-1}.\overline{r-2}.p^{r-3}(1-p)^2}{1.2.\left(1+\frac{r-2}{r}\epsilon\right)}+\text{etc.}\right]+\log h. \qquad (a)$$

If we subtract from this expression, that of the moral fortune of the merchant, when he exposes the sum ϵ on a single vessel, and if we obtain by making $r=1$ in the preceding, that which, setting aside $\log h$, reduces that here to $kp\int\frac{d\epsilon}{1+\epsilon}$, which is equal to

$$kp\int d\epsilon\left\{\frac{p^{r-1}}{1+\epsilon}+\frac{\overline{r-1}.p^{r-2}(1-p)}{1+\epsilon}+\frac{\overline{r-1}.\overline{r-2}.p^{r-3}(1-p)^2}{1.2.(1+\epsilon)}+\text{etc.}\right\},$$

the difference will be [436]

$$kp(1-p)\frac{r-1}{r}\int\frac{\epsilon d\epsilon}{1+\epsilon}\left[\frac{p^{r-2}}{1+\frac{r-1}{r}\epsilon}+\frac{\overline{r-2}.p^{r-3}(1-p)}{1+\frac{r-2}{r}\epsilon}+\text{etc.}\right];$$

this difference being positive, we see that there is morally the advantage to partition the sum ϵ on several vessels. This advantage is increased in measure as we increase the number r of vessels, and, if this number is very great, the moral advantage becomes nearly equal to the mathematical advantage.

In order to see this, let us take formula (a), and let us give to it this form,

$$kp\iint dx\,d\epsilon\,c^{-\left(1+\frac{\epsilon}{r}\right)x}\left(pc^{-\frac{\epsilon x}{r}}+1-p\right)^{r-1}+\log h; \qquad (a')$$

the integral relative to x being taken from x null to x infinity. In this interval, the coefficient of dx under the $\iint$ signs, has neither *maximum* nor *minimum*; because its differential taken with respect to x, is

$$-c^{\left(1+\frac{\epsilon}{r}\right)x}dx\left(pc^{-\frac{\epsilon x}{r}}+1-p\right)^{r-2}\left[p(1+\epsilon)c^{-\frac{\epsilon x}{r}}+(1-p)\left(1+\frac{e}{r}\right)\right];$$

this differential is constantly negative from $x=0$ to x infinity; thus the coefficient itself diminishes constantly in this interval. It is therefore here the case to make use of formula (A) of §22 of the first Book, in order to have, by a convergent approximation, the integral $\int y\,dx$, y being equal to

$$c^{-\left(1+\frac{\epsilon}{r}\right)x}\left(pc^{-\frac{\epsilon x}{r}}+1-p\right)^{r-1}.$$

The quantity that we have named ν in the section cited, becomes then

$$\nu=-\frac{y\,dx}{dy}=\frac{pc^{-\frac{\epsilon x}{r}}+1-p}{p(1+\epsilon)c^{-\frac{\epsilon x}{r}}+(1-p)\left(1+\frac{e}{r}\right)};$$

[437] that which gives

$$U=\frac{1}{1+p\epsilon+(1-p)\frac{e}{r}},$$

$$\frac{dU}{dx}=\frac{p(1-p)\epsilon^2\left(1-\frac{1}{r}\right)}{r\left[1+p\epsilon+(1-p)\frac{e}{r}\right]^2},$$

etc.;

U, $\frac{dU}{dx}$, etc. being that which ν, $\frac{d\nu}{dx}$, etc. become, when x is null. This premised, formula (A) cited, will give

$$\int dx\,c^{-\left(1+\frac{\epsilon}{r}\right)x}\left(pc^{-\frac{\epsilon x}{r}}+1-p\right)^{r-1}$$
$$=\frac{1}{1+p\epsilon+(1-p)\frac{e}{r}}\left\{1+\frac{p(1-p)\epsilon^2\left(1-\frac{1}{r}\right)}{r\left[1+p\epsilon+(1-p)\frac{e}{r}\right]}+\text{etc.}\right\}.$$

Formula (a') becomes thus, very nearly, when r is a great number,

$$k\int\frac{p\,d\epsilon}{1+p\epsilon}+\log h,$$

or

$$k\log(1+p\epsilon)+\log h.$$

Now let X be the physical fortune corresponding to this moral fortune; we have by that which precedes,

$$k\log X+\log h,$$

for the moral fortune corresponding to X; by comparing therefore these two expressions, we will have

$$X = 1 + p\epsilon.$$

In this case, the moral advantage is $p\epsilon$; it is therefore equal to the mathematical advantage.

Often the moral advantage of individuals is increased by the mean of the funds of assurance, at the same time as these funds produce to the assurers a certain benefit. Let us suppose, for example, that a merchant has a part ϵ of his fortune on a vessel of [438] which the probability of the arrival is p; and that he assures this part, by giving a sum to the assurance company. For perfect equality between the mathematical lots of the company and of the merchant, the latter must give $(1-p)\epsilon$ for price of assurance. By representing by unity, the fortune of the merchant, independently of his expectation ϵ, his moral fortune will be by that which precedes,

$$kp\log(1+\epsilon) + \log h,$$

in the case where one does not assure; and in the case where he assures, it will be

$$k\log(1+p\epsilon) + \log h;$$

now we have

$$\log(1+p\epsilon) > p\log(1+\epsilon),$$

or, that which reverts to the same,

$$\int \frac{p\,d\epsilon}{1+p\epsilon} > \int \frac{p\,d\epsilon}{1+\epsilon},$$

p being less than unity; the moral fortune of the merchant is therefore increased, by means of his assurance. He is able thus to make to the assurance company, a proper sacrifice to defray the expense of the establishment and to the benefit that it must make. If we name α this sacrifice, that is, if we suppose that the merchant gives to the company, for the price of his assurance, the sum $(1-p)\epsilon + \alpha$, we will have in the case of equality of the moral fortunes, when the merchant assures, and when he does not assure at all,

$$\log(1 - \alpha + p\epsilon) = p\log(1+\epsilon);$$

that which gives

$$\alpha = 1 + p\epsilon - (1+\epsilon)^p.$$

This is all that which the merchant can give to the company, without moral disadvantage; he will have therefore a moral advantage, by making a sacrifice less than this value of α, and at the same time, the company will have a benefit which, as we have seen, becomes certain, when its relations are very numerous. We see thence, how some establishments of this kind, well designed and sagely administered, can be [439] assured a real benefit, by procuring advantages to the persons who negotiate with them: this is in general the end of all the exchanges; but here, by a particular combination, the exchange holds between two objects of like nature, of which one is only probable, while the other is certain.

§42. The principle of which we just made use in order to calculate the moral expectation, has been proposed by Daniel Bernoulli, in order to explicate the difference between the result of the calculus of probabilities and the indication of common sense, in the following problem. Two players A and B play at *heads* and *tails*, with the condition that A pays to B two francs, if *heads* arrives at the first trial; four francs, if it arrives at the second trial; eight francs if it arrives at the third trial, and so forth to the n^{th} trial. We demand that which B must give to A in commencing the game.

It is clear that the advantage of B, relative to the first trial, is one franc; because he has $\frac{1}{2}$ of probability to win two francs at this trial. His advantage relative to the second trial, is similarly one franc; because he has $\frac{1}{4}$ of probability to win four francs at this trial, and so forth; so that the sum of all his advantages relative to the n trials, is n francs. He must therefore for the mathematical equality of the game, give to A, this sum which becomes infinite, if we suppose that the game continues to infinity.

However a person, in this game, will not risk with prudence, an even rather moderate sum, such as one hundred francs. If we reflect in the least on this kind of contradiction between the calculus, and that which common sense indicates; we see easily that it depends on this that if we suppose, for example, $n = 50$, that which gives 2^{50} for the sum that B can hope at the fiftieth trial, this immense sum produces to B not at all, a moral advantage proportional to its magnitude; in a manner that there is for him a moral disadvantage to expose a franc in order to obtain it with the excessively small probability $\frac{1}{2^{50}}$ to succeed. But the moral advantage that an expected sum can procure, depends on an infinity of circumstances proper to each individual, and that it is impossible to evaluate. The only general consideration that
[440] we are able to employ in this regard, is that the more one is rich, the less the very small sum can be advantageous, all things equal besides. Thus the most natural supposition that we can make, is that of a reciprocal moral advantage, to the wealth of the interested person. This is to that which the principle of Daniel Bernoulli is reduced, a principle which, as we have just seen, makes the results of the calculus coincide with the indications of common sense, and which gives the means to estimate with some exactitude, these always vague indications. His application to the problem of which we have just spoke, will furnish us a new example of it.

Let us name a the fortune of B before the game, and x that which he gives to player A. His fortune becomes $a-x+2$, if *heads* arrives at the first trial; it becomes $a-x+2^2$, if *heads* arrives at the second trial, and so forth to trial n, where it becomes $a-x+2^n$, if *heads* arrives only at the n^{th} trial. The fortune of B becomes $a-x$, if *heads* arrives not at all in the n trials, after which the game is supposed to end; but the probability of this last event is $\frac{1}{2^n}$. By multiplying the logarithms of these diverse fortunes by their respective probabilities and by k, we will have by that which precedes, the moral fortune of B, by virtue of the conditions of the game, equal to

$$\frac{1}{2}k\log(a-x+2)+\frac{1}{2^2}k\log(a-x+2^2)+\cdots$$
$$\cdots+\frac{1}{2^n}k\log(a-x+2^n)+\frac{1}{2^n}k\log(a-x)+\log h.$$

But before the game, his moral fortune was $k \log a + \log h$; by equating therefore these two fortunes, provided that B always conserves the same moral fortune, and passing again from the logarithms to the numbers, we will have, $a - x$ being supposed equal to a', and making $\frac{1}{a'} = \alpha$,

$$1 + \alpha x = (1 + 2\alpha)^{\frac{1}{2}} (1 + 2^2\alpha)^{\frac{1}{2^2}} \cdots (1 + 2^n\alpha)^{\frac{1}{2^n}}; \qquad (o)$$

the factors $(1+2\alpha)^{\frac{1}{2}}$, $(1+2^2\alpha)^{\frac{1}{2^2}}$ diminishing without ceasing, and their limit is unity; [441]
because we have

$$(1 + 2^i\alpha)^{\frac{1}{2^i}} > (1 + 2^{i+1}\alpha)^{\frac{1}{2^{i+1}}}.$$

In fact, if we raise to the power 2^{i+1}, the two members of this inequality, it becomes

$$1 + 2^{i+1}\alpha + 2^{2i}\alpha^2 > 1 + 2^{i+1}\alpha;$$

and under this form, the equality becomes evident. Moreover, the logarithm of $(1 + 2^i\alpha)^{\frac{1}{2^i}}$ is equal to $\frac{i \log 2}{2^i} + \frac{1}{2^i} \log\left(\alpha + \frac{1}{2^i}\right)$; and it is clear that this function is null in the case of i infinite, that which requires that in this case, $(1 + 2^i\alpha)^{\frac{1}{2^i}}$ is unity.

If we suppose n infinite in equation (o), we have the case where the game can be prolonged to infinity, that which is the most advantageous case to B. a' and consequently α being supposed known; we will take the sum of the tabular logarithms of a rather great number $i - 1$, of the first factors of the second member, in order that $2^i\alpha$ is at least equal to ten. The sum of the tabular logarithms of the following factors, to infinity, will be, very nearly equal to

$$\frac{\log \alpha}{2^{i-1}} + \frac{(i+1)\log 2}{2^{i-1}} + \frac{0,4342945}{3\alpha 2^{i-2}}.$$

The addition of these two sums will give the tabular logarithm of $a' + x$ or of a. Thus we will have for a physical fortune a, supposed B has before the game, the value of x which he must give to A at the beginning of the game, in order to conserve the same moral fortune. By supposing, for example, a' equal to one hundred, we find $a = 107^{\text{fr}}, 89$, whence it follows that the physical fortune of B being originally $107^{\text{fr}}, 89$, he must then risk prudently in this game, only $7^{\text{fr}}, 89$, instead of the infinite sum that the result of the calculus indicates, when we set aside all moral considerations. Having thus the value of a relative to $a' = 100$, it is easy to conclude from it in the following manner, its value relative to $a' = 200$; in fact we have, in this last case, [442]

$$a = (100 + 2)^{\frac{1}{2}} (100 + 2^2)^{\frac{1}{4}}.\text{etc.} = 2(100 + 1)^{\frac{1}{2}} (100 + 2)^{\frac{1}{4}} (100 + 4)^{\frac{1}{8}}.\text{etc.}$$

But we have just found

$$(100 + 2)^{\frac{1}{4}} (100 + 4)^{\frac{1}{8}}.\text{etc.} = (107, 89)^{\frac{1}{2}};$$

therefore

$$a = 2\sqrt{101.107,89} = 208,78.$$

Thus the physical fortune of B being originally $208,78$, he is not able to risk prudently in this game, beyond $8^{\text{fr}}, 78$.

§43. We will now extend the principle exposed above, to the things of which the existence is distant and uncertain. For this, let us consider two persons A and B, who wish to each invest, in a life annuity, a capital q. They can make it separately: they can partner and constitute a life annuity on their heads, in a manner that the pension is reversible to the one who survives the other. Let us examine what is the most advantageous part.

Let us suppose the two persons of the same age, and having the same annual fortune that we will represent by unity, independently of the capital that they wish to place. Let β be the life pension that this capital would produce to each of them, if they placed their capitals separately, so that their annual fortune becomes $1+\beta$. We will express, conformably to the principle of which there is concern, their corresponding annual moral fortune, by $k\log(1+\beta)+\log h$. But this fortune will take place only probably, in the x^{th} year; thus, by designating by y_x the probability that A will survive to the end of the x^{th} year, we must multiply his annual moral fortune relative to this year, by y_x; by adding therefore all these products, their sum, that we will designate by $[k\log(1+\beta)+\log h]\sum y_x$, will be that which I name here *life-annuity moral fortune.*

Let us suppose now that A and B place the sum $2q$ of their capitals, on their heads, and that that produces a life pension β', reversible to the survivor. So long as A and B will live, each of them will touch only $\frac{1}{2}\beta'$ of life annuity, and their annual
[443] moral fortune will be $k\log(1+\frac{1}{2}\beta')+\log h$. By multiplying it by the probability that they both will live to the end of year x, a probability equal to $(y_x)^2$; the sum of these products for all the values of x, will be the life-annuity moral fortune of A, relative to the supposition of their simultaneous existence; this fortune is therefore

$$\left[k\log\left(1+\frac{\beta'}{2}\right)+\log h\right]\sum(y_x)^2.$$

The probability that A will exist alone to the end of the x^{th} year, is $y_x-(y_x)^2$; his life-annuity moral fortune relative to his existence after the death of B, which renders his annual moral fortune equal to $1+\beta'$, is therefore

$$[k\log(1+\beta')+\log h]\ \sum\left[y_x-(y_x)^2\right].$$

The sum of these two functions,

$$k\log\left(1+\frac{\beta'}{2}\right)\sum(y_x)^2+k\log(1+\beta')\left[\sum y_x-\sum(y_x)^2\right]+\log h\sum y_x,$$

will be the life-annuity moral fortune of A under the hypothesis where A and B place conjointly their capital.

If we compare this fortune to that which we have just found in the case where they place their capitals separately; we see that there will be for A advantage or disadvantage to place conjointly, according as

$$\log\left(1+\frac{\beta'}{2}\right)\sum(y_x)^2+\log(1+\beta')\left[\sum y_x-\sum(y_x)^2\right]$$

will be greater or lesser than $\log(1+\beta')\sum y_x$. In order to know it, it is necessary to determine the ratio of β' to β; now we have, by §40,

$$q = \beta \sum p^x y_x,$$

$\frac{1-p}{p}$ being the annual interest on the money: we have next by the same section,

$$2q = \beta' \sum p^x \left[2y_x - (y_x)^2\right];$$

we have therefore

$$\beta' = \frac{2\beta \sum p^x y_x}{\sum p^x \left[2y_x - (y_x)^2\right]}.$$

The tables of mortality will give the values of $\sum y_x$, $\sum (y_x)^2$, $\sum p^x y_x$, $\sum p^x (y_x)^2$; we [444]
will be able thus to judge which of the two placements of which there is concern, is most advantageous.

Let us suppose β and β' some very small fractions; the quantity $\log(1+\beta)\sum y_x$ becomes very nearly $\beta \sum y_x$. The quantity

$$\log\left(1+\frac{\beta'}{2}\right)\sum (y_x)^2 + \log(1+\beta')\left[\sum y_x - \sum (y_x)^2\right]$$

becomes

$$\frac{\beta'}{2}\left[2\sum y_x - \sum (y_x)^2\right],$$

and by substituting for β' its preceding value, it becomes

$$\beta \frac{\left[2\sum y_x - \sum (y_x)^2\right]\sum p^x y_x}{2\sum p^x y_x - \sum p^x (y_x)^2};$$

there is therefore advantage to place conjointly, if

$$\left[2\sum y_x - \sum (y_x)^2\right]\sum p^x y_x$$

surpasses over

$$\left[2\sum p^x y_x - \sum p^x (y_x)^2\right]\sum y_x,$$

or if we have

$$\frac{\sum p^x (y_x)^2}{\sum p^x y_x} > \frac{\sum (y_x)^2}{\sum y_x};$$

it is in fact that which holds generally, p being smaller than unity.

The advantage to place conjointly the capitals, increases by the consideration that the increase $\frac{\beta'}{2}$ of revenue arrives to the survivor, at an ordinarily advanced age in which the greatest needs which are sensed, render it much more useful. This advantage increases yet on all the affections which the two individuals can attach to one another, and which make them desire the well being of the one who must survive. The establishments in which one is able thus to place his capitals, and by a slight sacrifice of his revenue, to assure the existence of his family for a time where one must fear no longer being sufficient to its needs, are therefore very advantageous to the dead, by favoring the softest penchants of nature. They offer not at all the inconvenience that we have noted in even the most equitable games, the one to render the loss

[445] more sensible than the gain; since to the contrary, they offer the means to exchange the superfluous, against some assured resources in the future. The Government must therefore encourage these establishments, and to respect them in their vicissitudes; because the expectations that they present, carrying onto an extended future, they are able to prosper only with shelter from all anxiety on their duration.

CHAPTER 11

On the probability of testimonies

§44. I will first consider a single witness. The probability of his testimony is composed of his veracity, of the possibility of his error and of the possibility of the fact in itself. In order to fix the ideas, let us imagine that we have extracted a ticket from an urn which contains the number n of them; and that a witness to the drawing announces that the n° i came out. The observed event is here the witness announcing the exit of the n° i. Let p be the veracity of the witness, or the probability that he will not at all seek to deceive: let further r be the probability that he is not deceived at all. This premised. [446]

We are able to form the following four hypotheses. Either the witness does not deceive at all and is not deceived at all; or he deceives not at all and is deceived; or he deceives and is not deceived at all; finally, or he deceives and is deceived at the same time. Let us see what is, *a priori*, under each of these hypotheses, the probability that the witness will announce the exit of the n° i.

If the witness does not deceive at all and is not deceived at all, the n° i will exit; but the probability of this exit, is *a priori*, $\frac{1}{n}$; by multiplying it by the probability pr of the hypothesis, we will have $\frac{pr}{n}$ for the entire probability of the observed event, under this first hypothesis.

If the witness does not deceive at all and is deceived, the n° i must not exit at all, in order that he announces its exit; the probability of that is $\frac{n-1}{n}$. But the error of the witness must carry over one of the non-drawn tickets. Let us suppose that it is able to carry equally over all: the probability that it will carry over the n° i, will be $\frac{1}{n-1}$; the probability that the witness not deceiving at all and being deceived, will [447]
announce the n° i, is therefore $\frac{n-1}{n}\cdot\frac{1}{n-1}$ or $\frac{1}{n}$. By multiplying it by the probability $p(1-r)$ of the hypothesis itself, we will have $\frac{p(1-r)}{n}$ for the probability of the observed event under this second hypothesis.

If the witness deceives and is not deceived at all, the n° i will not exit at all, and the probability of that is $\frac{n-1}{n}$; but the witness must choose among the $n-1$ tickets not drawn, the n° i. If we suppose that his choice is able equally to carry over each of them, $\frac{1}{n-1}$ will be the probability that his choice will be fixed on the n° i; $\frac{n-1}{n}\cdot\frac{1}{n-1}$ or $\frac{1}{n}$ is therefore the probability that the witness will announce the n° i. By multiplying it by the probability $(1-p)r$, of the hypothesis; we will have $\frac{(1-p)r}{n}$ for the entire probability of the observed event under this third hypothesis.

Finally, if the witness deceives and is deceived, the probability that he will not believe the n° i drawn, will be $\frac{n-1}{n}$, and the probability that he will choose it among the $n-1$ tickets that he will not believe drawn, will be $\frac{1}{n-1}$; $\frac{n-1}{n}\cdot\frac{1}{n-1}$ or $\frac{1}{n}$ will be therefore the probability that he will announce the exit of n° i. By multiplying it by the probability $(1-p)(1-r)$ of the hypothesis, we will have $\frac{(1-p)(1-r)}{n}$ for the probability of the observed event, under this fourth hypothesis.

This hypothesis contains one case in which the n° i exited; namely, the case in which the n° i being drawn, the witness believes it not drawm, and he chooses it among the $n-1$ tickets that he believes not drawn. The probability of that is the product of $\frac{1}{n}$ by $\frac{1}{n-1}$. By multiplying this product, by the probability $(1-p)(1-r)$ of the hypothesis, we will have $\frac{(1-p)(1-r)}{n(n-1)}$ for the probability of the case of which there is concern.

[448] We are able to arrive to the same results, in this manner. Let a, b, c, d, i, etc., be the n tickets. Since the witness is deceived, he must not believe drawn at all, the removed ticket; and since he deceives, he must not announce at all as drawn, the ticket that he believes drawn. Let us put therefore in the first place, the drawn ticket; in the second, the ticket that the witness believes drawn; and in the third, the ticket that he announces. Among all the possible combinations of the tickets three by three, without excluding those where they are repeated, there are compatibles with the present hypothesis, only those where the ticket which occupies the second place, occupies neither the first, nor the third; such are the combinations aba, abc, etc. Now it is easy to see that the number of combinations which satisfy the two preceding conditions, is $n.\overline{n-1}^2$; because the combination ab is able to be combined with the $n-1$ n$^{\text{os}}$ other than b; and the number of combinations ab, ba, ac, is $n.\overline{n-1}$. Now the combinations in which the n° i is announced, without being drawn, are of the form abi, bai, aci, etc., and the number of these combinations is $\overline{n-1}.\overline{n-2}$; thus the probability that one of these combinations will take place, is $\frac{n-2}{n.\overline{n-1}}$. The combinations in which the n° i being drawn, it is announced, are of the form iai, ibi, etc., and the number of these combinations is clearly $n-1$; the probability that one of these combinations will take place, is therefore $\frac{1}{n.\overline{n-1}}$. It is necessary to multiply all these combinations, by the probability $(1-p)(1-r)$ of the hypothesis, and then we will have the preceding results.

Now, in order to have the probability of the exit of the n° i, we must make a sum of all the preceding probabilities, relative to this exit, and to divide it by the sum of all these probabilities; that which gives, for this probability,

$$\frac{\frac{pr}{n}+\frac{(1-p)(1-r)}{n(n-1)}}{\frac{pr}{n}+\frac{p(1-r)}{n}+\frac{(1-p)r}{n}+\frac{(1-p)(1-r)}{n}}$$

or $pr+\frac{(1-p)(1-r)}{n-1}$.

[449] If r is equal to unity, or if the witness is not deceived at all, the probability of the exit of the n° i, will be p; that is the probability of the veracity of the witness.

If n is a very great number, this probability will be very nearly, pr, or the probability of the veracity of the witness, multiplied by the probability that he is not deceived at all.

We have supposed that the error of the witness, when he is deceived, can equally fall on all the non-drawn tickets; but this supposition ceases to hold, if some of them have more resemblance than the others, with the drawn ticket; because the mistake in this regard, is easier. We have further supposed that the witness, when he deceives, has no motive in order to choose one ticket rather than another; that which can not take place. But it will be very difficult to make enter into a formula, all these particular considerations.

§45. Let us suppose now that the urn contains $n-1$ black balls, and one white ball; and that by having extracted one ball, a witness of the drawing announces the exit of a white ball. Let us determine the probability of this exit. We will form the same hypotheses as we just made. In the first, the probability of the exit of the white ball, is, as above, $\frac{pr}{n}$. Under the second hypothesis, the witness is being deceived without deceiving, a black ball must be drawn, and the probability of that is $\frac{n-1}{n}$, and as the witness supposed truthful, must announce the exit of a white ball; by that alone that he is mistaken; the probability of this announcement will be therefore $\frac{n-1}{n}$, a probability that it is necessary to multiply by the probability $p(1-r)$ of the hypothesis, that which gives $\frac{p(1-r).\overline{n-1}}{n}$ for the probability of the event observed, under this hypothesis. Under the third hypothesis, the witness being supposed deceiving and not at all being deceived, a black ball must be drawn, and the probability of that is $\frac{n-1}{n}$. By multiplying it by the probability $(1-p)r$ of this hypothesis, we will have [450]
$\frac{(1-p)r.\overline{n-1}}{n}$ for the probability of the observed event, under this hypothesis. Finally, under the fourth hypothesis, the witness deceiving and being deceived, can announce the exit of the white ball, only as long as it will be drawn. The probability of this exit is $\frac{1}{n}$. By multiplying it by the probability $(1-p)(1-r)$ of the hypothesis, we will have $\frac{(1-p)(1-r)}{n}$ for the probability of the observed event, under this hypothesis.

Presently, if we unite among the preceding probabilities, those in which the white ball exited; we will have the probability of this exit, by dividing their sum, by the sum of all the probabilities, that which gives

$$\frac{pr+(1-p)(1-r)}{pr+(1-p)(1-r)+[p(1-r)+(1-p)r](n-1)}$$

for the probability of the exit of the white ball; consequently

$$\frac{[p(1-r)+(1-p)r](n-1)}{pr+(1-p)(1-r)+[p(1-r)+(1-p)r](n-1)}$$

is the probability that the fact attested by the witness of the drawing, has not taken place.

We are able to observe here, that if we name q, the probability that the witness announces the truth, we will have

$$q = pr + (1-p)(1-r);$$

because it is clear that he spoke true, in the case of which there is concern, either that he deceives not at all and is not deceived at all, or that he deceives and is deceived.

This expression of q gives

$$1 - q = p(1 - r) + (1 - p)r.$$

In fact, the probability $1 - q$ that he does not enunciate the truth, is the probability that he deceives not at all and is deceived, plus the probability that he deceives and is not deceived at all. The preceding expression of the probability that the attested fact is false, becomes thus

$$\frac{(1-q)(n-1)}{q + (1-q)(n-1)}.$$

[451] If the number $n - 1$ of black balls is very great; this probability becomes, very nearly, equal to unity or to certitude, if the error or the mistake of the witness is in the least probable. Then the fact that he attests, becomes extraordinary. Thus we see how the extraordinary facts weaken the belief due to the witnesses; the mistake or the error becoming so much more possible, as the attested fact is the more extraordinary in itself.

§46. Let us consider presently two urns A and B, of which the first contains a great number n of white balls; and the second, the same number of black balls. We draw from one of these urns, a ball that we replace into the other urn; next we draw a ball from this latter urn. A witness of the first drawing, attests that one white ball exited: a witness of the second drawing, attests similarly that he has seen a white ball extracted. Each of these testimonies, considered isolated, offers nothing of the unlikely. But the consequence which results from their collection, is that the same ball exited in the first drawing, has reappeared in the second; that which is a phenomenon so much more extraordinary, as n is a greater number. Let us see how the value of these testimonies, is weakened from it.

Let us name q the probability that the first witness enunciates the truth. We see by the preceding section, that in the present case, this probability is composed of the probability that the witness deceives not at all and is not deceived at all, added to the probability that he deceives and is deceived at the same time; because the witness, in these two cases, enunciates the truth. Let q' be the same probability relative to the second witness. We are able to form these four hypotheses: either the first and the second witness say the truth; or the first says the truth, the second does not say it at all; or the second witness says the truth, the first does not say the truth at all; or finally neither of the two say the truth. Let us determine *a priori*, under each of these hypotheses, the probability of the observed event.

This event is the announcement of the exit of one white ball at each drawing. The probability that one white ball exited at the first drawing, is $\frac{1}{2}$, since the ball
[452] extracted can be equally drawn from urn A or from urn B. In the case where it has been extracted from urn A, and put into urn B, $n + 1$ balls are contained in this last urn; and the probability to extract from it a white ball is $\frac{1}{n+1}$; the product of $\frac{1}{2}$ by $\frac{1}{n+1}$ is therefore the probability *a priori*, of the extraction of one white ball, in the two consecutive drawings. By multiplying it by the probability qq' that the two

witnesses say the truth; we will have

$$\frac{qq'}{2(n+1)}$$

for the probability of the observed event, under the first hypothesis.

Under the second hypothesis, the ball has been extracted from urn A, and put into urn B: the probability of this extraction is $\frac{1}{2}$. Moreover, since the second witness does not say the truth, a black ball has been extracted from urn B, and the probability of this extraction is $\frac{n}{n+1}$. By multiplying therefore $\frac{1}{2}$ by $\frac{n}{n+1}$, and the product by the probability $q(1+q')$, that the first witness says the truth, while the second does not say it, we will have

$$\frac{q(1-q')n}{2(n+1)}$$

for the probability of the observed event, under the second hypothesis.

Under the third hypothesis, a black ball has been extracted from urn B and put into urn A: the probability of this extraction is $\frac{1}{2}$. Moreover, a white ball has been further extracted from urn A, and the probability of this extraction is $\frac{n}{n+1}$; by multiplying therefore $\frac{1}{2}$ by $\frac{n}{n+1}$, and the product by the probability $(1-q)q'$, that the second witness says the truth, while the first does not say it, we will have

$$\frac{(1-q)q'n}{2(n+1)},$$

for the probability relative to the third hypothesis.

Finally, under the fourth hypothesis, a black ball has first been extracted from urn B, and the probability of this extraction is $\frac{1}{2}$. Next this black ball put into urn A, has been extracted from it in the second drawing, and the probability of this extraction is $\frac{1}{n+1}$; by multiplying therefore the product of these two probabilities, by the probability $(1-q)(1-q')$ that none of the witnesses says the truth, we will have [452]

$$\frac{(1-q)(1-q')}{2(n+1)}$$

for the probability relative to the fourth hypothesis.

Now the probability of the fact which results from the collection of the two testimonies, namely, that a white ball extracted in the first drawing has reappeared in the second drawing, is clearly equal to the probability relative to the first hypothesis, divided by the sum of the probabilities relative to the four hypotheses; this probability is therefore

$$\frac{qq'}{qq'+(1-q)(1-q')+[q(1-q')+q'(1-q)]n}.$$

The phenomenon of the reappearance of a white ball in the second drawing, becomes so much more extraordinary, as the number n of balls of each urn is more considerable; and then the preceding probability becomes very small. We see therefore that the probability of the fact resulting from the collection of the witnesses is extremely weak, when it is extraordinary.

§47. Let us consider simultaneous witnesses: let us suppose two witnesses in accord on a fact, and let us determine its probability. In order to fix the ideas, let us suppose that the fact is the extraction of the n° i, from an urn which contains the number n of them; such that the observed event is the accord of two witnesses of the drawing, to enunciate the exit of the n° i. Let us name p and p' their respective veracities; and let us suppose, in order to simplify, that they are not deceived at all. This premised, we are able to form only these two hypotheses: the witnesses say the truth: the witnesses deceive.

[454] Under the first hypothesis, the n° i exited, and the probability of this event is $\frac{1}{n}$. By multiplying it by the product of the veracities p and p' of the witnesses; we will have $\frac{pp'}{n}$ for the probability of the observed event, under this hypothesis.

In the second, the n° i is not drawn, and the probability of this event is $\frac{n-1}{n}$; but the two witnesses are agreed to choose the n° i among the $n-1$ non-drawn tickets. Now the number of different combinations which are able to result from their choice is $(n-1)^2$, and in this number, they must choose that where the n° i is combined with itself; the probability of this choice is therefore $\frac{1}{(n-1)^2}$. By multiplying it by the preceding probability $\frac{n-1}{n}$, and by the products of the probabilities $1-p$ and $1-p'$ that the witnesses deceive; we will have $\frac{(1-p)(1-p')}{n.\overline{n-1}}$ for the probability of the observed event, under the second hypothesis.

Now, we will have the probability of the exit of the n° i, by dividing the probability relative to the first hypothesis, by the sum of the probabilities relative to the two hypotheses; we will have therefore, for this probability,

$$\frac{pp'}{pp' + \frac{(1-p)(1-p')}{n-1}}; \tag{o}$$

if $n=2$, then the exit of the n° i is as probable as its non-exit; and the probability of its exit, resulting from the accord of the testimonies, is

$$\frac{pp'}{pp' + (1-p)(1-p')}.$$

This is generally the probability of a fact attested by two witnesses, when the existence of the fact is as probable as its nonexistence. If the two witnesses are equally truthful, that which gives $p'=p$, this probability becomes

$$\frac{p^2}{p^2 + (1-p)^2}.$$

[455] In general, If the number r of the equally truthful witnesses, affirm the existence of a fact of this kind; its probability resulting from the testimonies will be

$$\frac{p^r}{p^r + (1-p)^r}.$$

But this formula is applicable only in the case where the existence of the fact and its nonexistence are in themselves, equally probable.

If the number n of the tickets of the urn is very great, formula (o) becomes to very nearly unity; and consequently the exit of n° i is extremely probable. That holds to this that it is not very credible that the witnesses wishing to deceive, are agreed to enunciate the same ticket, when the urn contains a great number of them. Simple good sense indicates this result from the calculus; but we see at the same time that the probability of the exit of the n° i is much diminished, if the two witnesses seeking to deceive, have been able to hear one another.

Let us suppose now that the first witness affirms the exit of the n° i, and that the second witness affirms the exit of the n° i'. We can form then the following three hypotheses. The first witness says the truth and the second deceives. In this case, the n° i exited, and the probability of this event is $\frac{1}{n}$. Moreover, the second witness who deceives, must choose among the other non-drawn tickets, the n° i', and the probability of this choice is $\frac{1}{n-1}$. The product of these two probabilities, by the product of the probabilities p and $1-p'$, that the first witness not deceive and that the second deceive, will be the probability of the observed event, or of the enunciation of the exit of the n$^{\text{os}}$ i and i', under this hypothesis; a probability which is thus $\frac{p(1-p')}{n.\overline{n-1}}$.

Under the second hypothesis, the first witness deceives, and the second does not deceive. Then the n° i' exited; and the probability of this event is $\frac{1}{n}$. Moreover, the first witness chooses the n° i out of the $n-1$ non-drawn tickets, and the probability of this choice is $\frac{1}{n-1}$. By multiplying the product of these two probabilities, by the product of the probabilities $1-p$ and p', that the first witness deceives and that the second does not deceive, we will have $\frac{(1-p)p'}{n(n-1)}$. [456]

Finally, under the third hypothesis, the two witnesses deceive at the same time. Then none of the two tickets i and i' exited. The probability of this event is $\frac{n-2}{n}$. Moreover, the first witness must choose the n° i, and the second must choose the n° i', among the $n-1$ non-drawn tickets, and the probability of this composite event is $\frac{1}{(n-1)^2}$. By multiplying the product of these two probabilities, by the product of the probabilities $1-p$ and $1-p'$ that the first and the second witness deceive; we will have $\frac{\overline{n-2}.\overline{1-p}.\overline{1-p'}}{n(n-1)^2}$ for the probability of the observed event, under this hypothesis.

Now, we will have the probability of the exit of the n° i, by dividing the probability relative to the first hypothesis, by the sum of the probabilities relative to the three hypotheses; the probability of this exit is therefore

$$\frac{p(1-p')}{1-pp'-\frac{(1-p)(1-p')}{n-1}}.$$

If $n=2$, that is if the existence of each fact attested by the two witnesses, is *a priori*, as probable as its nonexistence; then the preceding probability becomes $\frac{1}{2}$, when $p=p'$; that which is clear besides, the two testimonies destroying themselves reciprocally. In general, if a fact of this kind is attested by r witnesses, and denied by r' witnesses, all equally truthful; it is easy to see that its probability will be

$$\frac{p^{r-r'}}{p^{r-r'}+(1-p)^{r-r'}};$$

that is, the same as if the fact was attested by $r - r'$ witnesses.

§48. Let us consider presently a traditional chain of r witnesses, and let us suppose
[457] that the fact transmitted is the exit of the n° i from an urn which contains n tickets. Let us designate by y_r its probability. The addition of a new witness will change this probability into y_{r+1}, a probability which will be formed, 1° from the product of y_r by the veracity of the new witness, a veracity that we will designate by p_{r+1}; 2° from the product of the probability $1 - p_{r+1}$ that this new witness deceives, by the probability $1 - y_r$ that the preceding witness has not said the truth, and by the probability $\frac{1}{n-1}$ that the new witness will choose the drawn ticket, in the number of the $n-1$ other tickets than the one which has been indicated to him by the preceding witness; therefore we will have

$$y_{r+1} = p_{r+1}y_r + \frac{1}{n-1}(1 - p_{r+1})(1 - y_r);$$

an equation of which the integral is

$$y_r = \frac{1}{n} + C\frac{(np_1 - 1)(np_2 - 1)\cdots(np_r - 1)}{(n-1)^r},$$

C being an arbitrary constant. In order to determine it, we will observe that the probability of the fact, after the first testimony, is, by that which precedes, equal to p_1; we have therefore $y_1 = p_1$; that which gives $C = \frac{n-1}{n}$; hence

$$y_r = \frac{1}{n} + \frac{n-1}{n} \cdot \frac{(np_1 - 1)(np_2 - 1)\cdots(np_r - 1)}{(n-1)^r}.$$

If n is infinite, we have

$$y_r = p_1p_2\dots p_r.$$

If $n = 2$, that is, if the existence of the fact is as probable as its nonexistence; we have

$$y_r = \tfrac{1}{2} + \tfrac{1}{2}(2p_1 - 1)(2p_2 - 1)\cdots(2p_r - 1).$$

In general, in measure as the traditional chain is prolonged, y_r approaches indefinitely to its limit $\frac{1}{n}$, a limit which is the probability *a priori*, of the exit of the n° i. The
[458] term $\frac{n-1}{n} \cdot \left(\frac{np_1-1}{n-1}\right)$ .etc. of the expression of y_r, is therefore that which the chain of witnesses adds to this probability. We see thus how the probability is weakened in measure as the tradition is prolonged. In truth, the monuments, printings and other causes are able to diminish this inevitable effect of time; but they are never able to entirely destroy it.

If we have two traditional chains, each of r witnesses, if we suppose the witnesses of these chains, equally truthful, and if the last witness of the one of the chains, accords with the last of the other, to affirm the exit of the n° i, we will have the probability of this exit, by substituting y_r for p and p', in formula (o) of the preceding section, which becomes thence

$$\frac{y_r^2}{y_r^2 + \frac{(1-y_r)^2}{n-1}}.$$

§49. Let us consider two witnesses of whom p and p' are the respective veracities. We know that both, or at least one of the them, without being contradicted by the other who, in this case, has not pronounced at all, affirm that the n° i exited from an urn which contains the number n of them. By supposing always that we have extracted only a single ticket, we demand the probability of the exit of the n° i.

Let r and r' be the respective probabilities that the witnesses pronounce. We are able to make here only the following four hypotheses: 1° the two witnesses pronounce and say the truth; 2° the two witnesses pronounce and deceive; 3° one of the witnesses pronounces and says the truth, and the other witness does not pronounce; 4° one of the witnesses pronounces and deceives, and the other does not pronounce at all.

Under the first hypothesis, the n° i exited, and the probability of this event is $\frac{1}{n}$. It is necessary to multiply it by the product of the probabilities r and r' that the two witnesses have pronounced, and by the product of the probabilities p and p' that they say the truth; we will have thus

$$\frac{pp'.rr'}{n}$$

for the probability of the observed event, under this hypothesis.

In the second, the n° i did not exit, and the probability of this event is $\frac{n-1}{n}$. But [459] if the two witnesses deceive without hearing one another, the probability that they will agree to enunciate the same n° i, is $\frac{1}{(n-1)^2}$. It is necessary to multiply the product of these probabilities by the probability rr' that the two witnesses pronounce at the same time, and by the probability $(1-p)(1-p')$ that they both deceive. We will have thus

$$\frac{(1-p)(1-p')rr'}{n.\overline{n-1}}$$

for the probability of the observed event under the second hypothesis.

Under the third, the n° i exited, and the probability of this event is $\frac{1}{n}$. It is necessary to multiply by the probability $pr(1-r')+p'r'(1-r)$ that one of the witnesses pronounces by saying the truth, while the other witness does not pronounce it at all. We will have thus

$$\frac{pr(1-r')+p'r'(1-r)}{n}$$

for the probability of the observed event under this hypothesis.

Finally, under the fourth, the n° i did not exit, and the probability of this event is $\frac{n-1}{n}$; but the witness who deceives, must choose it in the $n-1$ non-drawn tickets, and the probability of this choice is $\frac{1}{n-1}$. It is necessary to multiply the product of these probabilities by the probability $(1-p)r(1-r')+(1-p')r'(1-r)$ that one of the witnesses pronouncing, deceives, while the other witness does not pronounce at all. We have thus

$$\frac{(1-p)r(1-r')+(1-p')r'(1-r)}{n}$$

for the probability corresponding to the fourth hypothesis.

Now we will have the probability of the exit of the n° i, by dividing the sum of the probabilities relative to the first and to the third hypothesis, by the sum of the [460] probabilities relative to all the hypotheses; that which gives, for this probability,

$$\frac{pp'.rr' + pr(1-r') + p'r'(1-r)}{pp'.rr' + r(1-r') + r'(1-r) + \frac{(1-p)(1-p')rr'}{n-1}}.$$

These examples indicate sufficiently the method to subject to calculus of probabilities, the testimonies.

§50. We are able to assimilate the judgment of a tribunal which pronounces between two contradictory opinions, to the result of the testimonies of many witnesses of the extraction of a ticket from one urn which contains only two tickets. In expressing by p the probability that the judge pronounces the truth; the probability of the goodness of a judgment rendered by unanimity will be, by that which precedes,

$$\frac{p^r}{p^r + (1-p)^r},$$

r being the number of judges. We are able to determine p by the observation of the ratio of the judgments rendered by unanimity by the tribunal, to the total number of judgments. When this number is very great; by designating it by n, and by i the number of judgments rendered by unanimity; we will have very nearly

$$p^r + (1-p)^r = \frac{i}{n};$$

the resolution of this equation will give the veracity p of the judges. This equation is reduced to a degree less by half, by making $p = 1 + \sqrt{u}$. It becomes thus

$$(1+\sqrt{u})^r + (1-\sqrt{u})^r = \frac{i}{n};$$

an equation which developed is of degree $\frac{r}{2}$, or $\frac{r-1}{2}$, according as r is even or odd.

