THE ANALYTIC THEORY OF PROBABILITIES
Third Edition
Book I

Pierre-Simon Laplace

THÉORIE

ANALYTIQUE

DES PROBABILITÉS

BY THE

MARQUIS DE LAPLACE,

Peer of France, Grand Officer of the Legion of Honor; one of the forty of the French Academy; of the Academy of Sciences; Member of the Bureau of Longitudes of France; of the Royal Societies of London et of Göttingen, of the Academies of Sciences of Russia, of Denmark, of Sweden, de Prussia, of Holland, of Italy, etc.

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REVIEWED AND AUGMENTED BY THE AUTHOR.

TRANSLATION

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Preface

The *Théorie analytique des Probabilités*, henceforth referenced as the TAP, was published in 1812 with a dedication to Napoléon-le-Grand [**4**]. A second edition revised and augmented by the author appeared in 1814, a third with the introduction further expanded and the addition of four supplements was published in 1820 [**5**].

After the death of Laplace an edition of his works was published at national expense between 1843 and 1847. The seventh volume, dated 1847, contains the TAP. A printing of his complete works in 14 volumes was undertaken between 1872 and 1912 [**6**]. The TAP is again contained in Volume 7. This latter edition sometimes modernized the notation of earlier papers. The translation presented here is based upon the original third edition.

Returning then to the third edition of 1820, we find that it may be partitioned into five parts. These are

- **Introduction.** This has become known as the *Essai philosophique sur les probabilités*. The essay has passed through a number of editions. An English translation as *A Philosophical Essay on Probabilities* was made by Frederick Truscott and Frederick Emory in 1902 from the sixth French edition [**7**]. More recently, it has been retranslated by Andrew Dale as *Philosophical Essay on Probabilities* from the fifth French edition [**8**].
- **Book I.** This consists of two parts. Part 1 is essentially a reprint of the 1779 memoir "Mémoire sur les Suites" [**10**]. Part 2 similarly reprints his memoir of 1782 "Mémoire sur les Approximations des Formules qui sont fonctions de très grands nombres" [**11**].

  Part 1 treats of generating functions. Isaac Todhunter states that it has been superceded by the Calculus of Operations developed by George Boole. We disagree. Generating functions remain important in mathematics.

  Part 2 extends the theory of generating functions to two variables.
- **Book II.** Here Laplace presents applications of the theory. For the most part, he restricts himself to the most difficult questions. Many of the problems had been treated by Laplace in earlier memoirs. Consequently the TAP may be considered in one sense as a consolidation of his work in probability.
- **Additions.** The first addition offers a proof of Wallis' Theorem which gives a representation of $\pi/2$ as an infinite product. The second addition proves a formula for finite differences. The third gives a demonstration of formula (p) of §42 on page 135.
- **Supplements.**

This volume presents a translation of the following front matter: the dedication to Napoleon of the first edition, the forwards to the first, second and third editions, and the plan of the work taken from the first edition. The original table of contents has been moved to the front matter.

Parts 1 and 2 are followed by the Additions because these are related to Book I alone.

Book II and the Supplements are relegated to the second volume of this translation.

The errata for the TAP are incomplete. There are clearly printer errors in the 1820 edition which are not in the official list. Where these occur, a comparison has been made to both the 1847 and 1878 printings. On the other hand, new errors are sometimes introduced in these also or known ones left unchanged. Nonetheless, I believe the text here is as free as possible from them. All corrections are made silently.

The text is for the most part reproduced faithfully. The most notable exception is that I have generally suppressed the use of the period to separate factors in a product. His notation overall is very clumsy. Examples of this are the following: Superscripts are used whereas we would use subscripts. Primes are used to distinguish variables whereas we would use different letters and reserve primes for derivatives. Care should be taken to note that the differential dx is treated as an object and not just part of an operator.

The reader will further note that Laplace puts a great burden on the reader to understand and follow his arguments. He does not post guideposts except in a cursory way. A very careful reading, however, uncovers an order and one will see how each part develops from the previous.

The translation of the *Mécanique Céleste* of Laplace by Nathaniel Bowditch included a memoir of Bowditch written by his son, Nathaniel Ingersoll Bowditch. In it the son says,

> Dr. Bowditch himself was accustomed to remark, "Whenever I meet in La Place with the words 'Thus it plainly appears,' I am sure that hours, and perhaps days, of hard study will alone enable me to discover how it plainly appears." [**1**, Vol. IV, p. 62]

This applies no less to the work presented here.

Originally I had planned to comment on the TAP as Bowditch did the *Mécanique Céleste*. It does not seem worthwhile to do so for Book I. I refer the reader rather to Todhunter for his summaries [**12**].

DEDICATION

First Edition

To Napoleon-le-grand,

Lord,

The benevolence with which YOUR MAJESTY has deigned to recieve graciously the homage of my Traité de Mécanique Céleste, has inspired me the desire to dedicate this Work on the Calculus of Probabilities to You. This delicate calculus is extended to the most important questions of life, which are in fact, for the most part, only problems of probability. It must, in this respect, interest YOUR MAJESTY of whom genius knows to value so well and to encourage so worthily all that which can contribute to the progress of knowledge and of public prosperity. I dare to beg Him to approve this new homage dictated by the sharpest recognition, and by the profound sentiments of admiration and of respect, with which I am,

LORD,
OF YOUR MAJESTY,

The very humble and very obedient,
servant and faithful subject,
LAPLACE.

Forward

To the first edition

I myself propose to expose in this work, the analysis and the principles necessary in order to resolve the problems concerning probabilities. This analysis is composed of two theories that I have given, thirty years ago, in the *Mémoires de l'Académie des Sciences*. One of them is *the Theory of generating Functions*; the other is *the Theory of the approximation of Formulas functions of very great numbers*. They are the object of the first Book, in which I present them in a manner yet more general than in the Memoirs cited. Their comparison shows evidently, that the second is only an extension of the first, and that they are able to be considered as two branches of one same calculus, that I designate by the name of *Calculus of generating Functions*. This calculus is the foundation of my *Théorie des Probabilités*, which is the object of my second Book. The questions relative to events due to chance, amount most often with facility, to some linear equations in simple or partial differences: the first branch of the calculus of generating functions gives the most general method to integrate this kind of equations. But when the events that we consider, are in great number, the expressions to which we are led, are composed of a so great multitude of terms and factors, that their numerical calculation becomes impractical; it is therefore then indispensable to have a method which transforms them into convergent series. It is this that the second branch of the Calculus of generating Functions does with so much more advantage, as the method becomes more necessary.

My object being to present here the methods and the general results of the theory of probabilities, I treat especially the most delicate questions, the most difficult, and at the same time the most useful of this theory. I apply myself especially, to determine the probability of the causes and of the results indicated by the events considered in great number, and to seek the laws according to which that probability approaches its limits, in measure as the events are multiplied. This research merits the attention of the Geometers, by the analysis that it requires: it is there principally that the theory of approximation of the formulas functions of large numbers, finds its most important applications. This research interests observers, by indicating to them the means that they must choose among the results of their observations, and the probability of the errors that they have yet to fear. Finally, it merits the attention of the philosophers, by showing how the regularity completes by being established in the same things which appear to us entirely delivered by chance, and by revealing the hidden, but constant causes, on which this regularity depends. It is on this regularity of the mean results of the events considered in great number, that diverse establishments repose, such as life annuities, tontines, assurances, etc. The questions which are related to them, such as inoculation of vaccine, and to the decisions of electoral assemblies, offer no difficulty according to my theory. I limit myself here to resolve the most general; but the importance of these objects in civil life, the moral considerations of which they complicate themselves, and the numerous observations that they suppose, require a work apart.

If we consider the analytical methods to which the theory of probabilities has already given birth, and those that it is able to yet give birth; the justice of the principles which serve as foundation to it, the rigorous and delicate logic that their use requires in the solution of the problems; the establishments of public utility which depend on it: if we observe next that in the same things which are not able to be submitted to the calculation, this theory gives the most certain outline which is able to guide us in our judgments, and that it teaches to guard against illusions which often mislead us; we will see that there is no science more worthy of our meditations, and of which the results are more useful. It owes birth to two French Geometers of the seventeenth century, so fecund in great men and in great discoveries, and perhaps of all the centuries the one which gives most honor to the human spirit. Pascal and Fermat proposed and resolved some problems on probabilities. Huygens united these solutions, and extended them in a small treatise on this matter, which next had been considered in a more general manner by Bernoulli, Montmort, Moivre, and by many celebrated Geometers of these last times.

To the second edition

This Work has appeared in the course of 1812, namely, the first Part towards the beginning of the year, and the second Part some months after the first. Since that time, the Author has occupied himself especially in perfecting it, either by correcting slight faults which had slipped there, or by useful additions. The principal is a quite extended Introduction, in which the principles of the Theory of Probabilities and their most interesting applications are exposed without the help of the calculus. This Introduction, which serves as preface to the Work, appears further separately under this title: *Essai philosophique sur les Probabilités.* The theory of the probability of witnesses, omitted in the first edition, is here presented with the development that its importance requires. Many analytic theorems, to which the Author had arrived by some indirect paths, are demonstrated directly in the Additions, which contain, moreover, a short extract from the *Arithmetica infinitorum* of Wallis, one of the Works which have most contributed to the progress of Analysis and where we find the germ of the theory of definite integrals, one of the bases of this new Calculus of Probabilities. The Author desires that his Work, increased by at least a third by these diverse Additions, merits the attention of the geometers, and excites them to cultivate a branch so curious and so important in human knowledge.

To the third edition

This third Edition differs from the preceding: 1° by a new Introduction which has appeared last year, under this title: *Essai philosophique sur les Probabilités*, fourth Edition; 2° by three Supplements which are related to the application of the Calculus of Probabilities to the natural sciences and to geodesic operations. The first two have been published already separately; the third, relative to the operations of leveling, is terminated with the exposition of a general method of the Calculus of Probabilities, whatever be the number of the sources of error.

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Plan of the Work

It is divided into two Books: the first has for object, the Calculus of generating Functions, which serve as base to the Theory of Probabilities, exposed in the Work. This calculus is divided into two branches of which the one is the same theory of generating functions, and of which the other is the extension of this theory to the approximations of formulas ⟨which are⟩ functions of great numbers. The exposition of the principles of the theory of probabilities, and the application of these principles and of the analysis exposed in the first Book, to the most difficult and the most important questions of the probabilities, are the object of the second Book. (1812 Edition)

Part 1. General considerations on the elements of magnitudes 1

The notation of exponents, imagined by Descartes, has led Wallis and Newton, to the consideration of fractional exponents, positive and negative, and to the interpolation of series. Leibnitz has rendered these exponents variables, that which has given birth to the exponential calculus, and has completed the system of elements of finite functions. These functions are formed of exponential, algebraic and logarithmic quantities; quantities essentially distinct from one another. Integrals are not often reducible to finite functions. Leibnitz having adapted to his differential characteristic, of the exponents, in order to express the repeated differentiations; he has been led by the analogy of the powers and of the differences, an analogy that Lagrange has followed by way of induction, in all its developments. The theory of generating functions, extends this analogy to some unspecified characteristics, and

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u being any function of a variable t, and y_x being the coefficient of t^x in the development of this function, u is the *generating function* of y_x. If we multiply u, by any function s of $\frac{1}{t}$, we will have a new generating function which will be that of a function of y_x, y_{x+1}, etc. By designating by ∇y_x this last function, us^i will be the generating function of $\nabla^i y_x$, so that the exponent of s, in the generating function, becomes the one of the characteristic ∇ in the engendered function. .... §2 page 7

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$$'\triangle^n y_x = [(1+\triangle y_x)^i - 1]^n, \qquad '\Sigma^n y_x = [(1+\triangle y_x)^i - 1]^n,$$

$\triangle$ and Σ corresponding to the case where x varies by unity, and $'\triangle$ and $'\Sigma$ corresponding to the case where x varies by i. We deduce from these formulas, the following,

$$'\triangle^n y_x = \left(c^{\alpha\frac{dy_x}{dx}} - 1\right)^n, \qquad '\Sigma^n y_x = \left(c^{\alpha\frac{dy_x}{dx}} - 1\right)^{-n},$$

in which c designates the number of which the hyperbolic logarithm is unity, and $'\triangle$ and $'\Sigma$ correspond to the variation α, of x. We transform the expression of $'\triangle_n y_x$ into this here,

$$\left(c^{\frac{\alpha}{2}\frac{dy_{x+\frac{n\alpha}{2}}}{dx}} - c^{-\frac{\alpha}{2}\frac{dy_{x+\frac{n\alpha}{2}}}{dx}}\right)^n.$$

We arrive to these formulas

$$\frac{d^n y_x}{dx^n} = [\log(1+\triangle y_x)]^n,$$

$$\int^n y_x dx^n = [\log(1+\triangle y_x)]^{-n}.$$

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$$0 = \triangle^n y_{x,x'} + \frac{a}{\alpha}\triangle^{n-1}.'\triangle y_{x,x'} + \frac{b}{\alpha^2}\triangle^{n-2\prime}\triangle^2 y_{x,x'} + \text{etc.},$$

$\triangle$ corresponding to the variability of x of which unity is the difference, and $'\triangle$ corresponding to the variability of x' of which α is the difference. We deduce from it the integral of the equation in the infinitely small and finite partial differences, that we obtain by changing in the preceding, α into dx', and the characteristic $'\triangle$ into d. .... §17 page 52

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That which this series becomes, when the integral is taken between two limits which render null the function under the integral sign. Its value depends then on integrals of the form $\int t^r dt\, c^{-t^n}$, and taken from t null to t infinity. We establish this theorem

$$n^2 \int t^{r-2} dt\, c^{-t^n} \int t^{n-r} dt\, c^{-t^n} = \frac{\pi}{\sin\left(\frac{r-1}{n}\right)\pi},$$

π being the semi-circumference of which the radius is unity. We deduce from it this remarkable result

$$\int dt\, c^{-t^2} = \tfrac{1}{2}\sqrt{\pi}.$$

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This last result gives by the passage from the real to the imaginary,

$$\int dx \cos rx\, c^{-a^2x^2} = \frac{\sqrt{\pi}}{2a} c^{-\frac{r^2}{4a^2}},$$

the integral being taken from x null to x infinity. A direct method which leads to this equation, and from which we deduce the value of the integral, when the quantity under the sign $\int$ is multiplied by x^{2n}: value of the integral

$$\int x^{2n\pm1} dx\, \sin rx\, c^{-a^2x^2}$$

. . . . §25 page 78

We arrive to the formulas

$$\int \frac{dx \cos rx}{1+x^2} = \int \frac{x dx \sin rx}{1+x^2} = \pi\, c^{-r},$$

the integrals being taken from $x = -\infty$ to $x = +\infty$; and we deduce from it the integrals $\int \frac{M}{N} dx \begin{matrix}\cos\\ \sin\end{matrix} rx$, taken within the same limits, N being a rational and integral function of x, of a degree superior to M, and not having a real factor of first degree. . . . . §26 page 80

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$$S = Ay_s + B\triangle y_s + C\triangle^2 y_s + \text{etc.},$$

A, B, C being rational and integral functions of s. If the variable y_s is expressed by the definite integral $\int x^s \phi\, dx$ or by this here $\int c^{-sx} \phi\, dx$, ϕ being function of x; we have, by the formulas of the preceding chapter, the value of y_s, in very convergent series, when the index s is a large number. In order to determine ϕ, we substitute for y_s, its expression as definite integral, into the equation in the differences in y_x, which is partitioned into two others, of which the one is a differential equation in ϕ, which serves to determine this unknown; the other equation gives the limits of the definite integral. . . . . §29 page 89

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$$\frac{2\pi(-1)^{\frac{1}{2}-\mu}}{\int x^{\mu-1}dxe^{-x}} = \int \frac{dx\,c^{-x}}{x^\mu};$$

the first integral being taken from x null to x infinity, and the last integral being taken between the imaginary limits of x, which render null the function $\frac{c^{-x}}{x^\mu}$; that which gives an easy means to have the integral $\int \frac{dx \begin{smallmatrix}\cos\\ .\\ \sin\end{smallmatrix} x}{x^\mu}$, taken from x null to x infinity. This equation gives further the value of the integrals $\int \frac{d\varpi \, \cos\varpi}{1+\varpi^2}$, $\int \frac{d\varpi \, \sin\varpi}{1+\varpi^2}$, taken from ϖ null to ϖ infinity. We find $\frac{\pi}{2c}$ for these integrals; their accord with the results of the §26, proves the justice of these passages from the positive to the negative, and from the real to the imaginary: these diverse results have been given in the *Mémoires de l'Académie des Sciences* for the year 1782. . . . . . . §33 page 103

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Part 1

General considerations on the elements of magnitudes

§1. Magnitudes considered in general, are expressed commonly by the letters of the alphabet, and it is to Viète that this handy notation is due which transports the alphabets of the common languages to the analytic language.[1] The application that Viète makes of this notation, to Geometry, to the theory of equations and to the angular sections, forms one of the remarkable periods of the history of Mathematics. Some very simple signs express the correlations of magnitudes. The position of a magnitude one after another, suffices to express their product. If these magnitudes are the same, this product is the square or the second power of that magnitude. But, instead of
[4] writing it twice, Descartes imagined writing it only once, by giving to it the number 2 for exponent; and he expressed the successive powers, by increasing successively this exponent by one unit. This notation, by considering it only as an abbreviated way to represent these powers, seems a little thing; but such is the advantage of a well-made language, that its most simple notations have become often the source of the most profound theories; and it is that which has held for the exponents of Descartes. Wallis who applied himself especially to the line of induction and analogy, has been led by this means, to express the radical powers, by some fractional exponents; and likewise as Descartes expressed by the exponents 2, 3, etc., the second, third, etc. powers of a magnitude; he expressed its second, third, etc. roots by the fractional exponents $\frac{1}{2}$, $\frac{1}{3}$, etc. In general, he expressed by the exponent $\frac{m}{n}$ the root n of a magnitude raised to the power m. Indeed, following the notation of Descartes, this expression holds in the case where m is divisible by n; and Wallis, by analogy, extended it to all the cases. He noted next that the multiplication of the powers of one same magnitude, reverts to adding the exponents of those powers, that it is necessary to subtract in their division; so that the exponent $n - m$ indicates the quotient of the power n of a magnitude, divided by its power m; whence it follows that this quotient becomes unity, when m is equal to n, each magnitude having zero for exponent, is the unit itself. If m surpasses n, the exponent $n - m$ becomes negative, and the quotient becomes unity divided by the power $m - n$ of the magnitude. Wallis supposed therefore generally that the negative exponent $-\frac{m}{n}$ expresses the unit divided by the n^{th} root of the magnitude raised to the power m.

It was in his work entitled *Arithmetica infinitorum*,[2] that Wallis exposed those remarks which led him to sum x^n, x being supposed formed of an infinity of elements taken for unity; that which, according to the actual notations, reverts to integrating the differential $x^n dx$. He showed that this integral taken from x null, is $\frac{x^{n+1}}{n+1}$, that
[5] which gave to him the integral of a series formed of similar differentials. By considering thus the integral $\int dx(1-x^{\frac{1}{n}})^s$, when n and s are whole numbers, and when it is taken from x null to $x = 1$, he found that it is equal to $\frac{1.2.3\cdots n}{(s+1)(s+2)\cdots(s+n)}$. If the indices n and s are fractional and equal to $\frac{1}{2}$, this integral expresses the ratio of the surface of the

[1] There is an English translation of François Viète by T. Richard Witmer. *The Analytic Art*, Kent State U. Press (1983), reprinted by Dover Publications.

[2] The relevant passage may be found in *A Source Book in Mathematics, 1200–1800*, pages 244–253. See also *The Arithmetic of Infinitesimals: John Wallis 1656* by Jacqueline Stedall, Springer (2004).

circle to the square of its diameter. Wallis applied himself therefore to interpolate the preceding product, in the case where n and s are fractional numbers; a problem entirely new at the period where this illustrious Geometer busied himself with it, and that he arrived to resolve by a quite ingenious method which contains the germs of the theories of interpolations and definite integrals, of which the geometers have so much occupied themselves, and which are the object of a great part of this work. He obtained in this manner, the expression of the ratio of the surface of the circle to the square of its diameter, as a product of an infinity of factors, which give values more and more near to this ratio, in measure as we consider a greater number of these factors; a result one of the most singular of Analysis. But it is remarkable that Wallis who had so well considered the fractional indices of radical powers, had continued to note these powers, as one had done before him. We see the notation of radical powers, by fractional exponents, employed for the first time in the letters of Newton to Oldenburg, inserted into the *Commercium epistolicum*. By comparing by the path of induction of which Wallis had made such beautiful usage, the exponents of the powers of the binomial, with the coefficients of the terms of its development, in the case where these exponents are whole numbers; he determined the law of these coefficients, and he extended it by analogy, to fractional powers and to negative powers. These diverse results founded on the notation of Descartes, show the influence of handy notation on all analysis.

This notation has further the advantage to give the simplest and most just idea of logarithms, which are in fact, only whole and fractional exponents of one same magnitude of which the diverse powers represent all numbers. But the most important extension that this notation has received, is that of variable exponents; that which constitutes the exponential calculus, one of the most fertile branches of modern anal- [6]
ysis. Leibnitz has indicated first, in the *Actes de Leipzig* for 1682,[3] the transcendents to variable exponents, and thence he has completed the system of elements of which a finite function can be composed. For every explicit finite function is reduced in last analysis, to some simple magnitudes, added or subtracted from one another, multiplied or divided among them, raised to some constant or variable powers. The roots of equations formed of these elements, are implicit functions of them. It is thus that c being the number of which the hyperbolic logarithm is unity, the logarithm of a is the root of the transcendent equation $c^x - a = 0$. We can consider further the logarithmic quantities, as exponential functions of which the exponents are infinitely small. Thus $X \log X'$ is equal to $\frac{X'^{X\,dx}-1}{dx}$. All the modifications of magnitude that we can imagine to the exponents, are found therefore represented by the exponential, algebraic and logarithmic quantities. These quantities and their functions embrace consequently, all the explicit finite functions; and the roots of the equations formed of similar functions, embrace all the implicit finite functions.

These quantities are essentially distinct: the exponential a^x, for example, can never be identical with an algebraic function of x. For each algebraic function is

[3]"De vera proportione circuli ad Quadratum circumscriptum in Numeris rationalibus," *Acta Eruditorum*, 43–46. Reprinted *Mathematische Schriften*, Vol V (1858), 118–122.

reducible into a descending series of the form $kx^n + k'x^{n-n'} +$ etc.: now it is easy to demonstrate that a being supposed greater than unity, and x being infinite, a^x is infinitely greater than kx^n, however great that we suppose k and n. Similarly, it is easy to see that in the case of x infinite, x is infinitely greater than $k(\log x)^n$. The exponential, algebraic and logarithmic functions of an indeterminate variable, can therefore not re-enter into one another: the algebraic quantities hold the middle between the exponential and the logarithmic; the exponents, when the variable is infinite, can be considered as infinite in the exponentials, finite in the algebraic quantities, and infinitely small in the logarithmic quantities.

[7] We can further establish in principle, that a radical function of one variable, cannot be identical with a rational function of the same variable, or with another radical function. Thus $(1+x^3)^{\frac{1}{4}}$, is essentially distinct from $(1+x^3)^{\frac{1}{3}}$ and from $(1+x)^{\frac{1}{2}}$.

These principles founded on the nature itself of the functions, can be of great utility in analytic researches, by indicating the forms of which the functions that one intends to find, are susceptible, and by demonstrating their impossibility in a great number of cases; but then it is necessary to be quite certain to omit none of the possible forms. Thus differentiation leaving the exponential and radical quantities to subsist, and by making the logarithmic quantities vanish, only as long as they are multiplied by some constants; we must conclude from it that the integral of a differential function can contain no other exponential and radical quantities, than those which are contained in that function. By this means, I have recognized that we can not obtain in the form of an explicit or implicit finite function of the variable x, the integral $\int \frac{dx}{\sqrt{1+\alpha x^2+\beta x^4}}$. I have demonstrated similarly that the linear equations in partial differences of the second order among three variables, are not most often, susceptible of being integrated under a finite form; that which has led me to a general method in order to integrate them under this form, when it is possible. In the other cases, we can obtain a finite integral, only by means of definite integrals.

Leibnitz having adapted to the differential calculus, a very handy characteristic, he imagined giving to it the same exponents as to magnitudes; but then, these exponents, instead of indicating the repeated multiplications of one same magnitude, indicate the repeated differentiations of one same function.[4] This novel extension of the Cartesian notation, led Leibnitz to this remarkable theorem, namely, that the n^{th} differential of a product xyz etc., is equal to $(dx+dy+dz+ \text{ etc.})^n$, provided that in the development of this polynomial, we apply to the characteristic d, the exponents of the powers of dx, dy, dz, etc., and that thus we write $d^r x.d^{r'} y.d^{r''} z$ etc., instead of $(dx)^r.(dy)^{r'}.(dz)^{r''}$ etc., by taking care to change $d^0 x$, $d^0 y$, $d^0 z$, etc. into x, y, z, etc. This great
[8] Geometer observed moreover, that this theorem subsists, by supposing n negative there, provided that we change the negative differentials into integrals. Lagrange has followed this singular analogy of powers and of differences in all his developments; and by a sequence of very fine and fortunate inductions, he has deduced from it

[4] Laplace most likely refers here to "Symbolismus memorabilis calculi algebraici et infinitesimales in comparatione potentiarum et differentiarum," *Miscellanea Berolinensia*, pages 160–165, (1710).

general formulas as curious as useful, on the transformations of the differences and of the integrals into one another, when the variables have diverse finite increases, and when these increases are infinitely small. His Memoir[5] on this object, inserted into the *Recuiel de l'Academie de Berlin* for the year 1772, can be regarded as one of the most beautiful applications that one has made, on the method of inductions. The theory of generating functions extends the Cartesian notation to some unspecified characteristics; it shows at the same time, evidently, the analogy of the powers and of the operations indicated by these characteristics, and we are going to see all that which concerns series, and the integration of linear equations in differences, rising from it with an extreme ease.

[5] "Sur une nouvelle espece de calcul, rélatif à la differentiation & à l'intégration des quantités variables," *Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-Lettres*, pages 185–221.

CHAPTER 1

Concerning generating functions, in one variable

§2. Let y_x be any function whatever of x; if we form the infinite series [9]

$$y_0 + y_1 t + y_2 t^2 + y_3 t^3 \cdots + y_x t^x + y_{x+1} t^{x+1} \cdots + y_\infty t^\infty;$$

we can always imagine a function of t, which developed according to the powers of t, gives this series: this function is that which I name *generating function* of y_x.