The probability of the goodness of a new judgment rendered by unanimity, will be

$$1 - \tfrac{n}{i}(1-p)^r.$$

[461] If we suppose the tribunal formed of three judges, we will have

$$p = \frac{1}{2} \pm \sqrt{\frac{4i-n}{12n}}.$$

We will adopt the + sign; because it is natural to suppose to each judge, a greater probability for the truth than for error. If the half of the judgments rendered by the tribunal, have been rendered by unanimity; then $\frac{i}{n} = \frac{1}{2}$, and we find $p = 0,789$. The probability of a new judgment rendered by unanimity, will be $0,981$. If this judgment is rendered only by plurality, its probability will be p or $0,789$.

In general, we see that the probability $1 - \frac{n}{i}(1-p)^r$ of the goodness of a new judgment rendered by unanimity, is so much greater, as r is a greater number, and as the values of p and of $\frac{i}{n}$ are greater, that which depends on the wisdom of the judges. There is therefore a great advantage to form the tribunals of appeal, composed of a great number of judges chosen among the most enlightened persons.

First Supplement.

ON THE APPLICATION OF THE CALCULUS OF PROBABILITIES TO NATURAL PHILOSOPHY

The phenomena of nature are most often enveloped in so many strange circumstances; so great a number of perturbing causes mix their influence; that it is very difficult to recognize them. We can arrive there, only by multiplying the observations or experiments, so that the strange effects coming to be destroyed reciprocally, the mean results set into evidence, these phenomena and their diverse elements. The more the observations are numerous, and the less they deviate among themselves; the more their results approach to the truth. We fulfill this last condition, by the choice of methods, by the precision of the instruments, and by the care that we take to observe well. Next, we determine through the theory of probabilities, the most advantageous mean results, or those which lay us less open to error. But this does not suffice; it is more necessary to estimate the probability that the errors of these results, are comprehended within some given limits. Without this, we have only an imperfect knowledge of the degree of exactitude obtained. Some formulas proper to this object, are therefore a true perfection of the method of the sciences, and so it is quite important to add to this method. The analysis that they require, is the most delicate and the most difficult of the theory of probabilities. This is one of the things that I have had principally in view in my work, in which I am arrived to some formulas of this kind, which have the remarkable advantage to be independent of the law of probabilities of the errors, and to contain only quantities given by the observations themselves and by their expressions. I will recall here the principles. [3] [4]

Each observation has for analytic expression, a function of the elements that we wish to determine; and if these elements are nearly known, this function becomes a linear function of their corrections. By equating it to the same observation, we form that which we name the *equation of condition*. If we have a great number of similar equations, we combine them in a manner to obtain as many final equations, as there are elements of which we determine next the corrections, by resolving these equations. But what is the most advantageous manner to combine the equations of condition, in order to obtain the final equations? What is the law of the errors of which the elements that we deduce from it, are yet susceptible? It is this that the theory of probabilities makes known. The formation of a final equation by means of the equations of condition, reverts to multiplying each of these by an indeterminate factor, and to reunite these products; but it is necessary to choose the system of factors, which give the least error to fear. Now it is clear that if we multiply the possible errors of an

element, by their respective probabilities, the most advantageous system will be the one in which the sum of these products, all taken positively, is a *minimum*; because a positive or negative error, must be considered as a loss. By forming therefore this sum of products, the condition of the *minimum* will determine the system of factors that it is necessary to choose. We find thus that this system is the one of the coefficients of the elements in each equation of condition; so that we form a first final equation,
[5] by multiplying respectively each equation of condition, by its coefficient of the first element, and by reuniting all these equations thus multiplied. We form a second final equation, by employing likewise the coefficients of the second element, and so forth. In this manner, the elements and the laws of the phenomena, contained in the compilation of a great number of observations, are developed with the most evidence. I have given, in §21 of the second Book of my *Théorie analytique des Probabilités*, the expression of the mean error to fear respecting each element. This expression gives the probability of the errors of which the element is further susceptible, and which is proportional to the number of which the hyperbolic logarithm is unity, raised to a power equal to the square of the error taken negative, and divided by the square of the double of this expression, and by the ratio of the circumference to the diameter. The coefficient of the negative square of the error in this exponent, is able therefore to be considered as the modulus of the probability of the errors, since the error remaining the same, the probability decreases with rapidity, when it increases; so that the result obtained weighs, if I am able to say so, toward the truth, so much more as this modulus is greater. I will name, for this reason, this modulus, *weight* of the result. By a remarkable analogy of these weights with those of bodies compared to their common center of gravity, it happens that, if one same element is given by diverse systems composed, each, of a great number of observations; the most advantageous mean result of them altogether, is the sum of the products of each partial result, by its weight; this sum being divided by the sum of all the weights. Moreover, the total weight of the diverse systems is the sum of their partial weights; so that the probability of the mean result of them altogether, is proportional to the number which has unity for hyperbolic logarithm, raised to a power equal to the square of the error, taken negative and multiplied by the sum of all the weights. Each weight depends, in truth, on the law of probability of the errors in each system; and nearly always this law is unknown; but I am happily arrived to eliminate the factor which contains it, by means of the sum of the squares of the deviations of the observations of the system, from their
[6] mean result. It would be therefore to desire, in order to complete our understandings on the results obtained by the totality of a great number of observations, that we wrote beside each result, the weight which corresponds to it. In order to facilitate the calculation of these weights, I develop its analytic expression, when we have no more than three elements to determine. But this expression becoming more and more complicated, in measure as the number of elements increase; I give a quite simple way in order to determine the weight of a result, whatever be the number of elements. When we have obtained thus the exponential which represents the law of probability of the errors; we will have the probability that the error of the result is comprehended within some given limits, by taking within these limits, the integral of the product of

this exponential, by the differential of the error, and by multiplying it by the square root of the weight of the result, divided by the circumference of which the diameter is unity. Thence it follows that for one same probability, the errors of the results are reciprocals to the square roots of their weights; that which can serve to compare their respective precision.

In order to apply this method with success, it is necessary to vary the circumstances of the observations or of the experiments, in a manner to avoid the constant causes of the error. It is necessary that the observations be numerous, and that they be so many more, as there are more elements to determine; because the weight of the mean result increases as the number of the observations, divided by the number of elements. It is further necessary that the elements follow in these observations, a different march; because if the march of the two elements were rigorously the same, that which would render their coefficients proportionals in the equations of condition; these elements would form only a single unknown, and it would be impossible to distinguish them by these observations. Finally, it is necessary that the observations be precise. This condition, the first of all, increases much the weight of the result, of which the expression has for divisor, the sum of the squares of their deviations from this result. With these precautions, we will be able to make use of the preceding method, and to measure the degree of confidence which the results deduced from a great number of observations merit.

§1. A great advantage of this method, which permits evaluating numerically the expressions of it, is, as we have said, to be independent of the law of probability of the [7]
errors of the observations. The factor $\frac{2k''}{k}a^2s$, which depends on this law, has been eliminated from the formulas of §§19 and 21 of the second Book, by observing that this factor which is the sum of the squares of all the possible errors of the observations, multiplied by their respective probabilities, and which expresses thus the true mean of these squares, is very probably equal to the sum of the squares of the rest of the equations of condition, when we have substituted the elements determined by the most advantageous method. The importance of this method in natural philosophy, requires that the uncertainty that it can permit, is dissipated; and the only one which remains yet, is relative to the equality of which I just spoke. I will first clarify this delicate point of the theory of the probabilities, and show that the preceding equality can be employed without sensible error.

The sum of the squares of the errors of the observations of which the number is s, being supposed equal to $\frac{2k''}{k}a^2s + a^2r\sqrt{s}$; the probability that the value of r is comprehended within the given limits, is by §19 cited,

$$\frac{1}{2\sqrt{\pi}}\int \beta' dr\, c^{-\frac{\beta'^2 r^2}{4}},$$

the integral being taken within these limits. Let us represent the general equation of condition, of the elements z, z', etc., by this one

$$\epsilon^{(i)} = p^{(i)}z + q^{(i)}z' + \text{etc.} - \alpha^{(i)};$$

$\epsilon^{(i)}$ being the error of the observation. The elements z, z', etc., being determined by the most advantageous method; let us designate by u, u', etc., their errors; we will have, by naming $\epsilon'^{(i)}$ the remainder of the function

$$p^{(i)}z + q^{(i)}z' + \text{etc.} - \alpha^{(i)},$$

when we have substituted for z, z', etc. their values thus determined,

$$\epsilon^{(i)} = \epsilon'^{(i)} + p^{(i)}u + q^{(i)}u' + \text{etc.},$$

[8] that which gives

$$S\epsilon^{(i)2} = S\epsilon'^{(i)2} + 2S\epsilon'^{(i)}(p^{(i)}u + q^{(i)}u' + \text{etc.}) + S(p^{(i)}u + q^{(i)}u' + \text{etc.})^2;$$

the integral sign S being extended to all the values of i, from $i = 0$ to $i = s - 1$. But by the conditions of the most advantageous method, we have

$$Sp^{(i)}\epsilon'^{(i)} = 0, \quad Sq^{(i)}\epsilon'^{(i)} = 0, \quad \text{etc.};$$

we have therefore

$$S\epsilon^{(i)^2} = S\epsilon'^{(i)^2} + S(p^{(i)}u + q^{(i)}u' + \text{etc.})^2;$$

by comparing this value of $S\epsilon^{(i)^2}$ to its preceding value $\frac{2k''}{k}a^2s + a^2r\sqrt{s}$; we will have

$$a^2r\sqrt{s} = S\epsilon'^{(i)^2} - \frac{2k''}{k}a^2s + S(p^{(i)}u + q^{(i)}u' + \text{etc.})^2.$$

Let us make

$$S\epsilon'^{(i)^2} - \frac{2k''}{k}a^2s = t\sqrt{s},$$

$$u = \frac{\nu}{\sqrt{s}}, \quad u' = \frac{\nu'}{\sqrt{s}}, \quad u'' = \frac{\nu''}{\sqrt{s}}, \quad \text{etc.};$$

we will have

$$a^2r = t + \frac{S(p^{(i)}\nu + q^{(i)}\nu' + \text{etc.})^2}{s\sqrt{s}}.$$

The exponential $c^{-\frac{\beta'^2r^2}{4}}$ becomes thus

$$c^{-\frac{\beta'^2}{4a^2}\left\{t + \frac{S(p^{(i)}\nu + q^{(i)}\nu' + \text{etc.})^2}{s\sqrt{s}}\right\}^2};$$

thus the probability of t is proportional to this exponential.

The analysis of §21 of the second Book leads to this general theorem, namely that the probability of the simultaneous existence of the quantities u, u', u'', etc., is proportional to the exponential

$$c^{-\frac{k}{4k''a^2s}S(p^{(i)}\nu + q^{(i)}\nu' + \text{etc.})^2};$$

[9] the probability of the simultaneous existence of t, ν, ν', ν'', etc., is therefore proportional to

$$c^{-\frac{\beta'^2}{4a^4}\left\{t + \frac{S(p^{(i)}\nu + q^{(i)}\nu' + \text{etc.})^2}{s\sqrt{s}}\right\}^2 - \frac{k}{4k''a^2s}S(p^{(i)}\nu + q^{(i)}\nu' + \text{etc.})^2}.$$

By substituting for $\frac{4k''a^2s}{k}$, its value $2\text{S}\epsilon'^{(i)} - 2t\sqrt{s}$; this exponential is reduced by neglecting the terms of order $\frac{1}{s}$, in the following function:

$$\left\{1 - \frac{t\sqrt{s}}{2(S\epsilon'^{(i)2})^2}S(p^{(i)}\nu + q^{(i)}\nu' + \text{etc.})^2\right\}$$
$$\times c^{-\frac{\beta'^2}{4a^4}}\left\{t + \frac{S(p^{(i)}\nu + q^{(i)}\nu' + \text{etc.})^2}{s\sqrt{s}}\right\}^2 - \frac{S(p^{(i)}\nu + q^{(i)}\nu' + \text{etc.})^2}{2S\epsilon'^{(i)2}}$$

Now, in order to have the probability that the value of ν is comprehended within some given limits, it is necessary 1° to multiply this function by $dt\,d\nu\,d\nu'$.etc.; 2° to take the integral of the product, for all the possible values of t, ν', ν'', etc.; and with respect to ν, to integrate only within the given limits; 3° to divide the whole, by this same integral taken with respect to all the possible values of t, ν, ν', etc. By regarding $S\epsilon'^{(i)2}$ as a datum from observation, t varies only at the rate of the unknown value of $\frac{2k''a^2s}{k}$, and this value can vary from zero to infinity; t can therefore vary from $\frac{S\epsilon'^{(i)2}}{\sqrt{s}}$ to negative infinity; and as $S\epsilon'^{(i)2}$ is of the order of s, t can vary from negative infinity to a positive value of order $\sqrt{s}$. The preceding exponential becomes at that limit of the integral taken with respect to t, of the form c^{-Q^2s}, and will be able to be supposed null, because of the magnitude of s. Thus we can take the integral relative to t, from $t = -\infty$ to $t = \infty$. Similarly the integrals relative to ν', ν'', etc., can be taken within the same limits. If we make

$$t + \frac{S(p^{(i)}\nu + q^{(i)}\nu' + \text{etc.})^2}{s\sqrt{s}} = t',$$

the integral relative to t' will be able to be taken with respect to t' from $t' = -\infty$, to [10]
$t' = \infty$.

Thence it is easy to conclude that the probability that ν is comprehended within the given limits, is proportional to the integral

$$\int d\nu d\nu'\text{etc.}\left\{1 + \frac{\left[S(p^{(i)}\nu + q^{(i)}\nu' + \text{etc.})^2\right]^2}{(2S\epsilon'^{(i)2})^2 s}\right\}c^{-\frac{S(p^{(i)}\nu+q^{(i)}\nu'+\text{etc.})^2}{2S\epsilon^{(i)2}}},$$

the integrals being taken from ν', ν'', etc., equal to $-\infty$ to their positive infinite values, and with respect to ν, within the given limits; and being divided by the same integral extended to the positive and negative infinite values of ν, ν', ν'', etc.

The consideration of the difference which can exist between $\frac{2k''}{k}a^2s$ and $S\epsilon'^{(i)2}$ introduces therefore into the expression of the probability of which there is concern, only one term of order $\frac{1}{s}$, an order that I myself am permitted to neglect in my work. Thence, the preceding integral becomes

$$\int d\nu d\nu'.\text{etc.}c^{-\frac{S(p^{(i)}\nu+q^{(i)}\nu'+\text{etc.})^2}{2S\epsilon'^{(i)2}}}.$$

If we make

$$p_1^{(i)} = p^{(i)} - \frac{q^{(i)}Sp^{(i)}q^{(i)}}{Sq^{(i)2}},$$
$$r_1^{(i)} = r^{(i)} - \frac{q^{(i)}Sr^{(i)}q^{(i)}}{Sq^{(i)2}},$$
$$t_1^{(i)} = t^{(i)} - \frac{q^{(i)}St^{(i)}q^{(i)}}{Sq^{(i)2}},$$
etc.,

the exponential

$$c^{-\frac{S(p^{(i)}\nu+q^{(i)}\nu'+r^{(i)}\nu''+\text{etc.})^2}{2S\epsilon'^{(i)2}}}$$

will be able to be set under this form

$$c^{-\frac{S(p_1^{(i)}\nu+r_1^{(i)}\nu''+\text{etc.})^2}{2S\epsilon'^{(i)2}}-\frac{Sq^{(i)2}}{2S\epsilon'^{(i)2}}\left(\nu'+\left(\frac{\nu Sp^{(i)}q^{(i)}+\nu''Sr^{(i)}q^{(i)}+\text{etc.}}{Sq^{(i)2}}\right)\right)^2}.$$

[11] By multiplying this quantity by $d\nu'$, and by integrating it from $\nu' = -\infty$, to $\nu' = \infty$, we will have a quantity proportional to

$$c^{-\frac{S(p_1^{(i)}\nu+r_1^{(i)}\nu''+\text{etc.})^2}{2S\epsilon'^{(i)2}}},$$

and in which the variable ν' has disappeared. By following the same process, we will make the variables ν'', ν''', etc. vanish. We will arrive thus to an exponential of the form $c^{-\frac{\nu^2 Sp_{n-1}^{(i)2}}{2S\epsilon'^{(i)2}}}$, n being the number of elements. If we restore instead of ν, its value $u\sqrt{s}$, this exponential becomes

$$c^{-Pu^2},$$

by making

$$P = \frac{sSp^{(i)2}}{2S\epsilon'^{(i)2}}.$$

u being the error of the value of z, P is that which I name *weight* of this value. The probability that this error is comprehended within some given limits is therefore

$$\frac{\int du\sqrt{P}c^{-Pu^2}}{\sqrt{\pi}},$$

the integral being taken within these limits, and π being the circumference of which the diameter is unity. But it is simpler to apply the process of which we have just made use, to the final equations which determine the elements, in order to reduce them to one alone; that which gives an easy method to resolve these equations.

§2. Let us take the general equation of condition, and for more simplicity, let us limit it to the six elements z, z', z'', z''', z^{iv}, z^{v}; it becomes then

$$\epsilon^{(i)} = p^{(i)}z + q^{(i)}z' + r^{(i)}z'' + t^{(i)}z''' + \gamma^{(i)}z^{\text{iv}} + \lambda^{(i)}z^{\text{v}} - \alpha^{(i)}. \quad (1)$$

By multiplying it by $\lambda^{(i)}$, and reuniting all the products together; we will have

$$S\lambda^{(i)}\epsilon^{(i)} = zS\lambda^{(i)}p^{(i)} + z'S\lambda^{(i)}q^{(i)} + \text{etc.} - S\lambda^{(i)}\alpha^{(i)},$$

the integral sign S, extending to all the values of i, from $i = 0$ to $i = s - 1$, s being [12]
the number of observations employed. By the conditions of the most advantageous method, we have $S\lambda^{(i)}\epsilon^{(i)} = 0$; the preceding equation will give therefore

$$z^{\text{v}} = -z^{\text{iv}}\frac{S\lambda^{(i)}\gamma^{(i)}}{S\lambda^{(i)2}} - z'''\frac{S\lambda^{(i)}t^{(i)}}{S\lambda^{(i)2}} - z''\frac{S\lambda^{(i)}r^{(i)}}{S\lambda^{(i)2}} - z'\frac{S\lambda^{(i)}q^{(i)}}{S\lambda^{(i)2}} - z\frac{S\lambda^{(i)}p^{(i)}}{S\lambda^{(i)2}} + \frac{S\lambda^{(i)}\alpha^{(i)}}{S\lambda^{(i)2}}$$

If we substitute this value, into equation (1), and if we make

$$\begin{aligned}
\gamma_1^{(i)} &= \gamma^{(i)} - \lambda^{(i)}\frac{S\lambda^{(i)}\gamma^{(i)}}{S\lambda^{(i)2}},\\
t_1^{(i)} &= t^{(i)} - \lambda^{(i)}\frac{S\lambda^{(i)}t^{(i)}}{S\lambda^{(i)2}},\\
r_1^{(i)} &= r^{(i)} - \lambda^{(i)}\frac{S\lambda^{(i)}r^{(i)}}{S\lambda^{(i)2}},\\
q_1^{(i)} &= q^{(i)} - \lambda^{(i)}\frac{S\lambda^{(i)}q^{(i)}}{S\lambda^{(i)2}},\\
p_1^{(i)} &= p^{(i)} - \lambda^{(i)}\frac{S\lambda^{(i)}p^{(i)}}{S\lambda^{(i)2}},\\
\alpha_1^{(i)} &= \alpha^{(i)} - \lambda^{(i)}\frac{S\lambda^{(i)}\alpha^{(i)}}{S\lambda^{(i)2}},
\end{aligned}$$

we will have

$$\epsilon^{(i)} = p_1^{(i)}z + q_1^{(i)}z' + r_1^{(i)}z'' + t_1^{(i)}z''' + \gamma_1^{(i)}z^{\text{iv}} - \alpha_1^{(i)}; \tag{2}$$

by this means, the element z^{v} has disappeared from the equations of condition, that equation (2) represents. By multiplying this equation by $\gamma_1^{(i)}$ and reuniting all the products together; by observing next that we have $S\gamma_1^{(i)}\epsilon^{(i)} = 0$, by virtue of the equations

$$0 = S\lambda^{(i)}\epsilon^{(i)}, \quad 0 = S\gamma^{(i)}\epsilon^{(i)}$$

that the conditions of the most advantageous method give; we will have

$$0 = zS\gamma_1^{(i)}p_1^{(i)} + z'S\gamma_1^{(i)}q_1^{(i)} + z''S\gamma_1^{(i)}r_1^{(i)} + z'''S\gamma_1^{(i)}t_1^{(i)} + z^{\text{iv}}S\gamma_1^{(i)2} - S\gamma_1^{(i)}\alpha_1^{(i)};$$

whence we deduce [13]

$$z^{\text{iv}} = -z'''\frac{S\gamma_1^{(i)}t_1^{(i)}}{S\gamma_1^{(i)2}} - z''\frac{S\gamma_1^{(i)}r_1^{(i)}}{S\gamma_1^{(i)2}} - z'\frac{S\gamma_1^{(i)}q_1^{(i)}}{S\gamma_1^{(i)2}} - z\frac{S\gamma_1^{(i)}p_1^{(i)}}{S\gamma_1^{(i)2}} + \frac{S\gamma_1^{(i)}\alpha_1^{(i)}}{S\gamma_1^{(i)2}}.$$

If we substitute this value into equation (2), and if we make

$$t_2^{(i)} = t_1^{(i)} - \gamma_1^{(i)} \frac{S\gamma_1^{(i)} t_1^{(i)}}{S\gamma_1^{(i)2}},$$

$$r_2^{(i)} = r_1^{(i)} - \gamma_1^{(i)} \frac{S\gamma_1^{(i)} r_1^{(i)}}{S\gamma_1^{(i)2}},$$

$$q_2^{(i)} = q_1^{(i)} - \gamma_1^{(i)} \frac{S\gamma_1^{(i)} q_1^{(i)}}{S\gamma_1^{(i)2}},$$

$$p_2^{(i)} = p_1^{(i)} - \gamma_1^{(i)} \frac{S\gamma_1^{(i)} p_1^{(i)}}{S\gamma_1^{(i)2}},$$

$$\alpha_2^{(i)} = \alpha_1^{(i)} - \gamma_1^{(i)} \frac{S\gamma_1^{(i)} \alpha_1^{(i)}}{S\gamma_1^{(i)2}},$$

we will have

$$\epsilon^{(i)} = p_2^{(i)} z + q_2^{(i)} z' + r_2^{(i)} z'' + t_2^{(i)} z''' - \alpha_2^{(i)}. \tag{3}$$

By continuing thus, we will arrive to an equation of the form

$$\epsilon^{(i)} = p_5^{(i)} z - \alpha_5^{(i)}. \tag{4}$$

There results from §20 of the second Book, that if the value of z is determined by this equation, and if u is the error of this value; the probability of this error is

$$\sqrt{\frac{sSp_5^{(i)2}}{2S\epsilon'^{(i)2}\pi}} c^{-\frac{sSp_5^{(i)2}}{2S\epsilon'^{(i)2}} u^2},$$

$S\epsilon'^{(i)2}$ being the sum of the squares of the remainders of the equations of condition, when we have substituted there the elements determined by the most advantageous method. The weight P of this error is therefore equal to $\frac{sSp_5^{(i)2}}{2S\epsilon'^{(i)2}}$.

[14] The concern now is to determine $Sp_5^{(i)2}$. For this, we will multiply respectively each of the equations of condition, represented by equation (1), first by the coefficient of the first element, and we will take the sum of these products; next by the coefficient of the second element, and we will take the sum of these products; and thus of the rest. We will have by observing that by the conditions of the most advantageous method, $Sp^{(i)}\epsilon^{(i)} = 0$, $Sq^{(i)}\epsilon^{(i)} = 0$, etc., the six equations following:

$$\begin{aligned}
\overline{p\alpha} &= p^{(2)}.z + \overline{pq}.z' + \overline{pr}.z'' + \overline{pt}.z''' + \overline{p\gamma}.z^{\text{iv}} + \overline{p\lambda}.z^{\text{v}},\\
\overline{q\alpha} &= \overline{pq}.z + q^{(2)}.z' + \overline{qr}.z'' + \overline{qt}.z''' + \overline{q\gamma}.z^{\text{iv}} + \overline{q\lambda}.z^{\text{v}},\\
\overline{r\alpha} &= \overline{rp}.z + \overline{rq}.z' + r^{(2)}.z'' + \overline{rt}.z''' + \overline{r\gamma}.z^{\text{iv}} + \overline{r\lambda}.z^{\text{v}},\\
\overline{t\alpha} &= \overline{tp}.z + \overline{tq}.z' + \overline{tr}.z'' + t^{(2)}.z''' + \overline{t\gamma}.z^{\text{iv}} + \overline{t\lambda}.z^{\text{v}},\\
\overline{\gamma\alpha} &= \overline{\gamma p}.z + \overline{\gamma q}.z' + \overline{\gamma r}.z'' + \overline{\gamma t}.z''' + \gamma^{(2)}.z^{\text{iv}} + \overline{\gamma\lambda}.z^{\text{v}},\\
\overline{\lambda\alpha} &= \overline{\lambda p}.z + \overline{\lambda q}.z' + \overline{\lambda r}.z'' + \overline{\lambda t}.z''' + \overline{\lambda\gamma}.z^{\text{iv}} + \lambda^{(2)}.z^{\text{v}},
\end{aligned} \tag{A}$$

whence we must observe that we suppose

$$p^{(2)} = Sp^{(i)2}, \quad \overline{pq} = Sp^{(i)}q^{(i)}, \quad q^{(2)} = Sq^{(i)2}, \quad \overline{qr} = Sq^{(i)}r^{(i)}; \quad \text{etc.}$$

If we multiply similarly the equations of condition represented by equation (2), respectively by the coefficients of z, and if we add these products; next by the coefficients of z', by adding again these products, and so forth; we will have the following system of equations by observing that $Sp_1^{(i)}\epsilon^{(i)} = 0$, $Sq_1^{(i)}\epsilon^{(i)} = 0$, etc., by the conditions of the most advantageous method,

$$\begin{aligned}
\overline{p_1\alpha_1} &= p_1^{(2)}.z + \overline{p_1q_1}.z' + \overline{p_1r_1}.z'' + \overline{p_1t_1}.z''' + \overline{p_1\gamma}_1.z^{\text{iv}}, \\
\overline{q_1\alpha_1} &= \overline{p_1q_1}.z + q_1^{(2)}.z' + \overline{q_1r_1}.z'' + \overline{q_1t_1}.z''' + \overline{q_1\gamma_1}.z^{\text{iv}}, \\
\overline{r_1\alpha_1} &= \overline{p_1r_1}.z + \overline{q_1r_1}.z' + r_1^{(2)}.z'' + \overline{r_1t_1}.z''' + \overline{r_1\gamma_1}.z^{\text{iv}}, \\
\overline{t_1\alpha_1} &= \overline{p_1t_1}.z + \overline{q_1t_1}.z' + \overline{r_1t_1}.z'' + t_1^{(2)}.z''' + \overline{t_1\gamma}_1.z^{\text{iv}}, \\
\overline{\gamma_1\alpha_1} &= \overline{p_1\gamma_1}.z + \overline{q_1\gamma_1}.z' + \overline{r_1\gamma_1}.z'' + \overline{t_1\gamma_1}.z''' + \gamma_1^{(2)}.z^{\text{iv}},
\end{aligned} \tag{B}$$

where we must observe that [15]

$$\overline{p_1q_1} = Sp_1^{(i)}q_1^{(i)}, \quad p_1^{(2)} = Sp_1^{(i)2}, \quad \text{etc.}$$

By substituting instead of $p_1^{(i)}$, $q_1^{(i)}$, etc., their preceding values, we have

$$\overline{p_1q_1} = Sp_1^{(i)}q_1^{(i)} - \frac{S\lambda^{(i)}p^{(i)}S\lambda^{(i)}q^{(i)}}{S\lambda^{(i)2}}$$

or

$$\overline{p_1q_1} = \overline{pq} - \frac{\overline{\lambda p}\,\overline{\lambda q}}{\lambda^{(2)}};$$

we have similarly

$$p_1^{(2)} = p^{(2)} - \frac{\overline{\lambda p}^2}{\lambda^{(2)}}; \quad q_1^{(2)} = q^{(2)} - \frac{\overline{\lambda q}^2}{\lambda^{(2)}}; \quad \overline{p_1r_1} = \overline{pr} - \frac{\overline{\lambda p}\,\overline{\lambda r}}{\lambda^{(2)}}; \quad \text{etc.}$$
$$\overline{p_1\alpha_1} = \overline{p\alpha} - \frac{\overline{\lambda p}\,\overline{\lambda\alpha}}{\lambda^{(2)}}; \text{etc.}$$

Thus the coefficients of the system of equations (B) are deduced easily from the coefficients of the system of equations (A).

The equations of condition represented by equation (3) will give similarly, the following system of equations

$$\begin{aligned}
\overline{p_2\alpha_2} &= p_2^{(2)}.z + \overline{p_2q_2}.z' + \overline{p_2r_2}.z'' + \overline{p_2t_2}.z''', \\
\overline{q_2\alpha_2} &= \overline{p_2q_2}.z + q_2^{(2)}.z' + \overline{q_2r_2}.z'' + \overline{q_2t_2}.z''', \\
\overline{r_2\alpha_2} &= \overline{p_2r_2}.z + \overline{q_2r_2}.z' + r_2^{(2)}.z'' + \overline{r_2t_2}.z''', \\
\overline{t_2\alpha_2} &= \overline{p_2t_2}.z + \overline{q_2t_2}.z' + \overline{r_2t_2}.z'' + t_2^{(2)}.z''',
\end{aligned} \tag{C}$$

and we have

$$p_2^{(2)} = p_1^{(2)} - \frac{\overline{\gamma_1 p_1}^2}{\gamma_1^{(2)}}; \quad \overline{p_2 q_2} = \overline{p_1 q_1} - \frac{\overline{\gamma_1 p_1}\,\overline{q_1 \gamma_1}}{\gamma_1^{(2)}}; \quad \text{etc.}$$

$$\overline{p_2 \alpha_2} = \overline{p_1 \alpha_1} - \frac{\overline{\gamma_1 p_1}\,\overline{\gamma_1 \alpha_1}}{\gamma_1^{(2)}}; \quad \text{etc.}$$

We will have similarly the system of equations

$$\begin{aligned} \overline{p_3\alpha_3} &= p_3^{(2)}.z + \overline{p_3 q_3}.z' + \overline{p_3 r_3}.z'', \\ \overline{q_3\alpha_3} &= \overline{p_3 q_3}.z + q_3^{(2)}.z' + \overline{q_3 r_3}.z'', \\ \overline{r_3\alpha_3} &= \overline{p_3 r_3}.z + \overline{q_3 r_3}.z' + r_3^{(2)}.z'', \end{aligned} \qquad \text{(D)}$$

[16] by making

$$p_3^{(2)} = p_2^{(2)} - \frac{\overline{p_2 t_2}^2}{t_2^{(2)}}; \quad \overline{p_3 q_3} = \overline{p_2 q_2} - \frac{\overline{p_2 t_2}\,\overline{q_2 t_2}}{t_2^{(2)}}; \quad \text{etc.}$$

$$\overline{p_3 \alpha_3} = \overline{p_2 \alpha_2} - \frac{\overline{t_2 p_2}\,\overline{t_2 \alpha_2}}{t_2^{(2)}}; \quad \text{etc.}$$

we will have further

$$\begin{aligned} \overline{p_4\alpha_4} &= p_4^{(2)}.z + \overline{p_4 q_4}.z', \\ \overline{q_4\alpha_4} &= \overline{p_4 q_4}.z + q_4^{(2)}.z', \end{aligned} \qquad \text{(E)}$$

by making

$$p_4^{(2)} = p_3^{(2)} - \frac{\overline{p_3 r_3}^2}{r_3^{(2)}}; \quad \overline{p_4 q_4} = \overline{p_3 q_3} - \frac{\overline{p_3 r_3}\,\overline{q_3 r_3}}{r_3^{(2)}};$$

$$\overline{p_4 \alpha_4} = \overline{p_3 \alpha_3} - \frac{\overline{p_3 r_3}\,\overline{\alpha_3 r_3}}{r_3^{(2)}}; \quad \text{etc.}$$

Finally we will have

$$\overline{p_5\alpha_5} = p_5^{(2)}.z, \qquad \text{(F)}$$

by making

$$p_5^{(2)} = p_4^{(2)} - \frac{\overline{p_4 q_4}^2}{q_4^{(2)}}, \quad \overline{p_5 \alpha_5} = \overline{p_4 \alpha_4} - \frac{\overline{p_4 q_4}\,\overline{q_4 \alpha_4}}{q_4^{(2)}}; \text{etc.}$$

$p_5^{(2)}$ is the value of $Sp_5^{(i)2}$, and the weight P will be

$$\frac{sp_5^{(2)}}{2S\epsilon'^{(i)2}}.$$

We see by the sequence of the values of $p^{(2)}$, $p_1^{(2)}$, $p_2^{(2)}$, etc., that they diminish without ceasing, and that thus for the same number of observations, the weight P diminishes when the number of elements increase.

If we consider the sequence of equations which determine $\overline{p_5\alpha_5}$, we see that this function developed according to the coefficients of the system of equations (A), is of the form

$$\overline{p\alpha} + M\overline{q\alpha} + N\overline{r\alpha} + \text{etc.},$$

the coefficient of $\overline{p\alpha}$ being unity. It follows thence that if we resolve equations (A), by leaving $\overline{p\alpha}$, $\overline{q\alpha}$, $\overline{r\alpha}$, etc., as indeterminates; $\frac{1}{p_5^{(2)}}$ will be by virtue of equation (F), [17] the coefficient of $\overline{p\alpha}$ in the expression of z. Similarly $\frac{1}{q_5^{(2)}}$ will be the coefficient of $\overline{q\alpha}$, in the expression of z'; $\frac{1}{r_5^{(2)}}$ will be the coefficient of $\overline{r\alpha}$, in the expression of z''; and thus of the rest; that which gives a simple means to obtain $p_5^{(2)}$, $q_5^{(2)}$, etc.; but it is simpler yet to determine them thus.

First equation (F) gives the value of $p_5^{(2)}$, and of z. If in the system of equations (E) we eliminate z instead of z', we will have a single equation in z', of the form

$$\overline{q_5\alpha_5} = q_5^{(2)} z';$$

by making

$$q_5^{(2)} = q_4^{(2)} - \frac{\overline{p_4q_4}^2}{p_4^{(2)}}, \quad \overline{q_5\alpha_5} = \overline{q_4\alpha_4} - \frac{\overline{p_4q_4}\,\overline{p_4\alpha_4}}{p_4^{(2)}}.$$

If in the system of equations (D), we eliminate z instead of z'', in order to conserve at the end of the calculation, only z''; we will have $r_5^{(2)}$ by changing in the sequence of equations which, departing from this system, determine $p_5^{(2)}$, the letter p into the letter r, and reciprocally. We will have thus

$$r_4^{(2)} = r_3^{(2)} - \frac{\overline{p_3r_3}^2}{p_3^{(2)}}, \quad \overline{r_4q_4} = \overline{r_3q_3} - \frac{\overline{p_3q_3}\,\overline{p_3r_3}}{p_3^{(2)}}, \quad q_4^{(2)} = q_3^{(2)} - \frac{\overline{p_3q_3}^2}{p_3^{(2)}},$$

$$r_5^{(2)} = r_4^{(2)} - \frac{\overline{p_4q_4}^2}{q_4^{(2)}}, \text{etc.}$$

In order to have $t_5^{(2)}$, we will depart from the system of equations (C), by changing in the sequence of the values $p_3^{(2)}$, $\overline{p_3q_3}$, etc., $r_3^{(2)}$, $\overline{q_3r_3}$, etc., the letter p into the letter t, and reciprocally.

We will have similarly the value of $\gamma_5^{(2)}$, by departing from the system of equations [18] (B), and changing in the sequence of values of $p_2^{(2)}$, $p_3^{(2)}$, etc., the letter p into the letter γ, and reciprocally.

Finally, we will have the value of $\lambda_5^{(2)}$, by changing in the sequence of values of $p_1^{(2)}$, $p_2^{(2)}$, etc., the letter p into the letter λ, and reciprocally.

§3. The error of which the value of z is susceptible being u; its probability is, as we have seen,

$$\frac{\sqrt{P}.c^{-Pu^2}}{\sqrt{\pi}}.$$

By multiplying it by $u\,du$, and taking the integral from u null, to u infinity; we will have

$$\frac{1}{2\sqrt{\pi}\sqrt{P}}$$

for the mean positive error to fear respecting the value of z. This expression affected with the sign $-$ will be the mean negative error to fear, respecting this value. I have given in §21 of the second Book, the analytic expression of these mean errors, whatever be the number of elements. We will have therefore by comparing it to the preceding, the value of P; and it is easy to recognize the identity of these expressions. We find thus in the case of a single element

$$P = \frac{sp^{(2)}}{2S\epsilon'^{(i)2}}.$$

If we make generally, for any number of elements whatsoever,

$$P = \frac{s}{2S\epsilon'^{(i)2}}\frac{A}{B};$$

we find for two elements,

$$A = p^{(2)}q^{(2)} - \overline{pq}^2,$$
$$B = q^{(2)}.$$

[19] By applying these results to the equations (E), we will have the value of P relative to the element z.

We find for three elements,

$$A = p^{(2)}q^{(2)}r^{(2)} - p^{(2)}\overline{qr}^2 - q^{(2)}\overline{pr}^2 - r^{(2)}\overline{pq}^2 + 2\overline{pq}.\overline{pr}\,.\overline{qr},$$
$$B = q^{(2)}r^{(2)} - \overline{qr}^2.$$

These results applied to the equations (D) will give the value of P relative to the element z.

By continuing thus, we will have whatever be the number of elements, the weight relative to the first element z. By changing in its expression, p into q and q into p; we will have the weight relative to the second element z'. By changing in the expression of the weight of the first element, p into r and r into p; we will have the weight relative to the third element z'', and so forth. But when the number of elements surpasses three, it is much simpler to make use of the method of the previous section.

We will observe here that the mean error to fear respecting each element, being by §§20 and 21 of the second Book, smaller in the system of factors, which constitute the most advantageous method, than in every other system; the value of P is the greatest possible. Thus for one same error of an element in this method, the probability is smaller than in every other method, that which assures its superiority.

§4. All my analysis rests on the hypothesis that the facility of the errors is the same for the positive errors, and for the negative errors; that which renders null, the integral of the product of the error, by its probability and by its differential; the integral being taken in all the extent of the limits of the errors, and the origin of the errors being in the middle of the interval which separates these limits. But if the law of facility is different for the positive errors and for the negative errors; then the preceding integral becomes null only in the case where this origin is at the point of the abscissa through where passes the ordinate of the center of gravity of the curve of which the ordinates represent the law of facility of the errors represented themselves by the abscissas. For every other point, the mean error of the observation is this integral [20] divided by the interval of the limits; and, if we have a great number of observations, the mean of the errors of these observations will be by that which we have seen in the second Book, equal very nearly to this quotient. By making therefore so that the sum of the errors is null, we will be able to suppose null, the integral of which we just spoke, and then all my analysis subsists and becomes independent of the hypothesis of an equal facility of positive errors and of negative errors. We can always obtain this advantage, by adding to the equations of condition, an indeterminate element of which the coefficient is unity. It is that which holds of itself, in the equations of condition, relative to the movement of the planets in longitude; because the correction of the epoch there has for coefficient, unity. But the addition of an element, weakening, as we have said, the probability of the errors of the other elements, a probability which for the same number of observations, diminishes when the number of elements which are supported on them, is greater; it is necessary to recur to this addition, only when we can fear that a constant cause favors rather, the errors of one sign, than those of a contrary sign. Besides, we will be assured of it easily by making the sum of the positive remainders and that of the negative remainders of the equations of condition, when we will have substituted the values of the elements determined by the most advantageous method, without the addition of which we just spoke; and by seeing if the excess of one of these sums over the other, indicates a constant cause.

In order to leave no doubt on this object, I am going to apply the calculus. There results from §22 of the second Book, that the probability that the sum of the errors of the observations equals

$$\frac{ak'}{k}s + ar\sqrt{s}$$

is proportional to the exponential

$$c^{-\frac{k^2r^2}{2(kk''-k'^2)}}.$$

This sum is $S\epsilon^{(i)}$, and by §1 we have

$$S\epsilon^{(i)} = S\epsilon'^{(i)} + S(p^{(i)}u + q^{(i)}u' + \text{etc.}).$$

By the nature of the final equations, we have $S\epsilon'^{(i)} = 0$; we have therefore [21

$$\frac{ak'}{k}s + ar\sqrt{s} = S(p^{(i)}u + q^{(i)}u' + \text{etc.}).$$

If we make, as in this section, $u = \frac{\nu}{\sqrt{s}}$, $u' = \frac{\nu'}{\sqrt{s}}$, etc., we will have thus

$$r = -\frac{k'}{k}\sqrt{s} + \frac{1}{as}S(p^{(i)}\nu + q^{(i)}\nu' + \text{etc.}).$$

Thus $\frac{k'}{k}$ is of order $\frac{1}{\sqrt{s}}$, and its square is of order $\frac{1}{s}$; we can therefore neglect it, having regard to $\frac{k''}{k}$. The probability of the simultaneous existence of r, ν, ν', etc. is thus proportional to the exponential

$$c^{-\frac{k}{2k''}r^2-\frac{S(p^{(i)}\nu+q^{(i)}\nu'+\text{etc.})^2}{2S\epsilon'^{(i)2}}}.$$

By multiplying it by dr, $d\nu'$, etc., by integrating it with respect to r, ν', ν'', etc., from negative infinity to positive infinity; we will have a quantity proportional to the probability of ν. By multiplying therefore this quantity by $d\nu$, and by taking the integral within some given limits; by dividing next by this same integral taken from $\nu=-\infty$ to $\nu=+\infty$; we will have the probability that the value of ν is contained within these limits. We see thus that the consideration of the values that k' can have and on which depends the difference of probability of the positive and negative errors, has no sensible influence on the results of the general method exposed here above.

§5. Let us apply now this method to an example. For this, I have profited from the immense work that Bouvard has just finished on the movements of Jupiter and of Saturn, from which he has constructed very precise Tables. He has made use of all the oppositions observed by Bradley and by the astronomers who have followed him: he has discussed them anew and with the greatest care; that which has given to him 126 equations of condition for the movement of Jupiter in longitude, and 129 equations, for the movement of Saturn.

[22] In these last equations, Bouvard has made the mass of Uranus enter, as indeterminate. Here are the final equations that he has concluded by the most advantageous method.

$$\begin{aligned}
7212'',600 =&795938z-12729398z'+6788,2z''-1959,0z'''\\
&+696,13z^{\text{iv}}+2602z^{\text{v}},\\
-738297'',800 =&-12729398z+424865729z'-153106,5z''\\
&-39749,1z'''-5459z^{\text{iv}}+5722z^{\text{v}},\\
237'',782 =&6788,2z-153106,5z'+71,8720z''\\
&-3,2252z'''+1,2484z^{\text{iv}}+1,3371z^{\text{v}},\\
-40'',335 =&-1959,0z-39749,1z'-3,2252z''\\
&+57,1911z'''+3,6213\text{v}^{\text{iv}}+1,1128z^{\text{v}},\\
-343'',455 =&696,13z-5459z'+1,2484z''+3,6213z'''\\
&+21,543z^{\text{iv}}+46,310z^{\text{v}},\\
-1002'',900 =&2602z+5722z'+1,3371z''+1,1128z'''\\
&+46,310z^{\text{iv}}+129z^{\text{v}}.
\end{aligned}$$

In these equations, the mass of Uranus is supposed $\frac{1+z}{19504}$; the mass of Jupiter is supposed $\frac{1+z'}{1067{,}09}$; z'' is the product of the equation of the center, by the correction of the perihelion employed first by Bouvard; z''' is the correction of the equation of the center; z^{iv} is the secular correction of the mean movement; z^{v} is the correction of the epoch of the longitude at the beginning of 1750. The second of the decimal degree is taken for unity.

By means of the preceding equations contained in the system (A), I have concluded the following, contained in the system (B).

$$\begin{aligned}
27441'',68 =& 743454z - 12844814z' + 6761,23z'' \\
&- 1981,45z''' - 237,97z^{\text{iv}}, \\
-693812'',58 =& - 12844814z + 424611920z' - 153165,81z'' \\
&- 39798,46z''' - 7513,15z^{\text{iv}}, \\
248'',1772 =& 6761,23z - 153165,81z' + 71,8581z'' \\
&- 3,2367z''' + 0,7684z^{\text{iv}}, \\
-31'',6836 =& - 1981,45z - 39798,46z' - 3,2367z'' \\
&+ 57,1815z''' + 3,2218z^{\text{iv}}, \\
16'',5783 =& - 237,97z - 7513,15z' + 0,7684z'' \\
&+ 3,2218z''' + 4,9181z^{\text{iv}}.
\end{aligned}$$

From these equations, I have deduced the following four contained in the system (C). [23]

$$\begin{aligned}
28243'',85 =& 731939,5z - 1328350z' + 6798,41z'' \\
&- 1825,56z''', \\
-668486'',70 =& - 13208350z + 413134432z' - 1519920z'' \\
&- 34876,7z''', \\
245'',5870 =& 6798,41z - 151992,0z' + 71,7381z'' \\
&- 3,7401z''', \\
-42'',5434 =& - 1825,56z - 34876,7z' - 3,7401z'' \\
&+ 55,0710z''';
\end{aligned}$$

these last equations give the following contained in the system (D),

$$\begin{aligned}
26833'',55 =& 671414,7z - 14364541z' + 6674,43z'', \\
-695430'',0 =& - 14364541z + 391046861z' - 154360,6z'', \\
242'',6977 =& 6674,43z - 154360,6z' + 71,4841z''.
\end{aligned}$$

Finally I have concluded thence the following two equations contained in the system (E).

$$\begin{aligned}
4172'',95 &= 48442z + 48020z' \\
-171455'',2 &= 48020z + 57725227z'.
\end{aligned}$$

I stop myself at this system, because it is easy to conclude from it the values of the weight P, relative to the two elements z and z', that I desired particularly to know. The formulas of §3, give for z

$$P = \frac{s}{2S\epsilon'^{(i)2}}\left(48442 - \frac{(48020)^2}{57725227}\right),$$

[24] and for z',

$$P = \frac{s}{2S\epsilon'^{(i)2}}\left(57725227 - \frac{(48020)^2}{48442}\right).$$

The number s of the observations is here 129, and Bouvard has found

$$S\epsilon'^{(i)2} = 31096;$$

we have therefore for z,

$$\log P = 2,0013595,$$

and, for z',

$$\log P = 5,0778624.$$

The preceding equations give

$$z' = -\,0,00305,$$
$$z = 0,08916.$$

The mass of Jupiter is $\frac{1}{1067,09}(1 + z')$. By substituting for z' its preceding value, this mass becomes $\frac{1}{1070,35}$. The mass of the Sun is taken for unity. The probability that the error of z' is comprehended within the limits $\pm U$, is by §1

$$\frac{\sqrt{P}}{\sqrt{\pi}}\int du\, c^{-Pu^2},$$

the integral being taken from $u = -U$ to $u = U$. We find thus the probability that the mass of Jupiter is comprehended within the limits

$$\frac{1}{1070,35} \pm \frac{1}{100}\cdot\frac{1}{1067,09},$$

equal to $\frac{1000000}{1000001}$; so that there is odds one million very nearly against one that the value $\frac{1}{1070,35}$ is not in error of a hundredth of its value; or, that which reverts to quite nearly the same, that after a century of new observations added to the previous, and discussed in the same manner, the new result will not differ from the previous, by a hundredth of its value.

Newton had found by the observations of Pound on the elongations of the satellites of Jupiter, the mass of that planet, equal to the 1067$^{\text{th}}$ part of that of the Sun; that which differs very little from the result of Bouvard.

The mass of Uranus is $\frac{1+z}{19504}$. By substituting for z, its previous value; this mass
[25] becomes $\frac{1}{17907}$. The probability that this value is comprehended within the limits

$$\frac{1}{17907} \pm \frac{1}{4}\cdot\frac{1}{19504},$$

is equal to $\frac{2508}{2509}$; and the probability that that mass is comprehended within the limits

$$\frac{1}{17907} \pm \frac{1}{5} \cdot \frac{1}{19504}$$

is equal to $\frac{215,6}{216,6}$.

The perturbations that Uranus produces in the movement of Saturn, being not very considerable; we must not yet expect from the observations of this movement, a great precision in the value of its mass. But, after a century of new observations added to the previous and discussed in the same manner, the value of P will increase in a manner to give this mass, with a great probability that its value will be contained within some narrow limits; that which will be much preferable to the use than the elongations of the satellites of Uranus, because of the difficulty to observe these elongations.

Bouvard, by applying the previous method, to the 126 equations of condition which the observations of Jupiter have given to him, and by supposing the mass of Saturn equal to $\frac{1+z}{3534,08}$, has found

$$z = 0,00620$$

and

$$\log P = 4,8856829.$$

These values give the mass of Saturn, equal to $\frac{1}{3512,3}$, and the probability that this mass is comprehended within the limits

$$\frac{1}{3512,3} \pm \frac{1}{100} \cdot \frac{1}{3534,08}$$

is equal to $\frac{11327}{11328}$.

Newton had found by the observations of Pound, respecting the greatest elongation of the fourth satellite of Saturn, the mass of this planet equal to $\frac{1}{3012}$; that which surpasses by a sixth, the preceding result. There is odds of millions of billions against [26]
one, that the one of Newton is in error; and we will not at all be surprised, if we consider the difficulty to observe the greatest elongations of the satellites of Saturn. The facility to observe those of the satellites of Jupiter, has rendered, as we have seen, much more exact, the value that Newton has concluded from the observations of Pound.

On the probability of judgments.

I have compared in §50 of the second Book, the judgment of a tribunal which pronounces between two contradictory opinions, to the result of the testimonies of many witnesses of the extraction of a ticket from an urn which contains only two tickets. There is however between these two cases, this difference, namely, that the probability of the testimony is independent of the nature of the thing attested; because we suppose that the witness has not been able to be deceived on this thing; instead an object in litigation, can be surrounded by such obscurities that the judges, in their supposing all the good faith desirable, can be however, of contrary opinions.

The nature of the affair which is subject to them, must therefore influence on their judgment. I will make this consideration enter into the following investigations, by applying it to the judgments in criminal matter.

In order to condemn an accused, without doubt the strongest proof of his offense is necessary to the judges. But a moral proof is never but a probability, and experience has only too well made known the errors of which the criminal judgments, even those which appear to be most just, are yet susceptible. The possibility to repair these errors, is the most solid argument of the philosophers who have wished to proscribe the pain of death. We should therefore abstain ourselves from judging, if it was necessary we await mathematical evidence. But when the proofs have a force such that the product of the error to fear, by its feeble probability, is inferior to the danger which would result from the impunity of the crime; judgment is commanded by the interest of society. This judgment is reduced, if I do not deceive myself, to the solution
[27] of the following question: Has the proof of the offense of the accused the high degree of probability necessary in order that the citizens have less to fear the errors of the tribunals, if he is innocent and condemned, than his new attempts, and those of the unfortunate persons who the example of his impunity would embolden, if he was culpable and absolved? The solution of this question depends on many elements very difficult to know. Such is the imminence of danger which would menace society, if the accused criminal remained unpunished. Sometimes, this danger is so great, that the magistrate sees himself obliged to renounce the prudent forms established for the certainty of innocence. But that which renders nearly always the question of which there is concern, insoluble, is the impossibility to estimate exactly the probability of the offense, and to fix that which is necessary for the condemnation of the accused. Each judge in this regard, is forced to bring himself back to his own feeling. He forms his opinion, by comparing the diverse witnesses, and the circumstances of which the offense is accompanied, to the results of his reflections and of his experience; and, under this relation, a long habit of interrogating and judging the accused, gives much advantage in order to know the truth, in the midst of often contradictory indices.

The preceding question depends further on the magnitude of the punishment applied to the offense; because we require naturally in order to pronounce death, proofs much stronger, than to inflict a detention of some months. This is a reason to proportion the punishment to the offense; a grave punishment applied to a light offense, must inevitably to render absolved many guilty persons. The product of the probability of the offense, by its gravity, being the measure of the danger that absolution of the accused can make society experience; we would be able to think that the punishment must depend on this probability. This is that which we do indirectly in the tribunals where we retain during some times, the accused against whom are raised some very strong proofs, but insufficient to condemn him. In the view to acquire new understanding, we deliver him not at all immediately, into the midst of his fellow citizens, who would review it not without lively alarms. But the arbitrariness of this measure, and the abuse that we can make of it, has caused to reject it in the country where we attach a very great price to individual liberty.

Now, what is the probability that the decision of a tribunal which is able to condemn only by a given majority, will be just, that is, conformed to the true solution of the question posed above? This important problem well resolved will give the means to compare the diverse tribunals among themselves. The majority of a single vote in a numerous tribunal, indicates that the affair of which there is concern, is nearly doubtful; the condemnation of the accused would be therefore then contrary to the principles of humanity, protectors of innocence. The unanimity of the judges would give a very great probability of a just decision; but by being obliged, too many guilty persons would be absolved. It is necessary therefore, either to limit the number of judges, if we wish that they be unanimous, or to increase the majority necessary to condemn, when the tribunal becomes more numerous. I will test by applying the calculus to this object; persuaded that the applications of this kind, when they are well conducted and based on some data that good sense suggests to us, are always preferable to the most specious reasonings. [28]

The probability that the opinion of each judge is just, enters as principal element in this calculation. This probability is evidently relative to each affair. If in a tribunal of one thousand and one judges, five hundred one are of one opinion, and five hundred are of a contrary opinion; it is clear that the opinion[1] each judge surpasses quite little $\frac{1}{2}$; because by supposing it sensibly greater, a single vote of difference would be an unlikely event. But if the judges are unanimous; this indicates in the proofs, that degree of force which carries away the conviction. The probability of the opinion of each judge, is therefore then very near to unity or of certitude; not unless some passions or some common prejudgments mislead all the judges. Beyond these cases, the ratio of the votes for or against the accused, must alone determine this probability. I suppose thus that it can vary from $\frac{1}{2}$ to unity, but that it cannot be below $\frac{1}{2}$. If this were not, the decision of the tribunal would be insignificant as the lot: it has value, only as much as the opinion of the judge has more tendency to the truth, than to the error. It is next by the ratio of the numbers of votes favorable or contrary to the accused, that I determine the probability of this opinion.

These data suffice in order to have the general expression of the probability that the decision of the tribunal judging in a given majority, is just. In our special tribunals composed of eight judges, five votes are necessary for the condemnation of an accused: the probability of the error to fear respecting the justness of the decision, surpasses then $\frac{1}{4}$. If the tribunal were reduced to six members which would be able to condemn only with the plurality of four votes; the probability of the error to fear would be then below $\frac{1}{4}$; there would be therefore for the accused, an advantage in this reduction of the tribunal. In both cases, the majority required is the same, and equal to two. Thus, this majority remaining constant, the probability of the error increases with the number of judges. This is general, whatever be the majority required, provided that it remains the same. By taking therefore for rule, the arithmetic relation, the accused is found in a position less and less advantageous, in measure as the tribunal becomes more numerous. We would be able to believe that in a tribunal where we [29]

[1] Translator's note: The text should read: "it is clear that the probability of the opinion..."

would require a majority of twelve votes, whatever be the number of judges; the votes of the minority neutralizing a similar number of votes of the majority, the twelve votes remaining would represent the unanimity of a jury of twelve members, required in England, for the condemnation of the accused. But we would be in a great error. Good sense shows that there is a difference between the tribunal of two hundred twelve judges, of whom one hundred twelve condemn the accused, while one hundred absolve him, and that of a tribunal of twelve judges unanimous for condemnation. In the first case, the one hundred votes favorable to the accused, permit thinking that the proofs are far from attaining the degree of force which carry away the conviction. In the second case, the unanimity of the judges carry belief that they have attained this degree. But simple good sense does not suffice to estimate the extreme difference of the probability of error in these two cases. It is necessary then to recur to the calculus, and we find very nearly a fifth, for the probability of the error in the first case, and only $\frac{1}{8192}$ for this probability in the second case, a probability which is not a thousandth of the first. This is a confirmation of the principle that the arithmetic
[30] relation is unfavorable to the accused, when the number of judges increases. To the contrary, if we take for rule, the geometric relation, the probability of the error of the decision diminishes, when the number of the judges is increased. For example, in the tribunals which would be able to condemn only by the plurality of the two thirds of votes, the probability of the error to fear is nearly a fourth, if the number of judges is six: it is below $\frac{1}{4}$, if this number is raised to twelve. Thus we must be regulated, neither on the arithmetic relation, nor on the geometric relation, if we wish that the probability of error, is never above nor below a determined fraction.

But to what fraction must we be fixed? It is here that the arbitrary commences, and the tribunals offer in this regard great varieties. In the special tribunals where five votes out of eight, suffice for the condemnation of the accused, the probability of the error to fear respecting the goodness of the judgment, is $\frac{65}{256}$ or below $\frac{1}{4}$. The magnitude of this fraction is frightening; but that which must reassure a little, is the consideration that most often, the judge who absolves an accused, regards him not as innocent. He pronounces only that it is not attained by some sufficient proofs in order that he be condemned. We are especially reassured by the pity that nature has put into the heart of man, and which disposes the mind to see with difficulty a guilty person, in the accused submitted to his judgment. This sentiment more quick in those who have not at all the habit of criminal judgments, outweighs the inconveniences attached to the inexperience of juries. In a jury of twelve members, if the plurality required for the condemnation, is of eight votes out of twelve; the probability of error to fear is $\frac{1093}{8192}$, or a little less than an eighth: it is nearly $\frac{1}{22}$, if this plurality is of nine votes. In the cases of unanimity, the probability of error to fear is $\frac{1}{8192}$, that is more than one thousand times less than in our juries.

The solution of the problem that we just considered, does not suffice to fix the convenient majority, in a tribunal of any number of judges whatsoever. It is necessary for this, to know the probability of the offense, below which an accused cannot be condemned, without that the citizens having to dread more the errors of the tribunals,
[31] than the attacks which would be born from the impunity of a guilty one absolved.

It is necessary next to determine the probability of the offense, resulting from the decision of the tribunal, and to fix the majority, in a manner that these probabilities are equals. But it is impossible to obtain them. The first is, as we have said, relative to the position in which society is found, a variable position, very difficult to define well, and always too complicated in order to be submitted to the calculus. The second depends on a thing entirely unknown, the law of probability of the opinion of each judge, in the estimation that he makes of the probability of the offense. Seeing our ignorance of these two elements of the calculus, what is more reasonable than to depart from the solution of the single problem that we may resolve in this manner, the one of the probability of the error of the decision of a tribunal? This probability appears to me too high in our tribunals, and I think that in this regard, it is acceptable to approach to the English jury where it is only $\frac{1}{8192}$. By fixing it at the fraction $\frac{1}{1024}$, and in determining the majority necessary to attain it; we place the accused in the position where he would be vis-à-vis of a jury of nine members of which we would require unanimity; that which appears to me to guarantee sufficiently, the innocent, from the errors of the tribunals, and society, from the pains that impunity of the guilty persons would produce. It must be extremely rare then that an accused is condemned with a probability less than that which is necessary to his condemnation; because the majority who condemn him, declare that the probability of his offense, is at least equal to this necessary probability: the minority who absolve him, declare that the first of these probabilities appears to it inferior to the second; but it is natural to believe that this inferiority is not very considerable. It must rarely happen that the mean probability which results from the totality of the judgments of the members of the tribunal, is inferior to the probability required for the condemnation of the accused; if we reduce, by a convenient majority, the probability of the error to fear respecting the justice of the decision, to the fraction $\frac{1}{1024}$. The analysis furnishes in order to have this majority, some formulas which I will expose here, and that it is easy to reduce into a table dependent on the number of the judges. But a parallel table will appear too arbitrary to the common men who will prefer always one or the other of the arithmetic and geometric relations, which they are able to imagine easily.

§1. A judge must not in order to condemn an accused, expect the mathematical [32]
evidence that it is impossible to attain in moral things. But when the probability of the offense is such that the citizens had more to dread the attempts which would be able to be born from his impunity, than the errors of the tribunals; the interest of society requires the condemnation of the accused. I name a this degree of probability, and I suppose that the judge who condemns an accused, pronounces thence, that the probability of his offense is at least a. I name x the probability of this opinion of the judge, a probability that I will suppose equal or superior to $\frac{1}{2}$, and varying by some infinitely small degrees, equal to x and equally probable *a priori*. I suppose further that the tribunal is composed of $p+q$ judges of whom p condemn the accused, and q absolve him. The probability that the opinion of the tribunal is just, will be proportional to $x^p(1-x)^q$; and the probability that it is not will be proportional to $(1-x)^p x^q$; the probability of the goodness of the judgment will be therefore by §1 of

the second Book,

$$\frac{x^p(1-x)^q}{x^p(1-x)^q+(1-x)^p x^q}. \tag{a}$$

It is necessary to multiply this quantity, by the probability of the value of x, taken from the observed event. This event is that the tribunal is itself divided into two parts of which the one composed of p judges, condemn the accused; and of which the other formed of q judges, absolve him. The probability of x, is therefore the function $x^p(1-x)^q+(1-x)^p x^q$, divided by the sum of all the similar functions, relative to all the values of x, from $x=\frac{1}{2}$ to $x=1$; it is consequently

$$\frac{[x^p(1-x)^q+(1-x)^p x^q]\,dx}{\int x^p dx(1-x)^q}.$$

the integral of the denominator being taken from $x=0$, to $x=1$. By multiplying this function, by the function (a), we will have

$$\frac{x^p(1-x)^q dx}{\int x^p dx(1-x)^q}$$

for the probability of the goodness of the judgment, relative to x. The same
[33] probability relative to all the values of x, is therefore

$$\frac{\int x^p dx(1-x)^q}{\int x^p dx(1-x)^q}; \tag{b}$$

the integral of the numerator being taken from $x=\frac{1}{2}$ to $x=1$, and that of the denominator being taken from $x=0$ to $x=1$. It follows thence that the probability of the error to fear respecting the goodness of the judgment, is again expressed by formula (b), provided that we take the integral of the numerator, from $x=0$, to $x=\frac{1}{2}$. We find thus this last probability equal to

$$\frac{1}{2^{p+q+1}}\left\{\begin{array}{l} 1+\dfrac{p+q+1}{1}+\dfrac{(p+q+1)(p+q)}{1.2}+\dfrac{(p+q+1)(p+q)(p+q-1)}{1.2.3} \\ +\cdots+\dfrac{(p+q+1)(p+q)(p+q-1)\cdots(p+2)}{1.2.3\ldots q} \end{array}\right\} \tag{c}$$

If we require unanimity, q is null, and this expression becomes $\frac{1}{2^{p+q+1}}$.

§2. Let us determine presently the probability of the error to fear respecting the justice of the decision of the tribunal, when p and q are large numbers; that which renders formula (c) very difficult to evaluate in numbers. It is necessary to distinguish here two cases, one in which $p-q$ is considerable, the other in which $p-q$ is rather small. In the first case, we will make use of formula (o) of §28 of the second Book, which gives for the probability of error

$$\frac{(p+q)^{p+q+\frac{3}{2}}}{2^{p+q+\frac{3}{2}}p^{p+\frac{1}{2}}q^{q+\frac{1}{2}}(p-q)\sqrt{\pi}}\left\{1-\frac{p+q}{(p-q)^2}-\frac{[(p+q)^2-13pq]}{12pq(p+q)}\right\}, \tag{e}$$

π being the circumference of which the diameter is unity.

In the second case where $p-q$ is a small number relative to p; we will find easily by the analysis of §19 of the second Book, the probability of error to fear, equal to

$$\frac{\int dt\, c^{-t^2}}{\sqrt{\pi}}, \tag{f}$$

the integral being taken from [34]

$$t^2 = \frac{(p-q)^2(p+q)}{8pq}$$

to infinity.

In order to give an example of each of these formulas, let us suppose a tribunal formed of 144 judges, and that five-eighths is necessary for condemnation of the accused. Then we have

$$p = 90, \quad q = 54,$$

and formula (e) gives $\frac{1}{773}$ for the probability of the error to fear respecting the goodness of the decision of the tribunal. In the case of unanimity of a jury of eight members, the probability of the error to fear is $\frac{1}{512}$; the accused is therefore then in a more favorable position, than vis-à-vis a similar jury.

Let us suppose the tribunal formed of 212 judges, and that a majority of twelve votes suffices for condemnation. In this case

$$p + q = 212, \quad p - q = 12,$$

and formula (f) gives $\frac{1}{4{,}889}$ for the probability of the error to fear.

END.

Second Supplement.

APPLICATION OF THE CALCULUS OF PROBABILITIES TO THE GEODESIC OPERATIONS
February 1818

We determine the length of a great arc on the surface of the Earth, by a chain [3] of triangles which are supported on a base measured with exactitude. But whatever precision that we bring into the measure of the angles; their inevitable errors can, by accumulating, deviate sensibly from the truth, the value of the arc, that we have concluded from a great number of triangles. We know therefore only imperfectly this value, if we are not able to assign the probability that its error is comprehended within some given limits. The desire to extend the application of the Calculus of Probabilities to natural Philosophy, has made me seek the formulas proper to this object.

This application consists in deducing from the observations, the most probable results, and to determine the probability of the errors of which they are always sus- [4] ceptible. When these results being known very nearly, we wish to correct them from a great number of observations; the problem is reduced to determine the probability of one or many linear functions of the partial errors of the observations, the law of probability of these errors being supposed known. I have given, in the second Book of my *Théorie analytique des Probabilités*, a method and some general formulas for this object, and I have applied them in the first Supplement, to some interesting points of the *System of the World*. In questions of Astronomy, each observation furnishes in order to correct the elements, an equation of condition: when these equations are very manifold, my formulas give at the same time, the most advantageous corrections and the probability that the errors after these corrections, will be contained within some assigned limits, whatever be moreover the law of probability of the errors of each observation. It is so much the more necessary to be rendered independent of this law, as the simplest laws are always infinitely less probable, seeing the infinite number of those which are able to exist in nature. But the unknown law which the observations of which we make use, introduces into the formulas, an indeterminate which would permit not at all to reduce them in numbers, if we did not succeed to eliminate it. This is that which I have done by means of the sum of the squares of the remainders, when we have substituted into each equation of condition, the most probable corrections. The geodesic questions offering not at all similar equations; it was necessary to seek another means to eliminate from the formulas of probability, the indeterminate dependent on the law of probability of the errors of each partial

observation. The quantity by which the sum of the angles of each observed triangle surpasses two right angles plus the spherical excess, has furnished me this means, and I have replaced by the sum of the squares of these quantities, the sum of the squares of the remainders of the equations of condition. Thence, we are able to determine numerically, the probability that the final result of a long sequence of geodesic operations does not exceed a given quantity. By applying these formulas to the measure of a perpendicular to the meridian; they will estimate the errors, not only of the total arc, but also of the difference in longitude of its extreme points, concluded from the
[5] chain of triangles which unite them, and from the azimuths of the first and of the last side of this chain. If we diminish as much as it is possible, the number of triangles, and if we give a great precision to the measure of their angles, two advantages that the use of the repetitive circle and of the reflectors procure; this way to have the difference in longitude, of the extreme points of the perpendicular, will be one of the better of which we are able to make use.

In order to be assured of the exactitude of a great arc which is supported on a base measured toward one of its extremities; we measure a second base toward the other extremity, and we conclude from one of these bases, the length of the other. If the length thus calculated deviates very little from observation; there is everywhere to believe that the chain of triangles is quite nearly exact, just as the value of the great arc which results from it. We correct next this value, by modifying the angles of the triangles, in a manner that the bases calculated accord themselves with the measured bases; that which is able to be made in an infinity of ways. Those that we have until the present employed, are based on some vague and uncertain considerations. The methods exposed in the second book, lead to some very simple formulas in order to have directly the correction of the total arc, which results from the measures of many bases. These measures have not only the advantage to correct the arc, but further to increase that which I have named the *weight* of a result, that is to render the probability of its errors, more rapidly decreasing, so that the same errors become less probable with the multiplicity of the bases. I expose here the laws of probability of the errors, that the addition of new bases give birth to. The measure of a second base serves similarly to correct the difference in longitude, from the extreme points of a perpendicular to the meridian and to increase the weight of the value of this difference.

After we brought in the observations and in the calculations, the exactitude that we require now; we considered the sides of the geodesic triangles, as rectilinear, and we supposed the sum of their angles, equal to two right angles. Legendre has noted first, that the two errors that we commit thus, compensate themselves mutually, that is that by subtracting from each angle of a triangle, the third of the spherical excess,
[6] we can neglect the curvature of its sides, and to regard them as rectilinear. But the excess of the three observed angles over two right angles, is composed of the spherical excess and the sum of the errors of the measure of each of the angles. The analysis of the probabilities shows that we must yet subtract from each angle, the third of this sum, in order to have the law of probability of the errors of the results, most rapidly decreasing. Thus by the equal apportionment of the error of the observed

sum of the three angles of the triangle considered as rectilinear, we correct at the same time the spherical excess and the errors of the observations. The weight of the angles thus corrected increases, so that the same errors become by this correction less probable. There is therefore advantage to observe the three angles of each triangle, and to correct them, as we have just said. Simple good sense has a presentiment of this advantage; but the calculus of probabilities is able alone to estimate it, and to show that by this correction, it becomes the greatest that it is possible.

The formulas of which I just spoke, are related to some future observations: thus, when we apply them to some past observations, we set aside all the data that the comparison of these observations are able to furnish respecting the errors, data of which we are able to make use, when we know the law of probability of the errors of the partial observations. If this law is expressed by a constant less than unity, of which the exponent is the square of the error, then my formulas agree to the past observations as to the future observations, and they satisfy all the data of these observations, as I have shown in §25 of the second Book. In the case where the angles are measured by means of a repeating circle, each simple angle is the mean result of a great number of measures of the same angle, contained in the total arc observed; the error of the angle is therefore the mean of the errors of all these measures; and by §18 of the second Book, the probability of this error is expressed by a constant of which the exponent is equal to the square of the error. The employment of the repeating circle unites therefore to the benefit of giving a precise measure of the angles, the one to establish a law of probability of the errors which satisfies all the data of the observations.

In order to apply with success the formulas of probability to the geodesic obser- [7]
vations, it is necessary to return faithfully all those that we would admit if they were isolated, and to reject none of them by the sole consideration that it deviated a little from the others. Each angle must be uniquely determined by its measures, without regard to the two other angles of the triangle in which it belongs; otherwise, the error of the sum of the three angles would not be the simple result of the observations, as the formulas of probability supposes it. This remark seems to me important, in order to disentangle the truth in the middle from the slight uncertainties that the observations present.

§1. Let us conceive on a sphere, an arc of great circle $AA'A''$ etc., and let us suppose that we have formed about it, the chain of triangles ACC', $CC'C''$, $C'C''C'''$, $C''C'''C^{\text{iv}}$, etc.; of which the sides CC', $C'C''$, $C''C'''$, etc., cut this arc at A', A'', A''', etc. I do not give at all the figure, because it is easy to trace it according to these indications. Let A be the angle CAA'; $A^{(1)}$ the angle $C'A'A''$; $A^{(2)}$ the angle $C''A''A'''$; etc. Let further C be the angle ACC'; $C^{(1)}$ the angle $CC'C''$; $C^{(2)}$ the angle $C'C''C'''$; etc. We will have

$$A + A^{(1)} + C - \alpha = \pi + t;$$

α being the error of the observed angle C, t being the excess of the angles of the spherical triangle ACA' over π which expresses two right angles or the semi-circumference of which the radius is unity. We will have similarly

$$A^{(1)} + A^{(2)} + C^{(1)} - \alpha^{(1)} = \pi + t^{(1)};$$

$\alpha^{(1)}$ being the error of the observed angle $CC'C''$, and $t^{(1)}$ being the excess of the angles of the spherical triangle $A'C'A''$ over two right angles. We will form similarly the equations

$$A^{(2)} + A^{(3)} + C^{(2)} - \alpha^{(2)} = \pi + t^{(2)};$$
$$A^{(3)} + A^{(4)} + C^{(3)} - \alpha^{(3)} = \pi + t^{(3)};$$
$$\text{etc.};$$

whence we deduce easily

$$\begin{aligned} A^{(2i)} = A &+ C - C^{(1)} + C^{(2)} - C^{(3)} \cdots + C^{(2i-2)} - C^{(2i-1)} \\ &- \alpha + \alpha^{(1)} - \alpha^{(2)} + \alpha^{(3)} \cdots - \alpha^{(2i-2)} + \alpha^{(2i-1)} \\ &- t + t^{(1)} - t^{(2)} + t^{(3)} \cdots - t^{(2i-2)} + t^{(2i-1)}, \end{aligned}$$

[8]
$$\begin{aligned} A^{(2i-1)} = \pi &- A - C + C^{(1)} - C^{(2)} + C^{(3)} \cdots - C^{(2i-2)} \\ &+ \alpha - \alpha^{(1)} + \alpha^{(2)} - \alpha^{(3)} \cdots + \alpha^{(2i-2)} \\ &+ t - t^{(1)} + t^{(2)} - t^{(3)} \cdots + t^{(2i-2)}; \end{aligned}$$

by supposing therefore A well known, the error of the angle $A^{(n)}$ is

$$\alpha^{(n-1)} - \alpha^{(n-2)} + \alpha^{(n-3)} \cdots \pm \alpha$$

the superior sign having place if n is odd, and the inferior sign having place, if n is even. The values of t, $t^{(1)}$, etc., are quite small and are able to be determined with precision. The concern is now to have the probability that this error will be contained within given limits.

For this, I will suppose first that the probability of any error α is proportional to $c^{-h\alpha^2}$, c being the number of which the hyperbolic logarithm is unity. This supposition, the most natural and the most simple of all, results from the use of the repeating circle in the measure of the angles of the triangles. In fact, let us name $\phi(q)$ the probability of an error q, in the measure of a simple angle, this probability being supposed the same for the positive errors, and for the negative errors. Let us suppose further that s is the number of simple angles contained in all the series that we have made in order to determine this angle. The probability that the error of the mean result, or of the angle concluded from these series will be $\pm\frac{r}{\sqrt{s}}$, is by §18 of the second Book, proportional to

$$c^{-\frac{kr^2}{2k''}}$$

k being equal to $\int dq\,\phi(q)$, the integral being taken from q null to q equal to its greatest value that we are always able to suppose infinite; by making $\phi(q)$ discontinuous and null beyond the limit of q, k'' is equal to $\int q^2 dq\,\phi(q)$. By supposing therefore

$$r = \alpha\sqrt{s}, \quad h = \frac{ks}{2k''},$$

[9] $c^{-h\alpha^2}$ will be the probability of the error α. We will see at the end of this article, that the following results always hold, whatever be the probability of α.

Let β and γ be the errors of the two angles $AC'C$ and CAC' of the first triangle ACC'; the probability of the three errors α, β and γ will be proportional to

$$c^{-h\alpha^2-h\beta^2-h\gamma^2};$$

but the observation of these angles give the sum $\alpha+\beta+\gamma$ of the three errors; because the sum of the three angles must be equal to two right angles plus the area of the triangle ACC'; if we name T the excess of the three angles observed on this quantity, we will have

$$\alpha+\beta+\gamma = T;$$

the preceding exponential becomes thus

$$c^{-2h(\beta+\frac{1}{2}\alpha-\frac{1}{2}T)^2-\frac{3h}{2}(\alpha-\frac{1}{3}T)^2-\frac{h}{3}T^2}.$$

β being susceptible to all the values from $-\infty$, to ∞; it is necessary to multiply this exponential by $d\beta$, and take the integral within these limits, that which gives an integral which has for factor

$$c^{-\frac{3h}{2}(\alpha-\frac{1}{3}T)^2-\frac{h}{3}T^2};$$

the probability of α is therefore proportional to this factor. The value of α, most probable is evidently that which renders null, the quantity $\alpha-\frac{1}{3}T$; it is necessary therefore to correct the three angles of each triangle, by the third of the excess T of their observed sum, over two right angles plus the spherical excess. This is that which we do commonly.

Let us name $\bar{\alpha}$ and $\bar{\beta}$ the quantities $\alpha-\frac{1}{3}T$; and $\beta-\frac{1}{3}T$; the probability of $\bar{\alpha}$ will be proportional therefore to

$$c^{-\frac{3}{2}h\bar{\alpha}^2}.$$

If we diminish the angle C, by $\frac{1}{3}T$, that is if we employ the corrected angles of each triangle; by naming $\bar{C}$, $\bar{C}^{(1)}$, etc., that which the angles C, $C^{(1)}$, etc. become by these corrections; we will have

$$\begin{aligned} A^{(2i)} &= A+\bar{C}-\bar{C}^{(1)}+\bar{C}^{(2)}-\text{etc.}\\ &\quad -\bar{\alpha}+\bar{\alpha}^{(1)}-\bar{\alpha}^{(2)}+\text{etc.}\\ &\quad -t+t^{(1)}.\\ A^{(2i-1)} &= \pi-A-\bar{C}+\bar{C}^{(1)}-\text{etc.}\\ &\qquad +\bar{\alpha}-\bar{\alpha}^{(1)}+\text{etc.}\\ &\qquad +t-t^{(1)}+\text{etc.} \end{aligned} \tag{10}$$

The probability that the quantity

$$\bar{\alpha}^{(n-1)}-\bar{\alpha}^{(n-2)}-\cdots\cdots\pm\bar{\alpha},$$

or the error of the angle $A^{(n)}$ will be comprehended within the limits $\pm r\sqrt{n}$, will be by §18 cited,

$$\frac{2\sqrt{\frac{3}{2}h}}{\sqrt{\pi}}\int dr\, c^{-\frac{3}{2}hr^2}.$$

We are able to observe here the advantage that the observation of the three angles of each triangle produces, by the correction of these angles. Without this correction, the error of the angle $A^{(n)}$ would be

$$\alpha^{(n-1)} - \alpha^{(n-2)} - \cdots\cdots \pm \alpha,$$

and the probability that this error is comprehended within the limits $\pm r\sqrt{n}$ would be

$$\frac{2\sqrt{h}}{\sqrt{\pi}} \int dr\, c^{-hr^2},$$

a probability less than the preceding in which the weight of the result is $\frac{3}{2}h$, instead as it is here h.

Let us determine now, the value of h. Among the data of the observations, the quantities by which the sums of the angles of each triangle surpass two right angles plus the spherical excess, appear to be the most proper to make known this value. By that which precedes, the probability of the simultaneous existence of $\bar{\alpha}$ and of T is proportional to

$$c^{-\frac{h}{3}T^2 - \frac{3h}{2}\bar{\alpha}^2}.$$

By multiplying this exponential by $d\bar{\alpha}$, and taking the integral from $\bar{\alpha} = -\infty$, to $\bar{\alpha} = \infty$; the integral will have for factor $c^{-\frac{h}{3}T^2}$; and this factor will be proportional to the probability of T; this probability will be therefore

[11] $$\frac{dT c^{-\frac{h}{3}T^2}}{\int dT\, c^{-\frac{h}{3}T^2}},$$

the integral of the denominator being taken from $T = -\infty$, to $T = \infty$. It will be thus proportional to

$$\frac{\sqrt{\frac{1}{3}h}}{\sqrt{\pi}}\, c^{-\frac{h}{3}T^2}.$$

Here the observed event is that the sum of the angles of the first triangle, of the second, of the third, etc., surpass two right angles plus the spherical excess, respectively by the quantities T, $T^{(1)}$, $\ldots T^{(n-1)}$, n being the number of triangles; the probability of this event will be therefore proportional to

$$\left(\frac{\frac{1}{3}h}{\pi}\right)^{\frac{n}{2}} c^{-\frac{h}{3}\theta^2},$$

by making

$$\theta^2 = T^2 + T^{(1)2} + \cdots\cdots + T^{(n-1)2}.$$

Now if we consider the diverse values of h, as causes of the observed event; the probability of h, will be by the principle of the probability of the causes, deduced from observed events, equal to

$$\frac{h^{\frac{n}{2}} dh\, c^{-\frac{h}{3}\theta^2}}{\int h^{\frac{n}{2}} dh\, c^{-\frac{h}{3}\theta^2}},$$

the integral of the denominator being taken for all the values of h, that is from $h = 0$, to $h = \infty$. The value of h that it is necessary to choose is evidently the integral of the products of the values of h, multiplied by their probabilities; this value is therefore

$$\frac{\int h^{\frac{n+2}{2}} dh\, c^{-\frac{h}{3}\theta^2}}{\int h^{\frac{n}{2}} dh\, c^{-\frac{h}{3}\theta^2}},$$

the integrals being taken from $h = 0$, to $h = \infty$. The integral of the numerator is equal to [12]

$$\frac{3(n+2)}{2\theta^2} \int h^{\frac{n}{2}} dh\, c^{-\frac{h}{3}\theta^2}.$$

The preceding fraction becomes thus $\frac{3(n+2)}{2\theta^2}$; this is therefore the value of h that it is necessary to adopt. If we suppose n a great number, this value becomes very nearly $\frac{3n}{2\theta^2}$. This quantity is the value of h which renders the observed event, most probable; the probability of this event, *a priori*, being proportional to $h^{\frac{n}{2}} c^{-\frac{h}{3}\theta^2}$. By taking for h, the quantity $\frac{3n}{2\theta^2}$, the probability that the error of the angle $A^{(n)}$ will be comprehended within the limits $\pm r\sqrt{n}$, is

$$\frac{3\sqrt{n}}{\theta\sqrt{\pi}} \int dr\, c^{-\frac{9}{4}\frac{nr^2}{\theta^2}};$$

the probability that it will be contained in the limits $\pm\frac{2}{3}\theta r'$, is therefore

$$\frac{2\int dr'\, c^{-r'^2}}{\sqrt{\pi}},$$

the integral being taken from r' null.

§2. Let us suppose the arc $AA'A''$, etc., perpendicular to the meridian of the point A. Let ϕ be the angle formed by this meridian and by the one of the extreme point $A^{(n)}$; and V the smallest of the angles that this last meridian makes with the arc AA' etc.; we will have

$$\sin\phi = \frac{\cos V}{\sin l},$$

l being the latitude of point A. By designating therefore by $\delta\phi$ and δV, the errors of the angles ϕ and V, we will have

$$\delta\phi = -\frac{\delta V \sin V}{\sin l \cos\phi}.$$

If we have measured with a great exactitude, the angle that the last side of the chain of triangles forms at $A^{(n)}$ with the meridian of this point, it is easy to see that [13]

$$\delta V = \pm\delta A^{(n)},$$

$\delta A^{(n)}$ being the error of $A^{(n)}$; the preceding integral in r', is therefore the probability that the error $\delta\phi$, of the longitude ϕ concluded from the azimuths observed at A and $A^{(n)}$, will be comprehended within the limits $\cdots \pm \frac{2}{3}\theta r' \frac{\sin V}{\sin l \cos\phi}$.

There results from the analysis exposed in Chapter V of the third Book of the *Mécanique céleste*, that if there exists an eccentricity in the terrestrial parallels, it has no sensible influence on the value of ϕ concluded in this manner, provided that the measured arc is not very considerable. In measuring therefore with a great precision, the angles of the diverse triangles, and the amplitudes of the extreme points; we will have quite exactly the difference in longitude, of these points; and we will be able by the preceding formula, to estimate the probability of the small errors to fear respecting this difference.