The generating function of any variable y_x, is thus generally a function of t, which developed according to the powers of t, has this variable for the coefficient of t^x; and reciprocally, the corresponding variable of a generating function, is the coefficient of t^x in the development of this function according to the powers of t; so that the exponent of the power of t, indicates the rank that the variable y_x occupies in the series which we can imagine prolonged indefinitely to the left, relative to the negative powers of t.

It follows from these definitions, that u being the generating function of y_x, that of y_{x+r} is $\frac{u}{t^r}$; because it is clear that the coefficient of t^x in $\frac{u}{t^r}$ is equal to the one of t^{x+r} in u; and consequently it is equal to y_{x+r}.

The coefficient of t^x in $u\left(\frac{1}{t} - 1\right)$ is therefore equal to $y_{x+1} - y_x$, or to the difference of the two consecutive quantities y_{x+1} and y_x, a difference that we will designate by $\triangle y_x$, $\triangle$ being the characteristic of finite differences. We have therefore the generating function of the finite difference of a variable quantity, by multiplying by $\frac{1}{t} - 1$, [10] the generating function of the quantity itself. The generating function of the finite difference of $\triangle y_x$, a difference that we designate by $\triangle^2 y_x$, is thus $u\left(\frac{1}{t} - 1\right)^2$; that of the finite difference $\triangle^2 y_x$ or $\triangle^3 y_x$, is $u\left(\frac{1}{t} - 1\right)^3$; whence we can generally conclude that the generating function of the finite difference $\triangle^i y_x$ is $u\left(\frac{1}{t} - 1\right)^i$.

Similarly, the coefficient of t^x in the development of

$$u\left(a + \frac{b}{t} + \frac{c}{t^2} + \frac{e}{t^3} \cdots + \frac{q}{t^n}\right)$$

is

$$ay_x + by_{x+1} + cy_{x+2} + ey_{x+3} \cdots + qy_{x+n};$$

by naming therefore ∇y_x this quantity, its generating function will be

$$u\left(a + \frac{b}{t} + \frac{c}{t^2} \cdots + \frac{q}{t^n}\right).$$

If we name $\nabla^2 y_x$ that which ∇y_x becomes when we change y_x into ∇y_x there; if we name similarly $\nabla^3 y_x$ that which $\nabla^2 y_x$ becomes when we change ∇y_x into $\nabla^2 y_x$, and so forth, their corresponding generating functions will be

$$u\left(a+\frac{b}{t}+\frac{c}{t^2}\cdots+\frac{q}{t^n}\right)^2;$$
$$u\left(a+\frac{b}{t}+\frac{c}{t^2}\cdots+\frac{q}{t^n}\right)^3;$$
etc.,

and generally the generating function of $\nabla^i y_x$ will be

$$u\left(a+\frac{b}{t}+\frac{c}{t^2}\cdots+\frac{q}{t^n}\right)^i.$$

Thence it is easy to conclude generally that the generating function of $\triangle^i\nabla^s y_{x+r}$ is

$$\frac{u}{t^r}\left(a+\frac{b}{t}+\frac{c}{t^2}\cdots+\frac{q}{t^n}\right)^s\left(\frac{1}{t}-1\right)^i.$$

[11] We can generalize further these results, by supposing that ∇y_x represents any finite or infinite linear function, of y_x, y_{x+1}, y_{x+2}, etc.; that $\nabla^2 y_x$ is that which ∇y_x becomes, when we change y_x into ∇y_x there; that $\nabla^3 y_x$ is that which $\nabla^2 y_x$, becomes when we change ∇y_x into $\nabla^2 y_x$, and so forth; u being the generating function of y_x, us^i will be the generating function of $\nabla^i y_x$, s being that which ∇y_x becomes, when we change y_x into unity, y_{x+1} into $\frac{1}{t}$, y_{x+2} into $\frac{1}{t^2}$, etc. This is still true, when i is a negative number, or even fractional and incommensurable, by making however in this result, some convenient modifications.

Let us represent by Σ the characteristic of finite integrals, and let us name z the generating function of $\Sigma^i y_x$, u being the generating function of y_x; $z\left(\frac{1}{t}-1\right)^i$ will be by that which precedes, the generating function of y_x. But this function must, by having regard only to the positive powers of t, be reduced to u which contains only positive powers of t, if we extend the multiple integral $\Sigma^i y_x$ only to the positive values of x; we will have therefore then

$$z\left(\frac{1}{t}-1\right)^i = u+\frac{A}{t}+\frac{B}{t^2}+\frac{C}{t^3}\cdots+\frac{F}{t^i};$$

whence we deduce

$$z=\frac{ut^i+At^{i-1}+Bt^{i-2}+Ct^{i-3}\cdots+F}{(1-t)^i},$$

$A, B, C, \ldots, F$ being arbitrary constants which correspond to the i arbitrary constants that the i successive integrations of $\Sigma^i y_x$ introduce.

By setting aside these constants, the generating function of $\Sigma^i y_x$ is $u\left(\frac{1}{t}-1\right)^{-i}$; so that we obtain this generating function, by changing i into $-i$, in the generating function of $\triangle^i y_x$; $\triangle^{-i} y_x$ is therefore then equal to $\Sigma^i y_x$; that is that the negative differences are changed into integrals. But, if we have regard to the arbitrary constants, it is necessary, in passing from the positive powers of $\frac{1}{t}-1$ to its negative powers, to

augment u with the series $\frac{A}{t} + \frac{B}{t^2} + \frac{C}{t^3} + \text{etc.}$, prolonged until the number of its terms [12]
is equal the exponent of these powers. We can apply similar considerations, to the generating function of $\nabla^i y_x$.

We see by that which precedes, in what manner the generating functions are formed from the law of the corresponding variables. Let us see now how the variables are deduced from their generating functions. s being any function of $\frac{1}{t}$, if we develop s^i according to the powers of $\frac{1}{t}$, and if we designate by $\frac{k}{t^n}$ any term of this development; the coefficient of t^x in $\frac{ku}{t^n}$ will be ky_{x+n}; we will have therefore the coefficient of t^x in us^i, a coefficient that we have designated previously by $\nabla^i y_x$, 1° by substituting into s, y_x in place of $\frac{1}{t}$; 2° by developing that which s^i then becomes according to the powers of y_x, and by transporting to the index x, the exponent of the power of y_x; that is, by writing y_{x+1} instead of $(y_x)^1$; y_{x+2} instead of $(y_x)^2$, etc., and by multiplying the terms independent of y_x, and which can be counted to have $(y_x)^0$ for factor, by y_x. When the characteristic ∇ is changed into $\triangle$, s is, by that which precedes, equal to $\frac{1}{t} - 1$; we have therefore then

$$\triangle^i y_x = y_{x+i} - iy_{x+i-1} + \frac{i(i-1)}{1.2} y_{x+i-2} - \text{ etc.}$$

If, instead of developing s^i according to the powers of $\frac{1}{t}$, we develop it according to the powers of $\frac{1}{t} - 1$, and if we designate by $k\left(\frac{1}{t} - 1\right)^n$, any term of this development; the coefficient of t^x in $ku\left(\frac{1}{t} - 1\right)^n$ will be $k\triangle^n y_x$; we will have therefore $\nabla^i y_x$; 1° by substituting into s, $\triangle y_x$ in place of $\frac{1}{t} - 1$, or, that which reverts to the same, $1 + \triangle y_x$ in place of $\frac{1}{t}$; 2° by developing that which s^i then becomes according to the powers of $\triangle y_x$, and by applying to the characteristic $\triangle$, the exponents of the powers of $\triangle y_x$, that is by writing $\triangle y_x$ instead of $(\triangle y_x)^1$, $\triangle^2 y_x$ instead of $(\triangle y_x)^2$, etc., and by [13]
multiplying by $(\triangle y_x)^0$, or, that which is the same thing, by y_x the terms independent of $\triangle y_x$.

In general, if we consider s as a function of r, r being a function of $\frac{1}{t}$, such that the coefficient of t^x in ur, is $\square y_x$; we will have $\nabla^i y_x$, by substituting into s, $\square y_x$, in place of r; by developing next s^i according to the powers of $\square y_x$, and by applying to the characteristic $\square$, the exponents of $\square y_x$, that is, by writing $\square y_x$, in place of $(\square y_x)$, $\square^2 y_x$ in place of $(\square y_x)^2$, etc.; and by multiplying by y_x the terms independent of $\square y_x$.

The development of $\nabla^i y_x$ by a series ordered according to the successive variations $\square y_x$, $\square^2 y_x$, etc., is reduced therefore to the formation of the generating function of y_x, in the development of that function, according to the powers of a given function; finally, on the return of the generating function thus developed, to the corresponding variable coefficients; the exponents of the powers of the development of the generating function, becoming those of the characteristic of these coefficients. We see thus the analogy of the powers with the differences, or with every other combination of the consecutive variable coefficients. The passage from these coefficients to their generating functions, and the return of these developed functions to the coefficients constitute the *calculus of generating functions*. The following applications will make known the spirit and the advantages of them.

On the interpolation of the series in one variable, and on the integration of linear differential equations.

§3. All theory of the interpolation of series is reduced to determining, whatever be i, the value of y_{x+i} as a function of the terms which precede or which follow y_x. For this, we must observe that y_{x+i} is equal to the coefficient of t^{x+i} in the development of u, and consequently equal to the coefficient of t^x in the development of $\frac{u}{t^i}$; now we have

$$\frac{u}{t^i} = u\left(1+\frac{1}{t}-1\right)^i = u\left\{\begin{array}{l} 1+i\left(\frac{1}{t}-1\right)+\frac{i(i-1)}{1.2}\left(\frac{1}{t}-1\right)^2 \\ \quad +\frac{i(i-1)(i-2)}{1.2.3}\left(\frac{1}{t}-1\right)^3+\text{etc.}\end{array}\right\}.$$

[14] Moreover, the coefficient of t^x in the development of u, is y_x; this coefficient in the development of $u\left(\frac{1}{t}-1\right)$, is $\triangle y_x$; in the development of $u\left(\frac{1}{t}-1\right)^2$, it is equal to $\triangle^2 y_x$, and so forth; the preceding equation will give therefore, by passing again from the generating functions to the coefficients,

$$y_{x+i} = y_x + i\triangle y_x + \frac{i(i-1)}{1.2}\triangle^2 y_x + \frac{i(i-1)(i-2)}{1.2.3}\triangle^3 y_x + \text{ etc.}$$

This equation holding whatever be i, by supposing it even fractional, serves to interpolate the series of which the successive differences of the terms are decreasing.

If we have the equation in finite differences

$$\triangle^n y_x = 0;$$

the preceding series is terminated, and we have, whatever be i, by making x null,

$$y_i = y_0 + i\triangle y_0 + \frac{i(i-1)}{1.2}\triangle^2 y_0 \cdots + \frac{i(i-1)\dots(i-n+2)}{1.2.3\dots(n-1)}\triangle^{n-1} y_0.$$

This is the complete integral of the proposed equation in the differences, y_0, $\triangle y_0$, $\dots\triangle^{n-1} y_0$ being the n arbitrary constants of this integral.

All the ways of developing the power $\frac{1}{t^i}$, give as many different methods to interpolate the series. Let, for example,

$$\frac{1}{t} = 1 + \frac{\alpha}{t^r};$$

by developing $\frac{1}{t^i}$ according to the powers of α, by formula (p) of §**21** of the second book of the *Mécanique céleste*,[1] we will have

$$\frac{u}{t^i} = u\left\{\begin{array}{c} 1 + i\alpha + \dfrac{i(i+2r-1)}{1.2}\alpha^2 + \dfrac{i(i+3r-1)(i+3r-2)}{1.2.3}\alpha^3 \\ +\dfrac{i(i+4r-1)(i+4r-2)(i+4r-3)}{1.2.3.4}\alpha^4 + \text{etc.}\end{array}\right\}.$$

α being equal to $t^r\left(\frac{1}{t}-1\right)$, the coefficient of t^x in the development of $u\alpha$ is, by §2, [15]
$\triangle y_{x-r}$; this same coefficient in $u\alpha^2$ is $\triangle^2 y_{x-2r}$, and so forth. The preceding equation will give therefore, by passing again from the generating functions to the coefficients,

$$y_{x+i} = y_x + i\triangle y_{x-r} + \frac{i(i+2r-1)}{1.2}\triangle^2 y_{x-2r} + \frac{i(i+3r-1)(i+3r-2)}{1.2.3}\triangle^3 y_{x-3r} + \text{etc.}$$

§4. Here is now a general method of interpolation, which has the advantage of being applicable, not only to the series of which the differences of the terms conclude by being null, but further to the series of which the ultimate ratio of the terms is that of any recurrent series.

Let us suppose first that we have

$$t\left(\frac{1}{t}-1\right)^2 = z; \tag{1}$$

and let us seek the value of $\frac{1}{t^i}$ in a series ordered with respect to the powers of z. It is clear that $\frac{1}{t^i}$ is equal to the coefficient of θ^i in the development of the fraction $\frac{1}{1-\frac{\theta}{t}}$. If we multiply the numerator and the denominator of this fraction by $1-\theta t$, we will have this here

$$\frac{1-\theta t}{1-\theta\left(\frac{1}{t}+t\right)+\theta^2}.$$

Equation (1) gives

$$\frac{1}{t}+t = 2+z,$$

that which changes the preceding fraction into this one here

$$\frac{1-\theta t}{(1-\theta)^2 - z\theta};$$

[1]See Volume I of *Oeuvres de Laplace* (1843), p. 173. In his original paper of 1779 [**10**], Laplace credits Lagrange with this formula: "Recherches sur les suites recurrentes dont les termes varient de plusieurs manieres différentes, ou sur l'integration des équations linéaires aux différences finies et partielles; et sur l'usage de ces équations dans la théorie des hazards." This appeared in *Nouveaux Mémoires de l'Académie . . . Berlin* for the year 1775, published in 1777, [**3**, pages 183–272]. The formula appears there on page 115.

now we have

$$\frac{1}{(1-\theta)^2 - z\theta} = \frac{1}{(1-\theta)^2} + \frac{z\theta}{(1-\theta)^4} + \frac{z^2\theta^2}{(1-\theta)^6} + \text{etc.};$$

[16] besides the coefficient of θ^r in the development of $\frac{1}{(1-\theta)^s}$, is

$$\frac{s(s+1)(s+2)\dots(s+r-1)}{1.2.3\dots r},$$

whence it follows that the coefficient of θ^i is, 1° $i+1$ in the development of $\frac{1}{(1-\theta)^2}$; 2° $\frac{i(i+1)(i+2)}{1.2.3}$, in the development of $\frac{\theta}{(1-\theta)^4}$. 3° $\frac{(i-1)i(i+1)(i+2)(i+3)}{1.2.3.4.5}$, in the development of $\frac{\theta^2}{(1-\theta)^6}$, and thus of the rest; therefore if we name Z the coefficient of θ^i in the development of the function

$$\frac{1}{(1-\theta)^2 - z\theta},$$

we will have

$$Z = i+1+\frac{i(i+1)(i+2)}{1.2.3}z + \frac{(i-1)i(i+1)(i+2)(i+3)}{1.2.3.4.5}z^2 + \frac{(i-2)(i-1)i(i+1)(i+2)(i+3)(i+4)}{1.2.3.4.5.6.7}z^3 + \text{etc.},$$

or

$$Z = (i+1)\left\{1 + \frac{[(i+1)^2-1]z}{1.2.3} + \frac{[(i+1)^2-1][(i+1)^2-4]z^2}{1.2.3.4.5} + \text{etc.}\right\};$$

if we name next Z' the coefficient of θ^i in the development of

$$\frac{\theta}{(1-\theta)^2 - z\theta},$$

we will have Z' by changing i into $i-1$ in Z, that which gives

$$Z' = i\left[1 + \frac{(i^2-1)z}{1.2.3} + \frac{(i^2-1)(i^2-4)z^2}{1.2.3.4.5} + \text{etc.}\right];$$

we will have thus $Z - tZ'$ for the coefficient of θ^i in the development of the fraction

$$\frac{1-\theta t}{(1-\theta)^2 - z\theta};$$

this will be consequently the expression of $\frac{1}{t^i}$; therefore

$$\frac{u}{t^i} = u(Z - tZ').$$

[17] This premised, the coefficient of t^x in $\frac{u}{t^i}$, is y_{x+i}. This same coefficient, in any term of uZ, such as kuz^r or $kut^r\left(\frac{1}{t}-1\right)^{2r}$ is, by §2, $k\triangle^{2r}y_{x-r}$. In any term of utZ', such

as $kutz^r$ or $kut^{r+1}\left(\frac{1}{t}-1\right)^{2r}$, this coefficient is $k\triangle^{2r}y_{x-r-1}$; we will have therefore, by passing again from the generating functions to their coefficients,

$$y_{x+i}=(i+1)\left\{\begin{aligned}&y_x+\frac{(i+1)^2-1}{1.2.3}\triangle^2 y_{x-1}\\&\quad+\frac{[(i+1)^2-1][(i+1)^2-4]}{1.2.3.4.5}\triangle^4 y_{x-2}+\text{etc.}\end{aligned}\right\}$$
$$-i\left\{y_{x-1}+\frac{i^2-1}{1.2.3}\triangle^2 y_{x-2}+\frac{(i^2-1)(i^2-4)}{1.2.3.4.5}\triangle^4 y_{x-3}+\text{etc.}\right\}.$$

We can give the following forms to the preceding expression. Let Z'' be that which Z' becomes when we change i into $i-1$ there; and consequently, that which Z becomes when we change i into $i-2$. The equation

$$\frac{1}{t^i}=Z-tZ'$$

will give

$$\frac{1}{t^{i-1}}=Z'-tZ'';$$

hence,

$$\frac{1}{t^i}=\frac{Z'}{t}-Z''.$$

By adding these two values of $\frac{1}{t^i}$, and taking the half of their sum, we will have

$$\frac{1}{t^i}=\frac{1}{2}Z-\frac{1}{2}Z''+\frac{1}{2}(1+t)\left(\frac{1}{t}-1\right)Z';$$

now we have

$$\frac{1}{2}Z-\frac{1}{2}Z''=1+\frac{i^2}{1.2}z+\frac{i^2(i^2-1)}{1.2.3.4}z^2+\frac{i^2(i^2-1)(i^2-4)}{1.2.3.4.5.6}z^3+\text{etc.}$$

hence

$$\frac{u}{t^i}=u\left[1+\frac{i^2}{1.2}t\left(\frac{1}{t}-1\right)^2+\frac{i^2(i^2-1)}{1.2.3.4}t^2\left(\frac{1}{t}-1\right)^4+\text{etc.}\right]$$
$$+\frac{i}{2}u(1+t)\left\{\begin{aligned}&\frac{1}{t}-1+\frac{i^2-1}{1.2.3}t\left(\frac{1}{t}-1\right)^3\\&\qquad+\frac{(i^2-1)(i^2-4)}{1.2.3.4.5}t^2\left(\frac{1}{t}-1\right)^5+\text{etc.}\end{aligned}\right\};$$

[18] whence we conclude, by passing again from the generating functions to the coefficients,

$$\begin{aligned}y_{x+i} = y_x &+ \frac{i^2}{1.2}\triangle^2 y_{x-1} + \frac{i^2(i^2-1)}{1.2.3.4}\triangle^4 y_{x-2}\\ &+ \frac{i^2(i^2-1)(i^2-4)}{1.2.3.4.5.6}\triangle^6 y_{x-3} + \text{etc.}\\ &+ \frac{i}{2}(\triangle y_x + \triangle y_{x-1}) + \frac{i}{2}\cdot\frac{i^2-1}{1.2.3}(\triangle^3 y_{x-1} + \triangle^3 y_{x-2})\\ &+ \frac{i}{2}\frac{(i^2-1)(i^2-4)}{1.2.3.4.5}(\triangle^5 y_{x-2} + \triangle^5 y_{x-3}) + \text{ etc.}\end{aligned}$$

This formula[2] serves to interpolate between an odd number $2x+1$ of equidistant quantities; the common interval which separates them being taken for unity, y_x is the middle of the magnitudes y_0, y_1, y_2,... y_{2x}; and i is the distance from y_{x+i} to this middle. The preceding expression is then symmetric relative to these magnitudes; because $\triangle^2 y_{x-1}$, for example, is equal to $y_{x+1} - 2y_x + y_{x-1}$, and $\triangle y_x + \triangle y_{x-1}$ is equal to $y_{x+1} - y_{x-1}$. Thus the quantities placed above and below the mean y_x, enter in the same manner into this expression.

If we change i into $i+1$ in the last expression of $\frac{u}{t^i}$, and if we subtract from it that expression itself; we will have the expression of $\frac{u}{t^{i+1}} - \frac{u}{t^i}$, or of $\frac{u}{t^i}\left(\frac{1}{t}-1\right)$; by dividing next this value by $\frac{1}{t}-1$, we will have

$$\begin{aligned}\frac{u}{t^i} =& \frac{u}{2}(1+t)\left\{\begin{aligned}&1 + \frac{[(i+\frac{1}{2})^2 - \frac{1}{4}]}{1.2}t\left(\frac{1}{t}-1\right)^2\\ &\quad + \frac{[(i+\frac{1}{2})^2-\frac{1}{4}][(i+\frac{1}{2})^2-\frac{9}{4}]}{1.2.3.4}t^2\left(\frac{1}{t}-1\right)^4 + \text{etc.}\end{aligned}\right\}\\ &+ (i+\tfrac{1}{2})ut\left(\frac{1}{t}-1\right)\left\{\begin{aligned}&1 + \frac{[(i+\frac{1}{2})^2-\frac{1}{4}]}{1.2.3}t\left(\frac{1}{t}-1\right)^2\\ &\quad + \frac{[(i+\frac{1}{2})^2-\frac{1}{4}][(i+\frac{1}{2})^2-\frac{9}{4}]}{1.2.3.4.5}t^2\left(\frac{1}{t}-1\right)^4 + \text{etc.}\end{aligned}\right\}\end{aligned}$$

[19] By passing again from the generating functions to the coefficients, we will have

$$\begin{aligned}y_{x+i} =& \frac{1}{2}(y_x + y_{x-1}) + \frac{[(i+\frac{1}{2})^2-\frac{1}{4}]}{1.2}\cdot\frac{1}{2}(\triangle^2 y_{x-1} + \triangle^2 y_{x-2})\\ &+ \frac{[(i+\frac{1}{2})^2-\frac{1}{4}][(i+\frac{1}{2})^2-\frac{9}{4}]}{1.2.3.4}\cdot\frac{1}{2}(\triangle^4 y_{x-2} + \triangle^4 y_{x-3}) + \text{etc.}\\ &+ (i+\tfrac{1}{2})\left\{\begin{aligned}&\triangle y_{x-1} + \frac{[(i+\frac{1}{2})^2-\frac{1}{4}]}{1.2.3}\triangle^3 y_{x-2}\\ &\quad + \frac{[(i+\frac{1}{2})^2-\frac{1}{4}][(i+\frac{1}{2})^2-\frac{9}{4}]}{1.2.3.4.5}\triangle^5 y_{x-3} + \text{etc.}\end{aligned}\right\}.\end{aligned}$$

---

[2]In his original paper, Laplace states that this formula reverts to the one Newton gave in his *Methodus differentialis* in order to interpolate between an odd number of equidistant quantities [**10**].

This formula[3] serves to interpolate between an even number $2x$ of equidistant quantities, y_x and y_{x+1} being the two middle quantities.[4] It is disposed in a symmetric manner relative to the quantities equally distant from the middle of the interval which separates the extreme quantities: this middle is the origin of the values of $i+\frac{1}{2}$, which are positive above and negative below.

All these expressions of y_{x+i} are identical, and such that if we imagine a parabolic curve of which i is the abscissa, and y_{x+i} the ordinate, and of which the equation is that which gives the expression of y_{x+i}; this curve will pass through the extremities of the ordinates y_x, y_{x+1}, y_{x+2}, etc.; y_{x-1}, y_{x-2}, etc. We can thus, by taking the successive finite differences of any number of coordinates, make a parabolic curve pass through the extremities of these coordinates.

§5. Let us suppose generally

$$z = a + \frac{b}{t} + \frac{c}{t^2} + \frac{e}{t^3} \cdots + \frac{p}{t^{n-1}} + \frac{q}{t^n}; \qquad (a)$$

we will have

$$\frac{1}{t^n} = \frac{z-a}{q} - \frac{b}{qt} - \frac{c}{qt^2} \cdots - \frac{p}{qt^{n-1}},$$

that which gives

$$\frac{1}{t^{n+1}} = \frac{z-a}{qt} - \frac{b}{qt^2} - \frac{c}{qt^3} \cdots - \frac{p}{qt^n};$$

eliminating $\frac{1}{t^n}$ from the second member of this equation, by means of the proposed (a), we will have

$$\frac{1}{t^{n+1}} = -\frac{p(z-a)}{q^2} + \frac{pb + q(z-a)}{q^2 t} + \text{etc.}$$

This expression of $\frac{1}{t^{n+1}}$ contains only powers of $\frac{1}{t}$ of an order inferior to n. By [20]
multiplying it by $\frac{1}{t}$, we will have an expression of $\frac{1}{t^{n+2}}$, which will contain the power $\frac{1}{t^n}$; but by eliminating again this power, by means of the proposed (a), we will reduce the expression of $\frac{1}{t^{n+2}}$ to contain only powers of $\frac{1}{t}$ inferior to n. By continuing thus, we will arrive to an expression of $\frac{1}{t^i}$, which will contain only powers of $\frac{1}{t}$ less than n, and which will be consequently of the form

$$\frac{1}{t^i} = Z + \frac{1}{t}Z^{(1)} + \frac{1}{t^2}Z^{(2)} \cdots + \frac{1}{t^{n-1}}Z^{(n)},$$

Z, $Z^{(1)}$, $Z^{(2)}$, etc., being some rational and integral functions of z, of which the highest power of z does not surpass $\frac{i}{n}$.

This manner of determining $\frac{1}{t^i}$ would be very laborious, if i were a large number; it would lead besides with difficulty to the general expression of this quantity. We could arrive there directly by the following method.

[3]Again in his original paper, Laplace states that this formula reverts to the one given by Newton in *Methodus differentialis* [**10**].

[4]The original has "y_{x-1} and y_{x+1} being the two middle quantities." The errata replaces these values by y_x and $y_{x+\alpha}$. However, the change must be as above.