Let us determine presently the probability that the error of the measure of the line $AA'A''$, etc., will be comprehended within some given limits. For this, let us suppose that in the triangles CAC', $C'CC''$, etc. we had corrected the angles as one does ordinarily; that is by subtracting from each, the third of the quantity by which the sum of the three observed angles surpasses two right angles plus the spherical excess. If we lower the vertices C, C', C'', etc., from the perpendiculars CI, $C'I'$, $C''I''$, onto the line $AA'A''$, etc.; we will have very nearly,

$$AI = AC\cos IAC.$$

We will have next quite nearly,

$$II' = CC'\cos A^{(1)},$$

and, generally,

$$I^{(i)}I^{(i+1)} = C^{(i)}C^{(i+1)}\cos A^{(i+1)}.$$

By supposing therefore that δ is the characteristic of the errors, we will have

$$\frac{\delta.I^{(i)}I^{(i+1)}}{I^{(i)}I^{(i+1)}} = \frac{\delta.C^{(i)}C^{(i+1)}}{C^{(i)}C^{(i+1)}} - \delta A^{(i+1)}\tan A^{(i+1)}.$$

[14] We have by that which precedes,

$$\delta A^{(i+1)} = \bar\alpha^{(i)} - \bar\alpha^{(i-1)} + \bar\alpha^{(i-2)}\cdots \pm \bar\alpha;$$

next, we have in the $(i+1)^{\text{st}}$ triangle,

$$C^{(i)}C^{(i+1)} = \frac{C^{(i)}C^{(i-1)}\sin C^{(i+1)}C^{(i-1)}C^{(i)}}{\sin C^{(i-1)}C^{(i+1)}C^{(i)}};$$

that which gives

$$\begin{aligned}\frac{\delta.C^{(i)}C^{(i+1)}}{C^{(i)}C^{(i+1)}} = \frac{\delta.C^{(i)}C^{(i-1)}}{C^{(i)}C^{(i-1)}} &+ \delta C^{(i+1)}C^{(i-1)}C^{(i)}\cot C^{(i+1)}C^{(i-1)}C^{(i)}\\ &- \delta C^{(i-1)}C^{(i+1)}C^{(i)}\cot C^{(i-1)}C^{(i+1)}C^{(i)};\end{aligned}$$

but $\bar\alpha^{(i)}$ is, by that which precedes, the error of the angle $C^{(i)}$ or $C^{(i-1)}C^{(i)}C^{(i+1)}$, corrected by subtracting from it the third of the excess of the sum of the three observed angles of the triangle over two right angles. Let $\bar\beta^{(i)}$ be the error of the angle $C^{(i-1)}C^{(i+1)}C^{(i)}$, thus corrected; $-(\bar\alpha^{(i)} + \bar\beta^{(i)})$, will be the error of the third angle $C^{(i+1)}C^{(i-1)}C^{(i)}$. We will have therefore

$$\begin{aligned}\frac{\delta.C^{(i)}C^{(i+1)}}{C^{(i)}C^{(i+1)}} = \frac{\delta.C^{(i)}C^{(i-1)}}{C^{(i)}C^{(i-1)}} &+ (\bar\alpha^{(i)} + \bar\beta^{(i)})\cot C^{(i+1)}C^{(i-1)}C^{(i)}\\ &- \bar\beta^{(i)}\cot C^{(i-1)}C^{(i+1)}C^{(i)};\end{aligned}$$

that which gives, by observing that in the first triangle, the side $C^{(i-1)}C$, is AC that I supposed measured very exactly.

$$\frac{\delta.C^{(i)}C^{(i+1)}}{C^{(i)}C^{(i+1)}} = -S\left\{\begin{matrix}(\bar{\alpha}^{(i)}+\bar{\beta}^{(i)})\cot C^{(i+1)}C^{(i-1)}C^{(i)}\\ +\bar{\beta}^{(i)}\cot C^{(i-1)}C^{(i+1)}C^{(i)}\end{matrix}\right\},$$

the sign S serving to express the sum of all the quantities that it contains from $i = 0$, to i inclusively. We will have therefore thus the value of $\delta.I^{(i)}I^{(i+1)}$. By reuniting all these values, we will have for the entire error of their sum, or of the measured line, an expression of this form

$$p\bar{\alpha} + q\bar{\beta} + p^{(1)}\bar{\alpha}^{(1)} + q^{(1)}\bar{\beta}^{(1)} + \text{etc.} \tag{o}$$

The probability of the simultaneous values of $\bar{\alpha}$ and of $\bar{\beta}$, is, by that which precedes, proportional to

$$c^{-2h\left(\bar{\beta}+\frac{1}{2}\bar{\alpha}\right)^2-\frac{3}{2}h\alpha^2}.$$

By making [15]

$$\bar{\beta} + \tfrac{1}{2}\bar{\alpha} = \tfrac{1}{2}\underline{\alpha}\sqrt{3};$$

the preceding exponential becomes

$$c^{-\frac{3}{2}h\underline{\alpha}^2-\frac{3}{2}h\bar{\alpha}^2};$$

thus the laws of probability of the values of $\underline{\alpha}$ and of $\bar{\alpha}$ are the same. The function (o) takes then this form,

$$r\underline{\alpha} + r^{(1)}\bar{\alpha} + r^{(2)}\underline{\alpha}^{(1)} + r^{(3)}\bar{\alpha}^{(1)} + \cdots \tag{o'}$$

The probability that the error of this function and consequently of the function (o), is comprehended within the limits $\pm s$, is by §20 of the second Book,

$$\frac{2\int dt\, c^{-t^2}}{\sqrt{\pi}},$$

the integral being taken from t null, to t equal to

$$s\sqrt{\frac{\frac{3}{2}h}{r^2 + r^{(1)2} + r^{(2)2} + \text{etc.}}}$$

We have evidently

$$p\bar{\alpha} + q\bar{\beta} = \left(p - \tfrac{1}{2}q\right)\bar{\alpha} + \tfrac{1}{2}q\underline{\alpha}\sqrt{3};$$

that which gives by equating it to $r\underline{\alpha} + r^{(1)}\bar{\alpha}$,

$$r = \frac{1}{2}q\sqrt{3}, \quad r^{(1)} = p - \tfrac{1}{2}q;$$

the value of t will be therefore by substituting for h, its value $\frac{3n}{2\theta^2}$,

$$\frac{3s}{2\theta}\sqrt{\frac{n}{p^2 - pq + q^2 + p^{(1)2} - p^{(1)}q^{(1)} + q^{(1)2} + \text{etc.}}}.$$

The length of the measured arc makes known that of the osculating radius of the surface, at the point A of departure. Let $1+u$, be the radius drawn from the center of gravity of the Earth, to its surface; u being a function of the longitude and of the latitude, the semi-axis of the Earth being taken for unity; if we name R, the
[16] osculating radius of this point, in the sense AA', we will have by the Chapter cited from the third Book of the *Mécanique céleste*,

$$R = 1 + u - \left(\frac{du}{dl}\right)\tan l + \frac{\left(\frac{d\,du}{d\phi^2}\right)}{\cos^2 l};$$

and if we name ϵ, the length of the measured arc $AA^{(1)}$, we will have quite nearly

$$R = \frac{\epsilon}{\phi \cos l}(1 - \tfrac{1}{3}\epsilon^2 \tan^2 l);$$

that which gives quite nearly,

$$\delta R = \frac{\delta\epsilon}{\phi\cos l} - \frac{\epsilon\,\delta\phi}{\phi^2\cos l};$$

but we have by that which precedes,

$$\delta\epsilon = p\bar{\alpha} + q\bar{\beta} + \text{etc.},$$
$$\delta\phi = \frac{\mp\delta A^{(n)}}{\sin l} = \frac{\pm(\bar{\alpha} - \bar{\alpha}^{(1)} + \bar{\alpha}^{(2)} - \text{etc.})}{\sin l},$$

the inferior sign having place if n is even, and the superior sign if n is odd. By making therefore

$$\bar{p} = \frac{p}{\phi\cos l} \mp \frac{\epsilon}{\phi^2 \sin l \cos l}; \qquad \bar{q} = \frac{q}{\phi\cos l};$$
$$\bar{p}^{(1)} = \frac{p^{(1)}}{\phi\cos l} \pm \frac{\epsilon}{\phi^2 \sin l \cos l}; \qquad \bar{q}^{(1)} = \frac{q^{(1)}}{\phi\cos l};$$
etc.;

the probability that the error δR will be comprehended within the limits $\pm s$, will be

$$\frac{2\int dt\, c^{-t^2}}{\sqrt{\pi}},$$

the integral being taken from t null, to

$$t = \frac{3s}{2\theta}\sqrt{\frac{n}{\bar{p}^2 - \bar{p}\bar{q} + \bar{q}^2 + \bar{p}^{(1)2} - \bar{p}^{(1)}\bar{q}^{(1)} + \text{etc.}}}.$$

The difference in latitude, of the extreme points of the perpendicular, depend by the chapter cited from the *Mécanique céleste*, on the eccentricity of the terrestrial
[17] parallels which introduce into its expression, the quantity

$$-\phi\left\{\left(\frac{du}{d\phi}\right)\tan l + \left(\frac{d\,du}{d\phi\,dl}\right)\right\}; \qquad (u)$$

the part of this expression, which is independent of this eccentricity, is proportional to ϕ^2; thus the small error of which ϕ is susceptible, has no sensible influence at all, on the difference in latitude. By observing therefore with a great care this difference, the eccentricity of the terrestrial parallels must be manifest, as little as it is sensible.

If the geodesic line has been traced in the sense of the meridian; the azimuth, at the extremity of the measured arc, will make known the eccentricity of the terrestrial parallels; and it is remarkable that this azimuth is the function (u), by changing ϕ, into the difference in latitude of the extreme points of the measured arc, and by multiplying it by the sine of the latitude divided by the square of the cosine of the latitude at the origin of the arc.

The arc measured in the sense of the meridian will make known the osculating radius of the Earth, in this sense; and by the preceding formulas, we will have the probability of the errors of which its value is susceptible.

We will obtain more precision in all the results, by fixing toward the middle of the measured arc, the origin of the angles; because then, the superior powers of these angles, that we neglect, becomes much smaller.

§3. Let us suppose that in order to verify the operations, we measure toward the extremity $A^{(n)}$ of the arc $AA'A''$, etc. a second base. The expression of the error of this base, concluded from the chain of the triangles and from the base measured at the point A, will be, by that which precedes, of the form

$$l\bar{\alpha} + m\bar{\beta} + l^{(1)}\bar{\alpha}^{(1)} + m^{(1)}\bar{\beta}^{(1)} + \text{etc.}; \tag{p}$$

let λ be this error which will be known by the direct measure of the second base. If in the function (p) we make as previously

$$\bar{\beta} + \frac{1}{2}\bar{\alpha} = \frac{1}{2}\underline{\alpha}\sqrt{3};$$

it takes this form

$$f\underline{\alpha} + f^{(1)}\bar{\alpha} + f^{(2)}\underline{\alpha}^{(1)} + f^{(3)}\bar{\alpha}^{(1)} + \text{etc.}$$

By designating by s the value of the function (o) or of its equivalent (o') and observing [18] that the probabilities of $\underline{\alpha}$ and of $\bar{\alpha}$ follow the same law, and are proportionals to $c^{-\frac{3}{2}h\underline{\alpha}^2}$ and $c^{-\frac{3}{2}h\bar{\alpha}^2}$; the probability of the preceding function, will be proportional to

$$c^{-\frac{3}{2}h(\underline{\alpha}^2+\bar{\alpha}^2+\underline{\alpha}^{(1)2}+\bar{\alpha}^{(1)2}+\text{etc.})}.$$

By supposing the function equal to λ; this exponential becomes

$$c^{-\frac{3}{2}h\left[\left(\underline{\alpha}-\frac{f\lambda}{F}\right)^2+\left(\bar{\alpha}-\frac{f^{(1)}\lambda}{F}\right)^2+\text{etc.}+\frac{\lambda^2}{F}\right]},$$

F expressing the sum of the squares $f^2 + f^{(1)2} + f^{(2)2} + \text{etc.}$ The most probable values of $\underline{\alpha}, \bar{\alpha}, \underline{\alpha}^{(1)}$, etc., are evidently those which render the exponent of this exponential a *minimum*, that which gives

$$\underline{\alpha} = \frac{f\lambda}{F}; \quad \bar{\alpha} = \frac{f^{(1)}\lambda}{F}; \quad \underline{\alpha}^{(1)} = \frac{f^{(2)}\lambda}{F}; \quad \text{etc.}$$

If we observe next that we have by that which precedes

$$f = \tfrac{1}{2}m\sqrt{3}, \quad f^{(1)} = l - \tfrac{1}{2}m,$$
$$\bar{\beta} = \tfrac{1}{2}\underline{\alpha}\sqrt{3} - \tfrac{1}{2}\bar{\alpha},$$

we will have

$$\underline{\alpha} = \frac{(l - \frac{1}{2}m)\lambda}{F}; \qquad \bar{\beta} = \frac{(m - \frac{1}{2}l)\lambda}{F};$$
$$\bar{\alpha}^{(1)} = (l^{(1)} - \frac{1}{2}m^{(1)})\frac{\lambda}{F}; \qquad \bar{\beta}^{(1)} = \frac{(m^{(1)} - \frac{1}{2}l^{(1)})\lambda}{F};$$

etc.;

and F will become

$$l^2 - ml + m^2 + l^{(1)2} - m^{(1)}l^{(1)} + m^{(1)2} + \text{etc.}$$

If we substitute these values into the function (o) we will have the correction resulting from the measure of a second base, by affecting it with a contrary sign. But we are able to arrive to this result directly, by §21 of the second Book, according to which we see that s being the value of the function (o), its probability is proportional to

$$c^{-\frac{\frac{3}{2}h\left(s - \lambda\frac{Sr^{(i)}f^{(i)}}{Sf^{(i)2}}\right)}{Sr^{(i)2} - \frac{(Sr^{(i)}f^{(i)})^2}{Sf^{(i)2}}}},$$

[19] the sign S extending to all the values of i, from $i = 0$ inclusively. The most probable value of s, is that which renders null, the exponent of c; that which gives

$$s = \lambda\frac{Sr^{(i)}f^{(i)}}{Sf^{(i)2}};$$

it is necessary therefore to subtract from the measured arc $AA^{(1)} \ldots A^{(n)}$ this value of s; and if we name u, the error of the arc thus corrected; the probability of u, will be proportional to

$$c^{-\frac{\frac{3}{2}hu^2}{Sr^{(i)2} - \frac{(Sr^{(i)}f^{(i)})^2}{Sf^{(i)2}}}}.$$

We see by this expression, that the weight of the result is increased, by virtue of the measure of the second base; because before this measure, the coefficient of $-s^2$, was by the preceding section,

$$\frac{\frac{3}{2}h}{Sr^{(i)2}};$$

and by this measure, the coefficient of $-u^2$ becomes

$$\frac{\frac{3}{2}h}{Sr^{(i)2} - \frac{(Sr^{(i)}f^{(i)})^2}{Sf^{(i)2}}}.$$

The same error becomes therefore less probable by this measure and by the preceding correction of this arc.

We are able to observe here that the preceding values of r, $r^{(1)}$, f and $f^{(1)}$ give

$$r^2 + r^{(1)} = p^2 - pq + q^2;$$
$$f^2 + f^{(1)2} = l^2 - ml + m^2;$$
$$rf + r^{(1)}f^{(1)} = l(p - \tfrac{1}{2}q) + m(q - \tfrac{1}{2}p).$$

We will be able therefore to form easily $Sr^{(i)2}$ and $Sr^{(i)}f^{(i)}$ by means of the coefficients of $\bar{\alpha}$, $\bar{\beta}$, $\bar{\alpha}^{(1)}$, etc., in the functions of (o) and (p).

If we had measured some other bases, we would have by the analysis of §21 of the second book, the corrections which it would be necessary to make to the measured arc, and the law of its errors.

The measure of a new base is able to serve to correct, not only the measured arc, but also the difference in longitude of its extreme points, or the angle $A^{(n)}$. It will [20]
suffice to substitute into the function (o), this one

$$\pm(\bar{\alpha} - \bar{\alpha}^{(1)} + \bar{\alpha}^{(2)} - \text{etc.})$$

which expresses the error of $A^{(n)}$, the superior sign having place, if n is odd, and the inferior, if n is even. Then we have

$$p = \pm 1; \quad q = 0; \quad p^{(1)} = \mp 1; \quad q^{(1)} = 0; \quad \text{etc.};$$

thence it is easy to conclude that in order to correct the angle $A^{(n)}$, it is necessary to add to it the quantity

$$\frac{\mp\lambda \begin{pmatrix} l - l^{(1)} + l^{(2)} - \text{etc.} \\ -\frac{1}{2}m + \frac{1}{2}m^{(1)} - \text{etc.} \end{pmatrix}}{l^2 - ml + m^2 + l^{(1)2} - m^{(1)}l^{(1)} + m^{(1)2} + \text{etc.}}.$$

The probability that the error of $A^{(n)}$ thus corrected, is within the limits $\pm u$, will be

$$\frac{2\int dt\, c^{-t^2}}{\sqrt{\pi}};$$

the integral being taken from t null, to

$$t = \frac{u\sqrt{\frac{3}{2}h}}{\sqrt{n - \frac{(l - l^{(1)} + l^{(2)} - \text{etc.} - \frac{1}{2}m + \frac{1}{2}m^{(1)} - \text{etc.})^2}{l^2 - ml + m^2 + l^{(1)2} - \text{etc.}}}}$$

§4. We are arrived to the preceding results, by starting from the law of probability of the error α, proportional to $c^{-h\alpha^2}$; and we have proved that this law of probability is able to be admitted in regard to the angles measured with the repeating circle. We will show here, that these results hold generally, whatever be the law of probability of error α. Let $\phi(\alpha)$ be, this law. We will suppose it such that the same positive and negative errors are equally probable. We will suppose moreover that $\phi(\alpha)$ extends from $\alpha = -\infty$, to $\alpha = +\infty$: this supposition is always permitted; because if the probability becomes null beyond certain limits, the function $\phi(\alpha)$ is then discontinued and null beyond these limits. Let us seek now the probability of the values of the

function (o) of §1. This function has been calculated by correcting the angles of each
[21] triangle, by a third of the observed sum of their errors. Let us suppose generally that, in the first triangle, we correct the error α, by $(i+\frac{1}{3})T$; the error β, by $(i_1+\frac{1}{3})T$; and consequently the third error, by $(\frac{1}{3}-i-i_1)T$; by designating by $\underline{\alpha}$ and $\underline{\beta}$ the errors α and β thus corrected; we will have

$$\alpha = \underline{\alpha} + (i+\tfrac{1}{3})T, \quad \beta = \underline{\beta} + (i_1+\tfrac{1}{3})T.$$

By designating similarly by $\underline{\alpha}^{(1)}$ and $\underline{\beta}^{(1)}$, the errors $\alpha^{(1)}$ and $\beta^{(1)}$ respectively corrected by $(i^{(1)}+\frac{1}{3})T^{(1)}$, $(i_1^{(1)}+\frac{1}{3})T^{(1)}$; we will have

$$\alpha^{(1)} = \underline{\alpha}^{(1)} + (i^{(1)}+\tfrac{1}{3})T^{(1)}; \quad \beta^{(1)} = \underline{\beta}^{(1)} + (i_1^{(1)}+\tfrac{1}{3})T^{(1)};$$

and so forth. The function (o) is, by §1, equal to

$$p\bar{\alpha} + q\bar{\beta} + p^{(1)}\bar{\alpha}^{(1)} + q^{(1)}\bar{\beta}^{(1)} + \text{etc.};$$

next, we have

$$\alpha = \bar{\alpha} + \tfrac{1}{3}T = \underline{\alpha} + (i+\tfrac{1}{3})T;$$

that which gives

$$\bar{\alpha} = \underline{\alpha} + iT;$$

we have similarly

$$\bar{\beta} = \underline{\beta} + i_1T; \quad \bar{\alpha}^{(1)} = \underline{\alpha}^{(1)} + i^{(1)}T, \quad \text{etc.}$$

The function (o), becomes thus

$$p\underline{\alpha} + q\underline{\beta} + p^{(1)}\underline{\alpha}^{(1)} + q^{(1)}\underline{\beta}^{(1)} + \text{etc.} + S(pi+qi_1)T;$$

$S(pi+qi_1)T$ designating the sum

$$(pi+qi_1)T + (p^{(1)}i^{(1)} + q^{(1)}i_1^{(1)})T^{(1)} + \text{etc.}$$

The correction of the function (o), relative to the values of i, i_1, $i^{(1)}$, etc., is therefore

$$-S(pi+qi_1)T,$$

and then, this function thus corrected becomes

$$p\underline{\alpha} + q\underline{\beta} + p^{(1)}\underline{\alpha}^{(1)} + q^{(1)}\underline{\beta}^{(1)} + p^{(2)}\underline{\alpha}^{(2)} + \text{etc.}, \qquad (\epsilon)$$

In order to have the probability of the values of this last function, we will observe
[22] that the probability of the simultaneous existence of the values of α, β and T is

$$\frac{d\alpha\,d\beta\,dT.\phi(\alpha).\phi(\beta).\phi(T-\alpha-\beta)}{\iiint d\alpha\,d\beta\,dT.\phi(\alpha).\phi(\beta).\phi(T-\alpha-\beta)},$$

the integrals of the denominator being taken within their positive and negative infinite limits. Let us designate by k the integral $\int d\alpha.\phi(\alpha)$, taken within these limits; it is easy to see that this denominator will be equal to k^3. The preceding fraction becomes thus

$$\frac{d\alpha\,d\beta\,dT}{k^3}.\phi(\alpha).\phi(\beta).\phi(T-\alpha-\beta);$$

the probability of the simultaneous existence of the values of $\underline{\alpha}, \underline{\beta}$ and T, will be therefore

$$\frac{\underline{d\alpha}\,\underline{d\beta}\,dT}{k^3}\phi[\underline{\alpha}+(i+\tfrac{1}{3})T]\phi[\underline{\beta}+(i_1+\tfrac{1}{3})T]\phi[(\tfrac{1}{3}-i-i_1)T-\underline{\alpha}-\underline{\beta}]$$

T being supposed to be able to be varied from $-\infty$, to $+\infty$; we will have the probability of the simultaneous values of $\underline{\alpha}$ and $\underline{\beta}$, by integrating the preceding function with respect to T, within the infinite limits. Let us name $\frac{\underline{d\alpha}\,\underline{d\beta}}{k^3}\psi(\underline{\alpha},\underline{\beta})$ this integral. We see by §20 of the second book, that by designating by s, the value of the function (ϵ), the probability of s, will be proportional to

$$\int dw\, c^{-sw\sqrt{-1}}\left\{\begin{array}{l}\iint \underline{d\alpha}\,\underline{d\beta}\,\psi(\underline{\alpha},\underline{\beta})\cos(p\underline{\alpha}+q\underline{\beta})w\\ \times\iint \underline{d\alpha}^{(1)}\,\underline{d\beta}^{(1)}\,\psi(\underline{\alpha}^{(1)},\underline{\beta}^{(1)})\cos(p^{(1)}\underline{\alpha}^{(1)}+q^{(1)}\underline{\beta}^{(1)})w\\ \times\text{ etc.}\end{array}\right\}\qquad\text{(H);}$$

the integral relative to w being taken from $w=-\pi$ to $w=\pi$, and the integrals relative to $\underline{\alpha}$ and $\underline{\beta}$ being taken within their infinite limits. Let us develop into a series ordered with respect to the powers of w, the function comprehended within the parenthesis. The logarithm of $\iint \underline{d\alpha}\,\underline{d\beta}\,\psi(\underline{\alpha},\underline{\beta})\cos(p\underline{\alpha}+q\underline{\beta})w$ is equal to

$$\log\iint \underline{d\alpha}\,\underline{d\beta}\,\psi(\underline{\alpha},\underline{\beta})\times\frac{-\frac{w^2}{2}\iint \underline{d\alpha}\,\underline{d\beta}\,\psi(\underline{\alpha},\underline{\beta})(p\underline{\alpha}+q\underline{\beta})^2}{\iint \underline{d\alpha}\,\underline{d\beta}\,\psi(\underline{\alpha},\underline{\beta})}-\text{etc.}$$

Now we have [23]

$$\iint \underline{d\alpha}\,\underline{d\beta}\,\psi(\underline{\alpha},\underline{\beta})=$$
$$\iiint \underline{d\alpha}\,\underline{d\beta}\,dT\,\phi[\underline{\alpha}+(i+\tfrac{1}{3})T]\phi[\underline{\beta}+(i_1+\tfrac{1}{3})T]\phi\left(\begin{array}{r}(\tfrac{1}{3}-i-i_1)T\\ -\underline{\alpha}-\underline{\beta}\end{array}\right)$$

The integrals being taken within their infinite limits, it is easy to see by the known theory of multiple integrals, that the second member of this equation is equal to

$$\iiint d\alpha\,d\beta\,dT'\phi(\alpha)\phi(\beta)\phi(T'),$$

T' being equal to $T-\alpha-\beta$; it is therefore equal to k^3.

We have next

$$\begin{aligned}&\iint \underline{d\alpha}\,\underline{d\beta}\,\psi(\underline{\alpha},\underline{\beta})(p\underline{\alpha}+q\underline{\beta})^2\\ &=\iiint d\alpha\,d\beta\,dT'\phi(\alpha)\phi(\beta)\phi(T')(p\underline{\alpha}+q\underline{\beta})^2;\end{aligned}\qquad(u)$$

by substituting for $\underline{\alpha}$ and $\underline{\beta}$ their values in α, β, and T', in the quantity $(p\underline{\alpha}+q\underline{\beta})^2$. Now it follows from that which precedes that we have

$$\underline{\alpha} = (\tfrac{2}{3} - i)\alpha - (i + \tfrac{1}{3})\beta - (i + \tfrac{1}{3})T',$$
$$\underline{\beta} = (\tfrac{2}{3} - i)\beta - (i + \tfrac{1}{3})\alpha - (i_1 + \tfrac{1}{3})T'.$$

By substituting these values into the quantity $(p\underline{\alpha}+q\underline{\beta})^2$; we will be able, in its development, to neglect the terms dependent on the products $\alpha\beta$, $\alpha T'$, and $\beta T'$; because the triple integral

$$\iiint d\alpha\, d\beta\, dT'\phi(\alpha)\phi(\beta)\phi(T')(p\underline{\alpha}+q\underline{\beta})^2; \qquad (u)$$

being taken within its infinite limits, and the function $\phi(\alpha)$ being supposed the same for the values $+\alpha$ and $-\alpha$; it is clear that the elements of this integral, depending on $+\alpha\beta$, will be destroyed by the negative elements depending on $-\alpha\beta$. If we observe next that by designating $\int \alpha^2\, d\alpha\, \phi(\alpha)$ by k'', we have

$$\iiint \alpha^2 d\alpha\, d\beta\, dT'\phi(\alpha)\phi(\beta)\phi(T') = k^2k'';$$

the function (u) will become

$$k^2k''[\tfrac{2}{3}(p^2 - pq + q^2) + 3(pi + qi_1)^2];$$

[24] the logarithm of

$$\iint \underline{d\alpha}\, \underline{d\beta}\, \psi(\underline{\alpha},\underline{\beta})\cos(p\underline{\alpha}+q\underline{\beta})w$$

being thus

$$\log k^3 - \tfrac{k''}{2k}w^2[\tfrac{2}{3}(p^2 - pq + q^2) + 3(pi + qi_1)^2] - \text{etc.}$$

By passing again from logarithms to the numbers, and neglecting consistently with the analysis of §20 of the second book, the powers of w superior to the square; the integral (H) will take this form

$$k^{3n}\int dw\, c^{-sw\sqrt{-1}-\frac{k''w^2}{2k}[\frac{2}{3}S(p^2-pq+q^2)+3S(pi+qi_1)^2]},$$

$S(p^2-pq+q^2)$, representing the sum of the quantities $p^2-pq+q^2+p^{(1)2}-p^{(1)}q^{(1)}+$etc.; $S(pi+qi_1)^2$, representing the sum of the quantities $(pi+qi_1)^2+(p^{(1)}i^{(1)}+q^{(1)}i^{(1)})^2+$etc., and n being the number of triangles. Let us give to the preceding integral, this form,

$$k^{3n}\int dw\, c^{-Q\left(w+\frac{s\sqrt{-1}}{2Q}\right)^2-\frac{s^2}{4Q}},$$

Q being equal to

$$\frac{k''}{2k}[\tfrac{2}{3}S(p^2 - pq + q^2) + 3S(pi + qi_1)^2]$$

The integral must be taken from $w = -\pi$ to $w = \pi$, and we have seen, in the section cited from the second book, that it can be extended from $w = -\infty$ to $w = \infty$; then the preceding integral or the probability of s, becomes proportional to $c^{-\frac{s^2}{4Q}}$ or to

$$c^{-\frac{3ks^2}{4k''[S(p^2-pq+q^2)+\frac{9}{2}S(pi+qi_1)^2]}}$$

It is necessary now to determine the value of $\frac{k}{k''}$. For this we will make, as above, use of the observed values of T, $T^{(1)}$, $T^{(2)}$, etc. When these values are in great number; the sum of their squares divided by their number, will be quite nearly, by that which we have established in the second book, the mean value of T^2; by making therefore [25]

$$\theta^2 = T^2 + T^{(1)2} + T^{(2)2} + \text{etc.},$$

$\frac{\theta^2}{n}$ will be this mean value. Now, we have this value by multiplying each possible value of T^2, by its probability, and by taking the sum of all these products; the expression of the mean value of T^2, will be therefore

$$\frac{\iiint d\alpha\, d\beta\, dT.T^2\phi(\alpha)\phi(\beta)\phi(T-\alpha-\beta)}{\iiint d\alpha\, d\beta\, dT\phi(\alpha)\phi(\beta)\phi(T-\alpha-\beta)},$$

the integrals being taken within their infinite limits. Let there be as above

$$T' = T - \alpha - \beta,$$

the preceding fraction will become

$$\frac{\iiint (T'+\alpha+\beta)^2\; d\alpha\, d\beta\, dT'\phi(\alpha)\phi(\beta)\phi(T')}{\iiint d\alpha\, d\beta\, dT'\phi(\alpha)\phi(\beta)\phi(T')},$$

all these integrals being taken again within their infinite limits. It is easy to see by the preceding analysis, that the numerator of this fraction, is equal to $3k^2k''$, and that its denominator is equal to k^3; the fraction becomes thus $\frac{3k''}{k}$; by equating it to $\frac{\theta^2}{n}$, we will have

$$\frac{k''}{k} = \frac{\theta^2}{3n};$$

the probability of s, is therefore proportional to

$$c^{-\frac{9ns^2}{4\theta^2[S(p^2-pq+q^2)+\frac{9}{2}S(pi+qi_1)^2]}}$$

It is clear that the values of i and of i_1, which render this probability the most rapidly decreasing, are those which give $pi + qi_1 = 0$; and then the preceding correction of the measured arc, becomes null. The case of i and i_1 nulls, give therefore the law of probability of the geodesic errors, the most rapidly decreasing, a law which must be evidently adopted.

Thence, it is easy to conclude that the probability that the value of s, will be comprehended within the limits $\pm s$, is equal to [26]

$$\frac{2\int dt\, c^{-t^2}}{\sqrt{\pi}},$$

the integral being taken from t null, to

$$t = \frac{3s}{2\theta}\sqrt{\frac{n}{S(p^2 - pq + q^2)}};$$

that which is conformed to that which we have deduced in §1, from the particular law of probability of the errors α, proportional to $c^{-h\alpha^2}$.

Let us express as in §2, the error of a new base concluded from the first, by the function

$$l\bar{\alpha} + m\bar{\beta} + l^{(1)}\bar{\alpha}^{(1)} + m^{(1)}\bar{\beta}^{(1)} + \text{etc.}$$

By making as previously,

$$\underline{\alpha} = \bar{\alpha} - iT; \quad \underline{\beta} = \bar{\beta} - i_1 T; \quad \underline{\alpha} = \bar{\alpha}^{(1)} - i^{(1)}T^{(1)}; \quad \text{etc.},$$

the correction of this function, relative to the values of i, i_1, $i^{(1)}$, etc., will be $-S(li + mi_1)T$, and the error of the new base thus corrected will be

$$l\underline{\alpha} + m\underline{\beta} + l^{(1)}\underline{\alpha}^{(1)} + m^{(1)}\underline{\beta}^{(1)} + \text{etc.} \qquad (\lambda)$$

Let s' be the value of this function; the probability of the simultaneous existence of the values of s and s' of the functions (ϵ) and (λ), will be by §21 of the second book, proportional to

$$\iint dw\, dw' c^{-sw\sqrt{-1}-s'w'\sqrt{-1}-Qw^2-2Q_1ww'-Q_2w'^2},$$

the integrals being taken from w and w' equal to $-\infty$, to w and w' equal to $+\infty$. We see next by the analysis of the section cited, that we have

$$Qw^2 + 2Q_1ww' + Q_2w'^2$$
$$= \frac{\frac{1}{2}S\iiint\ d\alpha\, d\beta\, dT'\phi(\alpha)\phi(\beta)\phi(T')[(p\underline{\alpha} + q\underline{\beta})w + (l\underline{\alpha} + m\underline{\beta})w']^2}{\iiint\ d\alpha\, d\beta\, dT'\phi(\alpha)\phi(\beta)\phi(T')},$$

[27] the integrals relative to α, β and T', being taken within their infinite limits; that which gives by substituting for $\underline{\alpha}$ and $\underline{\beta}$ their previous values,

$$Q = \frac{1}{3}\frac{k''}{k}[S(p^2 - pq + q^2) + \tfrac{9}{2}S(pi + qi_1)^2],$$
$$Q_1 = \frac{1}{3}\frac{k''}{k}\left\{S\left[\left(p - \tfrac{q}{2}\right)l + \left(q - \tfrac{p}{2}\right)m\right] + \tfrac{9}{2}S((pi + qi_1)(li + mi_1)\right\},$$
$$Q_2 = \frac{1}{3}\frac{k''}{k}[S(l^2 - ml + m^2) + \tfrac{9}{2}S(li + mi_1)^2];$$

whence we conclude by the analysis of the section cited, that the probability of the simultaneous existence of the values of s and of s' is proportional to

$$c^{-\frac{(Q_2s^2 - 2Q_1ss' + Q_1s'^2)}{4(QQ_2 - Q_1^2)}}$$

or

$$c^{-\frac{Q_2(s - s'\frac{Q_1}{Q_2})^2}{4(QQ_2 - Q_1^2)} - \frac{s'^2}{4Q_2}}$$

The measure of the second base determines the value of s'; and, by naming it λ as above, the probability of s will be proportional to

$$c^{-\frac{Q_2(s-\frac{\lambda Q_1}{Q_2})^2}{4(QQ_2-Q_1^2)}}.$$

The most probable value of s is that which renders null the exponent of c; that which gives

$$s=\lambda\frac{Q_1}{Q_2};$$

by making therefore

$$s=\lambda\frac{Q_1}{Q_2}+u,$$

u will be the error of the arc measured and diminished by $\frac{\lambda Q_1}{Q_2}$; and the probability of this error will be proportional to

$$c^{-\frac{Q_2u^2}{4(QQ_2-Q_1^2)}}.$$

The values of i, i_1, $i^{(1)}$, etc., must be determined by the condition that the coefficient of u^2 in this exponential, is a *maximum*; let us see therefore what are the values of these quantities, which render the fraction [28]

$$\frac{Q_2}{QQ_2-Q_1^2}$$

a *maximum*. If we name Q' that which the expression of Q becomes, when we diminish the finite integral $S(pi+qi_1)^2$, by the element $(pi+qi_1)^2$, we will have

$$Q'=Q-\frac{3}{2}\frac{k''}{k}(pi+qi_1)^2.$$

If we name similarly Q'_1 that which the expression of Q_1 becomes, when we diminish the finite integral $S(pi+qi_1)(li+mi_1)$ by the element $(pi+qi_1)(li+mi_1)$; we will have

$$Q'_1=Q_1-\frac{3}{2}\frac{k''}{k}(pi+qi_1)(li+mi_1).$$

Finally if we name Q'_2 that which Q_2 becomes, when we diminish the finite integral $S(li+mi_1)^2$, by the element $(li+mi_1)^2$; we will have

$$Q'_2=Q_2-\frac{3}{2}\frac{k''}{k}(li+mi_1)^2.$$

The fraction

$$\frac{Q'_2}{Q'Q'_2-Q_1'^2}$$

surpasses the fraction

$$\frac{Q_2}{QQ_2-Q_1^2};$$

because by substituting into the first, instead of Q', Q'_1 and Q'_2 their values, and reducing to the same denominator, its excess over the second; the numerator of this excess, becomes

$$\frac{3}{2}\frac{k''}{k}[Q_2(pi + qi_1) - Q_1(li + mi_1)]^2.$$

Let us name further Q'', that which Q' becomes when we subtract $\frac{3}{2}\frac{k''}{k}(p^{(1)}i^{(1)} + q^{(1)}i_1^{(1)})^2$ from it; and consequently that which the expression of Q becomes when we diminish the integral $S(pi+qi_1)^2$, by the two elements $(pi+qi_1)^2+(p^{(1)}i^{(1)}+q^{(1)}i_1^{(1)})^2$. Let us name similarly Q''_1, that which Q'_1 becomes, when we subtract from it

$$\frac{3}{2}\frac{k''}{k}(p^{(1)}i^{(1)} + q^{(1)}i_1^{(1)})(l^{(1)}i^{(1)} + m^{(1)}i_1^{(1)});$$

finally, let us name Q''_2 that which Q'_2 becomes when we subtract from it

[29]
$$\frac{3}{2}\frac{k''}{k}(l^{(1)}i^{(1)} + m^{(1)}i_1^{(1)})^2;$$

we will see by the same process, that the fraction

$$\frac{Q''_2}{Q''Q''_2 - Q''^2_1},$$

surpasses the fraction

$$\frac{Q'_2}{Q'Q'_2 - Q'^2_1},$$

and consequently the fraction

$$\frac{Q_2}{QQ_2 - Q_1^2}.$$

By continuing thus, we see that this last fraction arrives to its *maximum*, when the finite integrals $S(pi + qi_1)^2$, $S(pi + qi_1)(li + mi_1)$, and $S(li + mi_1)^2$, are null, in the expressions of Q, Q_1 and Q_2; that which reverts to supposing null, the values of i, i_1, $i^{(1)}$, etc.; this supposition gives therefore the law of probability of the most rapidly decreasing values of Q, and then we have

$$Q = \frac{\theta^2}{9n}S(p^2 - pq + q^2);$$

$$Q_1 = \frac{\theta^2}{9n}S\left[\left(p - \tfrac{q}{2}\right)l + \left(q - \tfrac{p}{2}\right)m\right];$$

$$Q_2 = \frac{\theta^2}{9n}S(l^2 - ml + m^2).$$

The weight of the error u becomes thus

$$\frac{-\frac{9n}{4\theta^2}}{S(p^2 - pq + q^2) - \frac{\left[S\left(p-\frac{q}{2}\right)l+\left(q-\frac{p}{2}\right)m\right]^2}{S(l^2-ml+m^2)}}$$

It is easy to see that this result coincides with the analogous result of §2.

On the probability of the results deduced by any processes whatsoever, from a great number of observations. [30]

The true march of the natural sciences consists in showing through the path of induction, from the phenomena to the laws, and from the laws to the forces. We come down next from these forces, to the complete explication of the phenomena as far as their smallest details. The attentive inspection of a great assembly of observations, and their comparisons multiplied are a presentiment of the laws that it conceals. The analytic expression of these laws depends on constant coefficients that we name *elements*. We determine by the theory of probabilities, the most probable values of these elements; and if, by substituting them into the analytic expressions, these expressions satisfy all the observations, within the limits of the possible errors; we will be sure that these laws are those of nature, or at least they are very little different from them. We see thence, how much the application of the Calculus of Probabilities is useful to natural Philosophy, and how much it is essential to have methods in order to deduce from observations, the most advantageous results. These results are evidently those with which one same error is less probable than with each other result. Thus the condition that it is necessary to fulfill in the choice of a result, is that the law of probability of its errors is most rapidly decreasing. Before the application of the Calculus of Probabilities to this object, each calculator subjected the results of the observations, to the conditions which to him appeared to be most natural. Now if we have certain formulas in order to obtain the most advantageous result, he is no longer able to have uncertainty in this regard; at least when we make use of the factors. We can, not only determine this result, but further assign the probability of the errors of the results obtained by some other processes, and to compare these processes, to the most advantageous method. The excessive length of the calculations that this method requires, when we employ a very great number of observations, does not permit then to make use of it. But by grouping conveniently the equations of condition, and by applying this method, to the equations which result from each of these groups; we
can at the same time simplify considerably the calculations, and conserve a part of [31]
the advantages which are attached to it, as we will see in the following. Whatever be the process of which we make use, it is very useful to have a means to determine the probability of the results to which we arrive, especially when there is a question of the important elements. We will have easily this probability, by the following method.

§1. Let us consider first a quite simple case, the one of the angles measured by means of a repeating circle. Let us suppose that at the end of each partial observation, we read the corresponding division of the circle; we will have, by departing from the point of departure, a sequence of terms of which the first will be the angle itself; the second will be the double of this angle; the third will be the triple of it, and so forth. Let us designate by A_1, A_2,… A_n, these different terms, and by a_1, a_2,… a_n the n partial angles successively measured. We will have

$$\begin{aligned} A_1 &= a_1, \\ A_2 &= a_2 + a_1, \end{aligned}$$

$$A_3 = a_3 + a_2 + a_1,$$
etc.;

and if we name y, the true simple angle; we will have this sequence of equations;

$$\begin{aligned} y - a_1 + x_1 &= 0; \\ y - a_2 + x_2 &= 0; \\ y - a_3 + x_3 &= 0; \\ &\cdots\cdots \\ y - a_n + x_n &= 0; \end{aligned} \qquad (a)$$

x_1, x_2, x_3, etc., being the errors of the angles a_1, a_2, a_3, etc. We will have by §20 of the second book, the most advantageous result, by multiplying by unity, each of the preceding equations, and by adding them; that which gives

$$y = \frac{a_1 + a_2 + \cdots + a_n}{n} + \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

By supposing x_1, x_2, etc., null; we will have the result of the most advantageous method; and the error of this result, will be $\frac{x_1+x_2+\cdots+x_n}{n}$. By designating by u, this error; we see, by the section cited, that the probability of u, is proportional to $c^{-\frac{knu^2}{2k''}}$, k being equal to $\int dx\,\phi(x)$, and k'' being equal to $\int x^2\,dx\,\phi(x)$; $\phi(x)$ being the law
[32] of probability of the errors x of the partial observations, this law being supposed the same for the positive and negative errors, and being able to be extended to infinity. c is always the number of which the hyperbolic logarithm is unity.

Svangerg, in his excellent work on the degree of Lapland, exposes in order to determine y, a new process founded on the following considerations. Each term of the sequence A_1, A_2, etc., can give its value which can be equally determined by the difference $A_{s'} - A_s$ of any two terms whatsoever of this sequence, s' being greater than s. This difference divided by $s' - s$ gives a value of y so much more exact, as this divisor is greater. By multiplying it therefore by this divisor, we will render it preponderant by reason of its exactitude. If we make next a sum of these products, and if we divide it by the number of simple angles that it contains; we will have a value of y which concluded from all the combinations of the quantities A_1, A_2, etc., by giving to each of these combinations, the influence that it must have, seems ought to approach to the truth, the nearest that it is possible. This would be just, in fact, if all these values of y were independent. But their mutual dependence makes that the same simple angles are employed many times, and in a different manner for each of them, that which must change the respective probabilities of the values of y, and consequently the probability of the mean value. This is a new example of the illusions to which we are exposed in these delicate researches.

The process of which there is question, reverts to forming the sum of the differences $A_{s'} - A_s$, s' being greater than s, and must with this condition, be extended from $s' = 1$ to $s' = n$; s must be extended from $s = 0$ to $s = n - 1$, and we must make $A_0 = 0$. By dividing next this sum, by the number of simple angles that it contains, we have the value of y. It is easy to see that this value is

$$y = \frac{n.SA_n - 2.\mathrm{SS}A_{n-1}}{\frac{n\overline{n+1}.\overline{n+2}}{1.2.3}},$$

SA_n expressing the sum of the quantities A_1, $A_2 \ldots A_n$. SSA_{n-1} is the sum of the quantities,

$$\begin{aligned}&A_1,\\&A_1 + A_2,\\&A_1 + A_2 + A_3,\\&\ldots\ldots\ldots\ldots\\&A_1 + A_2 + \cdots + A_{n-1};\end{aligned}$$ [33]

the angle a_1 is contained $n - i + 1$ times in SA_n; it is contained $\frac{\overline{n-1}.\overline{n-i+1}}{1.2}$ times in the function SSA_{n-1}; it is therefore contained $\frac{i.\overline{n-i+1}}{\frac{n.\overline{n+1}.\overline{n+2}}{1.2.3}}$ times in the preceding expression of y. Thence it follows that this process reverts to multiplying the equations (a) respectively by the factors

$$\frac{n}{\frac{n.\overline{n+1}.\overline{n+2}}{6}};\quad \frac{2\overline{n-1}}{\frac{n.\overline{n+1}.\overline{n+2}}{6}};\quad \frac{3\overline{n-2}}{\frac{n.\overline{n+1}.\overline{n+2}}{6}};\quad \text{etc.};$$

and then we find, by §20 from the second book, that the probability of the error u in the preceding expression of y is proportional to

$$c^{-\frac{k}{k''}\frac{u^2}{SM_i^2}},$$

M_i being here equal to $\frac{i.\overline{n-i+1}}{\frac{n.\overline{n+1}.\overline{n+2}}{6}}$; the integral SM_i^2 must comprehend all the values of M_i^2 from $i = 1$ to $i = n$, inclusively. We have thus

$$SM_i^2 = \frac{6}{5} \cdot \frac{n^2 + 2n + 2}{n.\overline{n+1}.\overline{n+2}}.$$

n being supposed very great, this value of SM_i^2 is reduced very nearly to $\frac{6}{5}.n$; the probability of the error u is therefore proportional to

$$c^{-\frac{5}{6}\frac{k}{k''}nu^2}.$$

We have just seen that in the most advantageous method, the probability of a similar error of the result, is proportional to

$$c^{-\frac{knu^2}{2k''}}.$$

Thus in order that the same errors become equally probable, the observations must [34]
be in the process of Svanberg, more numerous than in the ordinary process, according to the ratio of six to five.

We could believe that the result obtained by the process of Svanberg, being a new datum from the observations; its combination with the result of the ordinary method, must give a more exact result and of which the law of probability of the errors is more rapidly decreasing. But the analysis proves that it is not. Let us consider in fact, the system of equations,

$$\begin{aligned} p_1y - a_1 + x_1 &= 0;\\ p_2y - a_2 + x_2 &= 0;\\ &\cdots\cdots\\ p_ny - a_n + x_n &= 0; \end{aligned} \tag{b}$$

x_1, x_2, etc., being as above, the errors of the observations. The most advantageous method prescribes to multiply these equations, respectively, by p_1, p_2, etc., and to add them, that which gives

$$y = \frac{Sp_ia_i}{Sp_i^2} - \frac{Sp_ix_i}{Sp_i^2},$$

the sign S comprehending, as above, all the values that it precedes, from $i = 1$ to $i = n$ inclusively. The first term of this expression, will be the value of y given by the most advantageous method, and its error will be $\frac{Sp_ix_i}{Sp_i^2}$; in designating it by u, its probability will be by §20 of the second book, proportional to

$$c^{-\frac{k}{2k''}u^2.Sp_i^2}.$$

If we multiply the equations (b) respectively by m_1, m_2, m_3, etc.; their sum will give

$$y = \frac{Sm_ia_i}{Sm_ip_i} - \frac{Sm_ix_i}{Sm_ip_i}.$$

The first term of this expression will be the value of y relative to the system of factors
[35] m_1, m_2, etc. and $\frac{Sm_ix_i}{Sm_ip_i}$ will be the error of this value, an error that we will designate by u'. If we make

$$l = Sp_ix_i, \quad l' = Sm_ix_i;$$

the probability of the simultaneous existence of l and of l' will be by §21 of the second book, proportional to

$$c^{-\frac{k}{2k''E}(l^2.Sm_i^2 - 2ll'.Sm_ip_i + l'^2.Sp_i^2)},$$

E being equal to $Sm_i^2.Sp_i^2 - (Sm_ip_i)^2$. Now we have

$$l = u.Sp_i^2; \quad l' = u'.Sm_ip_i;$$

the simultaneous existence of u and of u' is therefore proportional to

$$c^{-\frac{k}{2k''}\frac{Sp_i^2}{E}[u^2E + (u'-u)^2(Sm_ip_i)^2]}.$$

Let e be the difference of the preceding values from y; we have

$$e = \frac{Sp_ia_i}{Sp_i^2} - \frac{Sm_ia_i}{Sm_ip_i};$$

the equality of these values corrected respectively of their errors u and u', give

$$e = u - u';$$

the preceding exponential becomes thus

$$c^{-\frac{k}{2k''}Sp_i^2\left(u^2 + e^2\frac{(Sm_ip_i)^2}{E}\right)}.$$

e is a quantity given by the observations; the value of u which renders this exponential a *maximum*, is evidently $u = 0$; thus the consideration of the result given by the system of factors m_1, m_2, etc., add no correction to the result of the most advantageous method, and changes not at all the law of probability of its error u, which remains always proportional to

$$c^{-\frac{k}{2k''}u^2.Sp_i^2}.$$

If the very great number of equations of condition do not permit applying this method to them; there will be always advantage to apply it to some equations resulting [36] from groups of these equations. Let us suppose that we have r groups formed, each, of s equations, so that $n = rs$; we will have the following r equations:

$$\begin{aligned} &P_1 y - A_1 + X_1 = 0; \\ &P_2 y - A_2 + X_2 = 0; \\ &\dots\dots\dots\dots\dots\dots \\ &P_r y - A_r + X_r = 0; \end{aligned} \qquad \text{(V)}$$

and we have

$$\begin{aligned} &P_1 = p_1 + p_2 \cdots + p_s; \\ &A_1 = a_1 + a_2 \cdots + a_s; \\ &X_1 = x_1 + x_2 \cdots + x_s; \\ &P_2 = p_{s+1} + p_{s+2} \cdots + p_{2s}; \\ &\text{etc.} \end{aligned}$$

By applying to the equations (V), the process of the most advantageous method, we have

$$y = \frac{SP_tA_t}{SP_t^2} - \frac{SP_tX_t}{SP_t^2};$$

the sign S embraces all the quantities which it precedes, from $t = 1$ to $t = r$ inclusively. $\frac{SP_tX_t}{SP_t^2}$ is the error of the value $\frac{SP_tA_t}{SP_t^2}$ taken for y; by designating this error by u, its probability will be by §20 of the second book, proportional to

$$c^{-\frac{k}{2k''}\cdot\frac{u^2}{Sm_i^2}};$$

m_1, m_2, etc., being the coefficients of x_1, x_2, etc., in the expression of u; and the integral Sm_i^2 being extended from $i = 0$, to $i = n$ inclusively. Now it is easy to see that we have

$$\begin{array}{llll} m_1 = \dfrac{P_1}{SP_t^2}; & m_2 = \dfrac{P_1}{SP_t^2}; & \cdots & m_s = \dfrac{P_1}{SP_t^2}; \\ m_{s+1} = \dfrac{P_2}{SP_t^2}; & \dots\dots\dots\dots & \cdots & m_{2s} = \dfrac{P_2}{SP_t^2}; \\ m_{2s+1} = \dfrac{P_3}{SP_t^2}; & \text{etc.} & & \end{array}$$

Thence it is easy to conclude that we have

$$Sm_i^2 = \frac{s}{SP_t^2} = \frac{n}{r.SP_t^2};$$

[37] the probability of u is therefore proportional to

$$c^{-\frac{k}{2k''}\cdot\frac{r}{n}u^2.SP_t^2}.$$

If we reunited all the equations, into a single group; the probability of u would be proportional to

$$c^{-\frac{k}{2k''}\cdot\frac{u^2}{n}(\mathrm{S}p_i)^2};$$

because then r would become unity; P_1 would become Sp_i, P_2, P_3, etc., would be nulls. The weight of the result or the coefficient of $-u^2$, would be therefore, in the first case

$$\frac{k}{2k''}\cdot\frac{r}{n}\cdot SP_t^2,$$

and in the second case, it would be

$$\frac{k}{2k''n}(\mathrm{S}p_i)^2.$$

Now the first of these quantities surpasses the second; in fact

$$(\mathrm{S}p_i)^2 = (P_1 + P_2 + \cdots + P_r)^2.$$

If, in the development of this last square, we substitute instead of the product $2P_1P_2$ its value $P_2^1 + P_2^2 - (P_1 - P_2)^2$, and thus of the other products, we see that this square is equal to rSP_t^2, less a positive quantity; there is therefore advantage to partition the equations of condition, into many groups to which we apply the most advantageous method.

We see further that there is advantage to augment the number of groups. Because if we suppose r even and equal to $2r'$; the weight of the result relative to the number r' of groups will be proportional to

$$r'[(P_1 + P_2)^2 + (P_3 + P_4)^2 \cdots + (P_{2r'+1} + P_{2r'})^2];$$

and the weight of the result relative to $2r'$ groups will be proportional to

$$2r'(P_1^2 + P_2^2 \cdots + P_{2r'}^2).$$

[38] This last quantity surpasses the preceding, as we see by observing that

$$2(P_1^2 + P_2^2) > (P_1 + P_2)^2.$$

If the equations of condition contain many unknown elements, y, y', etc.; there will be always advantage to partition them into groups, in order to apply to the equations resulting from these groups, the most advantageous method. The more we will multiply these groups, the more we will augment the weight of the results.

But from whatever manner that we have obtained these results, we will be able always to determine by the following theorem, the probability of their errors. If we have, by any process whatsoever, deduced from the equations of condition, the equation $y - a = 0$; it is clear that we have multiplied the equations of condition, respectively by some factors M_1, M_2, M_3, etc., such that the unknowns have disappeared, with the exception of y which has unity for factor. The error u of the result $y = a$, is evidently $M_1x_1 + M_2x_2 + \text{etc.}$; the probability of this error will be therefore by §20 of the second book, proportional to

$$c^{-\frac{k}{2k''}\cdot\frac{u^2}{SM_i^2}},$$

the sign S being extended to all the values of i from $i = 1$ to $i = n$, n being the number of observations. All is reduced therefore to determine in the process that we have followed, the factors M_1, M_2, etc.

If, for example, the equations of condition contain two unknowns y and y'; and if in order to form the final two equations, we add together all these equations, 1° by changing the signs of the equations in which y has the sign $-$; 2° by changing the signs of the equations in which y' has the sign $-$; we will obtain by this process of which we have often made use, two equations that we will represent by the following:

$$Py + Ry' - A = 0,$$
$$P_1y + R_1y' - A_1 = 0.$$

In multiplying the first of these equations by

$$\frac{R_1}{PR_1 - P_1R},$$

and the second by [39]

$$\frac{-R}{PR_1 - P_1R};$$

we will have by adding them,

$$y - \frac{(AR_1 - A_1R)}{PR_1 - P_1R} = 0.$$

In the equations of condition, x_i has been multiplied by ± 1; the sign $-$ having place if, in order to form the final equations, we have changed the signs of the i^{th} equation. Thence it is easy to conclude that if we designate by s, the number of equations of condition in which the coefficients of y and of y' have the same sign; we will have

$$SM_i^2 = \frac{s(R_1 - R)^2 + (n - s)(R_1 + R)^2}{(PR_1 - P_1R)^2}.$$

We will simplify the calculation, by preparing the equations of condition, in a manner that in each, the coefficient of y has the sign $+$. We will form next a first final equation, by adding the s equations in which the coefficient of y' has the sign $+$. We will form a second final equation, by adding the $n - s$ equations in which the coefficient of y' has the sign $-$. Let

$$fy + gy' - h = 0;$$
$$f_1y + g_1y' - h_1 = 0;$$

be these two equations. By multiplying the first by $\frac{g_1}{fg_1+f_1g}$, and the second by $\frac{g}{fg_1+f_1g}$; we will have

$$y - \frac{(hg_1 + h_1g)}{fg_1 + f_1g} = 0,$$

and it is easy to see that

$$SM_i^2 = \frac{sg_1^2 + (n-s)g^2}{(fg_1 + f_1g)^2}.$$

These values of y and of SM_i^2 coincide with the preceding, as it is easy to see it by observing that we have

$$P = f + f_1; \qquad R = g - g_1; \qquad A = h + h_1;$$
$$P_1 = f - f_1; \qquad R_1 = g + g_1; \qquad A_1 = h - h_1.$$

[40] The equations of condition being represented generally by the following,

$$0 = x_i - a_i + p_iy + q_iy';$$

if we multiply them respectively by m_1, m_2, etc., and if we add them; we will have the final equation

$$0 = Sm_ix_i - Sm_ia_i + y.Sm_ip_i + y'.Sm_iq_i;$$

if we multiply next the same equations respectively by n_1, n_2, etc., we will have by adding them, the final equation

$$0 = Sn_ix_i - Sn_ia_i + y.Sn_ip_i + y'.Sn_iq_i.$$

By multiplying the first of these equations by $\frac{Sn_iq_i}{I}$, and the second by $-\frac{Sm_iq_i}{I}$, I being equal to

$$Sm_ip_i.Sn_iq_i - Sn_ip_i.Sm_iq_i;$$

we will have

$$0 = y - \frac{Sm_ia_i.Sn_iq_i - Sn_ia_i.Sm_iq_i}{I} + \frac{Sm_ix_i.Sn_iq_i - Sn_ix_i.Sm_iq_i}{I}.$$

This last term is the error of the value that we obtain for y, by supposing nulls x_1, x_2, etc.; we have therefore then

$$M_i = \frac{m_i.Sn_iq_i - n_i.Sm_iq_i}{I};$$

whence it is easy to conclude

$$c^{-\frac{k}{2k''}\cdot\frac{u^2}{SM_i^2}} = c^{-\frac{k}{2k''}\cdot u^2\cdot\frac{I^2}{H}},$$

by making

$$H = Sm_i^2(Sn_iq_i)^2 - 2.Sm_in_i.Sm_iq_i.Sn_iq_i + Sn_i^2(Sm_iq_i)^2;$$

a result which coincides with the one of §21 of the second book, in which we have proved that the *maximum* of the coefficient of $-u^2$ in this exponential takes place, when we suppose generally $m_i = p_i$, $n_i = q_i$; this supposition gives therefore the most advantageous result, or the one of which the weight is a *maximum*.

We will determine the value of $\frac{k}{2k''}$, by means of the squares of the remainders [41] which take place when we substitute into the equations of condition, the values determined for y and y'. By designating by ϵ_i this remainder in the i^{th} equation of condition,

$$0 = x_i - a_i + p_i y + q_i y';$$

and designating by u and u' the errors of these values; we will have

$$0 = x_i + \epsilon_i - p_i u - q_i u';$$

that which gives

$$\begin{aligned} S\epsilon_i^2 = Sx_i^2 - 2uSp_ix_i - 2u'Sq_ix_i \\ + u^2Sp_i^2 + 2uu'Sp_iq_i + u'^2Sq_i^2. \end{aligned}$$

We have, by §19 of the second book,

$$Sx_i^2 = \frac{k''}{k}n;$$

next, the values u and u' cease to be probable, when they surpass the quantities of order $\frac{1}{\sqrt{n}}$. The values of Sp_ix_i and Sq_ix_i cease to be probable, when they surpass quantities of order $\sqrt{n}$; the values of $-2u\,Sp_ix_i$ and $-2u'\,Sq_ix_i$ cease therefore to be probable, when they cease to be of a finite order, n being supposed infinitely great. Sp_i^2, Sp_iq_i and Sq_i^2 being of order n, the values of $u^2\,Sp_i^2$, $2uu'\,Sp_iq_i$, $u'^2\,Sq_i^2$ cease to be probable, when they cease to be finite quantities. We can therefore neglect all these quantities, and suppose, whatever be the process of which we make use,

$$S\epsilon_i^2 = \frac{k''}{k}n;$$

that which gives

$$\frac{k}{2k''} = \frac{n}{2S\epsilon_i^2}.$$

§2. The preceding methods are reduced to multiplying each equation of condition, by a factor; and to adding all these products, in order to form a final equation. But we can employ some other considerations in order to obtain the result sought. [42] For example, we can choose that of the equations of condition, which must most approach to the truth. The process that I have given in §40 of the third book of the *Mécanique céleste*, is of this kind. By supposing the equations (b) of the previous section, prepared in a manner that p_1, p_2, p_3, etc., are positive, and that the values $\frac{a_1}{p_1}$, $\frac{a_2}{p_2}$, etc. of y, given by these equations under the supposition of x_1, x_2, etc., nulls, form a decreasing sequence; the process of which there is question, consists in choosing the r^{th} equation of condition, such that we have

$$\begin{aligned} p_1 + p_2 \cdots + p_{r-1} < p_r + p_{r+1} \cdots + p_n, \\ p_1 + p_2 \cdots + p_r > p_{r+1} + p_{r+2} \cdots + p_n, \end{aligned}$$

and in supposing

$$y = \frac{a_r}{p_r}.$$

This value of y renders a *minimum*, the sum of all the deviations from the other values, taken positively; because by naming x_1, x_2, etc., these deviations, x_1, $x_2 \dots x_{r-1}$, will be positive, and x_{r+1}, x_{r+2}, $\dots x_n$ will be negative. If we increase the preceding value of y by the infinitely small quantity δy, the sum of the positive deviations x_1, x_2, $\dots x_{r-1}$ will diminish by the quantity

$$\delta y(p_1 + p_2 \cdots + p_{r-1});$$

but the sum of the negative deviations taken with the sign $+$, will increase by the quantity

$$\delta y(p_{r+1} + p_{r+2} \cdots + p_n);$$

the deviation x_r will become $p_r \delta y$. The sum of the deviations taken all positively,
[43] will be therefore increased by the quantity

$$\delta y(p_r + p_{r+1} \cdots + p_n - p_1 - p_2 \cdots - p_{r-1});$$

by the conditions to which the choice of the r^{th} equation is subject, this quantity is positive. We will see in the same manner, that if we diminish $\frac{a_r}{p_r}$ by δy, the sum of the deviations taken positively will be increased by the positive quantity,

$$\delta y(p_1 + p_2 \cdots + p_r - p_{r+1} - p_{r+2} \cdots - p_n).$$

Thus in the two cases of an increase and of a diminution of the value $\frac{a_r}{p_r}$ by y, the sum of the deviations taken positively, is increased. This consideration seems to give a great advantage to the preceding value of y, which, when there is a question to choose a middle among the results of an odd number of observations, becomes the result equidistant from the extremes. But the calculus of probabilities can alone estimate this advantage; I will therefore apply it to this delicate question.

The sole data of which we will make use, are that the equation of condition

$$0 = x_r - a_r + p_r y$$

gives, setting aside the errors, a value of y smaller, than the $r - 1$ anterior equations, and greater than the $n - r$ posterior equations; and that we have

$$p_1 + p_2 \cdots + p_{r-1} < p_r + p_{r+1} \cdots + p_n;$$
$$p_1 + p_2 \cdots + p_r > p_{r+1} + p_{r+1} \cdots + p_n.$$

We have

$$y = \frac{a_1}{p_1} - \frac{x_1}{p_1} = \frac{a_r}{p_r} - \frac{x_r}{p_r};$$

that which gives

$$\frac{x_1}{p_1} = \frac{a_1}{p_1} - \frac{a_r}{p_r} + \frac{x_r}{p_r}.$$

Thus $\frac{a_1}{p_1}$ surpassing $\frac{a_r}{p_r}$; $\frac{x_1}{p_1}$ surpasses $\frac{x_r}{p_r}$. It is the same of $\frac{x_2}{p_2}$, $\frac{x_3}{p_3}$, etc., to $\frac{x_{r-1}}{p_{r-1}}$. We will see in the same manner that $\frac{x_{r+1}}{p_{r+1}}$, $\frac{x_{r+2}}{p_{r+2}} \dots \frac{x_n}{p_n}$ are less than $\frac{x_r}{p_r}$. Thus the sole conditions to which we will subject the errors and the equations of condition, are the following:

$$\left\{\begin{matrix} s > r \\ \dfrac{x_s}{p_s} < \dfrac{x_r}{p_r} \end{matrix}\right\}; \qquad \left\{\begin{matrix} s < r, \\ \dfrac{x_s}{p_s} > \dfrac{x_r}{p_r} \end{matrix}\right\}; \tag{c}$$

$$p_1 + p_2 \dots + p_{r-1} < p_r + p_{r+1} \dots + p_n; \qquad [44]$$
$$p_1 + p_2 \dots + p_r > p_{r+1} + p_{r+1} \dots + p_n.$$

It is uniquely according to these data from the observations, that we will determine the probability of the error x_r. We will have besides no regard to the order that the first $r - 1$ equations of condition and the $n - r$ last observe among themselves, nor to the values of the quantities a_1, $a_2 \dots a_n$.

Let us represent as above by $\phi(x)$ the law of probability of the error x of the observations; and, in order to express that this probability is the same for the positive and negative errors, let us suppose $\phi(x)$ a function of x^2.

Now, if we suppose x_r positive; the probability that x_1 will surpass $p_1 \cdot \frac{x_r}{p_r}$, will be

$$\frac{1}{2} - \frac{\frac{1}{2}\int dx\,\phi(x)}{k},$$

the integral $\int dx\,\phi(x)$ being taken from $x = 0$, to $x = p_1 \cdot \frac{x_r}{p_r}$, and k being as above, this integral taken from x null, to x infinity. The probability that the quantities $\frac{x_1}{p_1}$, $\frac{x_2}{p_2} \dots \frac{x_{r-1}}{p_{r-1}}$, will be all greater than $\frac{x_r}{p_r}$, is therefore proportional to the product of the $r - 1$ factors

$$1 - \frac{\int dx\,\phi(x)}{k}; \quad 1 - \frac{\int dx\,\phi(x)}{k}; \quad \text{etc.};$$

the integral of the first factor being taken from $x = 0$, to $x = p_1 \cdot \frac{x_r}{p_r}$; the integral of the second factor being taken from $x = 0$, to $x = p_2 \cdot \frac{x_r}{p_r}$; and so forth.

Similarly all the quantities $\frac{x_{r+1}}{p_{r+1}}$, $\frac{x_{r+2}}{p_{r+2}} \dots \frac{x_n}{p_n}$ being supposed smaller than $\frac{x_r}{p_r}$; we see by the same reasoning, that the probability of this supposition, is proportional to the product of the $n - r$ factors

$$1 + \frac{\int dx\,\phi(x)}{k}; \quad 1 + \frac{\int dx\,\phi(x)}{k}; \quad \text{etc.}; \qquad [45]$$

the integral of the first factor, being taken from $x = 0$, to $x = p_{r+1} \cdot \frac{x_r}{p_r}$, that of the second factor being taken from $x = 0$, to $x = p_{r+2}\frac{x_r}{p_r}$, and so forth. The probability of the error x_r is $\phi(x_r)$; thus the probability that the error of the r^{th} observation will be x_r, and that the value of y given by the r^{th} equation, will be smaller than the values given by the preceding equations, and will surpass the values given by the following equations; this probability, I say, will be proportional to the product of the $n - 1$ preceding factors and of $\phi(x_r)$.

x being supposed very small, we have to the quantities near of order x^3,

$$\int dx\,\phi(x) = x\phi(0) + \tfrac{1}{2}x^2\phi'(0);$$

$\phi'(0)$ being that which $\frac{d\phi(x)}{dx}$ becomes, when x is null. In the present question, $\phi(x)$ being a function of x^2, we have $\phi'(0) = 0$, and then we have

$$\int dx\,\phi(x) = x\phi(0).$$

The preceding factors will become thus, by making $\frac{x_r}{p_r} = \zeta$,

$$1 - p_1\zeta\frac{\phi(0)}{k};$$
$$1 - p_2\zeta\frac{\phi(0)}{k};$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$1 - p_{r-1}\zeta\frac{\phi(0)}{k};$$
$$1 + p_{r+1}\zeta\frac{\phi(0)}{k};$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$1 + p_n\zeta\frac{\phi(0)}{k}.$$

If we designate by $\phi''(0)$, the value of $\frac{d^2\phi(x)}{dx^2}$, when x is null; $\phi(x_r)$ becomes

$$\phi(0) + \tfrac{1}{2}p_r^2\zeta^2\phi''(0).$$

[46] The sum of the hyperbolic logarithms of all these factors, is to the quantities near of order ζ^3, by dividing the factor $\phi(x_r)$ by $\phi(0)$,

$$\begin{aligned}
&-\zeta\frac{\phi(0)}{k}(p_1 + p_2 \cdots + p_{r-1} - p_{r+1} - p_{r+2} \cdots - p_n)\\
&-\frac{\zeta^2}{2}\left(\frac{\phi(0)}{k}\right)^2 (p_1^2 + p_2^2 \cdots + p_r^2 + p_{r+1}^2 \cdots + p_n^2)\\
&+\frac{1}{2}p_r^2\zeta^2\left[\frac{\phi''(0)}{k} + \left(\frac{\phi(0)}{k}\right)^2\right].
\end{aligned}$$

The probability of ζ is therefore proportional to the base c of the hyperbolic logarithms, elevated to a power of which the exponent is the preceding function. We must observe that by virtue of the conditions to which the choice of the r^{th} equation is subject, the quantity

$$p_1 + p_2 \cdots + p_{r-1} - p_{r+1} - p_{r+2} \cdots - p_n$$

is setting aside the sign, a quantity less than p_r; and that thus, by supposing ζ of order $\frac{1}{\sqrt{n}}$, the number n of the observations being supposed quite great; the term depending on the first power of ζ, in the preceding function, is of order $\frac{1}{\sqrt{n}}$; we are able therefore to neglect it, just as the last term of this function. By designating therefore by Sp_i^2 the entire sum

$$p_1^2 + p_2^2 \cdots + p_n^2,$$

the probability of ζ will be proportional to

$$c^{-\frac{\zeta^2}{2}\left(\frac{\phi(0)}{k}\right)^2 Sp_i^2},$$

ζ or $\frac{x_r}{p_r}$ being the error of the value $\frac{a_r}{p_r}$ given for y by the r^{th} equation. The value given by the most advantageous method is by the preceding section,

$$y = \frac{Sp_i a_i}{Sp_i^2},$$

and the probability of an error ζ in this result, is proportional to [47]

$$c^{-\frac{k}{2k''}\zeta^2 Sp_i^2},$$

k'' being always the integral $\int x^2\,dx\phi(x)$, taken from x null, to x infinity. The result of the method that we just examined, and that we will name method of *situation*, will be preferable to the one of the most advantageous method; if the coefficient of $-\zeta^2$, which is relative to it, surpasses the coefficient relative to the most advantageous method; because then the law of probability of the errors will be more rapidly decreasing there. Thus the method of situation must be preferred, if we have

$$\left(\frac{\phi(0)}{k}\right)^2 > \frac{k}{k''};$$

in the contrary case, the most advantageous method is preferable. If we have, for example,

$$\phi(x) = c^{-hx^2},$$

k becomes $\frac{\sqrt{\pi}}{2\sqrt{h}}$ and k'' becomes $\frac{\sqrt{\pi}}{4h\sqrt{h}}$; that which gives $\frac{k}{k''} = 2h$. The quantity $\left(\frac{\phi(0)}{k}\right)^2$ becomes $\frac{4h}{\pi}$; now we have $2h > \frac{4h}{\pi}$; the most advantageous method must therefore then be preferred.

By combining the results of these two methods, we can obtain a result of which the law of probability of the errors is more rapidly decreasing. Let us name always ζ the error of the result of the method of situation, and let us designate by ζ' the error of the result of the most advantageous method. The first of these results is, as we have seen, $\frac{a_r}{p_r}$, and the second is $\frac{Sp_i a_i}{Sp_i^2}$. If we designate $Sp_i x_i$ by l, $\frac{l}{Sp_i^2}$ will be the error of this last result; thus we will have $l = \zeta'.Sp_i^2$. The probability of the simultaneous existence of l and of ζ, is by §21 of the second book, proportional to

$$\int dw\, c^{-lw\sqrt{-1}}\phi(p_r\zeta)c^{p_r\zeta w\sqrt{-1}} \left\{\begin{array}{l} \int dx\phi(x)c^{p_1xw\sqrt{-1}} \\ \times \int dx\phi(x)c^{p_2xw\sqrt{-1}} \\ \times \text{ etc.} \end{array}\right\},$$

[48] the integral relative to w being taken from $w=-\pi$ to $w=\pi$. The integral relative to x in the factor $\int dx\phi(x)c^{p_1xw\sqrt{-1}}$, must be taken, by that which precedes, from $x=p_1\zeta$ to $x=\infty$. In developing this factor, according to the powers of w; it becomes

$$\int dx\phi(x)+p_1w\sqrt{-1}\int xdx\phi(x)-p_1^2\frac{w^2}{2}\int x^2dx\phi(x)+\text{etc.}$$

By taking the integral within the preceding limits, we have to the quantities near of order ζ^3,

$$\int dx\phi(x)=k-p_1\zeta\phi(0).$$

By neglecting similarly the quantities of the orders ζ^2w, ζ^3w^2, etc., we have

$$p_1w\sqrt{-1}\int xdx\phi(x)=k'p_1w\sqrt{-1};$$
$$\frac{-p_1^2}{2}w^2\int x^2dx\phi(x)=-\frac{k''}{2}p_1^2w^2;$$

k' being the integral $\int xdx\phi(x)$ taken from $x=0$ to x infinity. The factor of which there is question, becomes therefore by neglecting w^3, conformably to the analysis of the section cited from the second book,

$$k-p_1\zeta\phi(0)+k'p_1w\sqrt{-1}-\frac{k''}{2}p_1^2w^2.$$

Its hyperbolic logarithm is

$$-p_1\zeta\frac{\phi(0)}{k}+\frac{k'}{k}p_1w\sqrt{-1}-\frac{k''}{2k}p_1^2w^2-\frac{p_1^2}{2}\left(\zeta\frac{\phi(0)}{k}-\frac{k'}{k}w\sqrt{-1}\right)^2+\log k.$$

By changing p_1 successively into p_2, $p_3\ldots p_{r-1}$; we will have the logarithms of the factors following, to the factor relative to p_{r-1}.

In the factor $\int dx\phi(x)c^{p_{r+1}xw\sqrt{-1}}$, the integral must be taken from $x=-\infty$, to $x=p_{r+1}\zeta$; then $\int xdx\phi(x)$ becoming $-k'$, the logarithm of this factor is

$$p_{r+1}\zeta\frac{\phi(0)}{k}-\frac{k'}{k}p_{r+1}w\sqrt{-1}-\frac{k''}{2k}p_{r+1}^2w^2 \\ -\frac{p_{r+1}^2}{2}\left(\zeta\frac{\phi(0)}{k}-\frac{k'}{k}w\sqrt{-1}\right)^2+\log k.$$

We will have the logarithms of the factors following, by changing p_{r+1} successively into p_{r+2}, $p_{r+3}\dots p_n$. The factor $\phi(p_r\zeta)c^{p_r\zeta w\sqrt{-1}}$ is equal to [49]

$$\left[\phi(0)+\frac{p_r^2\zeta^2}{2}\right]\phi''(0)c^{p_r\zeta w\sqrt{-1}},$$

and its logarithm is

$$\frac{p_r^2}{2}\zeta^2\frac{\phi''(0)}{\phi(0)}+p_r\zeta w\sqrt{-1}+\log\phi(0).$$

Now if we reassemble all these logarithms; if we consider next the conditions (c) to which the r^{th} equation is subject; finally if we pass again from the logarithms to the numbers; we find by neglecting that which it is permissible to neglect, that the probability of the simultaneous existence of l and of ζ is proportional to

$$\int d\phi\, c^{-lw\sqrt{-1}-\left[\left(\zeta\frac{\phi(0)}{k}-\frac{k'}{k}w\sqrt{-1}\right)^2+\frac{k''}{k}w^2\right]^2\frac{Sp_i^2}{2}}$$

By making therefore

$$F=\left(\frac{k''}{k}-\frac{k'^2}{k^2}\right)\frac{Sp_i^2}{2};$$

the probability of the simultaneous existence of ζ and of ζ' will be proportional to

$$c^{-\frac{\zeta^2}{2}\left(\frac{\phi(0)}{k}\right)^2Sp_i^2-\frac{\left(\zeta'-\zeta\frac{k'}{k}\frac{\phi(0)}{k}\right)^2}{4F}(Sp_i^2)^2}$$
$$\times\int dw\, c^{-F\left\{w+\frac{\left(\zeta'-\zeta\frac{k'}{k}\frac{\phi(0)}{k}\right)\sqrt{-1}\,Sp_i^2}{2F}\right\}}.$$

By the analysis of §21 of the second book, the integral relative to w, can be taken from $w=-\infty$ to $w=\infty$; and then the preceding probability becomes proportional to

$$c^{-\frac{\zeta^2}{2}Sp_i^2\left(\frac{\phi(0)}{k}\right)^2-\frac{\left(\zeta'-\zeta\frac{k'}{k}\frac{\phi(0)}{k}\right)^2}{2\left(\frac{k''}{k}-\frac{k'^2}{k^2}\right)}Sp_i^2},$$

an expression that we are able to set yet under this form

$$c^{-\frac{k}{2k''}\zeta'^2Sp_i^2-\frac{k''}{k}\frac{\left(\zeta\frac{\phi(0)}{k}-\zeta'\frac{k'}{k}\right)^2}{2\left(\frac{k''}{k}-\frac{k'^2}{k^2}\right)}Sp_i^2}.$$

If we name e the excess of the value of y given by the most advantageous method, over that which the method of situation gives; we will have $\zeta=\zeta'-e$. Let us suppose [50]

$$\zeta'=u+\frac{e\frac{\phi(0)}{k}\left(\frac{\phi(0)}{k}-\frac{k'}{k''}\right)}{\frac{k}{k''}-\frac{k'^2}{k''^2}+\left(\frac{\phi(0)}{k}-\frac{k'}{k''}\right)^2};$$

the probability of u will be proportional to

$$c^{-\frac{u^2}{2}Sp_i^2\left\{\frac{k}{k''}+\frac{\frac{k''}{k}\left(\frac{\phi(0)}{k}-\frac{k'}{k''}\right)^2}{\frac{k''}{k}-\frac{k'2}{k2}}\right\}};$$

the result of the most advantageous method must therefore be diminished by the quantity

$$\frac{e\frac{\phi(0)}{k}\left(\frac{\phi(0)}{k}-\frac{k'}{k''}\right)}{\frac{k}{k''}-\frac{k'^2}{k''^2}+\left(\frac{\phi(0)}{k}-\frac{k'}{k''}\right)^2};$$

and the probability of the error u, in this result thus corrected, will be proportional to the preceding exponential. The weight of the new result will be augmented, if $\frac{\phi(0)}{k}-\frac{k'}{k''}$ is not null; there is therefore advantage to correct thus the result of the most advantageous method. Ignorance whereas one is, of the law of probability of the errors of the observations, renders this correction impractical; but it is remarkable that in the case where this probability is proportional to c^{-hx^2}, that is where we have $\phi(x)=c^{-hx^2}$, the quantity $\frac{\phi(0)}{k}-\frac{k'}{k''}$ is null. Then the result of the most advantageous method, receives no correction, from the result of the method of situation; and the law of probability of errors remains the same.

END.

Third Supplement.

APPLICATION OF THE GEODESIC FORMULAS OF PROBABILITY, TO THE MERIDIAN OF FRANCE

§1. The part of the meridian which extends from Perpignan to Formentera, is [3] supported on a base measured near Perpignan. Its length is around 466 thousand meters, and its last extremity is joined to the base of Perpignan, by a chain of twenty-six triangles. We can fear that so great a length which has not been verified at all by the measure of a second base, toward its other extremity, is susceptible to a sensible error arising from the errors of the twenty-six triangles employed to measure it. It is therefore interesting to determine the probability that this error not exceed forty or fifty meters. Mr. Damoiseau, lieutenant-colonel of artillery, who just gained the prize proposed by the Academy of Turin, on the return of the comet of 1759, has well wished at my request, to apply to this part of the meridian, my formulas of probability. Here the meridian does not cut all the triangles, as we have proposed, for [4] more simplicity. But it is easy to see that we are able to apply to the angles formed by the prolongations of the sides of the triangles with the meridian, that which I have said respecting the angles that these sides would form, if they were cut by the meridian. Mr. Damoiseau has found thus that departing from the latitude of the signal of Busgarach, a little more to the north than Perpignan, to Formentera, that which comprehends an arc of the meridian, of about 466006 meters, and by taking for unity, the base of Perpignan, we have (second Supplement, §1)

$$p^2 - pq + q^2 + p_1^{(1)^2} - p^{(1)}q^{(1)} + q^{(1)^2} \cdots + p^{(25)^2} - p^{(25)}q^{(25)} + q^{(25)^2} = 48350,606.$$

The probability that an error in the measure of this arc, is comprehended within the limits $\pm s$, becomes by the formulas of the same section,

$$\frac{2\int dt\, c^{-t^2}}{\sqrt{\pi}},$$

the integral being taken from t null, to the value of t, equal to

$$\frac{s}{2\theta} \cdot \frac{\sqrt{n+1}}{\sqrt{48350,606}};$$

$n+1$ being the number of triangles employed, and θ^2 being the sum of the squares of the errors observed in the sum of the three angles of each triangle: π is the ratio of the circumference to the diameter. By taking for unity, the sexagesimal second, we find

$$\theta^2 = 118,178.$$

But the number of triangles employed being only 26, it is preferable to determine by a great number of triangles, this constant θ^2 which depends on the unknown law of the partial observations. For this, we have made use of the one hundred seven triangles which have served to measure the meridian from Dunkirk to Formentera. The set of the sums of observed errors of the three angles of each triangle is in taking all of them positively, equal to 173, 82. The sum of the squares of these errors, is 445,217. By multiplying it by $\frac{26}{107}$, we will have, for the value of θ^2,

$$\theta^2 = 108,184.$$

[5] This value which differs little from the preceding, must be preferred. It is necessary to reduce θ, into parts of the radius taken for unity, that which we will make by dividing it by the number of sexagesimal seconds that this radius contains. We will have thus

$$t = s\,689,797,$$

s is a fraction of the base of Perpignan, taken for unity. This base is $11706^{\text{m}}, 40$ meters. By supposing therefore the error of 60 meters, we will have

$$t = \frac{60.689,797}{11706,40}.$$

This premised, we find for the probabilities of the errors of the arc of the meridian, of which there is question, are comprehended within the limits $\pm 60^{\text{m}}$, $\pm 50^{\text{m}}$, $\pm 40^{\text{m}}$, the following fractions:

$$\frac{1743695}{1743696},\quad \frac{32345}{32346},\quad \frac{1164}{1165}.$$

There is odds one against one that the error falls within the limits $8^{\text{m}}, 0757$.[2]

If the Earth were a spheroid of revolution and if the angles of all the triangles were exact; we would have exactly the inclination of the lasts side of the chain of the triangle onto its meridian, by supposing given, this inclination relative to the base. The probability that the error of the first of these inclinations, proceeding from the errors of the observed angles of the triangles, is comprehended within the limits $\pm\frac{2}{3}\theta t$ is, by that which precedes,

$$\frac{2\int dt\, c^{-t^2}}{\sqrt{\pi}};$$

[2]The comparison of this Memoir with the one which was published in the *Connaissance des Temps* for 1822 permits the identification of errors in this supplement. In the *Connaissance des Temps*, Laplace gives $\pm 8^{\text{m}}$,0937. In reality, it would be $\pm 8^{\text{m}}$,0940. We have $\frac{2}{\pi}\int c^{-t^2}dt = \frac{1}{2}$ for $t = 0,476936$ thereby giving even odds. From the previous relation here,

$$t = \frac{s\,689,797}{11706,40},$$

where $s = \pm 8^{\text{m}}$,0940. The number given by Laplace differs slightly, without doubt, because the value of t, deduced from the formulas of Laplace, has not been calculated with as much precision as that employed later.

the integral being taken from t null: these limits become by substituting for θ it preceding value, $\pm t.6'', 8997$, the seconds being sexagesimal. Thence it follows that there is odds one against one that the error falls within the limits $\pm 3'', 2908$. If the azimuthal observations were made with a great precision, we would determine by this means, the probability that they indicate an eccentricity in the terrestrial parallels. If we measured on the side of Spain, a base of verification, equal to the base of Perpignan, and if we joined it by two triangles, to the chain of triangles of [6]
the meridian; we find by the calculation, that there is odds one against one, that the difference between this base and its value concluded from the base of Perpignan, will not surpass a third of a meter: that is quite nearly, the difference of the measure of the base of Perpignan, to its value concluded from the base of Melun.

We have seen in the section cited, that, the angles of the triangles, having been measured by means of the repeating circle; we can suppose the probability of an error x in the observed sum of the three angles of each triangle, proportional to the exponential c^{-kx^2}, k being a constant. Thence it follows that the probability of this error is

$$\frac{dx\,\sqrt{k}\,c^{-kx^2}}{\sqrt{\pi}}.$$

By multiplying this differential by x, and integrating from x null to x infinity; the double of this integral will be the mean of all the errors taken positively. By designating therefore by ϵ, this mean error, we will have

$$\epsilon = \frac{1}{\sqrt{k\pi}}.$$

We will have the mean value of the squares of these errors, by multiplying by x^2, the preceding differential, and by integrating it from $x = -\infty$, to x infinity. By naming therefore ϵ' this value, we will have

$$\epsilon' = \frac{1}{2k};$$

thence we deduce

$$\epsilon' = \frac{\epsilon^2\pi}{2}.$$

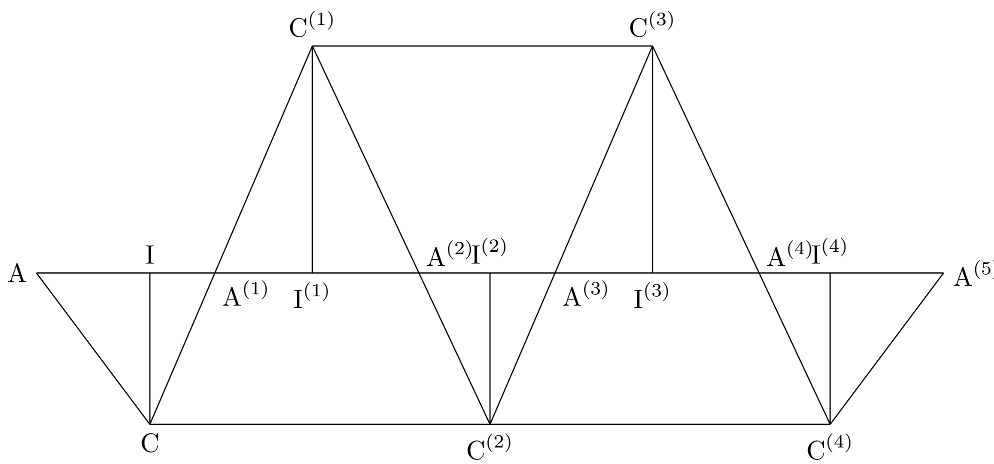
We can thus obtain θ^2, by means of the errors taken all positive, of the observed sums of the angles of each triangle. In the one hundred seven triangles of the meridian, the sum of the errors is $173, 82$; we are able thus to take for ϵ, $\frac{173,82}{107}$; that which gives for $26\epsilon'$, or for θ^2,

$$\theta^2 = 13\pi\left(\frac{173,82}{107}\right)^2 = 107,78;$$

this differs very little from the value $108, 134$, given by the sum of the squares of the [7]
errors of the observed sum of the angles of each of the one hundred seven triangles. This accord is remarkable.