$\frac{1}{t^i}$ is equal to the coefficient of θ^i in the development of the fraction $\frac{1}{1-\frac{\theta}{t}}$. If we multiply the numerator and the denominator of this fraction by

$$(a-z)\theta^n + b\,\theta^{n-1} + c\,\theta^{n-2}\cdots + p\,\theta + q;$$

and if in the numerator we substitute in the place of z, its value $a + \frac{b}{t} + \frac{c}{t^2} + \text{etc.}$, we will have

$$\frac{b\,\theta^{n-1}\left(1-\frac{\theta}{t}\right) + c\,\theta^{n-2}\left(1-\frac{\theta^2}{t^2}\right) + e\,\theta^{n-3}\left(1-\frac{\theta^3}{t^3}\right)\cdots + q\left(1-\frac{\theta^n}{t^n}\right)}{\left(1-\frac{\theta}{t}\right)\left(a\,\theta^n + b\,\theta^{n-1} + c\,\theta^{n-2}\cdots + p\,\theta + q - z\theta^n\right)};$$

[21] by dividing the numerator and the denominator of this fraction by $1-\frac{\theta}{t}$; it becomes

$$\frac{\left\{\begin{array}{l} b\theta^{n-1} + c\,\theta^{n-2} + e\,\theta^{n-3}\cdots + q \\ +\dfrac{\theta}{t}(c\,\theta^{n-2} + e\,\theta^{n-3}\cdots\cdots\cdots + q) \\ +\dfrac{\theta^2}{t^2}(e\,\theta^{n-3}\cdots\cdots\cdots\cdots\cdots + q) \\ +\text{ etc.} \\ +\dfrac{\theta^{n-1}}{t^{n-1}}q \end{array}\right\}}{a\,\theta^n + b\,\theta^{n-1} + c\,\theta^{n-2}\cdots + p\,\theta + q - z\theta^n}$$

The pursuit of the coefficient of θ^i in the development of this fraction, is reduced thus to determining, whatever be r, the coefficient of θ^r in the development of the fraction

$$\frac{1}{a\,\theta^n + b\,\theta^{n-1} + c\,\theta^{n-2}\cdots + p\,\theta + q - z\theta^n}.$$

For this, let us consider generally the fraction $\frac{P}{Q}$, P and Q being rational and integral functions of θ, the first being of an inferior order to the second. Let us suppose that Q has a factor $\theta-\alpha$ raised to a power s, so that we have

$$Q = (\theta-\alpha)^s R;$$

R being a rational and integral function of θ. We can decompose the fraction $\frac{P}{Q}$ into two others $\frac{A}{(\theta-\alpha)^s} + \frac{B}{R}$, A and B being rational and integral functions of θ, the first, of order $s-1$, and the second, of an order inferior to ⟨the one of⟩ R; because it is clear that by substituting for A and B, some functions of this nature, with some undetermined coefficients; by reducing next the two fractions to the same denominator, which becomes then equal to Q; by equating finally the sum of their numerators to P; the comparison of the powers similar to θ, will give as many equations as there are undetermined coefficients. This premised, the equation

$$\frac{A}{(\theta-\alpha)^s} + \frac{B}{R} = \frac{P}{(\theta-\alpha)^s R}$$

gives [22]

$$A = \frac{P}{R} - \frac{B(\theta-\alpha)^s}{R}.$$

If we consider A, B, P and R as some rational and integral functions of $\theta-\alpha$, A will be a function of order $s-1$, and consequently it will be equal to the development of $\frac{P}{R}$, into a series ordered with respect to the powers of $\theta-\alpha$, provided that we stop ourselves at the power $s-1$ inclusively. Let therefore

$$\frac{P}{R} = u_0 + u_1(\theta-\alpha) + u_2(\theta-\alpha)^2 + \text{etc.};$$

we will have

$$\frac{A}{(\theta-\alpha)^s} = \frac{u_0}{(\theta-\alpha)^s} + \frac{u_1}{(\theta-\alpha)^{s-1}} + \frac{u_2}{(\theta-\alpha)^{s-2}} + \text{etc.};$$

by rejecting the positive powers of $\theta-\alpha$; $\frac{A}{(\theta-\alpha)^s}$ is, consequently, equal to the coefficient of t^{s-1} in the development of the function

$$\frac{u_0 + u_1 t + u_2 t^2 + \text{etc.}}{\theta-\alpha-t}.$$

If we name P' and R' that which P and R become when we change $\theta-\alpha$ into t there, or, that which reverts to the same, θ into $t+\alpha$; we will have

$$\frac{P'}{R'} = u_0 + u_1 t + u_2 t^2 + \text{etc.};$$

hence, $\frac{A}{(\theta-\alpha)^s}$ is equal to the coefficient of t^{s-1} in the development of

$$\frac{P'}{R'(\theta-\alpha-t)};$$

it is therefore equal to

$$\frac{1}{1.2.3\ldots(s-1)dt^{s-1}} \cdot d^{s-1} \cdot \frac{P'}{R'(\theta-\alpha-t)},$$

provided that we suppose t null after the differentiations. Now, the coefficient of θ^r in

$$\frac{P'}{R'(\theta-\alpha-t)}$$

being equal to [23]

$$-\frac{P'}{R'(\alpha+t)^{r+1}},$$

this same coefficient in

$$\frac{1}{1.2.3\ldots(s-1)dt^{s-1}} \cdot d^{s-1}\frac{P'}{R'(\theta-\alpha-t)}$$

will be

$$-\frac{1}{1.2.3\ldots(s-1)dt^{s-1}} \cdot d^{s-1}\frac{P'}{R'(\alpha+t)^{r+1}},$$

t being supposed null after the differentiations; this last quantity is therefore the coefficient of θ^r in the development of $\frac{A}{(\theta-\alpha)^s}$. If we restore in P' and R', $\theta-\alpha$ in place of t, that which changes them into P and R, we will have

$$\frac{d^{s-1}\frac{P'}{R'(\alpha+t)^{r+1}}}{dt^{s-1}} = \frac{d^{s-1}\frac{P}{R\theta^{r+1}}}{d\theta^{s-1}},$$

provided that we suppose $\theta=\alpha$, after the differentiations in the second member of this equation; the function

$$-\frac{1}{1.2.3\ldots(s-1)}\frac{d^{s-1}\frac{P}{R\theta^{r+1}}}{d\theta^{s-1}}$$

is therefore, with this condition, the coefficient of θ^r in the development of the fraction $\frac{A}{(\theta-\alpha)^s}$.

It follows thence that if we suppose

$$Q = a(\theta-\alpha)^s(\theta-\alpha')^{s'}(\theta-\alpha'')^{s''}.\text{etc.},$$

the coefficient of θ^r in the development of the fraction $\frac{P}{Q}$, will be

$$\begin{aligned}
&-\frac{1}{1.2.3\ldots(s-1)d\theta^{s-1}}\cdot d^{s-1}\left(\frac{P}{a\theta^{r+1}(\theta-\alpha')^{s'}(\theta-\alpha'')^{s''}.\text{etc.}}\right)\\
&-\frac{1}{1.2.3\ldots(s'-1)d\theta^{s'-1}}\cdot d^{s'-1}\left(\frac{P}{a\theta^{r+1}(\theta-\alpha)^{s}(\theta-\alpha'')^{s''}.\text{etc.}}\right)\\
&-\frac{1}{1.2.3\ldots(s''-1)d\theta^{s''-1}}\cdot d^{s''-1}\left(\frac{P}{a\theta^{r+1}(\theta-\alpha)^{s}(\theta-\alpha')^{s'}.\text{etc.}}\right)\\
&-\ \text{etc.},
\end{aligned}$$

[24] by making $\theta=\alpha$ in the first term; $\theta=\alpha'$ in the second term; $\theta=\alpha''$ in the third term, and so forth.

Now, let there be

$$V = a(\theta-\alpha)(\theta-\alpha')(\theta-\alpha'').\text{etc.}$$

By developing the fraction

$$\frac{1}{V-z\theta^n}$$

into a series ordered with respect to the powers of z, we will have

$$\frac{1}{V}+\frac{z\theta^n}{V^2}+\frac{z^2\theta^{2n}}{V^3}+\frac{z^3\theta^{3n}}{V^4}+\text{etc.}$$

the coefficient of θ^r in the development of the fraction $\frac{1}{V^s}$ is, by that which precedes, equal to

$$-\frac{1}{1.2.3\ldots(s-1)a^s d\theta^{s-1}}\cdot d^{s-1}\left\{\begin{array}{l}\dfrac{1}{\theta^{r+1}(\theta-\alpha')^s(\theta-\alpha'')^s.\text{etc.}}\\ +\dfrac{1}{\theta^{r+1}(\theta-\alpha)^s(\theta-\alpha'')^s.\text{etc.}}\\ +\dfrac{1}{\theta^{r+1}(\theta-\alpha)^s(\theta-\alpha')^s.\text{etc.}}\\ +\text{ etc.}\end{array}\right\}; \qquad (o)$$

provided that after the differentiations, we suppose $\theta=\alpha$ in the first term; $\theta=\alpha'$ in the second term; $\theta=\alpha''$ in the third term, etc. If there is only a single factor $\theta-\alpha$, the function contained between the two parentheses, is reduced to $\frac{1}{\theta^{r+1}}$, θ must be changed into α after the differentiations, that which reduces the quantity (o) to

$$(-1)^s\cdot\frac{(r+1)(r+2)(r+3)\ldots(r+s-1)}{1.2.3\ldots(s-1)a^s}\frac{1}{\alpha^{r+s}}.$$

If in the expression of V, some of the factors $\theta-\alpha$, $\theta-\alpha'$, etc., are raised to some powers higher than unity; for example, if $\theta-\alpha$ is raised to the power m; it will be raised to the power $-ms$ in $\frac{1}{V^s}$; and then it is necessary to change the first term of the quantity (o) in the following, [25]

$$-\frac{1}{1.2.3\ldots(ms-1)a^s}\frac{d^{ms-1}}{d\theta^{ms-1}}\frac{1}{\theta^{r+1}(\theta-\alpha')^s(\theta-\alpha'')^s.\text{etc.}};$$

and in the other terms, it is necessary to change $(\theta-\alpha)^s$, into $(\theta-\alpha)^{ms}$.

Let us represent generally by $Z_r^{(s-1)}$, the quantity (o); the coefficient of θ^i, in the development of the fraction $\frac{1}{V-z\theta^n}$ will be

$$Z_i^{(0)}+Z_{i-n}^{(1)}z+Z_{i-2n}^{(2)}z^2+Z_{i-3n}^{(3)}z^3+\text{etc.};$$

we will have therefore for the coefficient of θ^i, in the development of the first fraction[5] on page [21], or for the value of $\frac{1}{t^i}$,

$$\begin{aligned}
\frac{1}{t^i} =& b\left[Z^{(0)}_{i-n+1} + zZ^{(1)}_{i-2n+1} + z^2Z^{(2)}_{i-3n+1} + z^3Z^{(3)}_{i-4n+1} + \text{etc.}\right]\\
&+ c\left[Z^{(0)}_{i-n+2} + zZ^{(1)}_{i-2n+2} + z^2Z^{(2)}_{i-3n+2} + z^3Z^{(3)}_{i-4n+2} + \text{etc.}\right]\\
&+ e\left[Z^{(0)}_{i-n+3} + zZ^{(1)}_{i-2n+3} + z^2Z^{(2)}_{i-3n+3} + z^3Z^{(3)}_{i-4n+3} + \text{etc.}\right]\\
&+ \text{etc.}\\
&+ \frac{1}{t}\left\{\begin{array}{l} \quad c\left[Z^{(0)}_{i-n+1} + zZ^{(1)}_{i-2n+1} + z^2Z^{(2)}_{i-3n+1} + \text{etc.}\right]\\ + e\left[Z^{(0)}_{i-n+2} + zZ^{(1)}_{i-2n+2} + z^2Z^{(2)}_{i-3n+2} + \text{etc.}\right]\\ + \text{etc.}\end{array}\right\} \qquad \text{(A)}\\
&+ \frac{1}{t^2}\left\{\begin{array}{l} \quad e\left[Z^{(0)}_{i-n+1} + zZ^{(1)}_{i-2n+1} + z^2Z^{(2)}_{i-3n+1} + \text{etc.}\right]\\ + \text{etc.}\end{array}\right\}\\
&+ \text{etc.}\\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&+ \frac{1}{t^{n-1}}q\left[Z^{(0)}_{i-n+1} + zZ^{(1)}_{i-2n+1} + z^2Z^{(2)}_{i-3n+1} + \text{etc.}\right].
\end{aligned}$$

Presently, if we designate by $\nabla \mathrm{y}_x$ the quantity

$$ay_x + by_{x+1} + cy_{x+2} \cdots + qy_{x+n};$$

by $\nabla^2\mathrm{y}_x$, that which ∇y_x becomes when we change y_x into ∇y_x there; by $\nabla^3\mathrm{y}_x$, that
[26] which $\nabla^2\mathrm{y}_x$ becomes when we change ∇y_x into $\nabla^2\mathrm{y}_x$, and so forth. It is clear by §2, that the coefficient of t^x in the development of $\frac{uz^s}{t^r}$ will be $\nabla^s y_{x+r}$; by multiplying therefore the preceding equation by u and by considering in each term only the coefficient of t^x, that is, by passing again from the generating functions to the coefficients;

[5]This refers to the fraction on 16 of this translation immediately after the page reference [21] in the margin.

we will have

$$\begin{aligned}
y_{x+i} =& y_x \left[bZ^{(0)}_{i-n+1} + cZ^{(0)}_{i-n+2} + eZ^{(0)}_{i-n+3} \cdots + qZ^{(0)}_{i}\right] \\
&+ \nabla y_x \left[bZ^{(1)}_{i-2n+1} + cZ^{(1)}_{i-2n+2} + eZ^{(1)}_{i-2n+3} \cdots + qZ^{(1)}_{i-n}\right] \\
&+ \nabla^2 y_x \left[bZ^{(2)}_{i-3n+1} + cZ^{(2)}_{i-3n+2} + eZ^{(2)}_{i-3n+3} \cdots + qZ^{(2)}_{i-2n}\right] \\
&+ \text{etc.} \\
&+ y_{x+1} \left[cZ^{(0)}_{i-n+1} + eZ^{(0)}_{i-n+2} \cdots + qZ^{(0)}_{i-1}\right] \\
&+ \nabla y_{x+1} \left[cZ^{(1)}_{i-2n+1} + eZ^{(1)}_{i-2n+2} \cdots + qZ^{(1)}_{i-n-1}\right] \\
&+ \text{etc.} \\
&+ y_{x+2} \left[eZ^{(0)}_{i-n+1} \cdots + qZ^{(0)}_{i-2}\right] \\
&+ \nabla y_{x+2} \left[eZ^{(1)}_{i-2n+1} \cdots + qZ^{(1)}_{i-n-2}\right] \\
&+ \text{etc.} \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&+ q y_{x+n-1} Z^{(0)}_{i-n+1} + q\nabla y_{x+n-1} Z^{(1)}_{i-2n+1} \\
&\qquad + q\nabla^2 y_{x+n-1} Z^{(2)}_{i-3n+1} + \text{etc.}
\end{aligned} \tag{B}$$

This formula will serve to interpolate the series of which the ultimate ratio of the terms is that of a recurrent series; because it is clear that in this case, ∇y_x, $\nabla^2 y_x$, etc. are always decreasing, and end by being null in the infinite.

§6. Formula (B) is arrested when we have $\nabla^r y_x = 0$, r being any positive whole number; and then the preceding expression of y_{x+i} becomes the integral of the equation in the finite differences $\nabla^r y_i = 0$, that which is analogous to what we have seen in §3 relative to the equation $\nabla^r y_i = 0$. Let us suppose $\nabla y_i = 0$, or, that which reverts to the same,

$$0 = ay_i + by_{i+1} + cy_{i+2} \cdots + qy_{i+n};$$

if we make x null in formula (B) of the preceding number, it becomes [27]

$$\begin{aligned}
y_i =& y_0 \left[bZ^{(0)}_{i-n+1} + cZ^{(0)}_{i-n+2} + eZ^{(0)}_{i-n+3} \cdots + qZ^{(0)}_{i}\right] \\
&+ y_1 \left[cZ^{(0)}_{i-n+1} + eZ^{(0)}_{i-n+2} \cdots\cdots\cdots + qZ^{(0)}_{i-1}\right] \\
&+ y_2 \left[eZ^{(0)}_{i-n+1} \cdots\cdots\cdots\cdots\cdots\cdots + qZ^{(0)}_{i-2}\right] \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&+ q y_{n-1} Z^{(0)}_{i-n+1};
\end{aligned}$$

y_0, y_1, y_2,. . . ,y_{n-1} are the first n values of y_i; these are the n arbitrary constants that the integral of the equation $\nabla y_i = 0$ introduces.

The value of $Z^{(0)}_{i-n+1}$ is equal to

$$-\frac{1}{a\alpha^{i-n+2}(\alpha-\alpha')(\alpha-\alpha'').\text{etc.}}-\frac{1}{a\alpha'^{i-n+2}(\alpha'-\alpha)(\alpha'-\alpha'').\text{etc.}}-\text{ etc.}$$

Thus V being equal to $a(\theta-\alpha)(\theta-\alpha')(\theta-\alpha'')$ etc.; the first of these terms becomes

$$-\frac{\alpha^{n-2}}{\alpha^i\frac{dV}{d\theta}},$$

provided that we change θ into α in $\frac{dV}{d\theta}$; by having regard therefore only to the term multiplied by $\frac{1}{\alpha^i}$, the preceding expression of y_i will become

$$y_i=-\frac{1}{\alpha^{i+1}\frac{dV}{d\theta}}\left\{\begin{array}{l} \quad y_0(b\alpha^{n-1}+c\alpha^{n-2}+e\alpha^{n-3}\cdots+q)\\ +y_1(c\alpha^{n-1}+e\alpha^{n-2}\cdots+q\alpha)\\ +y_2(e\alpha^{n-1}\cdots+q\alpha^2)\\ \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\ +y_{n-1}q\alpha^{n-1}\end{array}\right\}.$$

By changing successively in the second member of this equation, α into α', α'', etc., and reciprocally; we will have so many terms which, added to the preceding, will form the complete expression of y_i.

Let us name k the function comprehended between the two parentheses, so that
[28] this second member is $-\frac{k}{\alpha^{i+1}\frac{dV}{d\theta}}$. If the two roots α and α' are equals, V will be of this form $(\theta-\alpha)^2L$. We will suppose that α and α', instead of being rigorously equal, differ infinitely little, and that we have $\alpha'=\alpha+d\alpha$. Then the sum of the two terms of y_i relative to the roots α and α' will be

$$-\frac{1}{d\alpha}\left(\frac{k'}{\alpha'^{i+1}L'}-\frac{k}{\alpha^{i+1}L}\right),$$

k' being that which k becomes when we change α into α' there; L and L' being here, that which L becomes when we change θ into α and α'. This quantity is therefore equal to

$$-\frac{d\frac{k}{\alpha^{i+1}L}}{d\alpha};$$

but we have

$$L=\frac{1}{2}\frac{d^2V}{d\theta^2},$$

θ must be changed into α after the differentiations. The sum of the terms of the expression of y_i, relative to the two equal roots, is therefore

$$-\frac{d}{1.2d\alpha}\frac{k}{\alpha^{i+1}\frac{ddV}{d\theta^2}}.$$

We will find in the same manner, that if V contains three equal factors, the sum of the terms of the expression of y_i relative to these three factors is

$$-\frac{d^2}{1.2.1.2.3\,d\alpha^2}\frac{k}{\alpha^{i+1}\frac{d^3V}{d\theta^3}};$$

and so forth. $Z_i^{(0)}$ being, by that which precedes, the coefficient of θ^i in the development of $\frac{1}{V}$; there results from it that y_i is the coefficient of θ^i in the development of the function

$$\frac{\left\{\begin{array}{l} y_0(b\theta^{n-1}+c\theta^{n-2}\cdots+q) \\ +y_1(c\theta^{n-1}+e\theta^{n-2}\cdots+q\theta) \\ +y_2(e\theta^{n-1}\cdots\cdots\cdots+q\theta^2) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ +y_{n-1}q\theta^{n-1} \end{array}\right\}}{a\,\theta^n+b\,\theta^{n-1}+c\,\theta^{n-2}\cdots+p\,\theta+q}.$$

This function is therefore the generating function of y_i or of the principal variable of the equation in the differences $\nabla y_i = 0$. Formula (B) of the preceding section, will give similarly the value of y_i or the complete integral of the equation in the differences $\nabla^2 y_i = 0$; y^0, ∇y_0; y_1, ∇y_1; $\ldots$, y_{n-1}, ∇y_{n-1} will be the $2n$ arbitraries of this integral. The case of the equal roots will be resolved in the same manner as above. We will have by the same formula, the integral of the equations in the differences $\nabla^3 y_i = 0$, $\nabla^4 y_i = 0$, etc., that which shows the analogy which exists between interpolation of the series and the integration of the equations in the differences. [29]

Let $y_i = y'_i + y''_i$, and let us suppose that u' is the generating function of y'_i, and u'' that of y''_i, u being that of y_i; we will have $u = u' + u''$. Let further

$$u'' = \frac{\lambda}{z^s},$$

z being the signification that we have given to it in §5; and let us name X_i the coefficient of t^i in the development of λ; we will have, by §2,

$$X_i = \nabla^s y''_i.$$

Now, we have, by §5,

$$\frac{1}{z^s} = \frac{t^{ns}}{(a\,t^n + b\,t^{n-1} + c\,t^{n-2}\cdots + q)^s};$$

now the coefficient of t^i, in the development of the second member of this equation, is equal to the one of θ^{i-ns} in the development of

$$\frac{1}{(a\,\theta^n + b\,\theta^{n-1} + c\,\theta^{n-2}\cdots + q)^s};$$

and by the preceding article, this coefficient is equal to $Z_{i-ns}^{(s-1)}$; therefore the coefficient of t^i, in the development of $\frac{\lambda}{z^s}$ will be

$$X_{i-ns}Z_0^{(s-1)} + X_{i-ns-1}Z_1^{(s-1)} + X_{i-ns-2}Z_2^{(s-1)}\cdots + X_0 Z_{i-ns}^{(s-1)},$$

[30] or $\Sigma X_r Z^{(s-1)}_{i-ns-r}$, the integral being taken relative to r, from $r = 0$ to $r = i - ns + 1$; this will be the value of y''_i. This premised, if in formula (B) of the preceding section, we suppose $\nabla^s y_i = 0$; it will give, by observing that $y_i = y'_i + y''_i$,

$$\begin{aligned}
y'_i + \Sigma X_r Z^{(s-1)}_{i-ns-r} = y_0 &\left[bZ^{(0)}_{i-n+1} + cZ^{(0)}_{i-n+2} \cdots + qZ^{(0)}_{i}\right] \\
&+ \nabla y_0 \left[bZ^{(1)}_{i-2n+1} + cZ^{(1)}_{i-2n+2} \cdots + qZ^{(1)}_{i-n}\right] \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&+ \nabla^{s-1} y_0 \left[bZ^{(s-1)}_{i-sn+1} + cZ^{(s-1)}_{i-sn+2} \cdots + qZ^{(s-1)}_{i-sn+n}\right] \\
&+ y_1 \left[cZ^{(0)}_{i-n+1} \cdots + qZ^{(0)}_{i-1}\right] \\
&+ \nabla y_1 \left[cZ^{(1)}_{i-2n+1} \cdots + qZ^{(1)}_{i-n-1}\right] \qquad\qquad (C) \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&+ \nabla^{s-1} y_1 \left[cZ^{(s-1)}_{i-sn+1} \cdots + qZ^{(s-1)}_{i-sn+n-1}\right] \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&+ qZ^{(0)}_{i-n+1} y_{n-1} + qZ^{(1)}_{i-2n+1} \nabla y_{n-1} \cdots \\
&\cdots + qZ^{(s-1)}_{i-sn+1} \nabla^{s-1} y_{n-1},
\end{aligned}$$

$y_0, \nabla y_0, \ldots \nabla^{s-1} y_0$; $y_1, \nabla y_1$, etc. being the ns arbitraries of the integral of the equation $\nabla^s y_i = 0$ or

$$\nabla^s y'_i + \nabla^s y''_i = 0;$$

now $\nabla^s y''_i$ being equal to X_i, this equation becomes

$$0 = \nabla^s y'_i + X_i;$$

we will have therefore, by the preceding formula, the integral of the equations linear in the finite differences of which the coefficients are constants, in the case where they have a last term function of i.

The definite integral, relative to r $\Sigma X_r Z^{(s-1)}_{i-ns-r}$, can be easily transformed into a series of indefinite integrals, relative to i; because the general expression of $Z^{(s-1)}_{i-ns-r}$ is formed of ns terms of the form $Ir^\mu \alpha^r$, I being a function of i independent of the variable r; the preceding integral is therefore composed of integrals of the form
[31] $I\Sigma r^\mu \alpha^r X_r$; this last integral must be taken from r null to $r = i - ns + 1$, it is equal to the indefinite integral

$$I\Sigma(i - ns + 1)^\mu \alpha^{i-ns+1} X_{i-ns+1},$$

taken from $i = ns - 1$.

§7. We can give to the expression of $\frac{1}{t^i}$ an infinity of other forms of which many can be utile. Let us give to it, for example, this form

$$\frac{1}{t^i} = Z^{(0)} + \left(\frac{1}{t} - 1\right) Z^{(1)} + \left(\frac{1}{t} - 1\right)^2 Z^{(2)} \cdots + \left(\frac{1}{t} - 1\right)^{n-1} Z^{(n-1)}.$$

We will determine thus the values of $Z^{(0)}$, $Z^{(1)}$, $Z^{(2)}$, etc. We will put first the equation

$$z = a + \frac{b}{t} + \frac{c}{t^2} \cdots + \frac{q}{t^n}$$

under this form, by substituting $\left(\frac{1}{t} - 1 + 1\right)^r$ instead of $\frac{1}{t^r}$, and developing according to the powers of $\frac{1}{t} - 1$,

$$z = a' + b'\left(\frac{1}{t} - 1\right) + c'\left(\frac{1}{t} - 1\right)^2 \cdots + q'\left(\frac{1}{t} - 1\right)^n,$$

and we will have

$$\begin{aligned} a' &= a + b + c \cdots + q \\ b' &= b + 2c + 3e \cdots + nq \\ c' &= c + 3e \cdots\cdots + \frac{n(n-1)}{1.2} q \\ &\text{etc.} \end{aligned}$$

We will multiply next, as previously, the numerator and the denominator of the fraction $\frac{1}{1-\frac{\theta}{t}}$ by

$$(a - z)\theta^n + b\theta^{n-1} + c\theta^{n-2} \cdots + p\theta + q,$$

by observing to substitute into the numerator, 1° in place of z,

$$a' + b'\left(\frac{1}{t} - 1\right) + c'\left(\frac{1}{t} - 1\right)^2 + \text{etc.}$$

2° In place of $a\theta^n + b\theta^{n-1} + c\theta^{n-2} + \text{etc.}$, the quantity [32]

$$\theta^n \left[a' + b'\left(\frac{1}{\theta} - 1\right) + c'\left(\frac{1}{\theta} - 1\right)^2 + \text{ etc.}\right];$$

if moreover we make, for brevity,

$$\frac{1}{t} - 1 = \frac{1}{t'};$$

we will have

$$\frac{b'\theta^{n-1}\left(1 - \theta - \frac{\theta}{t'}\right) + c'\theta^{n-2}\left[(1-\theta)^2 - \frac{\theta^2}{t'^2}\right] \cdots + q\left[(1-\theta)^n - \frac{\theta^n}{t'^n}\right]}{\left(1 - \frac{\theta}{t}\right)\left(a\theta^n + b\theta^{n-1} + c\theta^{n-2} \cdots + p\theta + q - z\theta^n\right)};$$

by dividing the numerator and the denominator of the preceding fraction, by $1 - \frac{\theta}{t}$, it is reduced to this one,

$$\frac{\left\{\begin{array}{l}\theta^{n-1}\left[b'+c'\left(\frac{1}{\theta}-1\right)+e'\left(\frac{1}{\theta}-1\right)^2\cdots+q\left(\frac{1}{\theta}-1\right)^{n-1}\right]\\ +\frac{\theta^{n-1}}{t'}\left[c'+e'\left(\frac{1}{\theta}-1\right)\cdots+q\left(\frac{1}{\theta}-1\right)^{n-2}\right]\\ +\frac{\theta^{n-1}}{t'^2}\left[e'\cdots+q\left(\frac{1}{\theta}-1\right)^{n-3}\right]\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ +\frac{q\theta^{n-1}}{t'^{n-1}}\end{array}\right\}}{a\,\theta^n+b\,\theta^{n-1}+c\,\theta^{n-2}\cdots+p\,\theta+q-z\,\theta^n}$$

Thence is easy to conclude that if we conserve to $Z_r^{(s-1)}$, the same signification that we have given to it in §5, and if we consider that by designating q_i the coefficient of θ^i in the development of any function of θ, this same coefficient in the development
[33] of this function multiplied by $\left(\frac{1}{\theta}-1\right)^\mu$, will be, by §2, equal to $\triangle^\mu q_i$; we will have

$$\begin{aligned}\frac{1}{t^i} =& b'Z_{i-n+1}^{(0)}+b'zZ_{i-2n+1}^{(1)}+b'z^2Z_{i-3n+1}^{(2)}+\text{ etc.}\\ &+c'\triangle Z_{i-n+1}^{(0)}+c'z\triangle Z_{i-2n+1}^{(1)}+c'z^2\triangle Z_{i-3n+1}^{(2)}+\text{etc.}\\ &+e'\triangle^2 Z_{i-n+1}^{(0)}+e'z\triangle^2 Z_{i-2n+1}^{(1)}+e'z^2\triangle^2 Z_{i-3n+1}^{(2)}+\text{etc.}\\ &\quad\dots\dots\dots\dots\dots\dots\dots\dots\dots,\\ &+q\triangle^{n-1}Z_{i-n+1}^{(0)}+qz\triangle^{n-1}Z_{i-2n+1}^{(1)}+qz^2\triangle^{n-1}Z_{i-3n+1}^{(2)}+\text{etc.}\\ &+\frac{1}{t'}\left\{\begin{array}{l}\quad c'Z_{i-n+1}^{(0)}+c'zZ_{i-2n+1}^{(1)}+\text{etc.}\\ +e'\triangle Z_{i-n+1}^{(0)}+e'z\triangle Z_{i-2n+1}^{(1)}+\text{etc.}\\ +\text{ etc.}\end{array}\right\}\\ &+\frac{1}{t'^2}\left\{\begin{array}{l}\quad e'Z_{i-n+1}^{(0)}+e'zZ_{i-2n+1}^{(1)}+\text{etc.}\\ +\text{ etc.}\end{array}\right\}\\ &\quad\dots\dots\dots\dots\dots\dots\dots\dots\dots,\\ &+\frac{q}{t'^{n-1}}\left\{Z_{i-n+1}^{(0)}+zZ_{i-2n+1}^{(1)}+\text{etc.}\right\}.\end{aligned}$$

Presently, it is clear, by §2, that the coefficient of t^x in the development of the function $\frac{uz^s}{t'^r}$ is $\triangle^r\nabla^s y_x$; the preceding equation will give therefore, by multiplying its two members by u, and by passing again from the generating functions to their coefficients,

$$\begin{aligned} y_{x+i} = & y_x \left[b' Z^{(0)}_{i-n+1} + c' \triangle Z^{(0)}_{i-n+1} + e' \triangle^2 Z^{(0)}_{i-n+1} \cdots + q \triangle^{n-1} Z^{(0)}_{i-n+1}\right] \\ & + \nabla y_x \left[b' Z^{(1)}_{i-2n+1} + c' \triangle Z^{(1)}_{i-2n+1} + e' \triangle^2 Z^{(1)}_{i-2n+1} \cdots + q \triangle^{n-1} Z^{(1)}_{i-2n+1}\right] \\ & + \nabla^2 y_x \left[b' Z^{(2)}_{i-3n+1} + c' \triangle Z^{(2)}_{i-3n+1} + e' \triangle^2 Z^{(2)}_{i-3n+1} \cdots + q \triangle^{n-1} Z^{(2)}_{i-3n+1}\right] \\ & + \text{etc.} \\ & + \triangle y_x \left[c' Z^{(0)}_{i-n+1} + e' \triangle Z^{(0)}_{i-n+1} \cdots + q \triangle^{n-2} Z^{(0)}_{i-n+1}\right] \\ & + \triangle\nabla y_x \left[c' Z^{(1)}_{i-2n+1} + e' \triangle Z^{(1)}_{i-2n+1} \cdots + q \triangle^{n-2} Z^{(1)}_{i-2n+1}\right] \\ & + \triangle\nabla^2 y_x \left[c' Z^{(2)}_{i-3n+1} + e' \triangle Z^{(2)}_{i-3n+1} \cdots + q \triangle^{n-2} Z^{(2)}_{i-3n+1}\right] \\ & + \text{etc.} \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & + q Z^{(0)}_{i-n+1} \triangle^{n-1} y_x + q Z^{(1)}_{i-2n+1} \triangle^{n-1} \nabla y_x + q Z^{(2)}_{i-3n+1} \triangle^{n-1} \nabla^2 y_x + \text{etc.}, \end{aligned}$$

the characteristic ∇ is related to the variable x, and the characteristic $\triangle$ is related to the two variables x and i. [34]

§8. Let us suppose in the preceding formula, x and i infinitely great, in a way that we have

$$i = \frac{x'}{dx'}; \qquad x = \frac{\varpi}{dx'};$$

y_{x+i} will become a function of $\varpi + x'$, a function which we will designate by $\phi(\varpi + x')$. Let us suppose, moreover,

$$a' = a''; \quad b' = \frac{b''}{dx'}; \quad c' = \frac{c''}{dx'^2}; \quad \ldots \quad q = \frac{q''}{dx'^n};$$

the equation

$$0 = a' + b'\left(\frac{1}{\theta} - 1\right) + c'\left(\frac{1}{\theta} - 1\right)^2 + \text{etc.}$$

will become

$$0 = a'' + \frac{b''}{dx'}\left(\frac{1}{\theta} - 1\right) + \frac{c''}{dx'^2}\left(\frac{1}{\theta} - 1\right)^2 \cdots + \frac{q''}{dx'^n}\left(\frac{1}{\theta} - 1\right)^n.$$

This last equation gives for $\theta - 1$, n roots $f dx'$, $f' dx'$, $f'' dx'$, etc., and consequently for θ, the n values

$$\theta = 1 + f dx'; \quad \theta = 1 + f' dx'; \quad \theta = 1 + f'' dx'; \quad \text{etc.}$$

Now if we suppose $\theta = 1 + h\,dx'$, we will have, i being supposed infinite,

$$\frac{1}{\theta^i} = \frac{1}{(1 + h\,dx')^i} = 1 - ih\,dx' + \frac{i^2}{1.2} h^2 dx'^2 - \text{ etc.} = c^{-hx'},$$

c being the number of which the hyperbolic logarithm is unity. Besides the quantity a is, by the preceding section, equal to $a' - b' + c' -$ etc., and consequently equal to $a'' - \frac{b''}{dx'} \cdots \pm \frac{q''}{dx'^n}$, a value which is reduced to its last term, which surpasses infinitely the others; the expression of $Z_r^{(s-1)}$ of §5 becomes, by changing r into $i-1$,

$$Z_{i-1}^{(s-1)} = -\frac{dx'}{1.2.3\cdots(s-1)(\pm q'')^s dh^{s-1}} \cdot d^{s-1} \left\{ \begin{array}{l} \dfrac{c^{-hx'}}{(h-f')^s(h-f'')^s.\text{etc.}} \\ +\dfrac{c^{-hx'}}{(h-f)^s(h-f'')^s.\text{etc.}} \\ +\dfrac{c^{-hx'}}{(h-f)^s(h-f')^s.\text{etc.}} \\ +\text{etc.} \end{array} \right\},$$

[35] the difference d^{s-1} being taken by making only h vary and by substituting after the differentiations, f in place of h in the first term, f' in place of h in the second term, and so forth. Let us name $X^{(s-1)}dx'$ the preceding quantity; we will have, to the near infinitely small, μ being a finite number

$$Z_{i\pm\mu}^{(s-1)} = Z_{i-1}^{(s-1)} = X^{(s-1)}dx'.$$

Moreover we have $y_x = \phi(\varpi)$; and the characteristic $\triangle$ of the finite differences must be changed into the characteristic d of the infinitely small differences; so that the equation

$$\nabla y_x = ay_x + by_{x+1} + cy_{x+2} + \text{etc.}$$

or, that which returns to the same, this here

$$\nabla y_x = a'' + \frac{b''}{dx'}\triangle y_x + \frac{c''}{dx'^2}\triangle^2 y_x + \text{etc.}$$

becomes, by changing dx' into $d\varpi$ there;

$$\nabla y_x = a'' + b''\frac{d\phi(\varpi)}{d\varpi} + c''\frac{d^2\phi(\varpi)}{d\varpi^2} \cdots + q''\frac{d^n\phi(\varpi)}{d\varpi^n}.$$

The expression of y_{x+i} found in the preceding article, will become therefore

$$\begin{aligned} \phi(\varpi + x') = {} & \phi(\varpi)\left(b''X^{(0)} + c''\frac{dX^{(0)}}{dx'} + e''\frac{d^2X^{(0)}}{dx'^2} \cdots + q''\frac{d^{n-1}X^{(0)}}{dx'^{n-1}}\right) \\ & + \nabla\phi(\varpi)\left(b''X^{(1)} + c''\frac{dX^{(1)}}{dx'} + e''\frac{d^2X^{(1)}}{dx'^2} \cdots + q''\frac{d^{n-1}X^{(1)}}{dx'^{n-1}}\right) \\ & + \nabla^2\phi(\varpi)\left(b''X^{(2)} + c''\frac{dX^{(2)}}{dx'} + e''\frac{d^2X^{(2)}}{dx'^2} \cdots + q''\frac{d^{n-1}X^{(2)}}{dx'^{n-1}}\right) \\ & + \text{etc.} \\ & + \frac{d\phi(\varpi)}{d\varpi}\left(c''X^{(0)} + e''\frac{dX^{(0)}}{dx'} \cdots + q''\frac{d^{n-2}X^{(0)}}{dx'^{n-2}}\right) \end{aligned}$$

$$
\begin{aligned}
&+\frac{d\nabla\phi(\varpi)}{d\varpi}\left(c''X^{(1)}+e''\frac{dX^{(1)}}{dx'}\cdots+q''\frac{d^{n-2}X^{(1)}}{dx'^{n-2}}\right)\\
&+\text{etc.}\\
&+\frac{d^2\phi(\varpi)}{d\varpi^2}\left(e''X^{(0)}\cdots+q''\frac{d^{n-3}X^{(0)}}{dx'^{n-3}}\right)\\
&+\frac{d^2\nabla\phi(\varpi)}{d\varpi^2}\left(e''X^{(1)}\cdots+q''\frac{d^{n-3}X^{(1)}}{dx'^{n-3}}\right)\\
&+\text{etc.}\\
&\quad\dots\dots\dots\dots\dots\dots\dots\dots\\
&+q''\frac{d^{n-1}\phi(\varpi)}{d\varpi^{n-1}}X^{(0)}+q''\frac{d^{n-1}\nabla\phi(\varpi)}{d\varpi^{n-1}}X^{(1)}\\
&\qquad\qquad+q''\frac{d^{n-1}\nabla^2\phi(\varpi)}{d\varpi^{n-1}}X^{(2)}+\text{etc.}
\end{aligned}
$$

This formula will serve to interpolate the series of which the ultimate ratio of the terms is that of a linear equation in the infinitely small differences with constant coefficients. [36]

If we have

$$\nabla^s\phi(\varpi+x')=0,$$

the formula is terminated and gives the value of $\phi(\varpi+x')$, or the integral of the preceding differential equation; $\phi(\varpi)$, $\frac{d\phi(\varpi)}{d\varpi}$, etc.; $\nabla\phi(\varpi)$, $\frac{d\nabla\phi(\varpi)}{d\varpi}$, etc.; $\nabla^2\phi(\varpi)$, $\frac{d\nabla^2\phi(\varpi)}{d\varpi}$, etc. being the ns arbitraries of the integral.

Let us suppose that we have the differential equation

$$0=\nabla^s\phi(\varpi+x')-V_{x'},$$

$V_{x'}$ being a given function of x'; it is necessary, by §6, to add to the preceding expression of $\phi(\varpi+x')$, the term $\int V_r X^{(s-1)}_{x'-r}dr$, $X^{(s-1)}_{x'}$ being the same function of x' as $X^{(s-1)}$. The integral relative to r, must be taken from $r=0$ to $r=x'$. This definite integral can, by the section cited, be transformed into indefinite integrals relative to x'.

Concerning the transformation of series.

§9. The theory of generating functions can serve further to transform the series into others which follow a given law. Let us consider the infinite series

$$y_0+y_1\alpha+y_2\alpha^2\cdots+y_x\alpha^x+\text{etc.};\qquad\text{(V)}$$

and let us name, as above, u the sum of the infinite series

$$y_0+y_1\alpha t+y_2\alpha^2t^2\cdots+y_x\alpha^xt^x+\text{etc.};$$

the coefficient of t^x in the development of the fraction $\frac{u}{1-\frac{1}{t}}$, will be equal to the sum of the proposed series (V), taken from the term $y_x\alpha^x$ inclusively, to infinity. Let generally z be any function of $\frac{1}{t}$, and let us name $\Pi y_x\alpha^x$ the coefficient of t^x in uz. [37]

The coefficients of t^x in uz^2, uz^3, etc. will be $\Pi^2 y_x\alpha^x$, $\Pi^3 y_x\alpha^x$, etc. This premised, we will multiply the numerator and the denominator of the fraction $\frac{u}{1-\frac{1}{t}}$ by $k-z$, and we will take for k that which z becomes when we make t there equal to unity; $k-z$ will be divisible then by $1-\frac{1}{t}$. Let

$$h+\frac{h^{(1)}}{t}+\frac{h^{(2)}}{t^2}+\frac{h^{(3)}}{t^3}+\text{etc.}$$

be the quotient of this division; we will have

$$\begin{aligned}\frac{u}{1-\frac{1}{t}}=\frac{u.h}{k}&\left(1+\frac{z}{k}+\frac{z^2}{k^2}++\frac{z^3}{k^3}+\text{etc.}\right)\\ &+\frac{u.h^{(1)}}{kt}\left(1+\frac{z}{k}+\frac{z^2}{k^2}+\text{etc.}\right)\\ &+\frac{u.h^{(2)}}{kt^2}\left(1+\frac{z}{k}+\frac{z^2}{k^2}+\text{etc.}\right)\\ &+\text{etc.};\end{aligned}$$

that which gives, by passing again from the generating functions to the coefficients,

$$\begin{aligned}Sy_x\alpha^x=&\frac{hy_x\alpha^x}{k}+\frac{h\Pi(y_x\alpha^x)}{k^2}+\frac{h\Pi^2(y_x\alpha^x)}{k^3}+\text{etc.}\\ &+\frac{h^{(1)}y_{x+1}\alpha^{x+1}}{k}+\frac{h^{(1)}\Pi(y_{x+1}\alpha^{x+1})}{k^2}+\text{etc.}\\ &+\frac{h^{(2)}y_{x+2}\alpha^{x+2}}{k}+\frac{h^{(2)}\Pi(y_{x+2}\alpha^{x+2})}{k^2}+\text{etc.}\\ &+\text{etc.}\end{aligned}$$

The sign S designates the sum of the terms from x inclusively, to infinity. Let us suppose now

$$z=a+\frac{b}{\alpha t}+\frac{c}{\alpha^2t^2}+\frac{e}{\alpha^3t^3}+\text{etc.};$$

[38] we will have

$$\Pi(y_x\alpha^x)=\alpha^x(ay_x+by_{x+1}+cy_{x+2}+ey_{x+3}+\text{etc.}).$$

By designating by ∇y_x the quantity $ay_x+by_{x+1}+\text{etc.}$, we will have

$$\Pi(y_x\alpha^x)=\alpha^x\nabla y_x;$$

and generally we will have

$$\Pi^r(y_x\alpha^x)=\alpha^x\nabla^r y_x.$$

We have next

$$k=a+\frac{b}{\alpha}+\frac{c}{\alpha^2}+\frac{e}{\alpha^3}+\text{etc.};$$

this which gives

$$\begin{aligned} h &= \frac{b}{\alpha} + \frac{c}{\alpha^2} + \frac{e}{\alpha^3} + \text{etc.}, \\ h^{(1)} &= \frac{c}{\alpha^2} + \frac{e}{\alpha^3} + \text{etc.}, \\ h^{(2)} &= \frac{e}{\alpha^3} + \text{etc.}, \\ &\text{etc.}; \end{aligned}$$

we will have therefore

$$\begin{aligned} Sy_x\alpha^x = &\frac{\left(\frac{b}{\alpha} + \frac{c}{\alpha^2} + \frac{e}{\alpha^3} + \text{etc.}\right)}{k}\alpha^x \left(y_x + \frac{\nabla y_x}{k} + \frac{\nabla^2 y_x}{k^2} + \text{etc.}\right) \\ &+ \frac{\left(\frac{c}{\alpha} + \frac{e}{\alpha^2} + \text{etc.}\right)}{k}\alpha^x \left(y_{x+1} + \frac{\nabla y_{x+1}}{k} + \frac{\nabla^2 y_{x+1}}{k^2} + \text{etc.}\right) \\ &+ \frac{\left(\frac{e}{\alpha} + \text{etc.}\right)}{k}\alpha^x \left(y_{x+2} + \frac{\nabla y_{x+2}}{k} + \frac{\nabla^2 y_{x+2}}{k^2} + \text{etc.}\right) \\ &+ \text{etc.} \end{aligned}$$

By making $x = 0$, we will have one transformed from the series proposed, of which the terms follow another law; and if the quantities ∇y_x, $\nabla^2 y_x$, ... are decreasing, this series will be convergent. It will be terminated anytime that we have $\nabla^r y_x = 0$; that which will take place when the proposed will be a recurrent series. We will have therefore thus the sum of the recurrent series, by counting from any term $y_x\alpha^x$, and consequently we will have also the sum of their terms, comprehended between any two terms $y_x\alpha^x$ and $y_{x'}\alpha^{x'}$.

Theorems on the development of functions and of their differences, into series. [39]

§10. By applying to some particular functions, the general principles exposed in §1, we will have an infinity of theorems on the development of functions, into series. We are going to present here the most remarkable.

We have generally

$$u\left(\frac{1}{t^i} - 1\right)^n = u\left[\left(1 + \frac{1}{t} - 1\right)^i - 1\right]^n.$$

Now it is clear that the coefficient of t^x in the first member of this equation, is the n^{th} difference of y_x, x varying by i; because this coefficient in $u\left(\frac{1}{t^i} - 1\right)$ is $y_{x+i} - y_x$ or $'\triangle y_x$, by designating by the characteristic $'\triangle$, the finite differences, when x varies by the quantity i; whence it is easy to conclude that this same coefficient, in the development of $u\left(\frac{1}{t^i} - 1\right)^n$ is $'\triangle^n y_x$. Moreover if we develop $u\left[\left(1 + \frac{1}{t} - 1\right)^i - 1\right]^n$ according to the powers of $\frac{1}{t} - 1$, the coefficients of t^x in the developments of $u\left(\frac{1}{t} - 1\right)$, $u\left(\frac{1}{t} - 1\right)^2$, etc. are, by §2, $\triangle y_x$, $\triangle^2 y_x$, etc.; so that this coefficient, in $u\left[\left(1 + \frac{1}{t} - 1\right)^i - 1\right]^n$, is $[(1 + \triangle y_x)^i - 1]^n$, provided that in the development of this quantity, we apply to the characteristic $\triangle$, the exponents of the powers of $\triangle y_x$, and that thus in place of any power $(\triangle y_x)^r$, we write $\triangle^r y_x$; we will have therefore with this condition,

$$'\triangle^n y_x = [(1 + \triangle y_x)^i - 1]^n. \tag{1}$$

If we designate by the characteristic $'\Sigma$ the finite integral, when x varies by i, $'\Sigma^n y_x$ will be, by §2, the coefficient of t^x in the development of the function $u\left(\frac{1}{t^i} - 1\right)^{-n}$,
[40] by setting aside arbitrary constants which the integration introduces; now we have

$$u\left(\frac{1}{t^i} - 1\right)^{-n} = u\left[\left(1 + \frac{1}{t} - 1\right)^i - 1\right]^{-n};$$

moreover, the coefficient of t^x in $u\left(\frac{1}{t} - 1\right)^{-r}$ is $\Sigma^r y_x$, by setting aside arbitrary constants; this coefficient in $u\left(\frac{1}{t} - 1\right)^r$ is $\triangle^r y_x$; we will have therefore

$$'\Sigma^n y_x = [(1 + \triangle y_x)^i - 1]^{-n}; \tag{2}$$

provided that in the development of the second member of this equation, we apply to the characteristic $\triangle$, the exponents of the powers of $\triangle y_x$; that we change the negative differences into integrals, and that we substitute y_x in place of $\triangle^0 y_x$; and as this development contains the integral $\Sigma^n y_x$, which can be counted to contain n arbitrary constants; equation (2) is still true, by having regard to the arbitrary constants.

We can observe that this equation is deduced from equation (1), by making in that here, n negative and by changing the negative differences into integrals; that is, by writing $'\Sigma^n y_x$ in place of $'\triangle^n y_x$ in the first member; and generally in the development of the second member, $\Sigma^r y_x$ in place of $\triangle^{-r} y_x$.

Equations (1) and (2) would equally hold, if x, instead of varying by unity in $\triangle y_x$, varied by any quantity ϖ, provided that the variation of x in $'\triangle y_x$ is equal to $i\varpi$. Indeed, it is clear that if in y_x we make $x = \frac{x'}{\varpi}$, x' will vary by ϖ, when x will vary by unity; $\triangle y_x$ will be changed into $\triangle y_{x'}$, the variation of x' being ϖ; and $'\triangle y_x$ will be changed into $'\triangle y_{x'}$, the variation of x' being $i\varpi$. Now if after having substituted these quantities into equations (1) and (2), we suppose ϖ infinitely small and equal to dx', $\triangle y_{x'}$ will be changed into the infinitely small difference $dy_{x'}$. If moreover we make i infinite, and $idx' = \alpha$, α being a finite quantity; the variation of x' in $'\triangle y_{x'}$ will be α; we will have therefore

$$\begin{aligned} '\triangle^n y_{x'} &= \left[(1 + dy_{x'})^i - 1\right]^n; \\ '\Sigma^n y_{x'} &= \frac{1}{\left[(1 + dy_{x'})^i - 1\right]^n}; \end{aligned} \tag{q}$$

[41] now we have

$$\log(1 + dy_{x'})^i = i\log(1 + dy_{x'}) = i\,dy_{x'} = \alpha\frac{dy_{x'}}{dx'};$$

that which gives

$$(1 + dy_{x'})^i = c^{\alpha\cdot\frac{dy_{x'}}{dx'}},$$

c being the number of which the hyperbolic logarithm is unity; we have therefore

$$'\triangle^n y_{x'} = \left(c^{\alpha\cdot\frac{dy_{x'}}{dx'}} - 1\right)^n, \tag{3}$$

$$'\Sigma^n y_{x'} = \frac{1}{\left(c^{\alpha\cdot\frac{dy_{x'}}{dx'}} - 1\right)^n}; \tag{4}$$

by taking care to apply to the characteristic d, the exponents of the powers of $dy_{x'}$; by changing the negative differences into integrals, and the quantity $d^0 y_{x'}$ into $y_{x'}$.