We are able to estimate the relative exactitude of the instruments of which we make use in the geodesic observations, by the value of ϵ' concluded from a great number of triangles. This value concluded from the one hundred seven triangles of the meridian, is $\frac{445{,}217}{107}$, or $4,1609$. The same value concluded from the forty-three triangles employed by La Condamine, in the measure of the three degrees of the equator is $\frac{1718}{43}$, or $39,953$, and, consequently nearly ten times greater than the preceding. The equally probable errors, relative to the instruments employed in these two operations, are proportionals to the square roots of the values of ϵ'. Thence it follows that the limits $\pm 8^{\text{m}},0937$ between which we have just seen that there is odds one against one, that the error of the arc measured from Perpignan to Formentera falls, would have been $\pm 25^{\text{m}},022$, with the instruments employed by La Condamine. These limits would have surpassed $\pm 40^{\text{m}}$, with the instruments employed by La Caille and Cassini, in their measure of the meridian. We see thus how the introduction of the repeating circle, in the geodesic operations, has been advantageous.

§2. In order to give a very simple example of the application of the geodesic formulas,



[8] I will consider the straight line $AA^{(5)}$ of which we have determined the length by the chain of triangles $CC^{(1)}C^{(2)}$, $C^{(1)}C^{(2)}C^{(3)}$, etc. I will suppose all these triangles equal and isosceles, and such that their bases $CC^{(2)}$, $C^{(1)}C^{(3)}$, etc., are parallels to the line $AA^{(5)}$. We will have by lowering onto this line, the perpendiculars CI, $C^{(1)}I^{(1)}$, etc.

$$\begin{aligned}
II^{(1)} &= CC^{(1)} \cos A^{(1)},\\
C^{(1)}C^{(2)} &= \frac{CC^{(1)} \sin C^{(1)}CC^{(2)}}{\sin C^{(1)}C^{(2)}C},\\
I^{(1)}I^{(2)} &= C^{(1)}C^{(2)} \cos A^{(2)},\\
C^{(2)}C^{(3)} &= \frac{C^{(1)}C^{(2)} \sin C^{(2)}C^{(1)}C^{(3)}}{\sin C^{(2)}C^{(3)}C^{(1)}},
\end{aligned}$$

and generally

$$I^{(i)}I^{(i+1)} = C^{(i)}C^{(i+1)}\cos A^{(i+1)}$$

$$C^{(i+1)}C^{(i+2)} = \frac{C^{(i)}C^{(i+1)}\sin C^{(i+1)}C^{(i)}C^{(i+2)}}{\sin C^{(i+1)}C^{(i+2)}C^{(i)}}.$$

Let $\alpha^{(1)}$ and $\beta^{(1)}$ be the errors of the angles opposed to the sides $CC^{(1)}$ and $C^{(1)}C^{(2)}$, in the first triangle. Let $\alpha^{(2)}$ and $\beta^{(2)}$ be the errors of the angles opposed to the sides $C^{(1)}C^{(2)}$ and $C^{(1)}C^{(3)}$ of the second triangle, and so forth. By designating by δ, a variation relative to these errors, we will have

$$\begin{aligned}\frac{\delta I^{(i)}I^{(i+1)}}{I^{(i)}I^{(i+1)}} &= \frac{\delta C^{(i)}C^{(i+1)}}{C^{(i)}C^{(i+1)}} - \delta A^{(i+1)}\tan A^{(i+1)},\\ \frac{\delta C^{(i)}C^{(i+1)}}{C^{(i)}C^{(i+1)}} &= \frac{\delta C^{(i)}C^{(i-1)}}{C^{(i)}C^{(i-1)}} + \beta^{(i)}\cot C^{(i+1)}C^{(i-1)}C^{(i)}\\ &\qquad - \alpha^{(i)}\cot C^{(i)}C^{(i+1)}C^{(i-1)}.\end{aligned}$$

We have further by supposing the angles $A^{(i)}$ relative to the acute angles that the sides of the triangles form with the line $AA^{(1)}$, etc.

$$\delta A^{(i+1)} + \delta A^{(i)} + \delta C^{(i-1)}C^{(i)}C^{(i+1)} = 0;$$

we will suppose here that the errors $\alpha^{(i)}$ and $\beta^{(i)}$ of the angles $C^{(i+1)}C^{(i-1)}C^{(i)}$, $C^{(i)}C^{(i+1)}C^{(i-1)}$ of the triangle $C^{(i-1)}C^{(i)}C^{(i+1)}$ are those which remain, when we have subtracted from each angle of the triangle, the third of the sum of the errors of the three angles. Then we have

$$\delta C^{(i-1)}C^{(i)}C^{(i+1)} = -\alpha^{(i)} - \beta^{(i)},$$

that which gives

$$\delta A^{(i+1)} = -\delta A^{(i)} + \alpha^{(i)} + \beta^{(i)};$$

we will have therefore [9]

$$\begin{aligned}\delta A^{(i+1)} &= \alpha^{(i)} - \alpha^{(i-1)} + \alpha^{(i-2)} - \cdots \mp \alpha^{(1)}\\ &\qquad + \beta^{(i)} - \beta^{(i-1)} + \beta^{(i-2)} - \cdots \mp \beta^{(1)} \pm \delta A^{(1)},\end{aligned}$$

the superior sign having place, if i is even, and the inferior, if i is odd.

We will have next, by observing that

$$\cot C^{(i)}C^{(i-1)}C^{(i+1)} = \cot C^{(i)}C^{(i+1)}C^{(i-1)} = \cot A^{(i)}$$

and that $A^{(i)} = A^{(1)}$,

$$\frac{\delta C^{(i)}C^{(i+1)}}{C^{(i)}C^{(i+1)}} = \frac{\delta CC^{(1)}}{CC^{(1)}} + \left\{\begin{matrix} \beta^{(i)} + \beta^{(i-1)} \cdots + \beta^{(1)} \\ -\alpha^{(i)} - \alpha^{(i-1)} \cdots - \alpha^{(1)}\end{matrix}\right\}\cot A^{(1)};$$

we will have therefore

$$\frac{\delta I^{(i)}I^{(i+1)}}{I^{(i)}I^{(i+1)}} = \frac{\delta CC^{(1)}}{CC^{(1)}} + \left\{\begin{array}{l} \quad\beta^{(i)} + \beta^{(i-1)} \cdots + \beta^{(1)} \\ -\alpha^{(i)} - \alpha^{(i-1)} \cdots - \alpha^{(1)} \end{array}\right\} \cot A^{(1)}$$
$$- \left\{\begin{array}{l} \quad\alpha^{(i)} - \alpha^{(i-1)} \cdots \mp \alpha^{(1)} \\ +\beta^{(i)} - \beta^{(i-1)} \cdots \mp \beta^{(1)} \\ \pm\delta A^{(i)} \end{array}\right\} \tan A^{(1)}.$$

Let us suppose now that we have measured a base AC situated in a manner that the angle $CAC^{(1)}$ is equal to the angle $CA^{(1)}A$. The first of these angles determines the position of the line $AA^{(1)}$, with respect to the base, and it is supposed known. By naming α and β the errors of the angles $CC^{(1)}A$, and $CAC^{(1)}$; we will have

$$\delta A^{(1)} = \alpha + \beta,$$
$$\frac{\delta CC^{(1)}}{CC^{(1)}} = \beta \cot CAC^{(1)} - \alpha \cot\ CC^{(1)}A.$$

Let us make

$$\cot CAC^{(1)} = \cot A + h,$$
$$\cot\ CC^{(1)}A = \cot A + h';$$

we will have by designating by b the base AC, and by a the straight line $II^{(i)}$,

$$h = \frac{b}{2a\sin A} - \frac{1}{\sin 2A},$$
$$h' = \frac{a}{2b\sin A\cos^2\ A} - \frac{1}{\sin 2A}.$$

we will have next

[10] $$\delta A^{(i+1)} = \begin{array}{l} \alpha^{(i)} - \alpha^{(i-1)} \cdots \pm \alpha \\ +\beta^{(i)} - \beta^{(i-1)} \cdots \pm \beta; \end{array}$$

$$\frac{\delta I^{(i)}I^{(i+1)}}{I^{(i)}I^{(i+1)}} = \left\{\begin{array}{l} \quad\beta^{(i)} + \beta^{(i-1)} \cdots + \beta \\ -\alpha^{(i)} - \alpha^{(i-1)} \cdots - \alpha \end{array}\right\} \cot A$$
$$- \left\{\begin{array}{l} \quad\alpha^{(i)} - \alpha^{(i-1)} \cdots \pm \alpha \\ +\beta^{(i)} - \beta^{(i-1)} \cdots \pm \beta \end{array}\right\} \tan A + h\beta - h'\alpha.$$

The variation of the total length $II^{(i+1)}$ will be therefore

$$\delta II^{(i+1)} = \{(i+1)(\beta - \alpha) + i(\beta^{(i)} - \alpha^{(i)}) \cdots + (\beta^{(i)} - \alpha^{(i)})\} a \cot A$$
$$+ (i+1).ha\beta - (i+1)h'a\alpha$$
$$- \left\{\begin{array}{l} \quad\alpha^{(i)} + \alpha^{(i-2)} + \alpha^{(i-4)} + \text{etc.} \\ +\beta^{(i)} + \beta^{(i-2)} + \beta^{(i-4)} + \text{etc.} \end{array}\right\} a \tan A.$$

The quantity

$$p^2 - pq + q^2 + p^{(1)2} - p^{(1)}q^{(1)} + q^{(1)2} \cdots + p^{(i)2} - p^{(i)}q^{(i)} + q^{(i)2},$$

becomes thus, by neglecting the terms of order i,

$$\frac{\overline{i+1}.\overline{i+2}.\overline{2i+3}}{2} a^2 \cot^2 A + 3(h+h')(i+1)^2 a^2 \cot A \\ + (h^2 + hh' + h'^2)(i+1)^2 a^2.$$

Let us name Q this quantity; the probability that the error of the line $II^{(i+1)}$ is comprehended within the limits $\pm s$, will be by that which precedes,

$$\frac{2\int dt\, c^{-t^2}}{\sqrt{\pi}},$$

the integral being taken from t null, to

$$t = \frac{3s}{2\theta}\sqrt{\frac{i+1}{Q}},$$

θ^2 being the sum of the squares of the errors of the sum of the three angles of the $i+1$ triangles.

Let us suppose that we have, as for the part of the meridian of which we have spoken previously, 26 triangles, that which gives $i = 25$. Let us suppose further that the length $II^{(i+1)}$ is that of this part of the meridian, or of 466006^{m}; then, we will have

$$a = \frac{466006}{26}.$$

By taking for unity, the base measured near to Perpignan, which is of $11706^{\text{m}}, 40$, and by supposing right-angled, the isosceles triangles $CC^{(1)}C^{(2)}$, $C^{(1)}C^{(2)}C^{(3)}$, etc., that [11]
which gives $\tan A = \cot A = 1$; we find

$$Q = 48207, 6.$$

We have seen previously, that the twenty-six triangles which join the base of Perpignan to Formentera, give

$$Q = 48350, 6;$$

these two values of Q are not very different, and as the equally probable errors, are proportionals to the square roots of these values; we see that we are able to wager one against one that the errors of the entire measure are comprehended within the limits $\pm 8^{\text{m}}, 1$. Under this relation, the case that we examine, represents perfectly, the measure of the arc of the meridian from the base of Perpignan to Formentera.

§3. Let us suppose now that we measure toward the last extremity of the line $II^{(i+1)}$, a base $C^{(i+1)}A^{(i+2)}$, equal to the base CA, and put in a manner that the angle $C^{(i+1)}C^{(i)}A^{(i+2)}$ is equal to the angle $CC^{(1)}A$, and that the angle $C^{(i)}A^{(i+2)}C^{(i+1)}$ is equal to the angle $CAC^{(1)}$. In designating by $\alpha^{(i+1)}$ and $\beta^{(i+1)}$ the errors of the angles $C^{(i+1)}C^{(i)}A^{(i+2)}$, and $C^{(i)}A^{(i+2)}C^{(i+1)}$, the equation

$$C^{(i+1)}A^{(i+2)} = C^{(i+1)}C^{(i)}\frac{\sin C^{(i+1)}C^{(i)}A^{(i+2)}}{\sin C^{(i)}A^{(i+2)}C^{(i+1)}},$$

will give

$$\frac{\delta C^{(i+1)}A^{(i+2)}}{C^{(i+1)}A^{(i+2)}} = \frac{\delta C^{(i)}C^{(i+1)}}{C^{(i)}C^{(i+1)}} + \alpha^{(i+1)}\cot CC^{(1)}A - \beta^{(i+1)}\cot CAC^{(1)},$$

that which gives

$$\begin{aligned}\frac{\delta C^{(i+1)}A^{(i+2)}}{C^{(i+1)}A^{(i+2)}} = &\left\{\begin{matrix} \beta^{(1)} + \beta^{(2)} \cdots + \beta^{(i)} \\ -\alpha^{(1)} - \alpha^{(2)} \cdots - \alpha^{(i)}\end{matrix}\right\}\cot A \\ &+ \beta(h + \cot A) - \alpha(h' + \cot A) \\ &+ \alpha^{(i+1)}(h' + \cot A) - \beta^{(i+1)}(h + \cot A).\end{aligned}$$

That which we have designated in §2 of the second supplement by l, $l^{(1)}$, etc., m, $m^{(1)}$, etc., becomes

$$\begin{aligned} l &= -(1 + h')b, & m &= (1 + h)b, \\ l^{(i)} &= -b, & m^{(i)} &= b, \\ &\cdots\cdots, & &\cdots\cdots, \\ l^{(i)} &= -b, & m^{(i)} &= b, \\ l^{(i+1)} &= (1 + h')b, & m^{(i+1)} &= -(1 + h)b;\end{aligned}$$

[12] the quantity that we have designated by $Sf^{(i)2}$ in the section cited, or by

$$l^2 - ml + m^2 + l^{(1)2} - m^{(1)}l^{(1)} + m^{(1)2} + \text{etc.}$$

becomes here

$$3(i + 2)b^2 + 6(h + h')b^2 + 2(h^2 + hh' + h'^2)b^2.$$

The quantity that we have named $Sr^{(i)}f^{(i)}$, in the same section, or

$$l(p - \tfrac{1}{2}q) + m(q - \tfrac{1}{2}p) + l^{(1)}(p^{(1)} - \tfrac{1}{2}q^{(1)}) + m^{(1)}(q^{(1)} - \tfrac{1}{2}p^{(1)}) + \text{etc.},$$

becomes, by neglecting the terms which do not have i for coefficient,

$$\frac{3(i + 1)(i + 2)}{2}ab + 3(i + 1)(h + h')ab + (i + 1)(h^2 + hh' + h^{2'})ab;$$

by representing therefore, as above, by λ the excess of the measured base $C^{(i+1)}A^{(i+2)}$ on the calculated base, and by s the excess of the true length of the line $II^{(i+1)}$ over that calculated length, we will have

$$s = \frac{\lambda Sr^{(i)}f^{(i)}}{Sf^{(i)2}} = \frac{(i + 1)a\lambda}{2b};$$

it is necessary consequently, to add to the calculated length of the line $II^{(i+1)}$; the product of λ, by the ratio of the half of this line, to the base b; that which reverts

to calculating the first half of the line $II^{(i+1)}$ with the base AC, and the second half with the base $A^{(i+2)}C^{(i+1)}$. This process would be generally exact, whatever was the magnitude and the disposition of the triangles which unite the two bases, if the parts of $Sr^{(i)}f^{(i)}$ and of $Sf^{(i)2}$ corresponding to these halves, were respectively equal. This is the process that we adopted in the commission which fixed the length of the meter; and in the ignorance where we were then, of the true theory of these corrections, it was most convenient. But it did not make known the correction of the diverse parts of the total arc $II^{(i+1)}$. For this it is necessary to correct the angles of each triangle, or to determine the corrections α, β, $\alpha^{(1)}$, $\beta^{(1)}$, etc. which result from the excess λ of the second base observed over that base calculated after the first. I have given in the second Supplement, these corrections, by supposing the law of errors of the observations of the angles, proportional to the exponential $c^{-k\left(\alpha+\frac{1}{3}T\right)^2}$, k being a constant; T being the sum of the errors of the three angles of the triangle, $\alpha+\frac{1}{3}T$, [13]
$\beta+\frac{1}{3}T$ and $\frac{1}{3}T-\alpha-\beta$ being the errors of each of the angles. We have seen in the supplement cited, that the supposition of this law of probability, must be admitted, when the angles have been measured with the repeating circle, and that then we have

$$\alpha^{(s)} = \frac{\left(l^{(s)}-\frac{1}{2}m^{(s)}\right)}{F}\lambda; \quad \beta^{(s)} = \frac{\left(m^{(s)}-\frac{1}{2}l^{(s)}\right)}{F}\lambda,$$

by designating by F, the sum of all the quantities l^2-ml+m^2, $l^{(1)2}-m^{(1)}l^{(1)}+m^{(1)2}$, etc. I will demonstrate here that these corrections have place, whatever be the law of probability of the errors.

For this, I designate this law by $\phi(\alpha+\frac{1}{3}T)^2$: by supposing it the same for the positive errors and for the negative errors, its expression must contain only some even powers of these errors. The law of probability of the simultaneous values of α and β, will be thus proportional to the product

$$\phi(\alpha+\tfrac{1}{3}T)^2\phi(\beta+\tfrac{1}{3}T)^2\phi(\tfrac{1}{3}T-\alpha-\beta)^2.$$

If we develop this product, with respect to the powers of α and of β, by arresting ourselves at the squares and at the products of these quantities, we will have

$$\{\phi(\tfrac{1}{9}T)\}^3+(\alpha^2+\alpha\beta+\beta^2)\phi(\tfrac{1}{9}T^2)\left\{\begin{matrix} 2\phi(\frac{1}{9}T^2)\phi'(\frac{1}{9}T^2) \\ -\frac{4}{9}T^2\{\phi'(\frac{1}{9}T^2)\}^2 \\ +\frac{4}{9}T^2\phi(\frac{1}{9}T^2)\phi''(\frac{1}{9}T^2) \end{matrix}\right\};$$

$\phi'(x)$ expressing $\frac{d\phi(x)}{dx}$, and $\phi''(x)$ expressing, $\frac{d\phi'(x)}{dx}$. T being able to be supposed to vary from $-\infty$ to $T=\infty$, we will multiply the preceding function by dT, and we will integrate within these limits; we will have thus for the probability of the simultaneous values of α and β, a quantity of the form

$$H-H'(\alpha^2+\alpha\beta+\beta^2).$$

This probability will be therefore proportional to

$$1-\frac{H'}{H}(\alpha^2+\alpha\beta+\beta^2).$$

The probability of the simultaneous existence of α, β, $\alpha^{(1)}$, $\beta^{(1)}$, etc., will be propor-
[14] tional to the product of the quantities

$$1 - \frac{H'}{H}(\alpha^2 + \alpha\beta + \beta^2),$$
$$1 - \frac{H'}{H}(\alpha^{(1)2} + \alpha^{(1)}\beta^{(1)} + \beta^{(1)2}),$$
etc.

The logarithm of this product is, s being an indeterminate number,

$$-\frac{H'}{H}S(\alpha^{(s)2} + \alpha^{(s)}\beta^{(s)} + \beta^{(s)2}) - \text{etc.};$$

this product is at its *maximum*, if the preceding term is at its *minimum*, or if the function

$$S(\alpha^{(s)2} + \alpha^{(s)}\beta^{(s)} + \beta^{(s)2}),$$

is the smallest possible, the quantities α, β, $\alpha^{(1)}$, etc., satisfying besides the equation

$$\lambda = l\alpha + m\beta + l^{(1)}\alpha^{(1)} + m^{(1)}\beta^{(1)} + \text{etc.}$$

We are able to give to this function, the form

$$\frac{1}{4}S\left\{\left(2\beta^{(s)} + \alpha^{(s)} - \frac{3m^{(s)}\lambda}{2F}\right)^2 + \frac{3}{4}\left(\alpha^{(s)} - \frac{(l^{(s)} - \frac{1}{2}m^{(s)})\lambda}{F}\right)^2\right\} + \frac{3}{4}\frac{\lambda^2}{F};$$

this function is evidently at its *minimum*, if we suppose

$$2\beta^{(s)} + \alpha^{(s)} - \frac{3m^{(s)}\lambda}{2F} = 0; \qquad \alpha^{(s)} - \frac{(l^{(s)} - \frac{1}{2}m^{(s)})\lambda}{F} = 0;$$

whence we deduce generally

$$\alpha^{(s)} = (l^{(s)} - \tfrac{1}{2}m^{(s)})\frac{\lambda}{F}; \qquad \beta^{(s)} = (m^{(s)} - \tfrac{1}{2}l^{(s)})\frac{\lambda}{F}.$$

In the case that we just considered, we have

$$\alpha = -\frac{\lambda b}{F}(\tfrac{3}{2} + h' + \tfrac{1}{2}h); \qquad \beta = \frac{\lambda b}{F}(\tfrac{3}{2} + h + \tfrac{1}{2}h');$$
$$\alpha^{(1)} = \alpha^{(2)} = \cdots \alpha^{(i)} = -\frac{\frac{3}{2}b\lambda}{F}; \qquad \beta^{(1)} = \beta^{(2)} = \cdots \beta^{(i)} = \frac{\frac{3}{2}b\lambda}{F},$$
$$\alpha^{(i+1)} = \frac{\lambda b}{F}(\tfrac{3}{2} + h' + \tfrac{1}{2}h); \qquad \beta^{(i+1)} = -\frac{\lambda b}{F}(\tfrac{3}{2} + h + \tfrac{1}{2}h');$$

thus by these corrections, all the triangles other than those which have one of the bases for one of their sides, will remain right-angled.

[15] The probability of the error $\pm u$ of the line $II^{(i+1)}$, corrected by the second base, will be by the section cited in the second Supplement,

$$\frac{2\int dt\, c^{-t^2}}{\sqrt{\pi}},$$

the integral being taken from t null, to

$$t = \frac{3u}{2\theta}\sqrt{\frac{i+1}{Q\frac{(Sr^{(i)}f^{(i)})^2}{Sf^{(i)2}}}},$$

which becomes here

$$t = \frac{3u}{2\theta}\sqrt{\frac{i+1}{Q'}},$$

by designating by Q' the function

$$\frac{(i+1)(i+2)(i+3)}{4}a^2 + \tfrac{3}{2}(i+1)^2(h+h')a^2 + \tfrac{1}{2}(i+1)^2(h^2+hh'+h'^2)a^2.$$

The equally probable errors being proportionals to the square roots of Q and of Q', we see that they are diminished and nearly reduced to half, by the measure of a second base.

The probability of an error $\pm\lambda$ in the measure of a second base, is by the second Supplement,

$$\frac{2\int dt\, c^{-t^2}}{\sqrt{\pi}},$$

the integral being taken from t null, to

$$t = \frac{3u}{2\theta}\sqrt{\frac{i+1}{Sf^{(i)2}}};$$

and $f^{(i)^2}$ is equal to

$$3(i+1)b^2 + 6(h+h')b^2 + 2(h^2+hh'+h'^2)b^2.$$

In the present case where $i = 25$, this quantity becomes

$$86,8030b^2;$$

the equally probable errors in the measures of the arc $II^{(i+1)}$ and of a new base equal to the first, are therefore in the ratio of $\sqrt{Q}$ to $\sqrt{86,8030}$; whence it follows that there is odds one against one that the error of a new base will be comprehended within the limits $\pm 0^{\text{m}},34236$, or to very nearly $\pm\frac{1}{3}^{\text{m}}$. These are the same limits which result [16]
from the angles of the twenty-six triangles which reunite the base from Perpignan, to Formentera. Thus under this relation again, the hypothetical case which we have just examined, accords with that which this chain of triangles gives.

§4. I will consider now the zenithal distances of the vertices of the triangles, and the leveling which results from it. From one same vertex such as $C^{(2)}$, we are able to observe the four points C, $C^{(1)}$, $C^{(3)}$, $C^{(4)}$. Let us name f the distance $CC^{(1)}$, and h the base $CC^{(2)}$ of the isosceles triangle; all the triangles being supposed equal, if we name $x^{(i)}$ the height of $C^{(i)}$, above the level of the sea; the observed distance from

$C^{(i-2)}$ to the zenith of $C^{(i)}$, being designated by θ, the true distance will be quite nearly, the triangles being able to be supposed horizontal,

$$\theta + \frac{hu}{R} + \frac{h\epsilon}{R},$$

u being the factor by which we must multiply the angle $\frac{h}{R}$ in order to have the terrestrial refraction at the point $C^{(i)}$, R being the radius of the Earth and ϵ being the error of u. I take account here only of this error, as being much greater than that of θ. If we name similarly θ' the zenithal distance of $C^{(i)}$, observed from $C^{(i-2)}$, the true distance will be

$$\theta' + \frac{hu}{R} + \frac{h\epsilon'}{R},$$

ϵ' being the error of u in this observation. We will have

$$\theta + \theta' + \frac{2hu}{R} + \frac{h}{R}(\epsilon + \epsilon') = \pi + \frac{h}{R};$$

we will have next

$$x^{(i)} - x^{(i-2)} = \frac{h}{2}(\theta - \theta') + \frac{h^2}{2R}(\epsilon - \epsilon').$$

If we name similarly θ'' the zenithal distance of $C^{(i-1)}$ observed from $C^{(i)}$; the true distance will be

$$\theta'' + \frac{fu}{R} + \frac{f\epsilon''}{R},$$

ϵ'' being the error of u in this observation. By naming further θ''' and ϵ''', the same
[17] quantities relative to the zenithal distance of $C^{(i)}$, observed from $C^{(i-1)}$, we will have

$$\theta'' + \theta''' + \frac{2fu}{R} + \frac{f}{R}(\epsilon'' + \epsilon''') = \pi + \frac{f}{R},$$
$$x^{(i)} - x^{(i-1)} = \frac{f}{2}(\theta'' - \theta''') + \frac{f^2}{2R}(\epsilon'' - \epsilon''').$$

As I myself propose here only to examine what degree of confidence we must accord to this kind of leveling, I will make $h = f$; that which reverts to supposing all the triangles equilateral. I will take moreover $\frac{h^2}{2R}$ for unit of distance: by making next $\epsilon - \epsilon' = \lambda^{(i)}$, $\epsilon'' - \epsilon''' = \gamma^{(i)}$, we will have two equations of the form,

$$\begin{aligned} x^{(i)} - x^{(i-1)} &= \gamma^{(i)} + p^{(i)}, \\ x^{(i)} - x^{(i-2)} &= \lambda^{(i)} + q^{(i)}. \end{aligned} \tag{A}$$

The first of these equations extends from $i = 1$ to $i = n + 1$, n being the number of triangles. The second equation extends from $i = 2$ to $i = n+1$. It is necessary now to conclude from this system of equations, the most advantageous value of $x^{(n+1)} - x^{(0)}$, the elevation $x^{(0)}$ of the point C above the sea being supposed known. For this, we will multiply the first of the equations (A) by $f^{(i)}$ and the second by $g^{(i)}$, $f^{(i)}$ and $g^{(i)}$ being indeterminate constants. In the system of these equations added all together, the coefficient of $x^{(i)}$ will be $f^{(i)} - f^{(i+1)} + g^{(i)} - g^{(i+2)}$. By equating it to zero, and observing that $g^{(i+2)} - g^{(i)} = \Delta g^{(i+1)} + \Delta g^{(i)}$, Δ being the characteristic of finite differences; we will have by integrating,

$$f^{(i)} = a - g^{(i)} - g^{(i+1)},$$

a being a constant. But the values of $g^{(i)}$, beginning to take place only when $i = 2$, this expression of $f^{(i)}$ is able to serve only when $i = 2$. In order to have the value of $f^{(1)}$, we will observe that the equating to zero, of the coefficient of $x^{(1)}$, gives

$$f^{(1)} = f^{(2)} + g^{(3)};$$

substituting in the place of $f^{(2)}$, $a - g^{(2)} - g^{(3)}$, we will have

$$f^{(1)} = a - g^{(2)}.$$

Next, the preceding expression of $f^{(i)}$ extends only to $i = n$. But relatively to $i = n+1$, [18]
we must observe that the coefficient of $x^{(n+1)}$ must be unity, that which gives

$$f^{(n+1)} + g^{(n+1)} = 1;$$

or

$$f^{(n+1)} = 1 - g^{(n+1)};$$

the equality to zero, of the coefficient of $x^{(n)}$, gives $f^{(n)} = f^{(n+1)} - g^{(n)}$, or $f^{(n)} = 1 - g^{(n)} - g^{(n+1)}$. By comparing that expression to this one here $f^{(n)} = a - g^{(n)} - g^{(n+1)}$, we will have $a = 1$. The error of the value of $x^{(n+1)}$ will be thus

$$f^{(1)}\gamma^{(1)} + f^{(2)}\gamma^{(2)} \cdots + f^{(n+1)}\gamma^{(n+1)}$$
$$+ g^{(2)}\lambda^{(2)} + g^{(3)}\lambda^{(3)} \cdots + g^{(n+1)}\lambda^{(n+1)}.$$

The values of $\gamma^{(1)}$, $\gamma^{(2)}$, etc., $\lambda^{(1)}$, $\lambda^{(2)}$, etc., being evidently subject to the same law of probability; if we name s, this error, and if we make

$$H = f^{(1)2} + f^{(2)2} \cdots + f^{(n+1)2}$$
$$+ g^{(2)2} + g^{(3)2} \cdots + g^{(n+1)2},$$

the probability of the error s will be proportional by §20 of the second book, to an exponential of the form

$$c^{-\frac{Ks^2}{H}},$$

K being a constant dependent on the law of probability of $\gamma^{(i)}$ and $\lambda^{(i)}$.

It is necessary to determine the constants of H, in a manner that H is a *minimum.* Now we have

$$H = (1 - g^{(2)})^2 + (1 - g^{(2)} - g^{(3)})^2 \cdots + (1 - g^{(n)} - g^{(n+1)})^2$$
$$+ (1 - g^{(n+1)})^2 + g^{(2)2} + g^{(3)2} \cdots + g^{(n+1)2};$$

by equating to zero, the coefficient of the differential of $g^{(i)}$, we have

$$g^{(i+1)} + 3g^{(i)} + g^{(i-1)} = 2. \tag{1}$$

[19] This equation holds from $i = 3$, to $i = n$. The equality to zero, of the coefficient of $dg^{(2)}$, gives

$$g^{(3)} + 3g^{(2)} = 2,$$

and the equating to zero of the coefficient of $dg^{(n+1)}$, gives

$$3g^{(n+1)} + g^{(n)} = 2,$$

that which reverts to considering the general equation (1), as holding from $i = 2$, to $i = n + 1$, and to supposing null $g^{(1)}$ and $g^{(n+2)}$. The integration of equation (1) in the finite differences, gives

$$g^{(i)} = \tfrac{2}{5} + Al^{i-1} + A'l'^{i-1},$$

l and l' being the two roots, $-\frac{3}{2} - \frac{1}{2}\sqrt{5}$, $-\frac{3}{2} + \frac{1}{2}\sqrt{5}$, of the equation

$$y^2 + 3y + 1 = 0;$$

A and A' are two arbitraries such that $g^{(i)}$ becomes null, when $i = 1$, and when $i = n + 2$. We have therefore

$$Al^{n+1} + A'l'^{n+1} = -\tfrac{2}{5},$$
$$A + A' = -\tfrac{2}{5}.$$

l^{n+1} is an extremely great quantity, when n is a great number; and l'^{n+1} being $\frac{1}{l^{n+1}}$, we see that A is then an excessively small quantity, and that thus $A' = -\frac{2}{5}$. We have next

$$f^{(i)} = \tfrac{1}{5} - Al^{i-1}(1 + l) - A'l'^{i-1}(1 + l').$$

Thence it is easy to conclude that we have very nearly, and without fear $\frac{1}{25}$ of error.

$$H = \frac{n+1}{5},$$

and that thus the exponential proportional to the probability of error s, is

$$c^{-\frac{5Ks^2}{n+1}};$$

we are able therefore thus to determine this probability.

[20] We have concluded the value of $x^{(n+1)}$, of the system of equations (A), by the following process.

The system of equations (A) gives

$$x^{(1)} - x^{(0)} = p^{(1)} + \gamma^{(1)};$$

whence we deduce

$$x^{(1)} = p^{(1)} + x^{(0)} + \gamma^{(1)}.$$

We have next the two equations

$$x^{(2)} - x^{(1)} = p^{(2)} + \gamma^{(2)};$$
$$x^{(2)} - x^{(0)} = q^{(2)} + \lambda^{(2)};$$

that which gives

$$x^{(2)} = \tfrac{1}{2}x^{(1)} + \tfrac{1}{2}x^{(0)} + \tfrac{1}{2}(p^{(2)} + q^{(2)}) + \tfrac{1}{2}\gamma^{(2)} + \tfrac{1}{2}\lambda^{(2)}.$$

We have the two equations

$$x^{(3)} - x^{(2)} = p^{(3)} + \gamma^{(3)};$$
$$x^{(3)} - x^{(1)} = q^{(3)} + \lambda^{(3)};$$

that which gives

$$x^{(3)} = \tfrac{1}{2}x^{(2)} + \tfrac{1}{2}x^{(1)} + \tfrac{1}{2}(p^{(3)} + q^{(3)}) + \tfrac{1}{2}\gamma^{(3)} + \tfrac{1}{2}\lambda^{(3)}.$$

By continuing thus, we will have $x^{(n+1)}$. The quantities $\gamma^{(m)}$ and $\lambda^{(m)}$ commence to be introduced into this expression, only with the two values of $x^{(m)} - x^{(m-1)}$ and of $x^{(m)} - x^{(m-2)}$. Let us designate by $k^{(r)}$ the coefficient of $\gamma^{(m)}$, in the expression of $x^{(m+r)}$; this expression is

$$x^{(m+r)} = \tfrac{1}{2}x^{(m+r-1)} + \tfrac{1}{2}x^{(m+r-2)} + \tfrac{1}{2}(p^{(m+r)} + q^{(m+r)}) + \tfrac{1}{2}\gamma^{(m+r)} + \tfrac{1}{2}\lambda^{(m+r)};$$

by substituting for $x^{(m+r)}$, $x^{(m+r-1)}$, $x^{(m+r-2)}$ the parts of their values relative to $\gamma^{(m)}$, the comparison of the coefficients of this quantity, will give

$$k^{(r)} = \tfrac{1}{2}k^{(r-1)} + \tfrac{1}{2}k^{(r-2)};$$

whence we deduce by integrating

$$k^{(r)} = A + A'\left(-\tfrac{1}{2}\right)^{r-1},$$

A and A' being two arbitraries. In order to determine them, we will observe that r being null, we have $k^{(0)} = \frac{1}{2}$, and that, r being 1, we have

$$k^{(1)} = \tfrac{1}{2}k^{(0)} = \tfrac{1}{4};$$

thence we deduce [21]

$$A = \tfrac{1}{3}, \qquad A' = -\tfrac{1}{12},$$

thus in the value of $x^{(n+1)}$, or $r = n+1-m$, we will have for the coefficient $k^{(n+1-m)}$ of $\gamma^{(m)}$,

$$k^{(n+1-m)} = \tfrac{1}{3} - \tfrac{1}{12}\left(-\tfrac{1}{2}\right)^{n-m};$$

the coefficient of $\lambda^{(m)}$ in the same value, will be evidently the same. Thus the expression of $x^{(n+1)}$ will be a known quantity, plus the series

$$k^{(n)}\gamma^{(1)} + k^{(n-1)}(\gamma^{(2)} + \lambda^{(2)}) \cdots + k^{(0)}(\gamma^{(n+1)} + \lambda^{(n+1)}).$$

Let us designate by s this error, and by H, the sum of the squares of the coefficients of $\gamma^{(1)}$, $\gamma^{(2)}$, etc., $\lambda^{(2)}$, $\lambda^{(3)}$, etc.; the probability of s will be proportional to $c^{-\frac{Ks^2}{H}}$. We have very nearly,

$$H = \tfrac{2}{9}(n+1);$$

thus, the probability of s, is very nearly proportional to $c^{\frac{-9Ks^2}{2(n+1)}}$; the equally probable errors are therefore greater in this process, than according to the most advantageous method, and nearly in the ratio of $\sqrt{5}$ to $\sqrt{\frac{9}{2}}$; this process approaches therefore much the exactitude of the most advantageous method; and as the calculation of it is quite simple, we will determine the probability of the errors to which it exhibits, in the general case where the diverse triangles are neither equal nor equilateral.

If we represent by $m^{(i)}$, the square of $C^{(i-1)}C^{(i)}$, divided by $2R$, and by $n^{(i)}$ the square of $C^{(i-2)}C^{(i)}$, divided similarly by $2R$; the system of equations (A) will be changed into the following

$$\begin{aligned} x^{(i)} - x^{(i-1)} &= p^{(i)} + m^{(i)}\gamma^{(i)}, \\ x^{(i)} - x^{(i-2)} &= q^{(i)} + n^{(i)}\lambda^{(i)}, \end{aligned} \qquad (\text{A}')$$

The process that we have just examined, gives by following the preceding analysis,
[22] the coefficient of $\gamma^{(i)}$, in the expression of $x^{(n+1)}$, equal to

$$\tfrac{1}{3}m^{(i)} - \tfrac{1}{12}m^{(i)}\left(-\tfrac{1}{2}\right)^{n-i}.$$

Similarly the coefficient of $\lambda^{(i)}$ in the same expression, is

$$\tfrac{1}{3}n^{(i)} - \tfrac{1}{12}n^{(i)}\left(-\tfrac{1}{2}\right)^{n-i};$$

thence it follows that the value of H, is very nearly,

$$\tfrac{1}{9}S(m^{(i)2} + n^{(i)2}),$$

the integral sign S, extending to all the values of i to $i = n+1$; the probability of an error s, in the expression of $x^{(n+1)}$ is therefore proportional to

$$c^{\frac{-9Ks^2}{S(m^{(i)2}+n^{(i)2})}}.$$

If we apply to the equations (A$'$) the analysis that we have given above for the case of the most advantageous method, we will find by multiplying them respectively by $f^{(i)}$, and $g^{(i)}$, the following equation

$$f^{(i)} = 1 - g^{(i)} - g^{(i+1)},$$

and this equation will hold, from $i = 1$, to $i = n+1$, by supposing $g^{(i)}$ and $g^{(i+2)}$ nulls. We will have next the general equation

$$m^{(i)2}g^{(i+1)} + (n^{(i)2} + m^{(i)2} + n^{(i-1)2})g^{(i)} + m^{(i-1)2}g^{(i-1)} = m^{(i)2} + m^{(i-1)2}.$$

This equation holds from $i = 2$ to $i = n+1$. By combining it with the equations $g^{(1)} = 0$, $g^{(n+2)} = 0$, we will have the values of $f^{(1)}$, $f^{(2)}$, $\dots f^{(n+1)}$, $g^{(1)}$, $g^{(2)}$, $\dots g^{(n+2)}$; we will have next

$$H = S(f^{(i)2}m^{(i)2} + g^{(i)2}n^{(i)2});$$

the sign S comprehending all the values of $f^{(i)}m^{(i)2}$ and of $g^{(i)2}n^{(i)2}$; the probability of an error s, in the value of $x^{(n+1)}$ will be proportional to

$$c^{-\frac{Ks^2}{H}},$$

[23] §5. It is necessary now to determine the value of K. For this, we will observe that the factor u is determined by that which precedes, by means of the equation

$$u = \frac{\pi - \theta - \theta' + \frac{h}{R}}{\frac{2h}{R}},$$

and that the error of this expression is $\frac{\epsilon+\epsilon'}{2}$. Each double station furnishes a value of u, and the mean of these values is the value that it is necessary to adopt. If we name i, the number of these values, the error to fear will be $S\frac{(\epsilon+\epsilon')}{2i}$, the sign S corresponding to the i quantities $\frac{\epsilon+\epsilon'}{2i}$, related to each double station. Let s be the sum $S\frac{(\epsilon+\epsilon')}{2i}$; the probability of s will be, by §20 of the second book, proportional to an exponential of the form

$$c^{-\frac{K's^2}{i}},$$

and if we name q, the sum of the squares of the differences of each partial value to its mean value, we will have

$$K' = \frac{i}{2q}.$$

We have by that which precedes, the probability of the error of a value s' of the function $S\frac{(\epsilon-\epsilon')}{2}$, proportional to the exponential

$$c^{-\frac{4Ks'^2}{i}},$$

the sign S extending to i quantities of the form $\frac{\epsilon-\epsilon'}{2}$. Now the errors ϵ and $-\epsilon$ being supposed equally probable, it is clear that the same values of $S\frac{(\epsilon+\epsilon')}{2}$ and of $S\frac{(\epsilon-\epsilon')}{2}$ are equally probable; we have therefore

$$4K = K',$$

that which gives

$$K = \frac{i}{8q}.$$

The forty-five first values of u, given in the second volume of the *Base du Système métrique*, page 771, and which are founded on some observations made in the month of the year where we observe most often, give for its mean value, [24]

$$u = 0,07818,$$

and the sum q of the squares of the differences of these values to the mean is 0,04900629. i being here equal to 45, we have

$$K = \frac{45}{0,39205032} = 114,781.$$

If we suppose the number n of triangles equal to 25; and if we make all the sides equal to 20000^{m}; we will have 240000^{m} for the distance from $x^{(26)}$ to $x^{(0)}$: this is nearly the distance from Paris to Dunkirk. In this case, the quantity $\frac{f^2}{2R}$ taken for unit of distance, is $31^{\text{m}},416$. Thence we conclude that the odds is one against one, that the error respecting the height $x^{(26)}$ is comprehended within the limits $\pm 3^{\text{m}},1839$. There is odds nine against one that it is comprehended within the limits $\pm 7^{\text{m}},761$; we cannot therefore then respond with a sufficient probability, that this error will not exceed $\pm 8^{\text{m}}$.

The chain of triangles that we have just considered, is much more favorable to the determination of the height of its last point, than that of which Delambre has made use in the work cited, in order to determine the height of the Pantheon above the level of the sea. By considering this last chain, we see that we cannot respond with a sufficient probability, that the error respecting this height will not exceed $\pm 16^{\mathrm{m}}$.

§6. We see by that which precedes, that the great triangles which are very proper to the measure of terrestrial degrees, are too small, in order to determine the respective heights of the diverse stations. Thus in the case of a chain of equilateral triangles of which f is the length of each side, the equally probable errors of the difference of level of two extreme stations, being proportional to $\frac{f^2\sqrt{n+1}}{2R}$, n being the number of triangles; if we name a, the distance of these two stations; we will have by supposing
[25] $n+1$, even, $a = \frac{1}{2}(n+1)f$; $\frac{f^2\sqrt{n+1}}{2R}$ will be therefore proportional to $\frac{1}{(n+1)^{\frac{3}{2}}}$; the equally probable errors will be therefore proportional to this fraction. Thus, by quadrupling the number of triangles, they will become eight times smaller. But then the errors due to the observations of the angles, become comparable to the errors due to the variability of the terrestrial refractions. Let us examine how we are able to have regard at the same time to these two kinds of errors.

Let us consider a sequence of points C, $C^{(1)}$, $C^{(2)}$, etc. Let $h^{(0)}$ be the distance from C to $C^{(1)}$; $h^{(1)}$ the distance from $C^{(1)}$ to $C^{(2)}$; $h^{(2)}$ the distance from $C^{(2)}$ to $C^{(3)}$, and so forth. Let us imagine that from the point $C^{(i)}$, we observe $C^{(i+1)}$ and reciprocally. The zenithal distance from $C^{(i+1)}$ observed from $C^{(i)}$, will be by that which precedes,

$$\theta + \frac{h^{(i)}u}{R} + \frac{h^{(i)}\epsilon}{R} + \alpha,$$

ϵ being the error of u and α being that of the observed angle θ. The zenithal distance of $C^{(i)}$ observed from $C^{(i+1)}$, will be

$$\theta' + \frac{h^{(i)}u}{R} + \frac{h^{(i)}\epsilon'}{R} + \alpha',$$

ϵ' and α' being the errors of u and of θ', in the observation made at the point $C^{(i+1)}$. We will have therefore the two equations

$$\theta + \theta' + \frac{2h^{(i)}u}{R} + \frac{h^{(i)}u}{R}(\epsilon + \epsilon') + \alpha + \alpha' = \pi + \frac{h^{(i)}}{R};$$

$$x^{(i+1)} - x^{(i)} = \frac{(\theta - \theta')}{2}h^{(i)} + \frac{h^{(i)2}}{2R}(\epsilon - \epsilon') + \frac{1}{2}h^{(i)}(\alpha - \alpha').$$

Let us designate as above $\epsilon - \epsilon'$ by $\gamma^{(i)}$, and let us make $\alpha - \alpha'$ equal to $\lambda^{(i)}$; we will have for the elevation $x^{(n+1)} - x^{(0)}$ of the point $C^{(n+1)}$, above C, an expression of the form

$$x^{(n+1)} - x^{(0)} = M + S\frac{h^{(i)2}}{2R}\gamma^{(i)} + S.\tfrac{1}{2}h^{(i)}\lambda^{(i)},$$

the integral sign S, corresponding to all the values of i, from $i = 0$, to $i = n$. The error of this value of $x^{(n+i)}$ is

$$S\frac{h^{(i)2}\gamma^{(i)}}{2R} + S\frac{h^{(i)}}{2}\lambda^{(i)};$$

it is necessary now to determine the probability of this error that we will designate [26]
by s. Let there be generally

$$s = Sm^{(i)}\gamma^{(i)} + Sn^{(i)}\lambda^{(i)};$$

the probability of s will be by the analysis of §20 of the second book of the *Théorie analytique des Probabilités*, proportional to

$$\int d\varpi\, dx\, dy\, \phi(x)\psi(y)c^{-s\varpi\sqrt{-1}} \begin{cases} \cos(m^{(0)}x + n^{(0)}y)\varpi \\ \times\cos(m^{(1)}x + n^{(1)}y)\varpi \\ \times\cos(m^{(2)}x + n^{(2)}y)\varpi \\ \times\text{etc.} \end{cases}$$

$\phi(x)$ is the law of probability of a value x of $\gamma^{(0)}$; $\psi(y)$ is the law of probability of a value y of $\lambda^{(0)}$. The negative and positive errors are supposed equally probable: the integrals relative to x and y are taken from negative infinity to positive infinity, and the integral relative to ϖ, is taken from $\varpi = -\pi$, to $\varpi = \pi$. By making

$$2\int dx\, \phi(x) = k, \qquad \int x^2\, dx\, \phi(x) = k'',$$
$$2\int dy\, \psi(y) = \bar{k}, \qquad \int y^2\, dy\, \psi(y) = \bar{k}'',$$

the integrals being taken from x and y null, to x and y equal to infinity; the analysis of the section cited, will give the probability of s, proportional to

$$c^{\frac{-s^2}{\frac{4k''}{k}Sm^{(i)2} + \frac{4\bar{k}''}{\bar{k}}Sn^{(i)2}}}$$

It is easy to conclude generally from the same analysis, that if we make

$$s = Sm^{(i)}\gamma^{(i)} + Sn^{(i)}\lambda^{(i)} + Sr^{(i)}\delta^{(i)} + \text{etc.}$$

$\gamma^{(i)}$, $\lambda^{(i)}$, $\delta^{(i)}$, etc., being some errors deriving from different sources, the probability of s is proportional to the exponential

$$c^{\frac{-s^2}{\frac{4k''}{k}Sm^{(i)2} + \frac{4\bar{k}''}{\bar{k}}Sn^{(i)2} + \frac{4\bar{\bar{k}}''}{\bar{\bar{k}}}Sr^{(i)2} + \text{etc.}}}$$

by designating by $\pi(x)$ the probability of an error x due to the third source of error, [27]
and making

$$2\int dx\, \pi(x) = \bar{\bar{k}}, \qquad \int x^2\, dx\pi(x) = \bar{\bar{k}}'',$$

the integrals being taken from x null, to x infinity; and thus of the other errors.

In order to determine in the present question, the constants $\frac{4k''}{k}$ and $\frac{4\bar{k}''}{\bar{k}}$, I will suppose first the second null or very small, relatively to the first; as we can do in the great triangulations of the meridian. In this case, the probability of an error s, will be by making $m^{(i)} = 1$, proportional to

$$c^{\frac{-s^2}{\frac{4k''}{k}n}},$$

n being the number of intervals which separate the stations. The probability of a value s' of $S\frac{\epsilon-\epsilon'}{2}$, or of $S\frac{(\epsilon+\epsilon')}{2}$, that which corresponds to an error $2s'$ in the value of $S(\epsilon-\epsilon')$, will be proportional to

$$c^{\frac{-4s'^2}{\frac{4k''}{k}n}};$$

but by that which precedes, this probability is proportional to

$$c^{\frac{-i}{2q}\cdot\frac{s'^2}{n}};$$

we have therefore

$$\frac{2q}{i}=\frac{k''}{k} \quad \text{or} \quad \frac{4k''}{k}=\frac{8q}{i}=\frac{1}{114,781}.$$

If we suppose now $\frac{k''}{k}$ null, and $n^{(i)}=1$; the probability of a value s', of the sum $S\left(\frac{\alpha'-\alpha}{2}\right)$ will be proportional to

$$c^{\frac{-4s'^2}{\frac{4\bar{k}''}{\bar{k}}n}},$$

[28] and the probability of a same value s' of $S(\alpha+\alpha'+\alpha'')$ will be proportional to

$$c^{\frac{-2s'^2}{\frac{12\bar{k}''}{\bar{k}}n}}.$$

If we suppose this law of probability, the same as for the errors of the sum of the three angles of a spherical triangle, in the geodesic measures, and which, by §1 of the second Supplement, is able to be supposed proportional to

$$c^{\frac{-(i+2)}{2\theta^2}\frac{s'^2}{n}},$$

θ^2 being the sum of the squares of the excess observed in the sum of the errors of the three angles, in i triangles; we will have

$$\frac{4\bar{k}''}{\bar{k}}=\frac{4\theta^2}{3(i+2)}.$$

We have by that which we have seen,

$$\frac{i+2}{\theta^2}=\frac{109}{445,217};$$

hence

$$\frac{4\bar{k}''}{\bar{k}}=\frac{4}{3}\cdot\frac{445,217}{109},$$

a quantity that it is necessary to divide by the square of the number of sexagesimal seconds that this radius contains, and then we have

$$\frac{4\bar{k}''}{\bar{k}}=\frac{1,2801}{10^{10}}.$$

Let us suppose the distances of the consecutive stations, equal to twelve hundred meters; we will find that there is odds one against one that the error respecting the value of $x^{(n+1)}$ is not above $\pm 0^{\mathrm{m}},08555$, when $n=200$. There is odds one thousand against one that the error is not above $\pm 0^{\mathrm{m}},413$.

GENERAL METHOD [29]

On the calculation of probabilities, when there are many sources of errors.

The consideration of the two independent sources of error, which exist in the operations of the leveling, has led me to examine the general case of the observations subject to many sources of errors. Such are astronomical observations. The greater part are made by means of two instruments, the meridian lunette and the circle, of which the errors must not be supposed to have the same law of probability. In the equations of condition, that we deduce from these observations, in order to obtain the elements of the celestial movements, these errors are multiplied by some different coefficients for each source of error, and for each equation. The most advantageous systems of factors by which it is necessary to multiply these equations in order to have the final equations which determine the elements, are no longer, as in the case of a unique source of errors, the coefficients of each element in the equations of condition. The facility with which the analysis that I have given in the second book of my *Théorie des Probabilités*, is applied to this general case, will show the advantages of this analysis.

Let us suppose first that we have a system of equations of condition, represented by this here

$$p^{(i)}y = a^{(i)} + m^{(i)}\gamma^{(i)} + n^{(i)}\lambda^{(i)} + \text{etc.},$$

y being an element of which we seek the most advantageous value. If we multiply the preceding equation, by a factor $f^{(i)}$; the reunion of all these products, will give for y the expression

$$y = \frac{Sa^{(i)}f^{(i)}}{Sp^{(i)}f^{(i)}} + \frac{Sm^{(i)}f^{(i)}\gamma^{(i)} + Sn^{(i)}f^{(i)}\lambda^{(i)} + \text{etc.}}{Sp^{(i)}f^{(i)}}$$

The error of y will be

$$\frac{Sm^{(i)}f^{(i)}\gamma^{(i)} + Sn^{(i)}f^{(i)}\lambda^{(i)} + \text{etc.}}{Sp^{(i)}f^{(i)}}$$

By designating by s, this error; its probability will be proportional by the preceding section, to the exponential [30]

$$c^{\frac{-s^2(Sp^{(i)}f^{(i)})^2}{\frac{4k''}{k}Sm^{(i)2}f^{(i)2} + \frac{4\bar{k}''}{\bar{k}}Sn^{(i)2}f^{(i)2} + \text{etc.}}}$$

It is necessary to determine $f^{(i)}$ in a manner that

$$\frac{\frac{4k''}{k}Sm^{(i)2}f^{(i)2} + \frac{4\bar{k}''}{\bar{k}}Sn^{(i)2}f^{(i)2} + \text{etc.}}{(Sp^{(i)}f^{(i)})^2}$$

is a *minimum*; because it is clear that then the same error s, becomes less probable than in each other system of factors. If we name A the numerator of this fraction, and if we make $f^{(i)}$ vary, by a quantity dq; we will have through the condition of the *minimum*, by equating to zero, the differential of this fraction,

$$0 = \frac{\frac{k''}{k}m^{(i)2}f^{(i)} + \frac{\bar{k}''}{\bar{k}}n^{(i)2}f^{(i)} + \text{etc.}}{A} - \frac{p^{(i)}}{Sp^{(i)}f^{(i)}}$$

that which gives for $f^{(i)}$ an expression of this form

$$f^{(i)} = \frac{\mu p^{(i)}}{\frac{k''}{k} m^{(i)2} + \frac{\bar{k}''}{\bar{k}} n^{(i)2} + \text{etc.}}$$

We can make here $\mu = 1$, because this quantity being independent of i, it affects equally all the multipliers $f^{(i)}$; thus the quantity $f^{(i)}$ by which we must multiply each equation of condition, in order to have the most advantageous result, is

$$\frac{p^{(i)}}{\frac{k''}{k} m^{(i)2} + \frac{\bar{k}''}{\bar{k}} n^{(i)2} + \text{etc.}}$$

and the probability of an error s of this result, is proportional to the exponential

$$c^{\frac{-s^2}{4} S \frac{p^{(i)2}}{\frac{k''}{k} m^{(i)2} + \frac{\bar{k}''}{\bar{k}} n^{(i)2} + \text{etc.}}}$$

[31] We will have by the same analysis and by §22 of the second book, the factors by which we must multiply the equations of condition, in order to have the most advantageous results, whatever be the number of elements to determine, and the number of kinds of errors: we will have similarly the laws of probability of the errors of these results.

Let us suppose that we have between two elements x and y, the equation of condition

$$l^{(i)} x + p^{(i)} y = a^{(i)} + m^{(i)} \gamma^{(i)} + n^{(i)} \lambda^{(i)} + r^{(i)} \delta^{(i)} + \text{etc.},$$

$\gamma^{(i)}$, $\lambda^{(i)}$, $\delta^{(i)}$, etc. being some errors of which the sources are different. By multiplying first this equation, by a system $f^{(i)}$ of factors; the reunion of these products will give the final equation

$$xSl^{(i)} f^{(i)} + ySp^{(i)} f^{(i)} = Sa^{(i)} f^{(i)} + Sm^{(i)} f^{(i)} \gamma^{(i)} + Sn^{(i)} f^{(i)} \lambda^{(i)} + \text{etc.}$$

By multiplying next the equation of condition, by another system $g^{(i)}$ of factors, the reunion of the products will give a second final equation

$$xSl^{(i)} g^{(i)} + ySp^{(i)} g^{(i)} = Sa^{(i)} g^{(i)} + Sm^{(i)} g^{(i)} \gamma^{(i)} + \text{etc.}$$

We deduce from these two final equations,

$$x = \frac{Sa^{(i)} f^{(i)}.Sp^{(i)} g^{(i)} - Sa^{(i)} g^{(i)}.Sp^{(i)} f^{(i)}}{L} + \frac{Sm^{(i)} f^{(i)} \gamma^{(i)}.Sp^{(i)} g^{(i)} - Sm^{(i)} g^{(i)} \gamma^{(i)}.Sp^{(i)} f^{(i)} + \text{etc.}}{L},$$

L being equal to

$$Sl^{(i)} f^{(i)}.Sp^{(i)} g^{(i)} - Sl^{(i)} g^{(i)}.Sp^{(i)} f^{(i)}.$$

The coefficient of $\gamma^{(i)}$ in this value is

$$\frac{m^{(i)} f^{(i)}.Sp^{(i)} g^{(i)} - m^{(i)} g^{(i)}.Sp^{(i)} f^{(i)}}{L}.$$

By changing $m^{(i)}$ into $n^{(i)}$, $r^{(i)}$, etc., we will have the coefficients corresponding to $\lambda^{(i)}$, $\delta^{(i)}$, etc. By naming therefore s, the value of the part of x, dependent on the errors $\gamma^{(i)}$, $\lambda^{(i)}$, $\delta^{(i)}$, etc.; the probability of this value, will be by that which precedes, proportional to the exponential

$$c^{\frac{-s^2}{H}},$$

by making [32]

$$H = \frac{SM^{(i)}f^{(i)2}(Sp^{(i)}g^{(i)})^2 - 2SM^{(i)}f^{(i)}g^{(i)}.Sp^{(i)}f^{(i)}.Sp^{(i)}g^{(i)} + SM^{(i)}g^{(i)2}(Sp^{(i)}f^{(i)})^2}{L^2},$$

$M^{(i)}$ being equal to

$$\frac{4k''}{k}m^{(i)2} + \frac{4\bar{k}''}{\bar{k}}n^{(i)2} + \frac{4\bar{\bar{k}}''}{\bar{\bar{k}}}r^{(i)2} + \text{etc.}$$

It is necessary now to determine $f^{(i)}$ and $g^{(i)}$, in a manner that H is a *minimum.* For this we will make $f^{(i)}$ vary, and we will equate to zero, the coefficient of its differential; that which will give, by naming P the numerator of the expression of H,

$$\begin{aligned}0 =& M^{(i)}f^{(i)}(Sp^{(i)}g^{(i)})^2 - M^{(i)}g^{(i)}.Sp^{(i)}f^{(i)}.Sp^{(i)}g^{(i)} \\ &- p^{(i)}.SM^{(i)}f^{(i)}g^{(i)}.Sp^{(i)}g^{(i)} + p^{(i)}.SM^{(i)}g^{(i)2}.Sp^{(i)}f^{(i)} \\ &- \frac{P}{L}(l^{(i)}Sp^{(i)}g^{(i)} - p^{(i)}Sl^{(i)}g^{(i)}).\end{aligned}$$

It is easy to see that we satisfy this equation, by supposing

$$f^{(i)} = \frac{l^{(i)}}{M^{(i)}};\quad g^{(i)} = \frac{p^{(i)}}{M^{(i)}};$$

and we must conclude from it that we would satisfy by the same supposition, the corresponding equation that would give $dH = 0$, by making $g^{(i)}$ vary. We see that the same values of $f^{(i)}$ and $g^{(i)}$, satisfy the similar equations which result from the consideration of the element y.

If we have among the elements x, y, z, etc. some equations of condition represented by the general equation

$$l^{(i)}x + p^{(i)}y + q^{(i)}z + \text{etc.} = a^{(i)} + m^{(i)}\gamma^{(i)} + n^{(i)}\lambda^{(i)} + r^{(i)}\delta^{(i)} + \text{etc.},$$

$\gamma^{(i)}$, $\lambda^{(i)}$, $\delta^{(i)}$, etc. being the errors of diverse kinds; we will find by the preceding analysis that the factors by which we must multiply respectively this equation, in order to form the final equations which give the values of the most advantageous elements, are for the first final equation, represented by

$$\frac{l^{(i)}}{\frac{k''}{k}m^{(i)2} + \frac{\bar{k}''}{\bar{k}}n^{(i)2} + \frac{\bar{\bar{k}}''}{\bar{\bar{k}}}r^{(i)2} + \text{etc.}}$$

They are represented for the second final equation, by [33]

$$\frac{p^{(i)}}{\frac{k''}{k}m^{(i)2} + \frac{\bar{k}''}{\bar{k}}n^{(i)2} + \frac{\bar{\bar{k}}''}{\bar{\bar{k}}}r^{(i)2} + \text{etc.}}$$

and so forth. By applying therefore to the equations thus multiplied, the analysis of §2 of the first Supplement; we will have the values of the most advantageous elements, and the laws of probabilities of their errors.

In order to give an example of this application, let us consider only two elements x and y. If we make

$$M^{(i)} = \frac{k''}{k} m^{(i)2} + \frac{\bar{k}''}{\bar{k}} n^{(i)2} + \frac{\bar{\bar{k}}''}{\bar{\bar{k}}} r^{(i)2} + \text{etc.}$$

We will multiply the previous equation of condition, by $\frac{p^{(i)}}{M^{(i)}}$; and we will deduce from it

$$xS\frac{l^{(i)}p^{(i)}}{M^{(i)}} + yS\frac{p^{(i)2}}{M^{(i)}} = S\frac{a^{(i)}p^{(i)}}{M^{(i)}} + S\frac{p^{(i)}}{M^{(i)}}(m^{(i)}\gamma^{(i)} + n^{(i)}\lambda^{(i)} + \text{etc.});$$

but the condition of the most advantageous method, gives

$$0 = S\frac{l^{(i)}}{M^{(i)}}(m^{(i)}\gamma^{(i)} + n^{(i)}\lambda^{(i)} + \text{etc.}),$$

$$0 = S\frac{p^{(i)}}{M^{(i)}}(m^{(i)}\gamma^{(i)} + n^{(i)}\lambda^{(i)} + \text{etc.});$$

we will have therefore

$$y = \frac{S\frac{a^{(i)}p^{(i)}}{M^{(i)}} - xS\frac{p^{(i)}l^{(i)}}{M^{(i)}}}{S\frac{p^{(i)2}}{M^{(i)}}}.$$

Substituting this value of y, into the general equation of condition, and making

$$l_1^{(i)} = l^{(i)} - p^{(i)}\frac{S\frac{l^{(i)}p^{(i)}}{M^{(i)}}}{S\frac{p^{(i)2}}{M^{(i)}}},$$

$$a_1^{(i)} = a^{(i)} - p^{(i)}\frac{S\frac{l^{(i)}p^{(i)}}{M^{(i)}}}{S\frac{p^{(i)2}}{M^{(i)}}},$$

[34] we will have

$$x = \frac{S\frac{a_1^{(i)}l_1^{(i)}}{M^{(i)}}}{S\frac{l_1^{(i)2}}{M^{(i)}}};$$

and the probability of an error s, of this value, will be proportional to

$$c^{\frac{-s^2}{4}S\frac{l_1^{(i)2}}{M^{(i)}}}.$$

This analysis supposes knowledge of the constants $\frac{k''}{k}$ and $\frac{\bar{k}''}{\bar{k}}$. But we can obtain from it by the same observations, some very close values, in the following manner.

Let us imagine that we have determined the elements x, y, z, etc., by the method according to which we form the final equations, by multiplying each equation of condition successively by the corresponding coefficient of each element. If we substitute the values of the elements, thus determined, into the equation of condition

$$l^{(i)}x + p^{(i)}y + \text{etc.} - a^{(i)} = m^{(i)}\gamma^{(i)} + n^{(i)}\lambda^{(i)} + \text{etc.},$$

we will have an equation of this form

$$R^{(i)} = m^{(i)}\gamma^{(i)} + n^{(i)}\lambda^{(i)} + \text{etc.}$$

Let us suppose for greater simplicity, that we have only two kinds of errors $\gamma^{(i)}$ and $\lambda^{(i)}$: we will multiply first the preceding equation by $m^{(i)}$. By raising next each member to the square, and taking the sum of all the equations thus formed, we will have

$$Sm^{(i)2}R^{(i)2} = S(m^{(i)4}\gamma^{(i)2} + 2m^{(i)3}n^{(i)}\gamma^{(i)}\lambda^{(i)} + n^{(i)2}m^{(i)2}\lambda^{(i)2}).$$

The mean value of $m^{(i)4}\gamma^{(i)2}$ is evidently

$$\frac{m^{(i)4}\int \gamma^2\, d\gamma\phi(\gamma)}{\int d\gamma\,\phi(\gamma)},$$

the integrals being taken from $\gamma = -\infty$, to γ infinity; that which gives $\frac{2k''m^{(i)4}}{k}$. We have similarly $\frac{2\bar{k}''}{\bar{k}}m^{(i)2}n^{(i)2}$, for the mean value of $m^{(i)2}n^{(i)2}\lambda^{(i)2}$. We find in the [35] same manner that the mean value of $2m^{(i)3}n^{(i)}\gamma^{(i)}\lambda^{(i)}$ is null; we have therefore by substituting instead of the quantities, their mean values, that which we can make with so much more precision, as the number of observations is greater,

$$Sm^{(i)2}R^{(i)2} = \frac{2k''}{k}\,Sm^{(i)4} + \frac{2\bar{k}''}{\bar{k}}m^{(i)2}n^{(i)2}.$$

We will have similarly

$$Sn^{(i)2}R^{(i)2} = \frac{2k''}{k}\,Sm^{(i)2}n^{(i)2} + \frac{2\bar{k}''}{\bar{k}}\,Sn^{(i)4};$$

whence we deduce

$$\frac{4k''}{k} = \frac{2Sn^{(i)4}.Sm^{(i)2}R^{(i)2} - 2Sm^{(i)2}n^{(i)2}.Sn^{(i)2}R^{(i)2}}{Sm^{(i)4}.Sn^{(i)4} - (Sm^{(i)2}n^{(i)2})^2},$$

$$\frac{4\bar{k}''}{\bar{k}} = \frac{2Sm^{(i)4}.Sn^{(i)2}R^{(i)2} - 2Sm^{(i)2}n^{(i)2}.Sm^{(i)2}R^{(i)2}}{Sm^{(i)4}.Sn^{(i)4} - (Sm^{(i)2}n^{(i)2})^2};$$

be designating therefore by $2P$ and $2Q$ the numerators of these two expressions, the factors by which we must multiply the equation of condition will be

$$\frac{l^{(i)}}{m^{(i)2}P + n^{(i)2}Q}; \quad \frac{p^{(i)}}{m^{(i)2}P + n^{(i)2}Q}; \quad \text{etc.}$$

The concern now is to show that these values of $\frac{4k''}{k}$, $\frac{4\bar{k}''}{\bar{k}}$ are quite close. For this, let us consider only one element x: the equation of condition

$$l^{(i)}x = a^{(i)} + m^{(i)}\gamma^{(i)} + n^{(i)}\lambda^{(i)}$$

will give

$$x = \frac{Sa^{(i)}l^{(i)}}{Sl^{(i)2}} + \frac{Sl^{(i)}m^{(i)}\gamma^{(i)} + Sl^{(i)}n^{(i)}\gamma^{(i)}}{Sl^{(i)2}}.$$

Substituting this value in the equation of condition, we will have

$$R^{(i)} = \frac{l^{(i)} S a^{(i)} l^{(i)} - a^{(i)} S l^{(i)2}}{S l^{(i)2}},$$

$$R^{(i)} + l^{(i)} \frac{S(l^{(i)} m^{(i)} \gamma^{(i)} + l^{(i)} n^{(i)} \gamma^{(i)})}{S l^{(i)2}} = m^{(i)} \gamma^{(i)} + n^{(i)} \gamma^{(i)};$$

[36] but it is easy to see that the values of $S l^{(i)} m^{(i)} \gamma^{(i)}$ and of $S l^{(i)} n^{(i)} \lambda^{(i)}$ are nulls by the supposition of the negative errors as probable as the positive errors; we can therefore make, as above,

$$R^{(i)} = m^{(i)} \gamma^{(i)} + n^{(i)} \lambda^{(i)},$$

that which it was necessary to establish.

END.

Fourth Supplement.

§ 1. U being any function whatever of a variable t, if we develop it according to the powers of t; the coefficient of t^x, in this development, will be a function of x, that I will designate by y_x. U is that which I have named *generating function* of y_x. If we multiply U by a function T of t, similarly developed according to the ascending powers of t, the product UT, will be a new generating function of a function of x, derived from the function y_x according to a law which will depend on the function T. If T is equal to $\frac{1}{t} - 1$, it is easy to see that the derived will be $y_{x+1} - y_x$, or the finite difference of y_x. Let us designate generally, whatever be T, this derived by δy_x. If we multiply the product UT, by T, the derived of the product UT^2 will be a derived of δy_x similar to the derived of δy_x in y_x; we will be able therefore to designate by $\delta^2 y_x$ this second derived; whence it is clear generally that UT^n will be the generating function of $\delta^n y_x$. [1]

If we multiply U by another function Z of t, similarly developed according to the ascending powers of t; and if we designate by the characteristic $\triangle$, that which we have named δ relative to the function T; UZ^n will be the generating function of $\triangle^n y_x$.

We are able to imagine T as a function of Z. By developing this function into series with respect to the ascending powers of Z, we will have an expression of T of this form [2]

$$T = A^{(0)} + A^{(1)}Z + A^{(2)}Z^2 + \dots \text{etc.}$$

By multiplying this equation by U, and passing again from the generating functions, to the coefficients, we will have

$$\delta y_x = A^{(0)}y_x + A^{(1)}\triangle y_x + A^{(2)}\triangle^2 y_x + \dots \text{etc.}$$

We see thus that the same equation, which holds between T and Z, holds between their characteristics δ and $\triangle$, provided that in the development of this equation according to the powers of δ and of $\triangle$, we substitute, instead of any power δ^r, $\delta^r y_x$; instead of a power $\triangle^{r'}$, $\triangle^{r'} y_x$; instead of a product such as $\delta^r \triangle^{r'}$, $\delta^r \triangle^{r'} y_x$; and that we multiply by y_x the terms independent of δ and $\triangle$. Thus, by supposing T equal to $\frac{1}{t} - 1$, $Z = \frac{1}{t^i} - 1$, δy_x will be the finite difference of y_x, x varying by unity; $\triangle y_x$ will be the finite difference of y_x, x varying with i; we have next

$$Z = (1 + T)^i - 1,$$

and, consequently,

$$Z^n = [(1 + T)^i - 1]^n;$$

that which gives

$$\triangle^n = [(1 + \delta)^i - 1]^n,$$

provided that after the development, we place y_x after the powers of the characteristics. This equation will hold furthermore by making n negative; but then the differences are changed into integrals. The consideration of the generating functions show thus, in the most natural and most simple manner, the analogy of the powers and of the differences. We are able to consider this theory as the calculus of characteristics.

If we have $0 = \delta y_x$, we will have an equation in the finite differences. UT becomes then a polynomial which contains only some powers of t smaller than the highest of
[3] t in T. Let us designate by Q the polynomial in t, the most general of this nature; we will have

$$U = \frac{Q}{T}.$$

The coefficient of t^x in the development of U will be the integral y_x of the equation $0 = \delta y_x$; by this reason, I name U generating function of this equation.

If we imagine U, a function of two variables t and t'; the coefficient of the product $t^x t'^x$, in the development of U, will be a function of x and of x', that I designate by $y_{x,x'}$. T being a function developed in the same variables t and t', the product UT, will be the generating function of a derived from $y_{x,x'}$, that I will designate by $\delta y_{x,x'}$; and it is easy to conclude from it that UT^n will be the generating function of $\delta^n y_{x,x'}$.

If we have $0 = \delta y_{x,x'}$; we will have an equation in the partial finite differences. Let us represent this equation, by the following

$$\begin{aligned} 0 = {} & a y_{x,x'} + b y_{x,x'+1} + c y_{x,x'+2} + \dots \text{etc.}, \\ & + a' y_{x+1,x'} + b' y_{x+1,x'+1} + \dots \text{etc.}, \\ & + a'' y_{x+2,x'} + \dots \text{etc.}, \\ & + \dots \text{etc.}, \end{aligned}$$

it is easy to see that the generating function of the proposed equation will be

$$\frac{A + Bt' + Ct'^2 + \dots + Ht'^{n'-1} + A' + B't + C't^2 + \dots + H't^{n-1}}{\left\{ \begin{aligned} & a t^n t'^{n'} + b t^n t'^{n'-1} + c t^n t'^{n'-2} + \dots \text{etc.} \\ & + a' t^{n-1} t'^{n'} + b' t^{n-1} t'^{n'-1} + \dots \text{etc.} \\ & + a'' t^{n-2} t'^{n'} + \dots \text{etc.} \\ & + \dots \text{etc.}, \end{aligned} \right\}}$$

n and n' being the greatest increases of x and of x', in the proposed equation in partial differences. A, B, C,..., H are some arbitrary functions of t; A', B', C',..., H' are some arbitrary functions of t'. We will determine all these functions by means of the generating functions of

$$\begin{matrix} y_{0,x'}; & y_{1,x'}; & y_{2,x'}; & \dots; & y_{n-1,x'}; \\ y_{x,0}; & y_{x,1}; & y_{x,2}; & \dots; & y_{x,n'-1}. \end{matrix}$$

[4] One of the principal advantages of this manner to integrate the equations in partial differences, consists in this that the algebraic analysis furnishing diverse ways to develop the functions, we are able to choose the one which agrees best to the proposed question. The solution of the following problems, by the Count de Laplace,

my son, and the considerations that he has joined, will spread a new day on the calculus of generating functions.

§ 2. A player A draws from an urn containing some white and black balls, one ball, which he returns after the trial, with the probability p to bring forth a white ball, and the probability q to extract from it a black; a second player B draws next from another urn one ball, which he returns equally after the drawing, with the probabilities p' of a white ball and q' of a black. These two players continue thus to extract alternately each from their respective urn a ball, which they always take care to return. If one of the players brings forth a white ball, he counts a point; if on the contrary he draws a black ball, he counts nothing, and the turn of the player passes simply to the other. The players having settled, by the conditions of their game, the number of points that each must attain first in order to win the game, and having commenced to play, there is lacking yet to player A the number x points in order to win, and x' to player B; and the turn to play belongs to player A. We demand, in this position, what is the probability of each player to win the game.

Let $z_{x,x'}$ be the probability of second player B, and let us represent by $Y_{x,x'}$ his probability, if he were the first to play. Player A, by beginning, is able to bring forth a white ball, and the probability of B becomes $Y_{x-1,x'}$; or the first player draws a black, and then counts nothing, and the probability of the second is changed into $Y_{x,x'}$; but the probability of the first case is p, that of the second q; we will have therefore the equation

$$z_{x,x'} = pY_{x-1,x'} + qY_{x,x'},$$

by a similar reasoning, we will have further this one [5]

$$Y_{x,x'} = p'z_{x,x'-1} + q'z_{x,x'},$$

whence we deduce

$$Y_{x-1,x'} = p'z_{x-1,x'-1} + q'z_{x-1,x'},$$

and consequently(3)

$$z_{x,x'} = p(p'z_{x-1,x'-1} + q'z_{x-1,x'}) + q(p'z_{x,x'-1} + q'z_{x,x'})$$

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[3]We arrive again to this equation in partial differences by considering together the two successive drawings of A and B, as one trial, and by examining the different cases which are able to be presented after this trial played: now, they are in number of four; 1°. either the two players bring forth each one white ball, an event of which the probability is pp'; then the probability $z_{x,x'}$ will be changed into this one, $z_{x-1,x'-1}$; 2°. or the first player extracts a white ball, and the second a black; under this hypothesis, which has for probability pq', $z_{x,x'}$ will become $z_{x-1,x'}$; 3°. or on the contrary the first player withdraws a black ball, and the second a white; under this hypothesis, which has for probability $p'q$, $z_{x,x'}$ will become $z_{x,x'-1}$; 4°. or finally each player draws a black ball, an event of which the probability is qq', and then the probability $z_{x,x'}$ remains the same. We will have therefore, by the known principles of probabilities, the equation

$$z_{x,x'} = pp'z_{x-1,x'-1} + pq'z_{x-1,x'} + p'qz_{x,x'-1} + qq'z_{x,x'}.$$

We obtain the generating function of $z_{x,x'}$ in this equation in partial differences, by applying to this case the general rule which has just been exposed.

or

$$z_{x,x'} = \frac{pq'}{1-qq'} z_{x-1,x'} + \frac{p'q}{1-qq'} z_{x,x'-1} + \frac{pp'}{1-qq'} z_{x-1,x'-1},$$

and by making

$$\frac{pq'}{1-qq'} = m, \quad \frac{p'q}{1-qq'} = m', \quad \frac{pp'}{1-qq'} = n,$$

it will become

$$z_{x,x'} = mz_{x-1,x'} + m'z_{x,x'-1} + nz_{x-1,x'-1}.$$

[6] The generating function of $z_{x,x'}$, in this equation in partial differences, is

$$\frac{A + A'}{1 - mt - m't' - ntt'},$$

A being an arbitrary function of t, and A' another arbitrary function of t'; I observe first that by attributing to the function A' the term independent of t in the function A, the generating function above is able to be set under this form

$$\frac{A_1 t + A'_1}{1 - mt - m't' - ntt'},$$

A_1 and A'_1 being new arbitrary functions of t and of t', which the concern is to determine. Now, if we pay attention that $z_{0,x'}$ is null, whatever be x', the probability of player A is changed then to certitude, we see that the coefficient of t^0, in the development of the generating function, with respect to the powers of t, must be null, and we will have

$$\frac{A'_1}{1 - m't'} = 0 \qquad \text{or} \qquad A'_1 = 0.$$

Moreover, $z_{x,0}$ is null, when x is zero, and equal to unity, when x is either 1 or 2, or 3, etc., since then the probability of player B is changed into certitude; the generating function of $z_{x,0}$ is therefore $\frac{t}{1-t}$; it is the coefficient of t'^0 in the development of the generating function according to the powers of t'; we will have therefore

$$\frac{A_1 t}{1 - mt} = \frac{t}{1-t};$$

that which gives

$$A_1 t = \frac{t(1 - mt)}{1 - t};$$

consequently the generating function of $z_{x,x'}$ is

$$\frac{t(1 - mt)}{(1 - t)(1 - mt - m't' - ntt')}; \qquad (a)$$

[7] by setting it under this form

$$\frac{t}{1-t} \cdot \frac{1}{1 - \left(\frac{m' + nt}{1 - mt}\right) t'}$$

and developing it with respect to the powers of t', we have

$$\frac{t}{1-t}\left[1+\left(\frac{m'+nt}{1-mt}\right)t'+\left(\frac{m'+nt}{1-mt}\right)^2 t'^2+\left(\frac{m'+nt}{1-mt}\right)^3 t'^3+\ldots\text{etc.}\right].$$

The coefficient of $t'^{x'}$ in this series is

$$\frac{t}{1-t}\left(\frac{m'+nt}{1-mt}\right)^{x'},$$

and the one of t^x in the development of this last function, will be the expression of $z_{x,x'}$. Now, if we reduce first the expression $t\left(\frac{m'+nt}{1-mt}\right)^{x'}$ into a series ordered according to the powers of t, and if we multiply it next by the development of $\frac{1}{1-t}$, it is easy to see that the coefficient of t^x in this product, is that which the series becomes, by making $t=1$ in it and stopping ourselves at the power x of t, and we will find, for the value of this coefficient or of $z_{x,x'}$,

$$z_{x,x'}=m'^{x'}\left\{\begin{array}{l}1+\dfrac{x'}{1}\dfrac{n}{m'}+\dfrac{x'(x'-1)}{1.2}\dfrac{n^2}{m'^2}+\dfrac{x'(x'-1)(x'-2)}{1.2.3}\dfrac{n^3}{m'^3}+\ldots\text{etc.}+\dfrac{x'(x'-1)\ldots(x'-x+2)}{1.2.\ldots(x-1)}\dfrac{n^{x-1}}{m'^{x-1}}\\ +\dfrac{x'}{1}m\left[1+\dfrac{x'}{1}\dfrac{n}{m'}+\dfrac{x'(x'-1)}{1.2}\dfrac{n^2}{m'^2}+\ldots\text{etc.}+\dfrac{x'(x'-1)\ldots(x'-x+3)}{1.2.\ldots(x-2)}\dfrac{n^{x-2}}{m'^{x-2}}\right]\\ +\dfrac{x'(x'+1)}{1.2}m^2\left[1+\dfrac{x'}{1}\dfrac{n}{m'}+\ldots\text{etc.}+\dfrac{(x'-1)\ldots(x'-x+4)}{1.2.\ldots(x-3)}\dfrac{n^{x-3}}{m'^{x-3}}\right]\\ +\text{etc.}\\ +\dfrac{x'(x'+1)\ldots(x'+x-2)}{1.2.\ldots(x-1)}m^{x-1}\end{array}\right\}$$

By designating by $y_{x,x'}$ the probability of player A, we will be led, by the same reasonings, to a similar equation in the partial differences,

$$y_{x,x'}=my_{x-1,x'}+m'y_{x,x'-1}+ny_{x-1,x'-1},$$

which gives similarly for the variable $y_{x,x'}$ a generating function of the form [8]

$$\frac{A_1t+A_1'}{1-mt-m't'-ntt'}.$$

A_1 and A_1' being, as above, some arbitrary functions of t and of t', that we will determine by the same considerations. In fact, the generating function of $y_{0,x'}$ is $\frac{1}{1-t'}$; that of $y_{x,0}$ is unity: we will form therefore the equations

$$\frac{A_1'}{1-m't'}=\frac{1}{1-t'};$$

whence we deduce

$$A_1'=\frac{1-m't'}{1-t'},$$

and

$$\frac{A_1t+1}{1-mt}=1;$$

whence we conclude

$$A_1 t = -mt.$$

The generating function of $y_{x,x'}$ will be therefore

$$\frac{\frac{1-m't'}{1-t'} - mt}{1 - mt - m't' - ntt'} \dots \qquad (b)$$

which, developed according to the powers of t and of t', will give, by the coefficient of $t^x t'^{x'}$, the expression of $y_{x,x'}$, which will be of a form similar to that of $z_{x,x'}$, although a little more complicated.

By adding the two generating functions (a) and (b), their sum is reduced to that here

$$\frac{1}{(1-t)(1-t')},$$

in which the coefficient of $t^x t'^{x'}$ is unity; thus we have

$$y_{x,x'} + z_{x,x'} = 1;$$

[9] and effectively, the game must be necessarily won by one of the players, because both are certain to be able to extract each from their urn the determined numbers of white balls.

Now, let us suppose $p = 0$ and consequently $q = 1$, we have

$$m = 0, \quad m' = 1 \quad \text{and} \quad n = 0;$$

then, the expression of $z_{x,x'}$ becomes unity; that which is evident, since player B, having no more chances to lose, must always end by winning.