We can give to equation (3) this singular form which will be useful to us in the following.

$$'\triangle^n y_{x'} = \left(c^{\frac{\alpha}{2}\cdot\frac{dy_{x'+\frac{n\alpha}{2}}}{dx'}} - c^{-\frac{\alpha}{2}\cdot\frac{dy_{x'+\frac{n\alpha}{2}}}{dx'}}\right)^n.$$

Indeed, it gives

$$'\triangle^n y_{x'} = c^{\frac{n\alpha}{2}\cdot\frac{dy_{x'}}{dx'}}\left(c^{\frac{\alpha}{2}\cdot\frac{dy_{x'}}{dx'}} - c^{-\frac{\alpha}{2}\cdot\frac{dy_{x'}}{dx'}}\right)^n.$$

Let us consider any term of the development of $\left(c^{\frac{\alpha}{2}\cdot\frac{dy_{x'}}{dx'}} - c^{-\frac{\alpha}{2}\cdot\frac{dy_{x'}}{dx'}}\right)^n$, such as $k\left(\frac{dy_{x'}}{dx'}\right)^r$. By multiplying it by $c^{\frac{n\alpha}{2}\cdot\frac{dy_{x'}}{dx'}}$, and developing this last quantity, we will have

$$k\frac{d^r}{dx'^r}\left[y_{x'} + \frac{n\alpha}{2}\frac{dy_{x'}}{dx'} + \left(\frac{n\alpha}{2}\right)^2\frac{d^2 y_{x'}}{1.2.dx'^2} + \text{etc.}\right];$$

this quantity is equal to $k\frac{d^r y_{x'+\frac{n\alpha}{2}}}{dx'^r}$; whence it is easy to conclude

$$c^{\frac{n\alpha}{2}\frac{dy_{x'}}{dx'}}\left(c^{\frac{\alpha}{2}\frac{dy_{x'}}{dx'}} - c^{-\frac{\alpha}{2}\frac{dy_{x'}}{dx'}}\right)^n = \left(c^{\frac{\alpha}{2}\frac{dy_{x'+\frac{n\alpha}{2}}}{dx'}} - c^{-\frac{\alpha}{2}\frac{dy_{x'+\frac{n\alpha}{2}}}{dx'}}\right)^n = {}'\triangle^n y_{x'}.$$

If in equations (1) and (2), we suppose further i infinitely small and equal to dx; [42] we will have

$$'\triangle^n y_x = d^n y_x; \qquad '\Sigma^n y_x = \frac{1}{dx^n}\int^n y_x\, dx^n;$$

we have besides

$$(1+\triangle y_x)^i = e^{dx\log(1+\triangle y_x)} = 1 + dx\log(1+\triangle y_x);$$

equations (1) and (2) will become thus

$$\frac{d^n y_x}{dx^n} = [\log(1+\triangle y_x)]^n, \tag{5}$$

$$\int^n y_x dx^n = \frac{1}{[\log(1+\triangle y_x)]^n}. \tag{6}$$

We can observe here a singular analogy between the positive powers and the differences, and between the negative powers and the integrals. The equation

$$'\triangle y_x = e^{\alpha\cdot\frac{dy_x}{dx}} - 1 \tag{o}$$

is the translation of the known theorem of Taylor, when, in the development of its second member, according to the powers of $\frac{dy_x}{dx}$, we apply to the characteristic d, the exponents of these powers. By raising the two members of this equation to the power n, and applying to the characteristics $'\triangle$ and d, the exponents of the powers of $'\triangle y_x$ and of dy_x, we will have equation (3), whence results equation (4), by changing the negative differences into integrals.

The preceding equation gives

$$e^{\alpha \frac{dy_x}{dx}} = 1 + {'\triangle} y_x.$$

By taking the logarithms of each member, we will have

$$\alpha \cdot \frac{dy_x}{dx} = \log(1 + {'\triangle} y_x); \qquad (r)$$

Supposing next $\alpha = 1$, that which changes $'\triangle y_x$ into $\triangle y_x$, and raising the two mem-
[43] bers of that equation, to the power n, we will have equation (5), provided that we apply the exponents of the powers, to the characteristics. We will have equation (6), by making n negative, and changing the negative powers into integrals.

If in the preceding equation (r), we change α into i, we will have

$$\frac{dy_x}{dx} = \log(1 + {'\triangle} y_x)^{\frac{1}{i}};$$

and if we suppose there $\alpha = 1$, we will have

$$\frac{dy_x}{dx} = \log(1 + \triangle y_x).$$

The comparison of these two values of $\frac{dy_x}{dx}$, gives

$$\log(1 + \triangle y_x) = \log(1 + {'\triangle} y_x)^{\frac{1}{i}};$$

whence we deduce

$${'\triangle} y_x = (1 + \triangle y_x)^i - 1.$$

By raising each member to the power n, and applying the exponents of the powers to the characteristics; we will have equation (1), whence equation (2) results, by changing the negative differences into integrals. Equations (1), (2), (3), (4), (5) and (6) result therefore from the theorem of Taylor, set under the form of equation (o), by transforming that equation according to the rules of analysis, provided that in the results we apply to the characteristics, the exponents of the powers, that we change the negative differences into integrals, and that we substitute the variable y_x itself, in the place of its zero differences.[6]

This analogy of the positive powers with the differences, and of the negative powers with the integrals, becomes evident by the theory of generating functions. It holds, as we have seen, to this that the products of the function u, generator of y_x, by the powers $\frac{1}{t^i} - 1$ are the generating functions of the successive finite differences of y_x, x varying by any given quantity i; while the quotients of u, divided by these same powers, are the generating functions of the integrals of y_x.

[6] In his original paper, Laplace credits these same equations to Lagrange [**2**].

By considering, instead of the factor $\frac{1}{t^i} - 1$ and of its powers, the powers of any rational and integral function of $\frac{1}{t}$, we can conclude from it some theorems analogous to the preceding, on the successive *deriveds* of the functions. I name *derived* of a function y_x, each quantity derived from it, such as $ay_x + by_{x+1} + ey_{x+2} + \text{etc.}$ By regarding next this derived function as a new function that I designate by y'_x; the quantity $ay'_x + by'_{x+1} + ey'_{x+2} + \text{etc.}$ will be a second derived from the function y_x, and so forth. When the function $ay_x + by_{x+1} + \text{etc.}$ becomes $-y_x + y_{x+1}$, the derived becomes a finite difference. [44]

Now we have

$$u\left(a + \frac{b}{t} + \frac{e}{t^2} + \frac{h}{t^3} + \text{etc.}\right)^n$$
$$= u\left[a + b\left(1 + \frac{1}{t^{dx}} - 1\right)^{\frac{1}{dx}} + e\left(1 + \frac{1}{t^{dx}} - 1\right)^{\frac{2}{dx}} + \text{etc.}\right]^n \qquad (q)$$

we have next generally, by §2, by designating by ∇y_x the quantity $ay_x + by_{x+1} + ey_{x+2} + \text{etc.}$, $\nabla^n y_x$ for the coefficient of the generating function of the first member of this equation; moreover we have

$$u\left(1 + \frac{1}{t^{dx}} - 1\right)^{\frac{r}{dx}} = u\left[1 + \frac{r}{dx}\left(\frac{1}{t^{dx}} - 1\right) + \frac{r^2}{1.2.dx^2}\left(\frac{1}{t^{dx}} - 1\right)^2 + \text{etc.}\right].$$

The second member of this equation is the generating function of

$$y_x + r\frac{dy_x}{dx} + \frac{r^2}{1.2}\cdot\frac{d^2y_x}{dx^2} + \text{etc.},$$

or of $c^{r\frac{dy_x}{dx}}$; by applying to the characteristic d the exponents of the powers of $\frac{dy_x}{dx}$, and writing y_x in place of $\left(\frac{dy_x}{dx}\right)^0$. Thence we conclude that, under the same conditions, the second member of equation (q) is the generating function of

$$\left[a + bc^{\frac{dy_x}{dx}} + ec^{\frac{2dy_x}{dx}} + hc^{\frac{3dy_x}{dx}} + \text{etc.}\right]^n;$$

and that thus this equation gives, by passing again from the generating functions to the coefficients, [45]

$$\nabla^n y_x = \left[a + bc^{\frac{dy_x}{dx}} + ec^{\frac{2dy_x}{dx}} + hc^{\frac{3dy_x}{dx}} + \text{etc.}\right]^n. \qquad (7)$$

We can thus obtain an infinity of similar results. We will limit ourselves to the following, which will be useful to us in the sequel: $u\left(\frac{1}{\sqrt{t}} - \sqrt{t}\right)^n$ is the generating function of

$$y_{x+\frac{n}{2}} - ny_{x+\frac{n}{2}-1} + \frac{n(n-1)}{1.2}y_{x+\frac{n}{2}-2} - \text{ etc.},$$

or of $\triangle^n y_{x-\frac{n}{2}}$. Moreover, we have

$$u\left(\frac{1}{\sqrt{t}} - \sqrt{t}\right)^n = u\left[\left(1 + \frac{1}{t^{dx}} - 1\right)^{\frac{1}{2dx}} - \left(1 + \frac{1}{t^{dx}} - 1\right)^{-\frac{1}{2dx}}\right]^n;$$

whence we deduce, by passing again by the preceding analysis, from the generating functions to the coefficients

$$\triangle^n y_{x-\frac{n}{2}} = \left(c^{\frac{dy_x}{2dx}} - c^{-\frac{dy_x}{2dx}}\right)^n.$$

§11. I have considered until now, only one function alone y_x of x; but the consideration of the product of many functions of the same variable, leads to diverse curious and useful results of analysis. Let u be a function of t, and y_x the coefficient of t^x in the development of that function; let u' be a function of t', and y'_x the coefficient of t'^x in the development of that function; let further u'' be a function of t'', and y''_x the coefficient of t''^x in its development; and so forth. It is clear that $y_x y'_x y''_x$.etc. will be the coefficient of $t^x t'^x t''^x$.etc. in the development of the product $uu'u''$.etc.; this product will be therefore the generating function of $y_x y'_x y''_x$.etc. The generating function of $y_{x+1}y'_{x+1}y''_{x+1}$.etc. $-\, y_x y'_x y''_x$.etc., or of $\triangle y_x y'_x y''_x$ etc. will be thus

$$uu'u''.\text{etc.}\left(\frac{1}{tt't''.\text{etc.}} - 1\right);$$

[46] and the generating function of $\triangle^n y_x y'_x y''_x$.etc. will be

$$uu'u''.\text{etc.}\left(\frac{1}{tt't''.\text{etc.}} - 1\right)^n.$$

We will prove, as in §2, that the generating function of $\Sigma^n y_x y'_x y''_x$.etc. will be

$$uu'u''.\text{etc.}\left(\frac{1}{tt't''.\text{etc.}} - 1\right)^{-n};$$

that is that we can change n into $-n$ in the generating function of $\triangle^n y_x y'_x$.etc. provided that we change $\triangle^{-n}$ into Σ^n.

Let us apply these results to the two functions y_x and y'_x. The generating function of $\triangle^n y_x y'_x$ will be $uu'\left(\frac{1}{tt'} - 1\right)^n$. We can set it under this form

$$uu'\left[\frac{1}{t} - 1 + \frac{1}{t}\left(\frac{1}{t'} - 1\right)\right]^n;$$

by developing it, it becomes

$$uu'\left\{\left(\frac{1}{t} - 1\right)^n + \frac{n}{t}\left(\frac{1}{t} - 1\right)^{n-1}\left(\frac{1}{t'} - 1\right) + \frac{n(n-1)}{1.2.t^2}\left(\frac{1}{t} - 1\right)^{n-2}\left(\frac{1}{t'} - 1\right)^2 + \text{etc.}\right\};$$

the functions

$$uu'\left(\frac{1}{t} - 1\right)^n; \quad uu'\cdot\frac{1}{t}\left(\frac{1}{t} - 1\right)^{n-1}\left(\frac{1}{t'} - 1\right); \quad uu'\cdot\frac{1}{t^2}\left(\frac{1}{t} - 1\right)^{n-2}\left(\frac{1}{t'} - 1\right)^2; \quad \text{etc.};$$

are respectively generators of the products $y'_x\triangle^n y_x$; $\triangle y'_x\triangle^{n-1}y_{x+1}$; $\triangle^2 y'_x\triangle^{n-2}y_{x+2}$; etc. The equation

$$uu'\left(\frac{1}{tt'} - 1\right)^n = uu'\left[\left(\frac{1}{t} - 1\right)^n + \frac{n}{t}\left(\frac{1}{t} - 1\right)^{n-1}\left(\frac{1}{t'} - 1\right) + \text{etc.}\right]$$

will give therefore, by passing again from the generating functions to the coefficients,

$$\triangle^n y_x y'_x = y'_x \triangle^n y_x + n\triangle y'_x \triangle^{n-1} y_{x+1} + \frac{n(n-1)}{1.2}\triangle^2 y'_x \triangle^{n-2} y_{x+2} + \text{etc.} \tag{8}$$

By changing n into $-n$, we will have [47]

$$\Sigma^n y_x y'_x = y'_x \Sigma^n y_x - n\triangle y'_x \Sigma^{n+1} y_{x+1} + \frac{n(n+1)}{1.2}\triangle^2 y'_x \Sigma^{n+2} y_{x+2} - \text{ etc.} \tag{9}$$

In general, we have

$$\begin{aligned} &uu'u''.\text{etc.}\left(\frac{1}{tt't''.\text{etc.}} - 1\right)^n \\ &\quad = uu'u''.\text{etc.}\left[\left(1+\frac{1}{t}-1\right)\left(1+\frac{1}{t'}-1\right)\left(1+\frac{1}{t''}-1\right).\text{etc.} - 1\right]^n; \end{aligned}$$

that which gives, by passing again from the generating functions to the coefficients,

$$\triangle^n y_x y'_x y''_x.\text{etc.} = [(1+\triangle)(1+\triangle')(1+\triangle'').\text{etc.} - 1]^n, \tag{10}$$

provided that in each term of the development of the second member of this equation, we place immediately after each characteristic $\triangle$, $\triangle'$, $\triangle''$, etc., respectively y_x, y'_x, y''_x, etc., and that we multiply this term by the product of the functions of which the characteristic is not contained at all. Thus in the case of three variables, we will write, instead of $\triangle^r$, the quantity $y'_x y''_x \triangle^r y_x$; instead of $\triangle^r \triangle'^{r'}$, we will write $y''_x \triangle^r y_x \triangle^{r'} y'_x$; instead of $\triangle'^{r'} \triangle''^{r''}$, we will write $y_x \triangle^{r'} y'_x \triangle^{r''} y''_x$; and thus of the remainder.

By making n negative, equation (10) yet subsists, provided that we change the negative differences into integrals.

In the case of the infinitely small differences, the characteristics $\triangle$, $\triangle'$, $\triangle''$, etc. are changed into d, d', d'', etc. Equation (10) becomes thus, by neglecting the differentials of a superior order, relative to those of an inferior order,

$$d^n y_x y'_x y''_x.\text{etc.} = (d + d' + d'' + \text{etc.})^n.$$

This developed equation gives, relative to two functions y_x and y'_x,

$$d^n y_x y'_x = y'_x d^n y_x + n d y'_x d^{n-1} y_x + \frac{n(n-1)}{1.2} d^2 y'_x d^{n-2} y_x + \text{etc.}$$

By making n negative, the negative differences being changed into integrals, we will have [48]

$$\begin{aligned} \int^n y_x y'_x dx^n =& y'_x \int^n y_x dx^n - n\frac{dy'_x}{dx}\int^{n+1} y_x dx^{n+1} \\ &+ \frac{n(n+1)}{1.2}\frac{d^2 y'_x}{dx^2}\int^{n+2} y_x dx^{n+2} - \text{ etc.} \end{aligned}$$

We have

$$uu'u''.\text{etc.}\left(\frac{1}{t^i t'^i t''^i.\text{etc.}}-1\right)^n$$
$$= uu'u''.\text{etc.}\left[\left(1+\frac{1}{t}-1\right)^i\left(1+\frac{1}{t'}-1\right)^i\left(1+\frac{1}{t''}-1\right)^i.\text{etc.}-1\right]^n;$$

by designating therefore by $'\triangle^n y_x y'_x y''_x$.etc., the finite difference of the product $y_x y'_x y''_x$.etc., when x varies by i; the preceding equation will give, by passing again from the generating functions to the coefficients

$$'\triangle^n y_x y'_x y''_x.\text{etc.} = [(1+\triangle)^i(1+\triangle')^i(1+\triangle'')^i.\text{etc.}-1]^n; \tag{11}$$

by observing the conditions prescribed above relative to the characteristics $\triangle$, $\triangle'$, $\triangle''$, etc., and to their powers. This last equation subsists still, by making n negative, provided that we change the negative differences into integrals.

Let us suppose

$$x = \frac{x'}{dx'}, \qquad i = \frac{\alpha}{dx'};$$

y_x, y'_x, etc. will become functions of x', that we will designate by $y_{x'}$, $y'_{x'}$, etc.; equation (11) will give thus the following, by observing that the characteristics $\triangle$, $\triangle'$, etc. are changed into d, d', etc., and that we have

$$(1+dy_{x'})^{\frac{\alpha}{dx'}} = c^{\alpha\frac{dy_{x'}}{dx'}},$$
$$'\triangle^n y_{x'} y'_{x'} y''_{x'}.\text{etc.} = \left(c^{\alpha\frac{dy_{x'}}{dx'}+\alpha\frac{dy'_{x'}}{dx'}+\alpha\frac{dy''_{x'}}{dx'}+\text{etc.}}-1\right)^n; \tag{12}$$

an equation which subsists still by making n negative, and changing the negative differences into integrals.

Let us consider only two variables y_x and y'_x, and let us suppose $y'_x = p^x$; we will have

$$(1+\triangle')^i = p^x + i\triangle p^x + \frac{i(i-1)}{1.2}\triangle^2 p^x + \text{etc.};$$

[49] now we have generally, x varying by unity,

$$\triangle^r p^x = p^x(p-1)^r;$$

we will have therefore

$$(1+\triangle')^i = p^i p^x.$$

Equation (11) will become thus

$$'\triangle^n p^x y_x = p^x[p^i(1+\triangle y_x)^i - 1]^n; \tag{13}$$

by making n negative, we will have

$$'\Sigma^n p^x y_x = \frac{p^x}{[p^i(1+\triangle y_x)^i-1]^n} + ax^{n-1} + bx^{n-2} + \text{etc.}; \tag{14}$$

a, b, etc. being arbitrary constants due to the integration of $p^x y_x$ repeated n times. I add here these constants, to the second member of the preceding equation; because they are implicitly contained in its first term, only when $p = 1$.

If we make in the two preceding equations $x = \frac{x'}{dx'}$, $i = \frac{\alpha}{dx'}$, $p = 1 + dx' \log h$, we will have

$$'\triangle^n h^{x'} y_{x'} = h^{x'} \left[h^\alpha c^{\alpha \frac{dy_{x'}}{dx'}} - 1 \right]^n, \tag{15}$$

$$'\Sigma^n h^{x'} y_{x'} = \frac{h^{x'}}{\left[h^\alpha c^{\alpha \frac{dy_{x'}}{dx'}} - 1 \right]^n} + a' x'^{n-1} + b'^{n-2} + \text{etc.} \tag{16}$$

If in equations (13) and (14), we suppose i infinitely small and equal to dx; $'\triangle^n p^x y_x$ will be changed into $d^n p^x y_x$, and $'\Sigma^n p^x y_x$ will be changed into $\frac{1}{dx^n} \int^n p^x y_x dx^n$; we will have next

$$p^i (1 + \triangle y_x)^i = c^{dx \log[p(1+\triangle y_x)]};$$

we will have therefore

$$[p^i (1 + \triangle y_x)^i - 1]^n = dx^n \{\log[p(1 + \triangle y_x)]\}^n;$$

and equations (13) and (14) will become

$$\frac{d^n p^x y_x}{dx^n} = p^x \{\log[p(1 + \triangle y_x)]\}^n, \tag{17}$$

$$\int^n p^x y_x \, dx^n = \frac{p^x}{\{\log[p(1 + \triangle y_x)]\}^n} + a x^{n-1} + b x^{n-2} + \text{etc.} \tag{18}$$

CHAPTER 2

Concerning generating functions in two variables

§12. Let us name u a function of t and t'; let us suppose that by developing it according to the powers of t and t', it gives the infinite series [50]

$$\begin{array}{l} y_{0,0} \quad +y_{1,0}t \quad +y_{2,0}t^2 \quad \cdots + y_{x,0}t^x \quad +y_{x+1,0}t^{x+1} \quad \cdots + y_{\infty,0}t^{\infty} \\ +y_{0,1}t' +y_{1,1}tt' +y_{2,1}t^2t' \cdots + y_{x,1}t^xt' +y_{x+1,1}t^{x+1}t' \cdots + y_{\infty,1}t^{\infty}t' \\ +y_{0,2}t'^2+y_{1,2}tt'^2+y_{2,2}t^2t'^2\cdots + y_{x,2}t^xt'^2+y_{x+1,2}t^{x+1}t'^2\cdots + y_{\infty,2}t^{\infty}t'^2 \\ +\text{etc.} \end{array}$$

The coefficient of $t^x t'^{x'}$ will be $y_{x,x'}$; u will be therefore the generating function of $y_{x,x'}$.

If we designate by the characteristic $\triangle$, the finite differences, when x alone varies by unity, and by the characteristic $'\triangle$ the differences when x' alone varies by the same quantity, the generating function of $\triangle y_{x,x'}$ will be, by §1, $u\left(\frac{1}{t}-1\right)$ and that of $'\triangle y_{x,x'}$ will be $u\left(\frac{1}{t'}-1\right)$: whence it is easy to conclude that the generating function of $\triangle^i.'\triangle^{i'} y_{x,x'}$ will be

$$u\left(\frac{1}{t}-1\right)^i\left(\frac{1}{t'}-1\right)^{i'}.$$

In general, if we designate by $\nabla y_{x,x'}$ the quantity

$$\begin{array}{rrrr} Ay_{x,x'} + By_{x+1,x'} & +Cy_{x+2,x'} & +\text{etc.} \\ +B'y_{x,x'+1} & +C'y_{x+1,x'+1} & +\text{etc.} \\ & +C''y_{x,x'+2} & +\text{etc.} \\ & & +\text{etc.}; \end{array}$$

If we designate similarly by $\nabla^2 y_{x,x'}$ a function in which $\nabla y_{x,x'}$ enters in the same manner as $y_{x,x'}$ in $\nabla y_{x,x'}$; if we designate further by $\nabla^3 y_{x,x'}$ a function in which $\nabla^2 y_{x,x'}$ enters in the same manner as $y_{x,x'}$ in $\nabla y_{x,x'}$ and so forth; the generating function of $\nabla^n y_{x,x'}$ will be [51]

$$u\left\{\begin{array}{llll} A+\dfrac{B}{t} & +\dfrac{C}{t^2} & +\text{etc.} \\ \quad +\dfrac{B'}{t'} & +\dfrac{C'}{tt'} & +\text{etc.} \\ & +\dfrac{C''}{t'^2} & +\text{etc.} \\ & & +\text{etc.} \end{array}\right\}^n;$$

hence, the generating function of $\triangle^i.'\triangle^{i'}\nabla^n y_{x,x'}$ will be the preceding generating function, multiplied by $\left(\frac{1}{t}-1\right)^i\left(\frac{1}{t'}-1\right)^{i'}$.

s being supposed any function of $\frac{1}{t}$ and of $\frac{1}{t'}$, if we develop s^i according to the powers of these variables, and if we designate by $\frac{k}{t^m t'^{m'}}$ any term of this development, the coefficient of $t^x t'^{x'}$ in $\frac{ku}{t^m t'^{m'}}$ being $ky_{x+m,x'+m'}$, we will have the one of $t^x t'^{x'}$ in us^i, or, that which reverts to the same, we will have $\nabla^i y_{x,x'}$, 1° by substituting into s, y_x in the place of $\frac{1}{t}$, $y_{x'}$ in the place of $\frac{1}{t'}$; 2° by developing that which us^i then becomes according to the powers of y_x and of $y_{x'}$, and by applying respectively to the indices x and x' the exponents of these powers, that is by writing in the place of any term such as $k(y_x)^m(y_{x'})^{m'}$, $ky_{x+m,x'+m'}$ and consequently $ky_{x,x'}$ in the place of the total constant term k, or $k(y_x)^0(y'_{x'})^0$.

If, instead of developing s^i according to the powers of $\frac{1}{t}$ and $\frac{1}{t'}$, we develop it according to the powers of $\frac{1}{t}-1$ and $\frac{1}{t'}-1$, and if we designate by $k\left(\frac{1}{t}-1\right)^m\left(\frac{1}{t'}-1\right)^{m'}$ any term of this development, the coefficient of $t^x t'^{x'}$ in $ku\left(\frac{1}{t}-1\right)^m\left(\frac{1}{t'}-1\right)^{m'}$, being
[52] $k\triangle^m.'\triangle^{m'}y_{x,x'}$; we will have $\nabla^i y_{x,x'}$, 1° by substituting into s, $\triangle y_{x,x'}$ in the place of $\frac{1}{t}-1$ and $'\triangle y_{x,x'}$ in the place of $\frac{1}{t'}-1$; 2° by developing then s^i according to the powers of $\triangle y_{x,x'}$ and of $'\triangle y_{x,x'}$; and by applying to the characteristics $\triangle$ and $'\triangle$, the exponents of these powers, that is by writing, in the place of any term such as $k(\triangle y_{x,x'})^m('\triangle y_{x,x'})^{m'}$, this one $k\triangle^{m'}.'\triangle^{m'}y_{x,x'}$; and consequently $ky_{x,x'}$ in the place of the constant term k.

Let Σ be the characteristic of the finite integrals relative to x, and $'\Sigma$ that of the finite integrals relative to x'; let moreover z be the generating function of $\Sigma^i.'\Sigma^{i'}y_{x,x'}$; we will have $z\left(\frac{1}{t}-1\right)^i\left(\frac{1}{t'}-1\right)^{i'}$ for the generating function of $y_{x,x'}$. This function must, by having regard only to the positive or null powers of t and of t', be reduced to u; we will have thus, by §2,

$$z\left(\frac{1}{t}-1\right)^i\left(\frac{1}{t'}-1\right)^{i'} = u+\frac{a}{t}+\frac{b}{t^2}+\frac{c}{t^3}\cdots+\frac{q}{t^i} + \frac{a'}{t'}+\frac{b'}{t'^2}+\frac{c'}{t'^3}\cdots+\frac{q'}{t'^{i'}},$$

a, b, c,..., q being arbitrary functions of t', and a', b', c',..., q' being arbitrary functions of t; hence

$$z=\frac{ut^it'^{i'}+at^{i-1}t'^{i'}+bt^{i-2}t'^{i'}\cdots+qt'^{i'}+a't^it'^{i'-1}+b_1t^it'^{i'-2}\cdots+q't^i}{(1-t)^i(1-t')^{i'}}.$$

On the interpolation of series in two variables and on the integration of equations linear in partial differences.