If, to the contrary, we suppose $p = 1$ and $q = 0$, that is, if the first player A counts a point before each drawing of player B, then

$$m = q', \quad m' = 0 \quad \text{and} \quad n = p';$$

x' being greater than x, or equal, the expression $z_{x,x'}$ is reduced to zero; and, in fact, it is evidently impossible that, in this case, player B is able to win the game; but, when x is greater than x', the value of $z_{x,x'}$ takes this form

$$z_{x,x'} = p'^{x'}\left[1 + \frac{x'}{1}q' + \frac{x'(x'+1)}{1.2}q'^2 + \dots \text{etc.} + \frac{x'(x'+1)\dots(x-2)}{1.2\dots(x-x'-1)}q^{x-x'-1}\right].$$

Under this assumption, player B is able to win only so long as he will bring forth x' white balls before $x - x'$ black balls; otherwise, he is anticipated by player A, who counts a point at each trial: this expression of $z_{x,x'}$ is therefore the probability that player B will have drawn x' white balls before having extracted from it $x - x'$ blacks, and consequently the probability to win, if he made the wager with player A, who would count then a point with the exit of each black ball, while he counts one of them at the exit of a white, to attain x' points before his adversary has $x - x'$ of them; that which is the *problem of points.* ([4])

If we examine with attention the form of the general expression which gives $z_{x,x'}$, [10]
we will recognize that this problem is able yet to be resolved, and even with simplicity, by means of the theory of combinations: in fact, let a be the number of white balls contained in the urn of player A, and b that of the blacks; a' the number of white balls of player B, and b' that of the blacks; by considering, as we have already done, the set of two successive drawings of A and B as one trial,

aa' will be the number of combinations in which the players each bring forth one white ball;

ab' the one of the combinations which will give one white ball to A and one black to B;

$a'b$ the one of the combinations which will give, to the contrary, one black ball to A and one white to B;

bb' the one of the combinations in which both players draw a black ball;

And the sum $aa' + ab' + a'b + bb'$ will form the collection of all the combinations [11]
which are able to take place in a trial. The combinations where the players bring forth each one black ball, bring no change to their position, we are able to set it aside, and then we occupy ourselves only with the trials where there will be brought forth at least one white ball. It is clear that in $x + x'$ similar trials, one of the players has necessarily won, and the game must be decided: now the number of all the equally possible combinations, according to which these $x + x'$ trials are able to be presented, will be

$$(aa' + ab' + a'b)^{x+x'};$$

[4]The generating function of $z_{x,x'}$ is reduced in this case to

$$\frac{t(1 - q't)}{(1 - t)(1 - q't - p'tt')},$$

and the equation in the corresponding partial differences will be

$$z_{x,x'} = q'z_{x-1,x'} + p'z_{x-1,x'-1},$$

in which $z_{x,x'}$ is a function of x and of x' which we will designate by $\phi(x, x')$; if we make $x - x' = s$, we will have

$$\phi(x, x') = \phi(s + x', x'),$$

and, if we represent by $z_{s,x'}$ this last function, there results from it

$$z_{x,x'} = z_{s,x'}, \quad z_{x-1,x'} = z_{s-1,x'}, \quad z_{x-1,x'-1} = z_{s,x'-1};$$

and the equation in the partial differences is changed into that here

$$z_{s,x'} = q'z_{s-1,x'} + p'z_{s,x'-1},$$

an equation to which the problem of points would lead directly under the conditions enunciated above. By paying attention that in consequence of this transformation, $z_{s,0} = 1$ and $z_{0,x'} = 0$, and that $z_{0,0}$ is not able to take place, it is easy to see that the generating function of $z_{s,x'}$ will be

$$\frac{t(1 - q't)}{(1 - t)(1 - q't - p't')},$$

in the development of which the coefficient of $t^s t'^{x'}$ will be the expression of $z_{s,x'}$.

the question is reduced therefore to choose in all these combinations those which make player B win, that is, those in which this player will have x' white balls, before player A has brought forth x of them. In order to fix the ideas, let us suppose x' greater than x: we are able to form the following hypotheses; either player B will have won at the x'^{th} trial, that is, by drawing without interruption a white ball at each trial, and then the number of the preceding combinations which are corresponding to this case, is evidently

$$a'^{x'}\left[b^{x'}+\frac{x'}{1}ab^{x'-1}+\frac{x'(x'-1)}{1.2}a^2b^{x'-2}+\dots\text{etc.}\right.\\\left.+\frac{x'(x'-1)\dots(x'-x+2)}{1.2\dots(x-1)}a^{x-1}b^{x'-x+1}\right](aa'+ab'+a'b)^x;$$

and by dividing it by $(aa'+ab'+a'b)^{x+x'}$, the total number of combinations, we will have, for the probability of this hypothesis,

$$\frac{a'^{x'}b^{x'}}{(aa'+ab'+a'b)^{x'}}\left[1+\frac{x'}{1}\frac{a}{b}+\frac{x'(x'-1)}{1.2}\frac{a^2}{b^2}+\dots\text{etc.}+\frac{x'(x'-1)\dots(x'-x+2)}{1.2\dots(x-1)}\frac{a^{x-1}}{b^{x-1}}\right];$$

or player B will have won at the $(x'+1)^{\text{st}}$ trial, that is, by having drawn only a single black ball, for example, in commencing, and then the number of combinations favorable to this event is

$$b'a'^{x'}\left[b^{x'}+\frac{x'}{1}ab^{x'-1}+\frac{x'(x'-1)}{1.2}a^2b^{x'-2}+\dots\right.\\\left.+\frac{x'(x'-1)\dots(x'-x+3)}{1.2\dots(x-2)}a^{x-2}b^{x'-x+2}\right](aa'+ab'+a'b)^{x-1};$$

but this number is the same, if the black ball is brought forth in the first trial or in the second, etc., or in the x'^{th} trial; it is necessary therefore to multiply it by x' in order to have all the combinations relative to this hypothesis, of which the probability
[12] is by this means,

$$\frac{x'}{1}\frac{ab'a'^{x'}b^{x'}}{(aa'+ab'+a'b)^{x'+1}}\left[1+\frac{x'}{1}\frac{a}{b}+\frac{x'(x'-1)}{1.2}\frac{a^2}{b^2}+\dots\text{etc.}+\frac{x'(x'-1)\dots(x'-x+3)}{1.2\dots(x-2)}\frac{a^{x-2}}{b^{x-2}}\right];$$

or player B will have won at the $(x'+2)^{\text{nd}}$ trial, and we will see in the same manner that the probability of this hypothesis will be

$$\frac{x'(x'+1)}{1.2}\frac{a^2b'^2a'^{x'}b^{x'}}{(aa'+ab'+a'b)^{x'+2}}\left[1+\frac{x'}{1}\frac{a}{b}+\dots\text{etc.}+\frac{x'(x'-1)\dots(x'-x+4)}{1.2\dots(x-3)}\frac{a^{x-3}}{b^{x-3}}\right];$$

By continuing thus, we will have the probabilities of all the successive hypotheses which are able to be presented under the supposition of the gain of the game by player B, until that where he would win only at the $(x'+x-1)^{\text{st}}$ trial, an event of which the probability will be

$$\frac{x'(x'+1)\dots(x'+x-2)}{1.2\dots(x-1)}\frac{a^{x-1}b'^{x-1}a'^{x'}b^{x'}}{(aa'+ab'+a'b)^{x'+x-1}};$$

and effectively, in this case there are not able to be trials where the players bring forth at the same time a white ball.

The sum of all these probabilities will give evidently that of player B in order to win the game.

If we pay attention that

$$\frac{ab'}{aa'+ab'+a'b}=m, \quad \frac{a'b}{aa'+ab'+a'b}=m', \quad \text{and} \quad \frac{a}{b}=\frac{n}{m'},$$

we recover the expression of $z_{x,x'}$.

Let us imagine presently that there are in the urns some white balls bearing the n° 1, and other balls of the same color, which bear the n° 2; each ball diminishing by its numeral, on its withdrawal, the number of points which are lacking yet to the player to which it is favorable. The problem is no longer susceptible to be resolved generally by means of combinations, instead the calculus of generating functions will continue to furnish a general expression of which the development will contain the complete solution of the question and will be able, in certain cases, to be executed by some laws easy to know, as we will have occasion to see.

Let p be the probability player A to extract a ball labeled 1, p_1 that to extract a ball labeled 2, and q that to bring forth a black ball; p', p'_1 and q' the corresponding [13] probabilities for player B; and let always $z_{x,x'}$ be the probability of this last player in order to win the game. By following the same march as above, we will be led to the equation in partial differences

$$\begin{aligned} z_{x,x'} = & m z_{x-1,x'} + m_1 z_{x-2,x'} + m' z_{x,x'-1} + m'_1 z_{x,x'-2} \\ & + n z_{x-1,x'-1} + n_1 z_{x-2,x'-1} + n' z_{x-1,x'-2} + n'_1 z_{x-2,x'-2} \end{aligned}$$

in which we make

$$\begin{aligned} &\frac{pq'}{1-qq'}=m, &&\frac{p_1q'}{1-qq'}=m_1, &&\frac{p'q}{1-qq'}=m', &&\frac{p'_1q}{1-qq'}=m'_1,\\ &\frac{pp'}{1-qq'}=n, &&\frac{p_1p'}{1-qq'}=n_1, &&\frac{pp'_1}{1-qq'}=n', &&\frac{p_1p'_1}{1-qq'}=n'_1; \end{aligned}$$

the generating function of the variable $z_{x,x'}$, given by this equation, will be

$$\frac{A+Bt'+A'+B't}{1-mt-m_1t^2-m't'-m'_1t'^2-ntt'-n_1t^2t'-n'tt'^2-n'_1t^2t'^2}\cdots \tag{c}$$

A and B being some arbitrary functions of t, A' and B' some arbitrary functions of t', which will be determined by means of the generating functions of

$$z_{0,x'}, \quad z_{x,0}, \quad z_{1,x'}, \quad z_{x,1},$$

which are themselves given by the conditions of the game.

We find, as previously, that the generating function of $z_{0,z'}$ is zero, and that of $z_{x,0}$, $\frac{t}{1-t}$.

From the general equation, we deduce the equation in finite differences

$$z_{1,x'} = m' z_{1,x'-1} + m'_1 z_{1,x'-2},$$

which holds for all the values of x' from $x' = 2$ inclusively, and which gives consequently, for the generating function of $z_{1,x'}$

$$\frac{a + bt'}{1 - m't' - m'_1 t'^2},$$

[14] a and b being some constants that we determine by means of the values of $z_{1,0}$ and $z_{1,1}$; and as $z_{1,0}$ is equal to unity, $z_{1,1}$ is equal to $m' + m'_1$, and is at the same time the coefficient of t' in the development of the generating function, there results from it

$$a = 1 \quad \text{and} \quad b = m'_1;$$

the generating function of $z_{1,x'}$ is therefore

$$\frac{1 + m'_1 t'}{1 - m't' - m'_1 t'^2}.$$

Now, if in the preceding equation we put $1 - y_{x,x'}$ in the place of $z_{x,x'}$, $y_{x,x'}$ being always the probability of the first player A, it is reformed in the same manner with respect to this last variable, and we would deduce from it the equation in the finite differences

$$y_{x,1} = my_{x-1,1} + m_1 y_{x-2,1};$$

but we will see at the same time that it begins to hold only when x surpasses 2; because, x being 2, we will have

$$y_{2,1} = my_{1,1} + m_1 y_{0,1} + n_1 + n'_1.$$

It is necessary therefore to employ it only by departing from $x = 3$, and then the generating function of $y_{x,1}$ is of the form

$$\frac{a + bt + ct^2}{1 - mt - m_1 t^2},$$

a, b and c being some constants that we will determine, as previously, by means of the values of $y_{1,0}$, $y_{1,1}$ and $y_{2,1}$; now $y_{1,0}$ is unity; $y_{1,1}$ is equal to $1 - m' - m'_1$, and is the coefficient of t in the development of the generating function; $y_{2,1}$ has for value, as we have just seen,

$$m(1 - m' - m'_1) + m_1 + n_1 + n'_1;$$

this is the coefficient of t^2 in the development of the function. We will conclude from it

$$a = 1, \quad b = 1 - m - m' - m'_1, \quad \text{and} \quad c = n_1 + n'_1,$$

[15] and the generating function of $y_{x,1}$ will be therefore

$$\frac{1 + (1 - m - m' - m'_1)\,t + (n_1 + n'_1)\,t^2}{1 - mt - m_1 t^2};$$

consequently that of $z_{x,1}$ is

$$\frac{1}{1-t}-\frac{1+(1-m-m'-m'_1)\,t+(n_1+n'_1)\,t^2}{1-mt-m_1t^2}$$
$$=\frac{(m'+m'_1)\,t+(n+n')\,t^2+(n_1+n'_1)\,t^3}{(1-t)(1-mt-m_1t^2)}.$$

Let us resume actually the generating function (c); we are able always to restore it to this form

$$\frac{A_1t+B_1t^2t'+A'_1+B'_1tt'}{1-mt-m_1t^2-m't'-m'_1t'^2-ntt'-n_1t^2t'-n'tt'^2-n'_1t^2t'^2},$$

A_1 and B_1 being the arbitrary functions of t, A'_1 and B'_1 the arbitrary functions of t'; which we determine easily, by equating first the coefficient of t^0 in the development of this function to the generating function of $z_{0,x'}$ or zero; next the one of t'^0 to the generating function of $z_{x,0}$ or $\frac{t}{1-t}$, since the one of t to the generating function of $z_{1,x'}$, and finally the one of t' to the generating function of $z_{x,1}$, that which will give successively

$$A'_1=0,\quad A_1=\frac{1-mt-m_1t^2}{1-t},\quad B'_1=m'_1,\quad B_1=\frac{m'_1+n'+n'_1t}{1-t},$$

and, consequently, for the generating function of $z_{x,x'}$,

$$\frac{(1-mt-m_1t^2)t+m'_1tt'+n't^2t'+n'_1t^3t'}{(1-t)(1-mt-m_1t^2-m't'-m'_1t'^2-ntt'-n_1t^2t'-n'tt'^2-n'_1t^2t'^2)}\cdots \qquad (d)$$

If we suppose p and p' null, then we have

$$m=0,\quad m'=0,\quad n=0,\quad n_1=0,\quad \text{and}\quad n'=0,$$

and the function (d) takes this form

$$\frac{tt'(m'_1+n'_1t^2)}{(1-t)(1-m_1t^2)\left[1-\left(\frac{m'_1+n'_1t^2}{1-m_1t^2}\right)t'^2\right]}+\frac{t}{(1-t)\left[1-\left(\frac{m'_1+n'_1t^2}{1-m_1t^2}\right)t'^2\right]},$$

under which it is susceptible of the same developments as the function (a). It is to be noticed that we will recover the same coefficient for [16]

$$t^{2r}t'^{2r'},\quad t^{2r-1}t'^{2r'},\quad t^{2r}t'^{2r'-1},\quad t^{2r-1}t'^{2r'-1};$$

that which is seen *a priori*, by paying attention that the players always count two points with each white ball that they withdraw.

Let us suppose that player A has only some balls labeled 1 and 2, and that the other player has only some white balls marked 1, or which count to him only one point on exiting; then

$$p'_1=0,\quad \text{and hence}\quad m'_1=0,\quad n'=0,\quad n'_1=0;$$

the function (d) becomes

$$\frac{t(1-mt-m_1t^2)}{(1-t)(1-mt-m_1t^2-m't'-ntt'-n_1t^2t')}=\frac{t}{1-t}\cdot\frac{1}{1-\left[\frac{m'+(n+n_1t)t}{1-(m+m_1t)t}\right]t'};$$

by developing it according to the powers of t', the coefficient of $t'^{x'}$ will be

$$\frac{t\left[m'+(n+n_1t)t\right]^{x'}}{(1-t)\left[1-(m+m_1t)t\right]^{x'}},$$

an expression that the concern now is to develop with respect to the powers of t, in order to have the coefficient of t^x; now this coefficient will be the sum of all the coefficients of the powers of t inferior or equal to t^{x-1}, in the development of the expression

$$\frac{\left[m'+(n+n_1t)t\right]^{x'}}{\left[1-(m+m_1t)t\right]^{x'}},$$

which, by omitting the terms where the powers of t outside the binomials are superior to t^{x-1}, is able to be put under this form

$$m'^{x'}\left\{\begin{aligned}&1+\frac{x'}{1}\left(\frac{n+n_1t}{m'}\right)t+\frac{x'(x'-1)}{1.2}\left(\frac{n+n_1t}{m'}\right)^2t^2+\dots\text{etc.}+\frac{x'(x'-1)\dots(x'-x+2)}{1.2\dots(x-1)}\left(\frac{n+n_1t}{m'}\right)^{x-1}t^{x-1}\\&+\frac{x'}{1}(m+m_1t)t\left[1+\frac{x'}{1}\left(\frac{n+n_1t}{m'}\right)t+\dots\text{etc.}+\frac{x'(x'-1)\dots(x'-x+3)}{1.2\dots(x-2)}\left(\frac{n+n_1t}{m'}\right)^{x-2}t^{x-2}\right]\\&+\frac{x'(x'+1)}{1.2}(m+m_1t)^2t^2\left[1+\dots\text{etc.}+\frac{x'(x'-1)\dots(x'-x+4)}{1.2\dots(x-3)}\left(\frac{n+n_1t}{m'}\right)^{x-3}t^{x-3}\right]\\&+\dots\dots\text{etc.}\\&+\frac{x'(x'+1)\dots(x'-x+2)}{1.2\dots(x-1)}(m+m_1t)^{x-1}t^{x-1}.\end{aligned}\right\}$$

[17] If we reject further from this series all the powers of t superior to t^{x-1},which will result from the developments of the binomials, and if in that which remains, we make $t=1$, we will have the expression of $z_{x,x'}$.

Let us examine further the case where player A would be certain to extract at each trial a ball which would count to that player one point, that is where we would have

$$p=1,\quad p_1=0,\quad q=0,$$

and consequently

$$\begin{aligned}m&=q',&\quad m_1&=0,&\quad m'&=0,&\quad m_1'&=0,\\n&=p',&\quad n_1&=0,&\quad n'&=p_1',&\quad n_1'&=0.\end{aligned}$$

The generating function of $z_{x,x'}$, or the function (d) would be reduced then to

$$\frac{t(1-q't)+p_1't^2t'}{(1-t)(1-q't-p'tt'-p_1'tt'^2)},$$

and that of $y_{x,x'}$ would be, hence,

$$\frac{1}{(1-t)(1-t')} - \frac{t(1-q't)+p'_1t^2t'}{(1-t)(1-q't-p'tt'-p'_1tt'^2)},$$
$$= \frac{1}{1-t'} + \frac{tt'}{(1-t)(1-q't-p'tt'-p'_1tt'^2)}.$$

In this last expression, the first term represents the generating function of $y_{0,x'}$, which is equal to unity, whatever be x'; and the second will give, by developing it, with respect to the powers of t and of t', all the other values of $y_{x,x'}$; now the coefficient of t^x will be

$$\frac{t'[q'+(p'+p'_1t')t']^{x-1}}{1-t'};$$

whence it results that, if we reject from the development of the series

$$q'^{x-1}\left[t' + \frac{(x-1)}{1}\left(\frac{p'+p'_1t'}{q'}\right)t'^2 + \frac{(x-1)(x-2)}{1.2}\left(\frac{p'+p'_1t'}{q'}\right)^2 t'^3 + \ldots \text{etc.}\right]$$

all the powers of t' superior to $t'^{x'}$, and if we made in that which remains $t'=1$, we will have, by supposing x' even and equal to $2r+2$, the coefficient of $t^xt'^{x'}$, or [18]

$$y_{x,x'} = q'^{x-1}\left\{\begin{aligned}
&1 + \frac{(x-1)}{1}\left(\frac{p'+p'_1}{q'}\right) + \frac{(x-1)(x-2)}{1.2}\left(\frac{p'+p'_1}{q'}\right)^2 + \cdots + \frac{(x-1)(x-2)\ldots(x-r)}{1.2\ldots r}\left(\frac{p'+p'_1}{q'}\right)^r \\
&+ \frac{(x-1)(x-2)\ldots(x-r-1)}{1.2\ldots(r+1)}\frac{p'^{r+1}}{q'^{r+1}}\left[1 + \frac{(r+1)}{1}\frac{p'_1}{p'} + \frac{(r+1)r}{1.2}\frac{p'^2_1}{p'^2} + \cdots + \frac{(r+1)r\ldots 2}{1.2\ldots r}\frac{p'^r_1}{p'^r}\right] \\
&+ \frac{(x-1)(x-2)\ldots(x-r-2)}{1.2\ldots(r+2)}\frac{p'^{r+2}}{q'^{r+2}}\left[1 + \frac{(r+2)}{1}\frac{p'_1}{p'} + \cdots + \frac{(r+2)(r+1)r\ldots 4}{1.2\ldots(r-1)}\frac{p'^{r-1}_1}{p'^{r-1}}\right] \\
&+ \ \ldots\ldots \text{etc.} \\
&+ \frac{(x-1)(x-2)\ldots(x-2r-1)}{1.2\ldots(2r+1)}\frac{p'^{2r+1}}{q'^{2r+1}}
\end{aligned}\right\}$$

and, in the case of x' odd or equal to $2r+1$,

$$y_{x,x'} = q'^{x-1}\left\{\begin{aligned}
&1 + \frac{(x-1)}{1}\left(\frac{p'+p'_1}{q'}\right) + \frac{(x-1)(x-2)}{1.2}\left(\frac{p'+p'_1}{q'}\right)^2 + \cdots + \frac{(x-1)(x-2)\ldots(x-r)}{1.2\ldots r}\left(\frac{p'+p'_1}{q'}\right)^r \\
&+ \frac{(x-1)(x-2)\ldots(x-r-1)}{1.2\ldots(r+1)}\frac{p'^{r+1}}{q'^{r+1}}\left[1 + \frac{(r+1)}{1}\frac{p'_1}{p'} + \frac{(r+1)r}{1.2}\frac{p'^2_1}{p'^2} + \cdots + \frac{(r+1)r\ldots 3}{1.2\ldots(r-1)}\frac{p'^{r-1}_1}{p'^{r-1}}\right] \\
&+ \frac{(x-1)(x-2)\ldots(x-r-2)}{1.2\ldots(r+2)}\frac{p'^{r+2}}{q'^{r+2}}\left[1 + \frac{(r+2)}{1}\frac{p'_1}{p'} + \cdots + \frac{(r+2)(r+1)\ldots 5}{1.2\ldots(r-2)}\frac{p'^{r-2}_1}{p'^{r-2}}\right] \\
&+ \ \ldots\ldots \text{etc.} \\
&+ \frac{(x-1)(x-2)\ldots(x-2r)}{1.2\ldots 2r}\frac{p'^{2r}}{q'^{2r}}
\end{aligned}\right\}$$

It is clear that player B is able to expect to win only as long as x is greater than $r+1$, or that x' equal $2r+2$ or $2r+1$; and effectively, beyond this supposition, the preceding values of $y_{x,x'}$ become all equal to unity.

We will take notice also that player A has necessarily won the game when player B will have drawn $x-r-1$ black balls, before having attained x' points; but this last player is able yet to have lost before having drawn the totality of this number of black balls, that which makes that this question is not at all susceptible to return into that which is treated in the analytic Theory, following the *problem of points*, as previously a similar supposition has led us to this last problem.

§ 3. The problem of points having been the object of the researches of two great
[19] geometers of the XVII$^{\text{th}}$ century ([5]), and to some extent the first of this kind, subject to some analytic methods, one will be perhaps curious to see how this same problem is deduced again, as corollary, from another question of probability, of which the solution will offer besides a new application of the method of generating functions.

We draw successively from an urn, which contains a determined quantity of white and black balls, a ball that we do not return after the trial, and we demand, after a certain number of known drawings, what is the probability to complete the drawing of such given number of white balls before that of such other number, given equally, of black balls?

Let a and a' be the numbers of white and black balls, contained originally in the urn; n the number of white balls that we are proposed to attain, before having extracted another number n' of black balls; and let us suppose that after having drawn successively from the urn a ball without returning it, we have brought forth $n-x$ white balls and $n'-x'$ black balls, x and x' being then the number of white and black balls that there remain to withdraw in order to decide the question. Let us represent by $y_{x,x'}$ the probability to bring forth in the following drawings x white balls before x' black balls, or to attain the totality of n white balls before having extracted n' blacks; we will have, according to the known rules of probabilities, the equation

$$y_{x,x'} = \frac{a-n+x}{a+a'-n-n'+x+x'} y_{x-1,x'} + \frac{a'-n'+x'}{a+a'-n-n'+x+x'} y_{x,x'-1}.$$

Let us make

$$a-n+x=s, \quad a'-n'+x'=s' \quad \text{and} \quad y_{x,x'}=u_{s,s'};$$

the preceding equation becomes

$$u_{s,s'} = \frac{s}{s+s'} u_{s-1,s'} + \frac{s'}{s+s'} u_{s,s'-1},$$

and, by supposing

$$u = \frac{1.2.3\ldots s.1.2.3\ldots s'}{1.2.3\ldots(s+s')} z_{s,s'},$$

[20] it is restored to this form

$$z_{s,s'} = z_{s-1,s'} + z_{s,s'-1},$$

[5] Pascal and Fermat.

an equation in the partial differences with constant coefficients, which must hold for all the entire and positive values of s and of s', by departing from $s = a - n$ and from $s' = a' - n'$, and gives consequently for the generating function of $z_{s,s'}$

$$t^{a-n}t'^{a'-n'}\frac{A+A'}{1-t-t'},$$

A being an arbitrary function of t, and A' an arbitrary function of t'. We are able always to transform this expression into this one

$$t^{a-n}t'^{a'-n'}\frac{A_1+A_1't'}{1-t-t'},$$

in which A_1 and A_1' are new arbitrary functions of t and of t'. In order to determine them, we will observe that, $y_{0,0}$ not being able to take place and $y_{x,0}$ being equal to zero, whatever be the entire and positive values of x, we will have

$$0 = u_{s,a'-n'} = \frac{1.2.3\ldots s.1.2.3\ldots(a'-n')}{1.2.3\ldots(a'-n'+s)}z_{s,a'-n'};$$

consequently the generating function of $z_{s,a'-n'}$ will be null, that which gives

$$t^{a-n}t'^{a'-n'}\frac{A_1}{1-t} = 0, \quad \text{and hence} \quad A_1 = 0.$$

Moreover, $y_{0,x'}$ being equal to unity for all the values of x' from $x' = 1$, we will have similarly

$$1 = u_{a-n,s'} = \frac{1.2.3\ldots(a-n).1.2.3\ldots s'}{1.2.3\ldots(a-n+s')}z_{a-n,s'};$$

whence we deduce, for the value of $z_{a-n,s'}$ or the coefficient of $t^{a-n}t'^{s'}$ in the development of its generating function,

$$z_{a-n,s'} = \frac{(a-n+1)(a-n+2)\ldots(a-n+s')}{1.2.3\ldots s'},$$

that which gives [21]

$$\begin{aligned} t^{a-n}t'^{a'-n'}\frac{A_1't'}{1-t'} =& t^{a-n}t'^{a'-n'}\frac{(a-n+1)\ldots(a+a'-n-n'+1)}{1.2.3\ldots(a'-n'+1)} \\ &\times\left[\begin{aligned}&t' + \frac{(a+a'-n-n'+2)t'^2}{a'-n'+2} + \ldots\text{etc.} \\ &+\frac{(a+a'-n-n'+2)\ldots(a+a'-n-n'+x')t'^{x'}}{(a'-n'+2)\ldots(a'-n'+x')} + \ldots\text{etc.}\end{aligned}\right]\end{aligned}$$

The second member of this equation multiplied by $\frac{1}{1-\frac{t}{1-t'}}$ will be therefore the generating function of $z_{s,s'}$; by developing it with respect to the powers of t, and next with respect to those of t', it is easy to see that the coefficient of t^s or of t^{a-n+x} is

$$t'^{a'-n'}\frac{(a-n+1)\dots(a+a'-n-n'+1)}{1.2.3\dots(a'-n'+1)}$$
$$\times\left[t'+\frac{(a+a'-n-n'+2)}{a'-n'+2}t'^2+\dots\text{etc.}\right]$$
$$\times\left[1+\frac{x}{1}t'+\frac{x(x+1)}{1.2}t'^2+\dots\text{etc.}+\frac{x(x+1)\dots(x+x'-2)}{1.2\dots(x'-1)}t'^{x'-1}+\dots\text{etc.}\right],$$

and the one of $t'^{s'}$, or of $t'^{a'-n'+x'}$ in this last expression, or $z_{s,s'}$ is equal to

$$\frac{(a-n+1)\dots(a+a'-n-n'+1)}{1.2.3\dots(a'-n'+1)}$$
$$\times\left[\begin{array}{l}\frac{x(x+1)\dots(x+x'-2)}{1.2\dots(x'-1)}+\frac{a+a'-n-n'+2}{a'-n'+2}\times\frac{x(x+1)\dots(x+x'-3)}{1.2\dots(x'-2)}+\dots\text{etc.}\\ \qquad +\frac{(a+a'-n-n'+2)\dots(a+a'-n-n'+x')}{(a'-n'+2)\dots(a'-n'+x')}\end{array}\right].$$

Now, by multiplying this value of $z_{s,s'}$ by

$$\frac{1.2.3\dots(a'-n'+x')}{(a'-n'+x+1)\dots(a+a'-n-n'+x+x')},$$

we will have, after all the reductions, for the expression of $y_{x,x'}$,

$$y_{x,x'}=\frac{(a-n+x)\dots(a-n+1)}{(a+a'-n-n'+x+x')\dots(a+a'-n-n'+x'+1)}$$
$$\times\left[\begin{array}{l}1+\frac{x}{1}\frac{a'-n'+x'}{a+a'-n-n'+x'}+\frac{x(x+1)}{1.2}\frac{(a'-n'+x')(a'-n'+x'-1)}{(a+a'-n-n'+x')(a+a'-n-n'+x'-1)}+\dots\text{etc.}\\ +\frac{x(x+1)\dots(x+x'-2)}{1.2\dots(x'-1)}\frac{(a'-n'+x')\dots(a'-n'+2)}{(a+a'-n-n'+x')\dots(a+a'-n-n+2)}\end{array}\right]$$

[22] Let us imagine actually $a-n$ and $a'-n'$ in the ratio of p to q, so that we have $a-n=pk$ and $a'-n'=qk$, and let us imagine that k becomes a very great number or infinity; it is clear that the probability of the exit of a white ball or of a black ball in the successive drawings will become constant, and will be $\frac{p}{p+q}$ for a white ball and $\frac{q}{p+q}$ for a black, and the probability $y_{x,x'}$ will be reduced to this expression

$$y_{x,x'}=\left(\frac{p}{p+q}\right)^x\left[1+\frac{x}{1}\frac{q}{p+q}+\frac{x(x+1)}{1.2}\left(\frac{q}{p+q}\right)^2+\dots\text{etc.}\right.$$
$$\left.+\frac{x(x+1)\dots(x+x'-2)}{1.2\dots(x'-1)}\left(\frac{q}{p+q}\right)^{x'-1}\right];$$

such is the formula to which the *problem of points* leads, and effectively we return to the conditions of this problem, by the supposition of k infinite.

If we suppose n equal to a and n' equal to a', $y_{x,x'}$ will express then the probability of the exit of all the white balls remaining in the urn, before all the blacks had been depleted, and its expression will be changed into that here

$$\frac{1.2.3\ldots x}{(x+x')\ldots(x'+1)}\left[1+\frac{x}{1}+\frac{x(x+1)}{1.2}+\ldots\text{etc.}+\frac{x(x+1)\ldots(x+x'-2)}{1.2\ldots(x'-1)}\right],$$

which is reduced itself to

$$\frac{x'}{x+x'}.$$

The probability of extracting from the urn the totality of the white balls, before that of the blacks, is therefore to the contrary probability, in inverse ratio of the number of white balls to the one of the blacks.

We arrive to this last result, in an extremely simple manner, by means of combinations; in fact, the probability of the exit of all the balls from the urn, in any order, by color, will be

$$\frac{x(x-1)\ldots 2.1\, x'(x'-1)\ldots 2.1}{(x+x')(x+x'-1)\ldots 3.2.1}=\frac{1.2.3\ldots x'}{(x+1)\ldots(x+x')}.$$

But, in order that the white balls exit in totality first, it is necessary necessarily that a ball of the color black exit last: by combining $x'-1$ with $x'-1$ the $x+x'+1$ ranks [23] of exit which are found before the last, we will form as many different rankings for the balls of the color black, and as many orders of exit by color, which will comprehend all those where one black ball exits in last place; now the number of these combinations is

$$\frac{(x+x'-1)(x+x'-2)\ldots(x+1)}{1.2\ldots(x'-1)},$$

and by multiplying it by the probability common to each order of exit by color, we will have the sought probability equal to

$$\frac{1.2.3\ldots x'}{(x+1)\ldots(x+x')}\frac{(x+1)\ldots(x+x'-1)}{1.2.3\ldots(x'-1)}=\frac{x'}{x+x'}.$$

Remarks on generating functions.

§ 4. Let u be a generating function in one or many variables; each equation between this function and its variables, linear with respect to u, rational with respect to the variables, will subsist still if we pass from the generating functions to the coefficients, among these same coefficients, and will give place to an equation in the partial differences; but if, in this equation in partial differences, we pass again from the coefficients to the generating functions, we will no longer arrive to an equation rigorously exact, at least if we restore at the same time the functions of the variables which have been able to vanish in the first passage. Thus, in one of the questions that we have treated above, the equation in the partial differences

$$z_{x,x'}=mz_{x-1,x'}+m'z_{x,x'-1}+nz_{x-1,x'-1}$$

would give, by going up again simply from the coefficients to the generating functions, this one

$$u = mut + m'ut' + nutt',$$

[24] which is not at all exact; because it is easy to see that, according to the conditions of the problem, it would be necessary to add to the second member the generating function of $z_{x,0}$, less this same function multiplied by m. This function of t, which it is necessary to restore in the second member of the equation in order to complete it, is precisely the arbitrary function that we have had to determine in the solution of this question. In general, the functions to add in order to have still one equation in the passage from the coefficients to the generating functions are the same as the arbitrary functions which form the numerator of the generating function integral before it was developed.

For lack of having regard to these functions, we are able to fall into some grave errors, by serving ourselves in this manner in order to integrate the equations in the partial differences. For this same reason, the march followed in the solution of problems §§ 8 and 10 of Book II of the *Théorie analytique des Probabilités* is by no means rigorous, and seems to implicate contradiction in this that it established a liaison among the variables which are and must be always independent. Without entering into the particular considerations which have been able to make it succeed here, and which it is easy to know, we will show that the method of integration exposed at the beginning of this *Supplèment* is applied equally to these questions, and resolves them with no less simplicity.

In the problem of § 8, we have proposed to determine the lot of a number n of players A, B, C, etc. of whom p, q, r, ... represent the respective probabilities, that is, their probabilities to win a coup, when in order to win the game, there are lacking x coups to player A, x' coups to player B, x'' coups to player C, etc. By naming $y_{x,x',x'',\text{etc.}}$ the probability of player A winning the game, we have the equation in partial differences

$$y_{x,x',x'',\text{etc.}} = py_{x-1,x',x'',\text{etc.}} + qy_{x,x'-1,x'',\text{etc.}} + ry_{x,x',x''-1,\text{etc.}} + \text{etc.}$$

which gives for $y_{x,x',x'',\text{etc.}}$ this generating function

$$\frac{P + Q + R + \text{etc.}}{1 - pt - qt' - rt'' - \text{etc.}},$$

[25] in which P, Q, R, etc., are as many arbitrary functions of the variables t, t', t'', etc., as there are of these variables, by comprehending not at all t in the first, t' in the second, t'' in the third, etc. Now, this function is able to be put under this form

$$\frac{P' + Q't + R'tt' + S'tt't'' + \text{etc.}}{1 - pt - qt' - rt'' - st''' - \text{etc.}},$$

P', Q', R', etc., being, as above, arbitrary functions, the first of all the variables, with the exception of t; the second of all the variables, excepting t' from it, the third equally of all the variables, except t'', and so forth. In order to determine them, we will observe that, in $y_{x,x',x'',\text{etc.}}$, two of the indices x, x', x'', etc., or a greater number are not able to be nulls at the same time, since the game ceases when one of the players has attained his points: moreover, $y_{0,x',x'',\text{etc.}}$ is equal to unity, whatever be x', x'', etc., the generating function of this expression, or that which gives unity for the coefficient of any product whatsoever $t'^{x'}t''^{x''}t'''^{x'''}$ etc., is

$$\frac{t'}{1-t'}\frac{t''}{1-t''}\frac{t'''}{1-t'''}\text{etc.}$$

consequently, we will have

$$P' = \frac{t'}{1-t'}\frac{t''}{1-t''}\frac{t'''}{1-t'''}\text{etc.}\dots(1-qt'-rt''-st'''-\text{etc.}).$$

Each value of $y_{x,x',x'',\text{etc.}}$ in which another index than x is null being equal to zero, the corresponding generating function becomes null also; we will have therefore successively

$$Q' = 0, \quad R' = 0, \quad S' = 0, \quad \text{etc.},$$

hence the generating function of $y_{x,x',x'',\text{etc.}}$ will be

$$\frac{t'}{1-t'}\frac{t''}{1-t''}\text{etc.}\dots\frac{1-qt'-rt''-\text{etc.}}{1-pt-qt'-rt''-\text{etc.}},$$

and the coefficient of t^x, in the development of this function with respect to the powers of t,

$$\frac{t'}{1-t'}\frac{t''}{1-t''}\text{etc.}\dots\frac{p^x}{(1-qt'-rt''-\text{etc.}\dots)^x};$$

whence it is easy to deduce the coefficient of $t'^{x'}t''^{x''}$etc., or [26]

$$y_{x,x',x'',\text{etc.}} = p^x\left\{\begin{aligned}&1+\frac{x}{1}(q+r+\text{etc.})\\&+\frac{x(x+1)}{1.2}(q+r+\text{etc.})^2\\&+\frac{x(x+1)(x+2)}{1.2.3}(q+r+\text{etc.})^3\\&+\text{etc.}\end{aligned}\right\},$$

in taking care to reject the terms in which the power of q surpasses $x'-1$, those in which the power of r surpasses $x''-1$, etc.

In the problem of § 10, we consider two players A and B of whom the skills are p and q, and of whom the first has a tokens and the second b tokens; and we suppose that at each trial, the one who loses gives a token to his adversary, and that the game finishes only when one of the players will have lost all his tokens. We demand the probability that one of the players, A for example, will win the game before or at the n^{th} trial.

In representing by $y_{x,x'}$ the probability of this player to win the game, when he has x tokens, and when there are no more than x' trials to play in order to attain the n trials, we arrive, by the first principles of the probabilities, to the equation in the partial differences

$$y_{x,x'} = py_{x+1,x'-1} + qy_{x-1,x'-1},$$

which gives, for the generating function of $y_{x,x'}$,

$$\frac{A + A' + B't}{qt^2t' - t + pt'},$$

A being an arbitrary function of t, A' and B' two arbitrary functions of t'. In order to determine them more conveniently, we will transform this generating function into this one,

$$\frac{A_1t + A'_1 + B'_1tt'}{qt^2t' - t + pt'},$$

in which A_1, A'_1 and B'_1 are, as above, some arbitrary functions of t and of t'. Now $\frac{A'_1}{pt'}$
[27] is the coefficient of t^0 in the development of the function with respect to the powers of t, or the generating function of $y_{0,x'}$; but, by the conditions of the problem, $y_{0,x'}$ is null whatever be x'; consequently its generating function is also null; A'_1 is therefore equal to zero.

The coefficient of t'^0, in the development of the generating function, with respect to t', is $-A_1$, that which is at the same time the generating function of $y_{x,0}$, a quantity which is null so long as x is less than the sum of the tokens or $a+b$, and which becomes unity when $x = a + b$; A_1 is therefore a function of t which has for factor t^{a+b}, and of which we are able to take no account in the numerator of the generating function, because it must give only some powers of t superior to t^{a+b}, and we have seen it only to have a generating function composed of the powers inferior to t, since x is able to be extended only from $x = 0$ to $x = a + b$.

The generating function of $y_{x,x'}$, thus limited between these values, is reduced therefore to

$$\frac{B'_1tt'}{qt^2t' - t + pt'},$$

which we are able to put easily under this form

$$\left\{\begin{aligned} &\frac{B'_1}{p}t\frac{1}{\left(1 - \frac{\frac{1}{t'} + \sqrt{\frac{1}{t'^2} - 4pq}}{2p}t\right)\left(1 - \frac{\frac{1}{t'} - \sqrt{\frac{1}{t'^2} - 4pq}}{2p}t\right)} \\ &= \frac{B'_1}{p}\frac{t}{\sqrt{\frac{1}{t'^2} - 4pq}}\left\{\frac{\frac{1}{t'} + \sqrt{\frac{1}{t'^2} - 4pq}}{1 - \frac{\frac{1}{t'} + \sqrt{\frac{1}{t'^2} - 4pq}}{2p}t} - \frac{\frac{1}{t'} - \sqrt{\frac{1}{t'^2} - 4pq}}{1 - \frac{\frac{1}{t'} - \sqrt{\frac{1}{t'^2} - 4pq}}{2p}t}\right\}\cdots \end{aligned}\right. \qquad \text{(II)}$$

whence we deduce, for the coefficient of t^{a+b}, the expression

$$\frac{B'_1}{p}\frac{1}{(2p)^{a+b-1}}\frac{\left(\frac{1}{t'} + \sqrt{\frac{1}{t'^2} - 4pq}\right)^{a+b} - \left(\frac{1}{t'} - \sqrt{\frac{1}{t'^2} - 4pq}\right)^{a+b}}{2\sqrt{\frac{1}{t'^2} - 4pq}}.$$

[28] But this coefficient is the generating function of $y_{a+b,x'}$, a quantity which is equal to unity; because it is certain that player A has won the game, when he has won all the tokens from B: moreover, x' must be here zero or an even number, since the number

of trials in which A is able to win the game is equal to b plus an even number; and in fact, he must win all the tokens of B, and again win again each token that he has lost, that which requires two trials. The series

$$y_{a+b,0}t'^0 + y_{a+b,2}t'^2 + y_{a+b,4}t'^4 + \text{etc.},$$

which represents the coefficient of t^{a+b}, is therefore equal to $\frac{1}{1-t'^2}$, and we conclude from it

$$\frac{B'_1}{p}\frac{(2p)^{a+b-1}}{1-t'^2}\frac{2\sqrt{\frac{1}{t'^2}-4pq}}{\left(\frac{1}{t'}+\sqrt{\frac{1}{t'^2}-4pq}\right)^{a+b}-\left(\frac{1}{t'}-\sqrt{\frac{1}{t'^2}-4pq}\right)^{a+b}}.$$

Now the coefficient of t^a, deduced from the development of the function (Π), always with respect to the powers of t, will be

$$\frac{B'_1}{p}\frac{1}{(2p)^{a-1}}\frac{\left(\frac{1}{t'}+\sqrt{\frac{1}{t'^2}-4pq}\right)^{a}-\left(\frac{1}{t'}-\sqrt{\frac{1}{t'^2}-4pq}\right)^{a}}{2\sqrt{\frac{1}{t'^2}-4pq}},$$

and by substituting for $\frac{B'_1}{p}$ its value, we will have this coefficient, or the generating function of $y_{x,x'}$ equal to

$$\frac{2^b p^b}{1-t'^2}\frac{\left(\frac{1}{t'}+\sqrt{\frac{1}{t'^2}-4pq}\right)^{a}-\left(\frac{1}{t'}-\sqrt{\frac{1}{t'^2}-4pq}\right)^{a}}{\left(\frac{1}{t'}+\sqrt{\frac{1}{t'^2}-4pq}\right)^{a+b}-\left(\frac{1}{t'}-\sqrt{\frac{1}{t'^2}-4pq}\right)^{a+b}}$$

or

$$\frac{2^b p^b t'^b}{1-t'^2}\frac{\left(1+\sqrt{1-4pqt'^2}\right)^{a}-\left(1-\sqrt{1-4pqt'^2}\right)^{a}}{\left(1+\sqrt{1-4pqt'^2}\right)^{a+b}-\left(1-\sqrt{1-4pqt'^2}\right)^{a+b}},$$

that which is formula (o) of the *Théorie analytique*.

END.

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