§13. $y_{x+i,x'+i'}$ is evidently equal to the coefficient of $t^x t'^{x'}$ in the development of $\frac{u}{t^i t'^{i'}}$; now we have

$$\frac{u}{t^i t'^{i'}} = u\left(1+\frac{1}{t}-1\right)^i \left(1+\frac{1}{t'}-1\right)^{i'}$$

we will have therefore by the preceding section,

$$y_{x+i,x'+i'} = (1+\triangle y_{x,x'})^i (1+{}'\triangle y_{x,x'})^{i'};$$

by developing the second member of this equation, we will have [53]

$$\begin{aligned} y_{x+i,x'+i'} = y_{x,x'} + i\triangle y_{x,x'} &+ \frac{i(i-1)}{1.2}\triangle^2 y_{x,x'} &&+\text{etc.}\\ + i'.{}'\triangle y_{x,x'} &+ ii'\triangle.{}'\triangle y_{x,x'} &&+\text{etc.}\\ &+ \frac{i'(i'-1)}{1.2}{}'\triangle^2 y_{x,x'} &&+\text{etc.}\\ &&&\text{etc.}\end{aligned}$$

Let us now suppose that instead of interpolating according to the differences of the function $y_{x,x'}$, we wish to interpolate according to other laws. For this, let

$$\begin{aligned} z = A &+\tfrac{B}{t} &+\tfrac{C}{t^2} &+\tfrac{D}{t^3} &+\text{etc.}\\ &+\tfrac{B'}{t'} &+\tfrac{C'}{tt'} &+\tfrac{D'}{t^2t'} &+\text{etc.}\\ &&+\tfrac{C''}{t'^2} &+\tfrac{D''}{tt'^2} &+\text{etc.}\\ &&&+\tfrac{D'''}{t'^3} &+\text{etc.}\\ &&&&+\text{etc.}\end{aligned}$$

If we make

$$\begin{aligned} A+ \tfrac{B'}{t'}+ \tfrac{C''}{t'^2}+ \tfrac{D'''}{t'^3} &+\text{etc.} = a,\\ B+ \tfrac{C'}{t'}+ \tfrac{D''}{t'^2} &+\text{etc.} = b,\\ C+ \tfrac{D'}{t'} &+\text{etc.} = c,\\ &\text{etc.,}\end{aligned}$$

we will have for z an expression of this form

$$z = a + \frac{b}{t} + \frac{c}{t^2}\cdots + \frac{l}{t^n}.$$

We suppose here that the coefficient l of the highest power of $\frac{1}{t}$ is constant or independent of t', and that this power is equal or greater than the sum of the powers of $\frac{1}{t}$ and of $\frac{1}{t'}$ in each of the other terms of z. It is easy to conclude from the preceding [54] equation, as in §5, the successive values of $\frac{1}{t^{n+1}}$, $\frac{1}{t^{n+2}}$, $\frac{1}{t^{n+3}}$, etc., as functions of a, b, c, etc. and z; and it is clear that in each term of the expression of $\frac{1}{t^i}$, the highest power of $\frac{1}{t}$ will be less than n, and the sum of the powers of $\frac{1}{t}$ and of $\frac{1}{t'}$ will not surpass i.

Let us consider now formula (A) of §5, and let us suppose that by developing it according to the powers of $\frac{1}{t'}$ the quantity

$$\begin{aligned} &bZ^{(0)}_{i-n+1} + bzZ^{(1)}_{i-2n+1} + \text{etc.} \\ &+cZ^{(0)}_{i-n+2} + czZ^{(1)}_{i-2n+2} + \text{etc.} \\ &+eZ^{(0)}_{i-n+3} + ezZ^{(1)}_{i-2n+3} + \text{etc.} \\ &+ \text{etc.}, \end{aligned}$$

we have

$$\begin{aligned} M + Nz + \text{etc.} \ &+ \frac{1}{t'}(M^{(1)} + N^{(1)}z + \text{etc.}) \\ &+ \frac{1}{t'^2}(M^{(2)} + N^{(2)}z + \text{etc.}) \cdots + \frac{1}{t'^i}M^{(i)}; \end{aligned}$$

the ulterior powers of $\frac{1}{t'}$ vanish of themselves in this development, since the expression of $\frac{1}{t^i}$ must not contain them at all. Let us suppose similarly that by developing the quantity

$$\begin{aligned} &cZ^{(0)}_{i-n+1} + czZ^{(1)}_{i-2n+1} + \text{etc.} \\ &+eZ^{(0)}_{i-n+2} + ezZ^{(1)}_{i-2n+2} + \text{etc.} \\ &+ \text{etc.} \end{aligned}$$

we have

$$M_1 + N_1z + \text{etc.} \ + \frac{1}{t'}(M_1^{(1)} + N_1^{(1)}z + \text{etc.}) \cdots + \frac{1}{t'^{i-1}}M_1^{(i-1)}.$$

Let us suppose further that by developing the quantity

$$\begin{aligned} eZ^{(1)}_{i-n+1} &+ \text{etc.} \\ &+ \text{etc.}, \end{aligned}$$

[55] we have

$$M_2 + N_2z + \text{etc.} \ + \frac{1}{t'}(M_2^{(1)} + N_2^{(1)}z + \text{etc.}) \cdots + \frac{1}{t'^{i-2}}M_2^{(i-2)};$$

and so forth. Formula (A) of §5 will give

$$
\begin{aligned}
\frac{1}{t^i} =& M + Nz + \text{etc.}\\
&+ \frac{1}{t'}(M^{(1)} + N^{(1)}z + \text{etc.})\\
&+ \frac{1}{t'^2}(M^{(2)} + N^{(2)}z + \text{etc.})\\
&\cdots\cdots\cdots\cdots\\
&+ \frac{1}{t'^i}M^{(i)}\\
&+ \frac{1}{t}\left\{\begin{array}{l} M_1 + N_1 z + \text{etc.}\\ +\frac{1}{t'}(M_1^{(1)} + N_1^{(1)}z + \text{etc.})\\ \cdots\cdots\cdots\cdots\\ +\frac{1}{t'^{i-1}}M_1^{(i-1)}\end{array}\right\}\\
&+ \frac{1}{t^2}\left\{\begin{array}{l} M_2 + N_2 z + \text{etc.}\\ +\frac{1}{t'}(M_2^{(1)} + N_2^{(1)}z + \text{etc.})\\ \cdots\cdots\cdots\cdots\\ +\frac{1}{t'^{i-2}}M_2^{(i-2)}\end{array}\right\}\\
&\cdots\cdots\cdots\cdots\\
&+ \frac{1}{t^{n-1}}\left\{\begin{array}{l} M_{n-1} + N_{n-1} z + \text{etc.}\\ +\frac{1}{t'}(M_{n-1}^{(1)} + N_{n-1}^{(1)}z + \text{etc.})\\ \cdots\cdots\cdots\cdots\\ +\frac{1}{t'^{i-n+1}}M_{n-1}^{(i-n+1)}\end{array}\right\}.
\end{aligned}
$$

This premised, if we name $\nabla y_{x,x'}$ the quantity

$$
\begin{aligned}
Ay_{x,x'} + By_{x+1,x'} &+ Cy_{x+2,x'} &&+ \text{etc.}\\
+ B'y_{x,x'+1} &+ C'y_{x+1,x'+1} &&+ \text{etc.}\\
&+ C''y_{x,x'+2} &&+ \text{etc.}\\
& &&+ \text{etc.};
\end{aligned}
$$

the coefficient of $t^x t'^{x'}$ in the development of the quantity $\frac{uz^\mu}{t^r t'^{r'}}$ will be, by the preceding [56]
article, $\nabla^\mu y_{x+r,x'+r'}$; the preceding equation will give consequently, by multiplying it

by u, and by passing from the generating functions to their coefficients,

$$
\begin{aligned}
y_{x+i,x'} = {} & \left\{ \begin{array}{l} \;\; My_{x,x'} + N\nabla y_{x,x'} + \text{etc.} \\ +M^{(1)}y_{x,x'+1} + N^{(1)}\nabla y_{x,x'+1} + \text{etc.} \\ \quad \cdots\cdots\cdots\cdots\cdots\cdots \\ +M^{(i)}y_{x,x'+i} \end{array} \right\} \\
& + \left\{ \begin{array}{l} \;\; M_1 y_{x+1,x'} + N_1\nabla y_{x+1,x'} + \text{etc.} \\ +M_1^{(1)}y_{x+1,x'+1} + N_1^{(1)}\nabla y_{x+1,x'+1} + \text{etc.} \\ \quad \cdots\cdots\cdots\cdots\cdots\cdots \\ +M_1^{(i-1)}y_{x+1,x'+i-1} \end{array} \right\} \\
& \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + \left\{ \begin{array}{l} \;\; M_{n-1} y_{x+n-1,x'} + N_{n-1}\nabla y_{x+n-1,x'} + \text{etc.} \\ +M_{n-1}^{(1)}y_{x+n-1,x'+1} + N_{n-1}^{(1)}\nabla y_{x+n-1,x'+1} + \text{etc.} \\ \quad \cdots\cdots\cdots\cdots\cdots\cdots \\ +M_{n-1}^{(i-n+1)}y_{x+n-1,x'+i-n+1} \end{array} \right\}.
\end{aligned}
$$

§14. If we suppose $\nabla y_{0,x'} = 0$, the preceding equation will give, by making $x = 0$,

$$
\begin{aligned}
y_{i,x'} = {} & My_{0,x'} + M^{(1)}y_{0,x'+1} + M^{(2)}y_{0,x'+2} \cdots + M^{(i)}y_{0,x'+i} \\
& + M_1 y_{1,x'} + M_1^{(1)}y_{1,x'+1} + M_1^{(2)}y_{1,x'+2} \cdots + M_1^{(i-1)}y_{1,x'+i-1} \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + M_{n-1}y_{n-1,x'} + M_{n-1}^{(1)}y_{n-1,x'+1} \cdots\cdots\cdots + M_{n-1}^{(i-n+1)}y_{n-1,x'+i-n+1}
\end{aligned}
$$

$M^{(r)}$, $M_1^{(r)}$, $M_2^{(r)}$, etc. being functions of i and of r. The preceding expression of $y_{i,x'}$ can be set under this very simple form

$$
y_{i,x'} = \Sigma \left\{ \begin{array}{l} M^{(r)}y_{0,x'+r} + M_1^{(r-1)}y_{1,x'+r-1} + M_2^{(r-2)}y_{2,x'+r-2} \\ \qquad \cdots + M_{n-1}^{(r-n+1)}y_{n-1,x'+r-n+1} \end{array} \right\}; \qquad (\lambda)
$$

[57] the integral being taken from $r = 0$ to $r = i+1$ with respect to the first term; from $r = 1$ to $r = i+1$ with respect to the second term, and so forth. This expression of $y_{i,x'}$ will be the complete integral of the equation $\nabla y_{x,x'} = 0$, or

$$
\begin{aligned}
0 = Ay_{i,x'} + By_{i+1,x'} & + Cy_{i+2,x'} \quad \cdots + ly_{i+n,x'} \\
+ B'y_{i,x_{1+1}} & + C'y_{i+1,x'+1} \cdots \\
& + C''y_{i,x'+2} \quad \cdots \\
& \qquad\qquad \cdots \\
& \qquad\qquad + hy_{i,x'+n}.
\end{aligned}
$$

It is clear that $y_{0,x'}$, $y_{1,x'}$, $y_{2,x'}, \ldots$, $y_{n-1,x'}$ are the n arbitrary functions that the integration of the equation $\nabla y_{i,x'} = 0$ introduces. In order to determine them, it is necessary to know immediately, or at least to be able to conclude from the conditions of the problem, the first n vertical ranks of the following table:

$$\begin{array}{llllllllll}
y_{0,0}, & y_{1,0}, & y_{2,0}, & y_{3,0}, & \cdots & y_{i,0}, & y_{i+1,0}, & \cdots & y_{\infty,0}, \\
y_{0,1}, & y_{1,1}, & y_{2,1}, & y_{3,1}, & \cdots & y_{i,1}, & y_{i+1,1}, & \cdots & y_{\infty,1}, \\
y_{0,2}, & y_{1,2}, & y_{2,2}, & y_{3,2}, & \cdots & y_{i,2}, & y_{i+1,2}, & \cdots & y_{\infty,2}, \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
y_{0,x'}, & y_{1,x'}, & y_{2,x'}, & y_{3,x'}, & \cdots & y_{i,x'}, & y_{i+1,x'}, & \cdots & y_{\infty,x'}, \\
y_{0,x'+1}, & y_{1,x'+1}, & y_{2,x'+1}, & y_{3,x'+1}, & \cdots & y_{i,x'+1}, & y_{i+1,x'+1}, & \cdots & y_{\infty,x'+1}, \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
y_{0,\infty}, & y_{1,\infty}, & y_{2,\infty}, & y_{3,\infty}, & \cdots, & y_{i,\infty}, & y_{i+1,\infty}, & \cdots & y_{\infty,\infty}.
\end{array} \qquad (Q)$$

In a great number of problems, the first n vertical ranks are given by some equations in linear finite differences, and consequently by a sequence of terms of the form $Ap^{x'}$. Let us suppose that the expression of $y_{0,x'}$ contains the term $Ap^{x'}$, the corresponding part of $y_{i,x'}$ given by formula (λ) will be

$$Ap^{x'}(M + M^{(1)}p + M^{(2)}p^2 \cdots + M^{(i)}p^i);$$

but the function

$$M + \frac{M^{(1)}}{t'} + \frac{M^{(2)}}{t'^2} \cdots + \frac{M^{(i)}}{t'^i}$$

is the development of

$$bZ^{(0)}_{i-n+1} + cZ^{(0)}_{i-n+2} + \text{etc.},$$

according to the powers of $\frac{1}{t'}$; by changing therefore in this last quantity, $\frac{1}{t'}$ into p, and [58]
naming P that which it then becomes; we will have $APp^{x'}$, for the part of $y_{i,x'}$ which corresponds to the term $Ap^{x'}$. It follows thence that if the value of $y_{0,x'}$ is equal to $Ap^{x'} + A'p'^{x'} + A''p''^{x'} +$ etc., and that if we name P', P'', etc. that which P becomes, by changing there p into p', p'', etc., we will have for the corresponding part of $y_{i,x'}$,

$$APp^{x'} + A'P'p'^{x'} + A''P''p''^{x'} + \text{etc.}$$

We will find similarly that, if the value of $y_{1,x'}$ is expressed by $Bq^{x'} + B'q'^{x'} + B''q''^{x'} +$ etc. and if we name Q, Q', Q'', etc. that which the quantity

$$cZ^{(0)}_{i-n+1} + eZ^{(0)}_{i-n+2} + \text{etc.},$$

becomes when we change successively $\frac{1}{t'}$ into q, q', q'', etc., the corresponding part of $y_{i,x'}$ will be

$$BQq^{x'} + B'Q'q'^{x'} + B''Q''q''^{x'} + \text{etc.},$$

and so forth. The union of all these terms will give the expression of $y_{i,x'}$ the simplest to which we can arrive.

§15. The value of $y_{i,x'}$ given by formula (λ) of the preceding section, depending on the knowledge of $M^{(r)}$, $M_1^{(r-1)}$, etc.; it is clear that these quantities will be known, when we will have the coefficient of $\frac{1}{t'^r}$ in the development of $Z_i^{(0)}$; all is reduced therefore to determining this coefficient. We have by §5,

$$\begin{aligned} Z_i^{(0)} = &- \frac{1}{a\alpha^{i+1}(\alpha - \alpha')(\alpha - \alpha'') \text{ etc.}} \\ &- \frac{1}{a\alpha'^{i+1}(\alpha' - \alpha)(\alpha' - \alpha'') \text{ etc.}} \\ &- \frac{1}{a\alpha''^{i+1}(\alpha'' - \alpha)(\alpha'' - \alpha') \text{ etc.}} \\ &- \text{ etc.,} \end{aligned}$$

[59] α, α', α'', etc. being functions of $\frac{1}{t'}$. If we make $\frac{1}{t'} = s$, and if we differentiate the preceding expression of $Z_i^{(0)}$, n times in sequence with respect to s, we will have with the preceding equation, $n+1$ equations, by means of which, by eliminating the undetermined powers $\frac{1}{\alpha^{i+1}}$, $\frac{1}{\alpha'^{i+1}}$, $\frac{1}{\alpha''^{i+1}}$, etc., we will arrive to a linear equation among $Z_i^{(0)}$, $\frac{dZ_i^{(0)}}{ds}$, $\frac{d^2Z_i^{(0)}}{ds^2}$, etc., of which the coefficients will be functions of α, α', α'', etc. and of their differentials taken with respect to s; now it is clear that α, α', α'', etc. must enter in the same manner into these coefficients that we can thus obtain rational and integral functions of them from the coefficients of the equation which give the values of α, α', α'', etc. and from the differences of these coefficients, and consequently as rational functions of s. By making next the denominators of these functions disappear, we will have a linear equation between $Z_i^{(0)}$ and its differentials, an equation of which the coefficients will be rational and integral functions of s. This premised, let us consider any term of this equation, such as $ks^m\frac{d^\mu Z_i^{(0)}}{ds^\mu}$, and let us name λ_r the coefficient of $\frac{1}{t'^r}$ in the development of $Z_i^{(0)}$ according to the powers of $\frac{1}{t'}$; this coefficient in the development of $ks^m\frac{d^\mu Z_i^{(0)}}{ds^\mu}$ will be

$$k(r+\mu-m)(r+\mu-m-1)(r+\mu-m-2)\cdots(r-m+1)\lambda_{r+\mu-m}.$$

By thus passing again from the generating functions to their coefficients, the equation between $Z_i^{(0)}$ and its differences, will give an equation among λ_r, λ_{r+1}, etc. of which the coefficients will be some rational functions of r and of which the integral will be the value of λ_r.

It follows thence that integration of every linear equation in finite partial differences, of which the coefficients are constants, depends: 1° on the integration of a linear equation in finite differences of which the coefficients are variables; 2° on a
[60] *definite* integral. The definite integral on which the value of $y_{i,x'}$ depends in formula (λ) is relative to r, and must be extended to $r = i+1$.

Relative to the equation in the partial differences of first order

$$\begin{aligned} 0 = Ay_{i,x'} &+ By_{i+1,x'} \\ &+ B'y_{i,x'+1}, \end{aligned}$$

we have

$$Z_i^{(0)} = -\frac{1}{a\alpha^{i+1}};$$

we have moreover

$$a = A + B's,$$

$$\alpha = -\frac{B}{a},$$

that which gives

$$Z_i^{(0)} = -\frac{(A+B's)^i}{(-B)^{i+1}},$$

whence we deduce this differential equation

$$0 = \frac{dZ_i^{(0)}}{ds}(A+B's) - iB'Z_i^{(0)};$$

that which gives the equation in finite differences

$$0 = (r+1)A\lambda_{r+1} - (i-r)B'\lambda_r;$$

we have next

$$M^{(r)} = B\lambda_r.$$

Formula (λ) of the preceding article will become therefore

$$y_{i,x'} = B\Sigma\lambda_r y_{0,x'+r},$$

The finite integral being taken from $r = 0$ to $r = i$. It is the complete integral of the preceding equation in partial differences of the first order.

The equation in the differences in λ_r give by integrating it

$$\lambda_r = \frac{Hi(i-1)(i-2)\ldots(i-r+1)}{1.2.3\ldots r}\frac{B'^r}{A^r},$$

H being an arbitrary constant; and the denominator being unity when r is null. In order to determine this constant, we will observe that the coefficient independent of $\frac{1}{t'}$ [61]
in $Z_i^{(0)}$ is $-\frac{A^i}{(-B)^{i+1}}$; it is the value of λ_0, and consequently of H; we will have therefore

$$y_{i,x'} = -\Sigma\frac{i(i-1)(i-2)\ldots(i-r+1)}{1.2.3\ldots r}\frac{A^{i-r}B'^r}{(-B)^i}y_{0,x'+r}.$$

In passing from the finite to the infinitely small, the preceding method will give the integral of the equations linear in infinitely small partial differences of which the coefficients are constants, 1° by integrating a linear equation in infinitely small differences; 2° by means of a definite integral. But this is not the place here to expand myself on this object that I have considered elsewhere extensively.

We must make here an important remark relative to the number of arbitrary functions which the general expression of $y_{i,x'}$ contains. This number, in formula (λ) of the preceding section, is equal to n; but it becomes smaller in the case where the value of z of §13 containing only powers of $\frac{1}{t'}$ less than n, the highest power n' of $\frac{1}{t'}$ has a coefficient constant or independent of $\frac{1}{t}$. Then by following the preceding analysis, and determining by its means the value of $\frac{1}{t'^{x'}}$, as we have determined that of

$\frac{1}{t^i}$; by passing again from the generating functions to their coefficients, we will arrive to a formula analogous to formula (λ); alone, the definite integral, instead of being extended to $r = i+1$ must be extended to $r = x'+1$. This new expression of $y_{i,x'}$, will no longer depend but on the n' arbitrary functions $y_{i,0}$, $y_{i,1}$, $y_{i,2}$, $\ldots y_{i,n'-1}$; and while the first supposes the knowledge of the first n vertical ranks of Table (Q) of §14; this one requires only the knowledge of the first n' horizontal ranks of the same table. Thus the n arbitrary functions $y_{0,x'}$, $y_{1,x'}$, $y_{2,x'}$, $\ldots$, $y_{n-1,x'}$ of formula (λ) are equivalent only to n' arbitrary distinct functions. Indeed, the proposed equation in partial differences, gives $y_{i,n'}$ by means of the values of $y_{i\pm r,0}$, $y_{i\pm r,1}$, $\ldots$, $y_{i\pm r,n'-1}$, r being a whole number. It gives similarly $y_{i,n'+1}$ by means of $y_{i\pm r,0}$, $y_{i\pm r,1}$, $\ldots$, $y_{i\pm r,n'}$,
[62] and eliminating $y_{i\pm r,n'}$ by means of its expression, we have $y_{i,n'+1}$ by means of $y_{i\pm r,0}$, $y_{i\pm r,1}$, $\ldots$, $y_{i\pm r,n'-1}$; by continuing thus, we see that the general expression of $y_{i,x'}$ depends only on the arbitraries $y_{i\pm r,0}$, $y_{i\pm r,1}$, $\ldots$, $y_{i\pm r,n'-1}$; we can therefore, by means of the first n' horizontal ranks of Table (Q), form all its vertical ranks, which are, each, functions of x', in which i is invariable.

By passing from the finite to the infinitely small, we see evidently, that the number of arbitrary functions of the equations in partial differentials can be less than the highest degree of the differential in these equations.

§16. Although the formulas given in §13 and 14 have a great generality, there are however some cases which are not comprehended. These cases take place, when the equation $z = 0$ gives the expression of $\frac{1}{t^i}$ in $\frac{1}{t'}$ by an infinite series, that which arrives all the time that the highest power of $\frac{1}{t}$ is multiplied by a rational function of $\frac{1}{t'}$. In order to have then the expression of $y_{x,x'}$ in finite terms, it is necessary to resort to some artifices of analysis that we are going to expose, by applying them to the following equation:

$$z = \frac{1}{tt'} - \frac{a}{t'} - \frac{b}{t} - c. \tag{a}$$

This equation gives

$$\frac{1}{t} = \frac{\frac{a}{t'} + c + z}{\frac{1}{t'} - b},$$

consequently

$$\frac{u}{t^x t'^{x'}} = \frac{u\left(\frac{a}{t'} + c + z\right)^x}{\left(\frac{1}{t'} - b\right)^x t'^{x'}}.$$

By developing the second member of this last equation, and passing again from the generating functions to the coefficients, we will have the expression of $y_{x,x'}$; because
[63] this quantity is the coefficient of $t^0 t'^0$ in the development of the generating function $\frac{u}{t^x t'^{x'}}$; and the coefficient $t^0 t'^0$ in any term of the development of the second member, such as $u\frac{kz^\mu}{t'^{x'+r}}$ is $K\nabla^\mu y_{0,x'+r}$, $\nabla y_{x,x'}$ being the coefficient of the generating function uz, a coefficient which is here equal to

$$y_{x+1,x'+1} - ay_{x,x'+1} - by_{x+1,x'} - cy_{x,x'}.$$

If we have $0 = \nabla y_{x,x'}$, the coefficients of the affected terms of z will vanish, and then we will have the expression of $y_{x,x'}$ as function of $y_{0,x'}$, $y_{0,x'+1}$, $y_{0,x'+2}$, etc.; This expression will be the integral of the equation

$$0 = y_{x+1,x'+1} - ay_{x,x'+1} - by_{x+1,x'} - cy_{x,x'}. \tag{b}$$

In order to have this expression, z can be considered as null, since we must have regard only to the terms independent of z; equation (a) becomes thus

$$0 = \frac{1}{tt'} - \frac{a}{t'} - \frac{b}{t} - c;$$

this is that which I name *generating equation* of equation (b) in the partial differences. Indeed, we obtain this last equation by multiplying the preceding by u, and passing again from the generating functions to the coefficients.

The expression that we obtain by the preceding analysis for $y_{x,x'}$ is an infinite series. We will arrive in this manner to a finite expression. Let us take the value of $\frac{u}{t^x t'^{x'}}$, and let us give to it this form

$$\frac{u}{t^x t'^{x'}} = \frac{u\left(\frac{1}{t'} - b + b\right)^{x'}\left[c + ab + a\left(\frac{1}{t'} - b\right)\right]^x}{\left(\frac{1}{t'} - b\right)^x}.$$

If we develop the second member of this equation, with respect to the powers of $\frac{1}{t'} - b$, we will have

$$\begin{aligned}\frac{u}{t^x t'^{x'}} =& u\left\{\left(\frac{1}{t'} - b\right)^{x'} + x'b\left(\frac{1}{t'} - b\right)^{x'-1} + \frac{x'(x'-1)}{1.2}b^2\left(\frac{1}{t'} - b\right)^{x'-2} + \text{etc.}\right\}\\ &\times\left\{a^x + x(c+ab)\frac{a^{x-1}}{\frac{1}{t'} - b} + \frac{x(x-1)}{1.2}(c+ab)^2\frac{a^{x-2}}{\left(\frac{1}{t'} - b\right)^2} + \text{etc.}\right\}.\end{aligned}$$

Let there be [64]

$$\begin{aligned}V &= a^x,\\ V^{(1)} &= x'ba^x + x(c+ab)a^{x-1},\\ V^{(2)} &= \frac{x'(x'-1)}{1.2}b^2a^x + x'xb(c+ab)a^{x-1} + \frac{x(x-1)}{1.2}(c+ab)^2a^{x-2},\\ V^{(3)} &= \frac{x'(x'-1)(x'-2)}{1.2.3}b^3a^x + \frac{x'(x'-1)}{1.2}xb^2(c+ab)a^{x-1}\\ &\qquad + x'\frac{x(x-1)}{1.2}b(c+ab)^2a^{x-2}\\ &\qquad + \frac{x(x-1)(x-2)}{1.2.3}(c+ab)^3a^{x-3},\end{aligned}$$

etc.;

we will have

$$\frac{u}{t^x t'^{x'}} = u \left\{ \begin{array}{l} V\left(\frac{1}{t'}-b\right)^{x'} + V^{(1)}\left(\frac{1}{t'}-b\right)^{x'-1} + V^{(2)}\left(\frac{1}{t'}-b\right)^{x'-2} \cdots + V^{(x')} \\ + \frac{V^{x'+1}}{\frac{1}{t'}-b} + \frac{V^{(x'+2)}}{\left(\frac{1}{t'}-b\right)^2} \cdots + \frac{V^{(x'+x)}}{\left(\frac{1}{t'}-b\right)^x} \end{array} \right\}.$$

Now the equation

$$\frac{1}{tt'} - \frac{a}{t'} - \frac{b}{t} - c = 0$$

gives

$$\frac{1}{\frac{1}{t'}-b} = \frac{\frac{1}{t}-a}{c+ab};$$

hence

$$\frac{u}{t^x t'^{x'}} = u \left\{ \begin{array}{l} V\left(\frac{1}{t'}-b\right)^{x'} + V^{(1)}\left(\frac{1}{t'}-b\right)^{x'-1} \cdots + V^{(x')} \\ + \frac{V^{x'+1}}{c+ab}\left(\frac{1}{t'}-a\right) + \frac{V^{(x'+2)}}{(c+ab)^2}\left(\frac{1}{t'}-a\right)^2 \cdots + \frac{V^{(x'+x)}}{(c+ab)^x}\left(\frac{1}{t'}-a\right)^x \end{array} \right\}.$$

In order to pass again now from the generating functions to the coefficients, we will observe, 1° that the coefficient of $t^0 t'^0$ in $\frac{u}{t^x t'^{x'}}$, is $y_{x,x'}$; 2° that this same coefficient, in any term, such as $u\left(\frac{1}{t'}-b\right)^r$ or $ub^r\left(\frac{1}{bt'}-1\right)^r$, is $br.'\triangle^r\left(\frac{y_{0,x'}}{b^{x'}}\right)$, the characteristic $'\triangle$ of the differences corresponding to the variability of x', and this variable must
[65] be supposed null after the differentiations; 3° that this coefficient in $u\left(\frac{1}{t}-a\right)^r$, is $a^r\triangle^r\left(\frac{y_{x,0}}{a^x}\right)$, the characteristic $\triangle$ corresponding to the variability of x, and this variable must be supposed null after the differentiations; we will have therefore, with these conditions,

$$\begin{aligned} y_{x,x'} =& V b^{x'}.'\triangle^{x'}\left(\frac{y_{0,x'}}{b^{x'}}\right) + V^{(1)} b^{x'-1}.'\triangle^{x'-1}\left(\frac{y_{0,x'}}{b^{x'}}\right) \cdots + V^{(x')} y_{0,0} \\ &+ \frac{a}{c+ab} V^{(x'+1)}\triangle\left(\frac{y_{x,0}}{a^x}\right) + \frac{a^2}{(c+ab)^2} V^{(x'+2)}\triangle^2\left(\frac{y_{x,0}}{a^x}\right) \cdots \\ &\cdots + \frac{a^x}{(c+ab)^x} V^{(x'+x)}\triangle^x\left(\frac{y_{x,0}}{a^x}\right); \end{aligned}$$

this is the complete integral of equation (b) in partial differences. It is clear that this integral supposes that we know the first horizontal rank and the first vertical rank of Table (Q) of §14.

§17. The preceding expression of $y_{x,x'}$ offers this of the remarkable, namely, that the characteristics $\triangle$ and $'\triangle$ of the finite differences, have for exponents, the variables

x and x'. Here is another example. Let us consider the equation in the partial differences

$$0 = \triangle^n y_{x,x'} + \frac{a}{\alpha}\triangle^{n-1}.'\triangle y_{x,x'} + \frac{b}{\alpha^2}\triangle^{n-2}.'\triangle^2 y_{x,x'} + \text{etc.},$$

the characteristic $\triangle$ corresponding to the variable x of which unity is the difference, and the characteristic $'\triangle$ corresponding to the variable x' of which α is the difference. The corresponding generating equation will be, by the preceding section,

$$0 = \left(\frac{1}{t} - 1\right)^n + \frac{a}{\alpha}\left(\frac{1}{t} - 1\right)^{n-1}\left(\frac{1}{t'^\alpha} - 1\right) + \frac{b}{\alpha^2}\left(\frac{1}{t} - 1\right)^{n-2}\left(\frac{1}{t'^\alpha} - 1\right)^2 + \text{etc.}$$

This equation gives the following n:

$$\frac{1}{t} - 1 = \frac{q}{\alpha}\left(1 - \frac{1}{t'^\alpha}\right),$$
$$\frac{1}{t} - 1 = \frac{q'}{\alpha}\left(1 - \frac{1}{t'^\alpha}\right),$$
$$\frac{1}{t} - 1 = \frac{q''}{\alpha}\left(1 - \frac{1}{t'^\alpha}\right),$$
$$\text{etc.}$$

q, q', q'', etc. being the n roots of the equation [66]

$$0 = z^n - az^{n-1} + bz^{n-2} - \text{ etc.}$$

The equation

$$\frac{1}{t} - 1 = \frac{q}{\alpha}\left(1 - \frac{1}{t'^\alpha}\right)$$

gives

$$\begin{aligned}\frac{u}{t^x t'^{x'}} &= \frac{u}{t'^{x'}}\left(1 + \frac{q}{\alpha} - \frac{q}{\alpha}\frac{1}{t'^\alpha}\right)^x \\ &= \frac{u}{t'^{x'}}(-1)^x\left\{\begin{matrix}\frac{q^x}{\alpha^x}\frac{1}{t'^{\alpha x}} - x\frac{q^{x-1}}{\alpha^{x-1}}\left(1 + \frac{q}{\alpha}\right)\frac{1}{t'^{\alpha(x-1)}} \\ + \text{etc.}\end{matrix}\right\}.\end{aligned}$$

By passing again from the generating functions to the coefficients, we will have

$$y_{x,x'} = (-1)^x\left\{\frac{q^x}{\alpha^x}y_{0,x'+\alpha x} - x\frac{q^{x-1}}{\alpha^{x-1}}\left(1 + \frac{q}{\alpha}\right)y_{0,x'+\alpha(x-1)} + \text{etc.}\right\}.$$

The second member of this equation can be set under the form

$$\left(1 + \frac{\alpha}{q}\right)^{x + \frac{x'}{\alpha}}\left(-\frac{q}{\alpha}\right)^x.'\triangle^x\left[\left(\frac{q}{\alpha + q}\right)^{\frac{x'}{\alpha}}y_{0,x'}\right].$$

By designating therefore by the arbitrary function $\phi(x')$ the quantity $\left(\frac{q}{\alpha+q}\right)^{\frac{x'}{\alpha}} y_{0,x'}$, the expression of $y_{x,x'}$ will become

$$y_{x,x'} = \left(1+\frac{\alpha}{q}\right)^{x+\frac{x'}{\alpha}} \left(-\frac{q}{\alpha}\right)^x .'\triangle^x \phi(x').$$

This value satisfies therefore the proposed equation in the partial differences. It is clear that each of the roots q', q'', etc., furnish a similar value, in which we can introduce another arbitrary. We will designate by $\phi_1(x')$, $\phi_2(x')$, etc. these new arbitraries. The union of all these values will satisfy the proposed equation, because
[67] it is linear, and this union will be the complete integral of it, which is thus,

$$\begin{aligned} y_{x,x'} = {} & \left(1+\frac{\alpha}{q}\right)^{x+\frac{x'}{\alpha}} \left(-\frac{q}{\alpha}\right)^x .'\triangle^x \phi(x') \\ & + \left(1+\frac{\alpha}{q'}\right)^{x+\frac{x'}{\alpha}} \left(-\frac{q'}{\alpha}\right)^x .'\triangle^x \phi_1(x') \\ & + \text{etc.} \end{aligned}$$

If we suppose α infinitely small and equal to dx'; if we observe moreover that

$$\left(1+\frac{dx'}{q}\right)^{x+\frac{x'}{dx'}} = c^{\frac{x'}{q}},$$

as it is easy to be convinced of it, by taking the logarithms of each member of this equation, we will have

$$y_{x,x'} = c^{\frac{x'}{q}}(-q)^x \left[\frac{d^x\phi(x')}{dx'^x}\right] + c^{\frac{x'}{q'}}(-q')^x \left[\frac{d^x\phi_1(x')}{dx'^x}\right] + \text{etc.};$$

it is the complete integral of the equation in the finite and infinitely small partial differences,

$$0 = \triangle^n y_{x,x'} + a\triangle^{n-1}\left(\frac{dy_{x,x'}}{dx'}\right) + b\triangle^{n-2}\left(\frac{d^2 y_{x,x'}}{dx'^2}\right) + \text{etc.}$$

All the equations in the partial differences that we have examined until here, have no last term independent of the principle value. If they had, we would have regard, and we would integrate these equations by the method that we have given for this object, relative to the equations in the simple differences, and that it is easy to apply to the equations in partial differences.

Theorems on the development into series, of functions of many variables.

§18. If we apply to the functions of many variables, the method of §11; we will have from the development of these functions into series, some theorems analogous
[68] to those of §10. Let us consider the generating function $u\left[\frac{1}{tt't''\text{ etc.}} - 1\right]^n$, and let us give to it this form

$$u\left[\left(1+\frac{1}{t}-1\right)\left(1+\frac{1}{t'}-1\right)\left(1+\frac{1}{t''}-1\right)\text{ etc.}-1\right]^n,$$

u being supposed a function of t, t', t'', etc., in the development of which $y_{x,x',x'',\text{ etc.}}$ is the coefficient of $t^x t'^{x'} t''^{x''}$ etc. This coefficient in the development of $u\left[\frac{1}{tt't''\text{ etc.}}-1\right]^n$ will be $\triangle^n y_{x,x',x'',\text{ etc.}}$, x, x', x'', etc. being supposed to vary by unity in $y_{x,x',x'',\text{ etc.}}$. This same coefficient, in the development of the generating function

$$u\left(\frac{1}{t}-1\right)^r\left(\frac{1}{t'}-1\right)^{r'}\left(\frac{1}{t''}-1\right)^{r''}\text{ etc.},$$

will be

$$'\triangle^r.''\triangle^{r'}.'''\triangle^{r''}\text{ etc. } y_{x,x',x'',\text{ etc.}},$$

the characteristics $'\triangle$, $''\triangle$, $'''\triangle$, etc. corresponding respectively to the variables x, x', x'', etc.; we will have therefore, by passing again from the generating functions to their coefficients,

$$\triangle^n y_{x,x',x'',\text{ etc.}}=\left\{\begin{matrix}(1+{}'\triangle y_{x,x',x'',\text{ etc.}})(1+{}''\triangle y_{x,x',x'',\text{ etc.}})\\ \times(1+{}'''\triangle y_{x,x',x'',\text{ etc.}})\text{ etc.}-1\end{matrix}\right\}^n;$$

provided that in the development of the second member of this equation, we apply to the characteristics $'\triangle$, $''\triangle$, etc. the exponents of the powers of $'\triangle y_{x,x',x'',\text{ etc.}}$, $''\triangle y_{x,x',x'',\text{ etc.}}$, etc.

By changing n into $-n$, the same equation subsists further, provided that we change, as in the §§10 and 11, the characteristics $\triangle$, $'\triangle$, $''\triangle$, etc., when they have a negative exponent, into corresponding finite integrals, the signs Σ, $'\Sigma$, $''\Sigma$, etc. being the characteristics of the integrals, corresponding to the characteristics $\triangle$, $'\triangle$, $''\triangle$, etc. of the differences.

It is clear that $u\left[\frac{1}{t^i t'^{i'} t''^{i''}\text{ etc.}}-1\right]^n$ is the generating function of the n^{th} finite difference of $y_{x,x',x'',\text{ etc.}}$, x varying by i, x' varying by i', x'' varying by i'', etc.. Now we have [69]

$$u\left(\frac{1}{t^i t'^{i'} t''^{i''}\text{ etc.}}-1\right)^n=u\left[\left(1+\frac{1}{t}-1\right)^i\left(1+\frac{1}{t'}-1\right)^{i'}\left(1+\frac{1}{t''}-1\right)^{i''}\text{ etc.}-1\right]^n,$$

by designating therefore by $\bar{\triangle}$ the characteristic of the differences, when x varies by i, x' by i', x'' by i'', etc., and by $\bar{\Sigma}$ the corresponding integral characteristic, we will have

$$\bar{\triangle}^n y_{x,x',x'',\text{ etc.}}=[(1+{}'\triangle y_{x,x',x'',\text{ etc.}})^i(1+{}''\triangle y_{x,x',x'',\text{ etc.}})^{i'}\text{ etc.}-1]^n,$$

$$\bar{\Sigma}^n y_{x,x',x'',\text{ etc.}}=\frac{1}{[(1+{}'\triangle y_{x,x',x'',\text{ etc.}})^i(1+{}''\triangle y_{x,x',x'',\text{ etc.}})^{i'}\text{ etc.}-1]^n},$$

provided that in the development of the second member of these equations, we apply to the characteristics $'\triangle$, $''\triangle$, etc., the exponents of the powers of $'\triangle y_{x,x',x'',\text{ etc.}}$, $''\triangle y_{x,x',x'',\text{ etc.}}$, etc., and that we change the negative differences into integrals. We can thus dispense with indicating the arbitraries that the finite integral $\bar{\Sigma}^n$ must introduce, because they are counted contained in the integrals that the development of its expression gives.

The two preceding equations yet hold, by supposing that in the differences ${}'\triangle y_{x,x',x'',\text{ etc.}}$, ${}''\triangle y_{x,x',x'',\text{ etc.}}$, etc., x, x', x'', etc., instead of varying by unity, vary by any quantity ϖ; provided that in the difference $\bar{\triangle} y_{x,x',x'',\text{ etc.}}$, x varies by $i\varpi$, x' by $i'\varpi$, x'' by $i''\varpi$, etc. Now, if we suppose ϖ infinitely small, the differences ${}'\triangle y_{x,x',x'',\text{ etc.}}$, ${}''\triangle y_{x,x',x'',\text{ etc.}}$, etc., will be changed, the first into $dx\left(\frac{dy_{x,x',\text{ etc.}}}{dx}\right)$, the second into $dx'\left(\frac{dy_{x,x',\text{ etc.}}}{dx}\right)$, etc. Moreover, if we make i, i', i'', etc. infinitely great, and such that we have

$$i\,dx = \alpha, \qquad i'dx' = \alpha', \qquad \text{etc.};$$

we will have

$$(1 + {}'\triangle y_{x,x',x'',\text{ etc.}})^i = \left\{1 + dx\left(\frac{dy_{x,x',\text{ etc.}}}{dx}\right)\right\}^{\frac{\alpha}{dx}} = c^{\alpha\left(\frac{dy_{x,x',\text{ etc.}}}{dx}\right)},$$

c being always the number of which the hyperbolic logarithm is unity. We will have similarly

$$(1 + {}''\triangle y_{x,x',\text{ etc.}})^{i'} = c^{\alpha'\left(\frac{dy_{x,x',\text{ etc.}}}{dx'}\right)},$$

[70] and so forth; hence

$$\bar{\triangle} y_{x,x',\text{ etc.}} = \left[c^{\alpha\left(\frac{dy_{x,x',\text{ etc.}}}{dx}\right)+\alpha'\left(\frac{dy_{x,x',\text{ etc.}}}{dx'}\right)+\text{etc.}} - 1\right]^n,$$

$$\bar{\Sigma} y_{x,x',\text{ etc.}} = \frac{1}{\left[c^{\alpha\left(\frac{dy_{x,x',\text{ etc.}}}{dx}\right)+\alpha'\left(\frac{dy_{x,x',\text{ etc.}}}{dx'}\right)+\text{etc.}} - 1\right]^n},$$

x varying by α, x' by α', etc., in the first two members of these equations.

If, instead of supposing ϖ infinitely small, we suppose it equal to unity, and i infinitely small and equal to dx; if we suppose further i', i'', etc. infinitely small and respectively equal to dx', dx'', etc., we will have

$$(1 + {}'\triangle y_{x,x',\text{ etc.}})^i = (1 + {}'\triangle y_{x,x',\text{ etc.}})^{dx} = 1 + dx\log(1 + {}'\triangle y_{x,x',\text{ etc.}});$$

we will have similarly

$$(1 + {}''\triangle y_{x,x',\text{ etc.}})^{i'} = 1 + dx'\log(1 + {}''\triangle y_{x,x',\text{ etc.}});$$
$$\text{etc.}$$

moreover $\bar{\triangle}^n y_{x,x',\text{ etc.}}$ is changed then into $d^n y_{x,x',\text{ etc.}}$; we will have therefore

$$d^n y_{x,x',\text{ etc.}} = [dx\log(1 + {}'\triangle y_{x,x',\text{ etc.}}) + dx'\log(1 + {}''\triangle y_{x,x',\text{ etc.}}) + \text{etc.}]^n;$$

an equation which by making n negative, subsists yet, provided that we change the negative differences into integrals. These diverse results are analogous to those that we have found in §10, relative to the functions of one variable alone; and we find again the analogy that we have observed between the positive powers and the differences, and between the negative powers and the integrals.

Considerations on the passage from the finite to the infinitely small.

§19. The passage from the finite to the infinitely small, consists in neglecting the infinitely small differences, with respect to the finite quantities, and generally the infinitely small of an order superior relative to those of an order inferior. This omission seems to remove from this passage, geometric rigor; but, in order to be convinced of its entire exactitude, it suffices to consider it as the result of the comparison of the [71] homogeneous powers of an indeterminate variable, in the development of the terms of an equation which subsists, whatever be that indeterminate; because it is clear that the terms affected of the same power must be mutually destroyed.

In order to render that sensible by an example, let us consider the following equation that equation (q) of §10 gives, by making $n = 1$,

$$'\triangle y_x = (1 + dy_{x'})^{\frac{\alpha}{dx'}} - 1,$$

$'\triangle$ is the characteristic of the finite differences, x' varying by α, and d is the characteristic of the differences, x' varying by dx'. The preceding equation developed gives, by applying conformably to the analysis of the section cited, the exponents of the powers of $dy_{x'}$ to the characteristic d,

$$'\triangle y_{x'} = \frac{\alpha}{dx'} dy_{x'} + \frac{(\alpha^2 - \alpha dx')}{1.2.dx'^2} d^2 y_{x'} + \text{etc.};$$

$dy_{x'}$ is equal to $y_{x'+dx'} - y_{x'}$. Let us suppose that by developing the function of $x' + dx'$, represented by $y_{x'+dx'}$, we have

$$y_{x'+dx'} = y_{x'} + dx' y'_{x'} + dx'^2 z_{x'} + \text{etc.};$$

we will have

$$dy_{x'} = dx' y'_{x'} + dx'^2 z_{x'} + \text{etc.};$$

whence we deduce

$$d^2 y_{x'} = dx' dy'_{x'} + dx'^2 dz_{x'} + \text{etc.}$$

Let us develop similarly $y'_{x'+dx'}$, $z_{x'+dx'}$, etc. according to the powers of dx', and let us suppose that we have

$$y'_{x'+dx'} = y'_{x'} + dx' y''_{x'} + dx'^2 s_{x'} + \text{etc.},$$
$$z_{x'+dx'} = z_{x'} + dx' z'_{x'} + \text{etc.};$$

we will have

$$dy'_{x'} = dx' y''_{x'} + dx'^2 s_{x'} + \text{etc.},$$
$$dz_{x'} = dx' z'_{x'} + \text{etc.},$$

hence

$$d^2 y_{x'} = dx'^2 dy''_{x'} + dx'^3 s_{x'} + \text{etc.}$$
$$+ dx'^3 z'_{x'} + \text{etc.}$$

[72] The preceding expression of $'\triangle y_{x'}$ will become thus,

$$'\triangle y_{x'} = \alpha y'_{x'} + \frac{\alpha^2}{1.2} y''_{x'} + \text{etc.}$$
$$+ dx' \begin{Bmatrix} \alpha(z_{x'} - \frac{1}{2} y''_{x'} + \text{etc.}) \\ +\alpha^2 (s_{x'} + z''_{x'} + \text{etc.}) \\ + \text{ etc.} \end{Bmatrix} \qquad (o)$$
$$+ dx'^2 \text{ etc.},$$

dx' being undetermined; the terms independent of dx' must be equal separately among them; we have therefore

$$'\triangle y_{x'} = \alpha y'_{x'} + \frac{\alpha^2}{1.2} y''_{x'} + \text{etc.}$$

Now, $y'_{x'}$ is the coefficient of dx' in the development of $y_{x'+dx'}$; it is that which we designate in the differential Calculus, by $\frac{dy_{x'}}{dx'}$. Similarly $y''_{x'}$ is the coefficient of dx' in the development of $y'_{x'+dx'}$; it is that which we designate by $\frac{dy'_{x'}}{dx'}$, or by $\frac{d^2 y_{x'}}{dx'^2}$, and so forth; by substituting therefore, in the preceding equation, $y_{x'+\alpha} - y_{x'}$ instead of $'\triangle y_{x'}$, we will have the following theorem:

$$y_{x'+\alpha} = y_{x'} + \alpha \frac{dy_{x'}}{dx'} + \frac{\alpha^2}{1.2} \frac{d^2 y_{x'}}{dx'^2} + \frac{\alpha^3}{1.2.3} \frac{d^3 y_{x'}}{dx'^3} + \text{etc.}$$

Considered as a result of the comparison of the terms independent of dx', this theorem leaves no doubt on its rigorous exactitude, and it is clear by the preceding analysis, that this comparison returns to neglecting the terms multiplied by dx' and its powers, relative to the finite quantities; this omission removes therefore nothing from the rigor of the differential Calculus. But we see moreover, *a priori*, that the terms affected of the same power of the indeterminate dx' must be mutually destroyed, that which we can verify *a posteriori*; thus that which we neglect as infinitely small is rigorously null; so that the omission of the infinitely small, relative to the finite quantities, is at base only a easy way to eliminate the superfluous terms which must vanish in the final result.

[73] This bringing together of the calculus in finite differences, and of the differential calculus, puts into evidence the rigor of the results of this last calculation, and gives its true metaphysics; but its applications to extent, duration and movement supposes moreover, the principle of limits. We can, by a similar bringing together, clear up diverse points of the infinitesimal analysis, which have been subjects of dispute among geometers: such is the discontinuity of arbitrary functions in the integrals of equations in the partial differences. Those who have rejected this discontinuity, based themselves on this that the ordinary analysis of infinitely small differences, suppose

that the successive differentials of a function, must be infinitely small relative to the previous, that which does not hold when the function is discontinuous. In order to clarify this delicate question, it is necessary to consider it in the finite differences, and to observe that which arrives in the passage from these differences to the infinitely small differences.

Let us take for example the following equation in partial finite differences:

$$(y_{x+1,x'} - 2y_{x,x'} + y_{x-1,x'}) - (y_{x,x'+1} - 2y_{x,x'} + y_{x,x'-1}) = 0; \qquad (a)$$

its generating equation is, by §16,

$$t\left(\frac{1}{t} - 1\right)^2 - t'\left(\frac{1}{t'} - 1\right)^2 = 0;$$

and by following the analysis given previously, it is easy to conclude from it that the complete integral of the proposed equation (a) is

$$y_{x,x'} = \phi(x + x') + \psi(x - x'),$$

$\phi(x+x')$ being an arbitrary function of $x+x'$, and $\psi(x-x')$ being an arbitrary function of $x - x'$. It is easy moreover to be assured that this value satisfies the proposed, and that it is the complete integral, since it contains two arbitrary functions.

Let us suppose presently that, in the following Table,

$$\begin{array}{lllllll} y_{0,0}, & y_{1,0}, & y_{2,0}, & y_{3,0}, & \cdots & y_{n-1,0}, & y_{n,0}, \\ y_{0,1}, & y_{1,1}, & y_{2,1}, & y_{3,1}, & \cdots & y_{n-1,1}, & y_{n,1}, \\ y_{0,2}, & y_{1,2}, & y_{2,2}, & y_{3,2}, & \cdots & y_{n-1,2}, & y_{n,2}, \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ y_{0,\infty}, & y_{1,\infty}, & y_{2,\infty}, & y_{3,\infty}, & \cdots & y_{n-1,\infty}, & y_{n,\infty}, \end{array} \qquad (Z)$$

we know the first two horizontal ranks comprehended between the two extreme vertical columns [74]

$$\begin{array}{lllll} y_{0,0}, & y_{0,1}, & y_{0,2}, & \cdots & y_{0,\infty}, \\ y_{n,0}, & y_{n,1}, & y_{n,2}, & \cdots & y_{n,\infty}, \end{array}$$

and that we know moreover all the terms of these two columns; we could determine all the values of $y_{x,x'}$ which fall between these two columns. Because if we wish to form the third horizontal rank, we will observe that equation (a) gives

$$y_{x,x'+1} = y_{x+1,x'} + y_{x-1,x'} - y_{x,x'-1}.$$

By making in this last equation, $x' = 1$, and successively $x = 1$, $x = 2$, $x = 3$, ..., $x = n-1$, we will have the values of $y_{1,2}$, $y_{2,2}$, $y_{3,2}$, ..., $y_{n-1,2}$, or the third horizontal rank, by means of the first two horizontal ranks. We will form in the same manner the fourth horizontal rank, and so forth to infinity. But, if we wish to determine the values of $y_{x,x'}$, which fall outside of Table (Z), the preceding conditions do not suffice, and it is necessary to add others to them.

Let us take the integral

$$y_{x,x'} = \phi(x + x') + \psi(x - x');$$

and let us suppose that the second horizontal rank which determines one of the two arbitrary functions, be such that we have

$$\psi(x-x') = \phi(x-x');$$

we will have

$$y_{x,x'} = \phi(x+x') + \phi(x-x').$$

By making $x'=0$, we will have $\phi(x) = \frac{1}{2}y_{x,0}$; hence

$$y_{x,x'} = \tfrac{1}{2}y_{x+x',0} + \tfrac{1}{2}y_{x-x',0}.$$

It is easy to see that this equation satisfies the proposed equation (a); but it is only a particular integral, which corresponds to the case where the second horizontal rank is formed from the first, by means of the equation

$$y_{x,1} = \tfrac{1}{2}y_{x+1,0} + \tfrac{1}{2}y_{x-1,0}.$$

As much as $x+x'$ will be equal or less than n, and as $x-x'$ will be positive or null,
[75] we will have the value of $y_{x,x'}$, by means of the first horizontal rank. But, when x' increasing, $x+x'$ will become greater than n or when $x-x'$ will become negative; it will be necessary to determine the values of $y_{x+x',0}$ and of $y_{x-x',0}$ by means of the two extreme vertical columns. Let us suppose that all the terms of these columns are null, and that we have thus $y_{0,x'}=0$ and $y_{n,x'}=0$. By making x null in the equation

$$y_{x,x'} = \tfrac{1}{2}y_{x+x',0} + \tfrac{1}{2}y_{x-x',0}.$$

we will have

$$y_{-x',0} = -y_{x',0}.$$

By making next $x=n$ in the same equation, we will have

$$y_{n+x',0} = -y_{n-x',0}.$$

If we change next in this last equation, x' into $n+x'$, we will have

$$y_{2n+x',0} = -y_{-x',0} = y_{x',0};$$

by changing next x' into $n+x'$, we will have

$$y_{3n+x',0} = y_{n+x',0} = -y_{n-x',0};$$

whence generally we will have

$$y_{2rn+x',0} = y_{x',0},$$
$$y_{(2r+1)n+x',0} = -y_{n-x',0}.$$

We will thus be able, by means of these two equations, to continue the values of $y_{x,x'}$ to infinity, on the side of the positive values of x, and we will conclude from them those which correspond to x negative, by means of the equation

$$y_{-x',0} = -y_{x',0}.$$

Thence results the following construction. Let us represent the values of $y_{x,0}$ from $x=0$ to $x=n$, by the ordinates drawn at the angles of a polygon of which the abscissa is x and of which the two extremities, that I designate by A and B, lead to the points where $x=0$ and $x=n$. We will carry this polygon from $x=n$ to $x=2n$,

by giving a position to it contrary to the one which it had from $x = 0$ to $x = n$, that is, a position such, that the parts which were above the axis of the abscissas x, are found below, the point B remaining moreover in this second position, in the same place as in the first, and the point A corresponding thus to the abscissa $x = 2n$. We [76]
will place next this same polygon, from $x = 2n$ to $x = 3n$, by giving to it a position contrary to the second, and consequently similar to the first, in a manner that the point A, in this third position, conserves the place that it had in the second, and that thus the point B corresponds to the abscissa $x = 3n$. By continuing to place thus this polygon alternately above and below the axis of the abscissas; the ordinates drawn at the angles of this sequence of polygons, will be the values of $y_{x,0}$ which correspond to x positive.

Similarly, we will place this polygon from $x = 0$ to $x = -n$, by giving it a position contrary to that which it had from $x = 0$ to $x = n$, A remaining moreover in the same place in these two positions. We will place next this polygon from $x = -n$ to $x = -2n$, by giving to it a position contrary to the second, the point B conserving the same place, and so forth to infinity. The ordinates of these polygons represent the values of $y_{x,0}$ which correspond to x negative. We will have next the value of $y_{x,x'}$ by taking the half-sum of the two ordinates which correspond to the abscissas $x + x'$ and $x - x'$.

This geometric construction is general, whatever be the nature of the polygon which we just considered. It will serve to determine all the values of $y_{x,x'}$ comprehended from $x = 0$ to $x = n$, and from $x' = 0$ to $x' = \infty$, provided that we have $y_{0,x'} = 0$ and $y_{n,x'} = 0$, and that moreover the second horizontal rank of Table (Z) is such, that we have

$$y_{x,1} = \tfrac{1}{2}y_{x+1,0} + \tfrac{1}{2}y_{x-1,0}.$$

We have by that which precedes,

$$y_{x,x'+n} = \tfrac{1}{2}y_{x+x'+n,0} + \tfrac{1}{2}y_{x-x'-n,0};$$

moreover,

$$y_{x+x'+n,0} = -y_{n-x-x',0}, \qquad y_{x-x'-n,0} = -y_{n-x+x',0};$$

therefore

$$y_{x,x'+n} = -\tfrac{1}{2}y_{n-x-x',0} - \tfrac{1}{2}y_{n-x+x',0} = -y_{n-x,x'};$$

it follows thence that in Table (Z), the $(n+x')^{\text{th}}$ horizontal rank, is the x'^{th} rank taken [77]
with a contrary sign and in a reversed order, so that the r^{th} term of the $(n + x')^{\text{th}}$ rank is the same as the $(n - r)^{\text{th}}$ term of the x'^{th} rank taken with a contrary sign. We have next

$$y_{x,2n+x'} = \tfrac{1}{2}y_{2n+x+x',0} + \tfrac{1}{2}y_{x-x'-2n,0};$$

we have besides, by that which precedes,

$$y_{2n+x+x',0} = y_{x+x',0};$$

$$y_{x-x'-2n,0} = -y_{2n+x'-x,0} = -y_{x'-x,0} = y_{x-x',0};$$

hence

$$y_{x,2n+x'} = \tfrac{1}{2}y_{x+x',0} + \tfrac{1}{2}y_{x-x',0} = y_{x,x'};$$

whence it follows that the $(2n+x')^{\text{th}}$ horizontal rank is exactly equal to the x'^{th} rank.

Let us consider presently the vibrations of a taut cord of which the initial figure is anything, provided that it is very near in all its points, to the axis of the abscissas. Let us name x the abscissa, t the time, $y_{x,t}$ the ordinate of any point of the cord after time t. Let us imagine moreover the abscissa x divided into an infinity of parts equal to dx, and that we take for unity; that which returns to considering x as an infinite number. This premised, we will have, by the principles of dynamics,

$$\left(\frac{ddy_{x,t}}{dt^2}\right) = \frac{a^2}{dx^2}(y_{x+1,t} - 2y_{x,t} + y_{x-1,t});$$

a being a constant coefficient depending on the tension and on the thickness of the cord. If we make $t = \frac{x'}{a}$, we will have $dt = \frac{dx'}{a}$, and $y_{x,t}$ will become a function of x and of x', which we will designate by $y_{x,x'}$; now, the magnitude of dt being arbitrary, we can suppose it such, that the variation of x' is equal to that of x, which we have taken for unity; the preceding equation will become thus

$$y_{x,x'+1} - 2y_{x,x'} + y_{x,x'-1} = y_{x+1,x'} - 2y_{x,x'} + y_{x-1,x'},$$

x and x' being infinite numbers. This equation is the same as that which we just considered; thus the geometric construction which we have given previously, can be
[78] employed in this case: the polygon of which the ordinates of the angles are represented by $y_{x,0}$, is here the initial figure of the cord; but it is necessary for this to suppose the length n divided into an infinity of parts equal to dx. It is necessary moreover that the cord be fixed at its extremities, finally that we have $y_{0,x'} = 0$ and $y_{n,x'} = 0$. Moreover the equation of condition

$$y_{x,1} = \tfrac{1}{2}y_{x+1,0} + \tfrac{1}{2}y_{x-1,0});$$

or, that which reverts to the same,

$$y_{x,1} - y_{x,0} = \tfrac{1}{2}(y_{x+1,0} - 2y_{x,0} + y_{x-1,0})$$

is changed into this one

$$dt\left(\frac{dy_{x,0}}{dt}\right) = \frac{1}{2}dx^2\left(\frac{d^2y_{x,0}}{dx^2}\right);$$

that which gives

$$\left(\frac{dy_{x,0}}{dt}\right) = 0.$$

Now $\left(\frac{dy_{x,0}}{dt}\right)$ is the initial velocity of the cord; this velocity must therefore be null at the origin of the movement. Every time that these conditions will hold, the preceding construction will give always the movement of the cord, whatever be its initial figure, provided however that in all its points, $y_{x+2,0} - 2y_{x+1,0} + y_{x,0}$ is an infinitely small quantity of the second order, that is that two contiguous elements of the cord, do not form a finite angle. This condition is necessary in order that the differential equation of the problem can subsist, and in order that this here

$$dt\left(\frac{dy_{x,0}}{dt}\right) = \frac{1}{2}(y_{x+1,0} - 2y_{x,0} + y_{x-1,0})$$

gives $\left(\frac{dy_{x,0}}{dt}\right) = 0$. But besides it is evident, by that which precedes, that the initial figure of the cord can be discontinuous and composed of any number of arcs of different curves, provided that these arcs are touching.

The different situations of the cord in its movement, are represented by the horizontal ranks of Table (Z); and as the ranks which correspond to the values of x', [79] $x' + 2n$, $x' + 4n$, etc. are the same by that which precedes, there results from it that the cord returns to the same situation after time t, $t + \frac{2n}{a}$, $t + \frac{4n}{a}$, etc.

We see next by the geometric construction given above, that if we imagine a sequence of cords linked among them, and placed alternatively above and below the axis of the abscissas, as in this construction; all the cords will vibrate in the same manner, so that their initial figures being the same, their figures will be constantly parallel. We can likewise fix only the two extremities of this sequence, and leave their nodes entirely free; because the elements of the two cords at the point of their junction, being in a straight line and equally taut, this point has no tendency to be moved and must consequently remain immobile, that which experience confirms.

This analysis of the vibrating cords, establishes in an incontestable manner, the possibility of admitting discontinuous functions into this problem, and we must generally conclude from it that these functions can be employed in all the problems which depend on equations in partial infinitely small differences, provided that they may subsist with these equations and with the conditions of the problem. We can indeed consider these equations, as some particular cases of equations in finite differences, in which we suppose that the variables become infinite; now nothing being neglected in the theory of equations in the partial finite differences, it is clear that the arbitrary functions of their integrals, are not at all subject to the law of continuity, and that the constructions of these equations, by means of the polygons, hold whatever be the nature of these polygons. Now, when we pass from the finite to the infinitely small, these polygons are changed into some curves which, consequently, can be discontinuous; thus the law of continuity is necessary neither in the arbitrary functions of the integrals, nor in the geometric constructions which represent them. It is necessary only to observe that if the equation in the partial differentials in $y_{x,x'}$ is of order n, it must not have a jump between two consecutive values of $\left(\frac{d^{n-r}y_{x,x'}}{dx^s\, dx'^{n-r-s}}\right)$, r and s being [80] positive whole numbers, s being able to be null; that is that the differential of this quantity must be infinitely small with respect to that quantity itself.

This condition is indispensable in order that the proposed differential equation may subsist, because every partial differential equation supposes that the partial differentials of $y_{x,x'}$ from which it is formed, and divided by the respective powers of dx and dx', are finite quantities and comparable among themselves; but nothing obliges admitting the same condition relative to the differences of $y_{x,x'}$ of order n or of a superior order. By taking for arbitrary functions, the most elevated differences of the arbitrary functions which enter into the integral of an equation in the partial differences; this integral will contain no more then but some arbitrary functions and their successive integrals which are continuous, because in general the integral $\int ds\phi(s)$ is continuous in the case even where the function $\phi(s)$ is not. The preceding condition is reduced therefore to this that the $(n-1)^{\text{th}}$ difference of each arbitrary function is continuous, that is that its differential is infinitely small. It must not therefore have a jump between two consecutive tangents of the curve which represents the arbitrary function of the integral of an equation in the partial differentials of the second order; thus, in the problem of the vibrating cords that we just discussed, it is necessary and it suffices that any two contiguous elements of the initial figure of the cord, form between them an angle infinitely little different from two right angles. It must not have a jump between two consecutive osculatory radii of the curve which represents the continuous arbitrary function in the integral, if the equation in the partial differences is of third order, and so forth.

General considerations on generating functions.

§20. It is often useful to know the generating function of a quantity given by an equation in finite differences, ordinary or partial; because, analysis offering diverse means to develop the functions into series, we can thus obtain in a quite simple [81] manner the value of the sought quantity. There results from §5, that the quantity y_x, given by the equation in the finite differences

$$0 = a\,y_x + b\,y_{x+1} + c\,y_{x+2} \cdots + p\,y_{x+n-1} + q\,y_{x+n},$$

is the coefficient of t^x in the development of the function

$$\frac{A + Bt + Ct^2 \cdots + Ht^{n-1}}{a\,t^n + b\,t^{n-1} + c\,t^{n-2} \cdots + p\,t + q},$$

A, B, C,..., H being arbitrary constants. Indeed, if we compare that function to this here,

$$y_0 + y_1 t + y_2 t^2 \cdots + y_x t^x + y_{x+1} t^{x+1} \cdots + y_\infty t^\infty,$$

we will have, by making the denominator vanish, and by virtue of the equation in the differences in y_x,

$$\begin{aligned} A + Bt + Ct^2 \cdots + Ht^{n-1} &= t^{n-1}(b\,y_0 + c\,y_1 + \text{etc.}) \\ &\quad + t^{n-2}(c\,y_0 + e\,y_1 + \text{etc.}) \\ &\quad + \text{etc.}; \end{aligned}$$

by equating next the homogeneous powers of t, we will have the values of A, B, C, etc. by means of the n values y_0, y_1, ..., y_{n-1}; we will have therefore thus the generating function of y_x.

If we suppose $\Sigma^i y_x = y'_x$, we will have $y_x = \triangle^i y'_x$; and then the equation

$$0 = a\,y_x + b\,y_{x+1} + c\,y_{x+2} \cdots + q\,y_{x+n}$$

becomes

$$0 = a\triangle^i y'_x + b\triangle^i y'_{x+1} \cdots + q\triangle^i y'_{x+n};$$

that which gives, by integrating,

$$ay'_x + by'_{x+1} \cdots + qy'_{x+n} = Mx^{i-1} + Nx^{i-2} + \text{etc.},$$

M, N, etc. being arbitrary constants. By §2, u being the generating function of y_x, that of y'_x is

$$\frac{ut^i + A't^{i-1} + B't^{i-2} + \text{etc.}}{(1-t)^i};$$

the generating function of y'_x or of the quantity given by the preceding equation in y'_x [82]
is therefore

$$\frac{(A + Bt + Ct^2 \cdots + Ht^{n-1})t^i + (A't^{i-1} + B't^{i-2} + \text{etc.})(at^n + bt^{n-1} \cdots + q)}{(1-t)^i(a\,t^n + b\,t^{n-1} + c\,t^{n-2} \cdots + p\,t + q)}.$$

Let us imagine now that a, b, c, etc. are rational and entire functions of t' of order n, and that A, B, C, etc. are arbitrary functions of the same quantity; y_x will be a function of x and of t'. By developing it with respect to the powers of t', we will name $y_{x,x'}$ the coefficient of $t'^{x'}$ in this development. This premised, if we suppose

$$\begin{aligned}
a &= a'\ \ t'^n + b'\ \ t'^{n-1} + c'\ t'^{n-2} + \text{etc.}\\
b &= a''\ t'^n + b''\ t'^{n-1} + c''t'^{n-2} + \text{etc.}\\
c &= a'''t'^n + \text{etc.}\\
&\text{etc.}
\end{aligned}$$

The preceding differential equation in y_x will give, by comparing the coefficients of the power $t'^{x'+n}$, the following equation in the partial differences in $y_{x,x'}$,

$$\begin{aligned}
0 = a'y_{x,x'} + b'y_{x,x'+1} &+ c'y_{x,x'+2} &&+ \text{etc.}\\
a''y_{x+1,x'} &+ b''y_{x+2,x'+1} &&+ \text{etc.}\\
&+ a'''y_{x+2,x'} &&+ \text{etc.}\\
& &&+ \text{etc.};
\end{aligned}$$

the generating function of the variable $y_{x,x'}$ of this equation will be therefore

$$\frac{A + Bt + Ct^2 \cdots + Ht^{n-1}}{\begin{aligned}a't^nt'^n + b't^nt'^{n-1} &+ c't^nt'^{n-2} + \text{etc.}\\ + a''t^{n-1}t'^n &+ b''t^{n-1}t'^{n-1} + \text{etc.}\\ &+ a'''t^{n-2}t'^n + \text{etc.}\\ &+ \text{etc.}\end{aligned}}$$

A, B, C,... being arbitrary functions of t', they will give by their development, the arbitrary functions which must enter into the expression of $y_{x,x'}$.

We can further determine the generating functions of the equations in finite differences, in which the coefficients are variables. Let us consider for this the equation in the differences

$$\begin{aligned} 0 = \quad & a\ y_x + b\ y_{x+1} + c\ y_{x+2} \cdots + q\ y_{x+n} \\ & + x(a'\ y_x + b'\ y_{x+1} + c'\ y_{x+2} \cdots + q'\ y_{x+n}) \\ & + x^2(a'' y_x + b'' y_{x+1} + c'' y_{x+2} \cdots + q'' y_{x+n}) \\ & + \text{etc.} \end{aligned}$$

[83] If we name u the generating function of y_x, we will have, by virtue of the preceding equation,

$$\begin{aligned} & u\left(a + \frac{b}{t} + \frac{c}{t^2} \cdots + \frac{q}{t^n}\right) \\ & \quad + t\frac{d}{dt}\left\{u\left(a' + \frac{b'}{t} + \frac{c'}{t^2} \cdots + \frac{q}{t^n}\right)\right\} \\ & \quad + t\frac{d}{dt}\left\{t\frac{d}{dt}\left\{u\left(a'' + \frac{b''}{t} + \frac{c''}{t^2} \cdots + \frac{q''}{t^n}\right)\right\}\right\} \\ & \quad + \text{etc.} \\ & = A + Bt + Ct^2 \cdots + Ht^{n-1}, \end{aligned}$$

A, B, C, $\dots$, H being arbitrary constants, which depend on the values of y_0, y_1, y_2,$\dots$, y_{n-1}. Indeed, if we substitute into this equation, the preceding value of u in series; we see that by virtue of the proposed differential equation, all the coefficients of the same power of t, vanish when this power is equal or greater than n; and the comparison of the inferior powers give a number n of equations which determine the constants A, B, C, etc., by means of the values y_0, y_1, y_2,$\dots$, y_{n-1}.

The preceding differential equation is generally integrable, only in the case where it is of the first order, and then the coefficients of the equation in finite differences in y_x contain only the first power of x: in this last case, we can obtain the generating function u by quadratures.

§21. The knowledge of generating functions of differential equations, gives the expression of the integrals of these equations, by means of defined quadratures. Let us take for this, the equation

$$u = y_0 + y_1 t + y_2 t^2 \cdots + y_x t^x + y_{x+1} t^{x+1} \cdots + y_\infty t^\infty.$$

Let us substitute into its two members $c^{x\varpi\sqrt{-1}}$ instead of t^x, c being always the number of which the hyperbolic logarithm is unity; and let us name U, that which u then becomes. By multiplying the equation by $c^{-x\varpi\sqrt{-1}}d\varpi$ and integrating, we will have

$$\int U d\varpi c^{-x\varpi\sqrt{-1}} = \int d\varpi \left\{ \begin{aligned} & y_0 c^{-x\varpi\sqrt{-1}} + y_1 c^{-(x-1)\varpi\sqrt{-1}} \cdots \\ & \qquad \cdots + y_x + y_{x+1} c^{\varpi\sqrt{-1}} \text{ etc.} \end{aligned} \right\}$$

If we substitute, for $c^{\pm r\varpi\sqrt{-1}}$, its value $\cos r\varpi \pm \sqrt{-1}\sin r\varpi$, and if we take the integral from $\varpi = -\pi$ to $\varpi = \pi$, 2π being the circumference, the second member is reduced to $2\pi y_x$; we have therefore [84]

$$y_x = \frac{1}{2\pi}\int U d\varpi(\cos x\varpi - \sqrt{-1}\sin x\varpi);$$

but this formula has the inconvenience of introducing imaginaries from which we can be disencumbered in the following manner.

Let us consider the equation

$$\begin{aligned}0 = {} & a y_x + b y_{x+1} \cdots + q y_{x+n} \\ & + x(a' y_x + b' y_{x+1} \cdots + q' y_{x+n}),\end{aligned}$$

and let us suppose

$$y_x = \int t^{-x-1} T dt,$$

T being a function of t that it is a question of determining, as well as the limits of the integral. By substituting for y_x this value into the differential equation in y_x, and observing that we have

$$x\int t^{-x-1} dt \frac{T}{t^r} = -t^{-x}\frac{T}{t^r} + \int t^{-x} d\left(\frac{T}{t^r}\right),$$

that which makes the variable coefficient x vanish; we will have

$$\begin{aligned}0 = {} & -Tt^{-x}\left(a' + \frac{b'}{t} \cdots + \frac{q'}{t^n}\right) \\ & + \int t^{-x-1} dt \left\{\begin{aligned} & T\left(a + \frac{b}{t} \cdots + \frac{q}{t^n}\right) \\ & + t\frac{d}{dt}\left[T\left(a' + \frac{b'}{t} \cdots + \frac{q'}{t^n}\right)\right]\end{aligned}\right\}.\end{aligned} \qquad (h)$$

By equating to zero the part under the sign $\int$, we will have

$$\begin{aligned}0 = {} & T\left(a + \frac{b}{t} \cdots + \frac{q}{t^n}\right) \\ & + t\frac{d}{dt}\left[T\left(a' + \frac{b'}{t} \cdots + \frac{q'}{t^n}\right)\right].\end{aligned}$$

This equation integrated gives T as function of t. It is the same as the differential equation in u of the preceding section, by neglecting in the latter the term independent of u. The value of T is therefore the part of u which is independent of this term.

In order to have the limits of the integral $\int t^{-x-1} T dt$, we will equate to zero the part outside the $\int$ sign, in equation (h); that which gives

$$0 = Tt^{-x}\left(a' + \frac{b'}{t} \cdots + \frac{q'}{t^n}\right).$$

[85] This equation is satisfied by supposing t infinite, and by supposing it equal to one of the roots of the equation

$$0 = a' + \frac{b'}{t} \cdots + \frac{q'}{t^n};$$

we will have thus $n+1$ limits of the integral $\int t^{-x-1} T dt$; by multiplying next each integral comprehended between one of these limits, and the n other limits, by an arbitrary constant; the sum of these products will be the complete value of y_x.

We can extend this method, to the equations in finite and infinitely small partial differences, as we will show in the second part of this Book.

We see by that which precedes, the analogy which exists between the generating functions of the variables, and the definite integrals by means of which these variables can be expressed. In order to render it yet more sensible, let us consider the equation

$$y_x = \int T dt\, t^{-x},$$

T being a function of t, and the integral being taken within some determined limits. We will have, x varying by α,

$$\triangle y_x = \int T dt\, t^{-x} \left(\frac{1}{t^\alpha} - 1 \right),$$

and, generally,

$$\triangle^i y_x = \int T dt\, t^{-x} \left(\frac{1}{t^\alpha} - 1 \right)^i;$$

by making i negative, the characteristic $\triangle$ is changed into the integral sign Σ. If we suppose α infinitely small and equal to dx; we will have

$$\frac{1}{t^\alpha} = 1 + dx \log \frac{1}{t};$$

we will have therefore, by observing that then $\triangle^i y_x$ is changed into $d^i y_x$,

$$\frac{d^i y_x}{dx^i} = \int T dt\, t^{-x} \left(\log \frac{1}{t} \right)^i.$$

We will find in the same manner, and by adopting the denominations of §2,

$$\nabla^i y_x \int T dt\, t^{-x} \left(a + \frac{b}{t} \cdots + \frac{q}{t^n} \right)^i.$$

[86] Thus the same analysis which gives the generating functions of the successive deriveds of the variables, gives the functions, under the $\int$ sign, of the definite integrals which express these deriveds. The characteristic ∇^i expresses, strictly speaking, only a number i of consecutive operations; the consideration of the generating functions reduces these operations to some elevations of a polynomial to its diverse powers; and the consideration of the definite integrals gives directly the expression $\nabla^i y_x$, in the same case where we would suppose i a fractional number.

But the great advantage of this transformation of the analytic expressions, into definite integrals, is to furnish an approximation as handy as convergent, of these expressions, when they are formed of a great number of terms and of factors; this is that which takes place in the theory of probabilities, when the number of the events that we consider is very great. Then the numerical calculus of the results to which we are led by the solution of the problems, become impractical, and it is indispensable to have for this calculation, a method of approximation so much more convergent, as these results are more complicated.

Their expression in definite integrals, procures this advantage, and the one to give the laws according to which the probability of the results indicated by the events, approaches certitude in measure as the events are multiplied, laws of which the knowledge is one of the most interesting objects of the theory of probabilities. It was on the occasion of a problem of this kind, of which the solution depended on the expression of the middle term of the binomial raised to a great power, that Stirling transformed this expression into a very convergent series: his result can be regarded as one of the most ingenious things that we have found on series. It is especially remarkable, in this that in an inquiry which seems to admit only some algebraic quantities, it introduces a transcendental quantity, namely, the square root of the ratio of the circumference to the diameter. But the method of Stirling, based on a theorem of Wallis and on the interpolation of series, left desiring a direct method which is extended to all functions composed of a great number of terms and of factors. Such is the method of which I [87]
just spoke, and that I have given first[1] in the *Mémoires de l'Académie des Sciences* for the year 1778, and next[2] more extensively, in the Memoirs of the same academy, for the year 1782. The development of this method will be the object of the second Part of this Book, and will complete thus the Calculus of generating functions.

The series to which this method leads, contains most often, the square root of the ratio of the circumference to the diameter; and it is the reason for which Stirling has encountered it in the particular case that he has considered; but sometimes they depend on other transcendentals of which the number is infinite.

The limits of the definite integrals that this method reduces into convergent series, are, as we just saw, given by the roots of an equation that we can name *equation of the limits*. But a very important remark in this analysis, and which permits extending it to the functions that the theory of probabilities presents most often, is that the series to which we arrive, hold equally in the same case where, by some changes of sign in the coefficients of the equation of the limits, its roots become imaginaries. These passages from the positive to the negative, and from the real to the imaginary, of [88]
which the first applications have appeared, if I do not deceive myself, in the Memoirs cited, have led me in these Memoirs, to the values of many definite integrals, which offer that of the remarkable, namely, that they depend at the same time on these two transcendentals, the ratio of the circumference to the diameter, and the number of which the hyperbolic logarithm is unity. We can therefore consider these passages, as means of discovering, similar to the induction of which geometers made long time use.

[1] "Mémoire sur les probabilités," *Mém. Acad. R. Sci. Paris*, 1778 (1781), [**9**, pages 227-332] *Oeuvres* **9**, p. 383–485.

[2] "Mémoire sur les approximations des formules qui sont fonctions de trés-grands nombres," *Mém. Acad. R. Sci. Paris*, 1782 (1785), [**11**, pages 1–88] and *Oeuvres* **10**, p.209–291.

But these means, although employed with much precaution and reserve, leave always desiring some demonstrations of their results. Their bringing together of the direct methods, serving to confirm them and to show the great generality of the analysis, and being able by this reason, to interest the geometers; I have insisted particularly on those passages that Euler, considered at the same time as myself, and of which he has made many curious applications, but which have appeared only since the publication of the Memoirs cited.

Part 2

Theory of the approximations of formulas which are functions of large numbers.

CHAPTER 1

On the integration by approximation of the differentials which contain factors raised to great powers

§22. We just saw that we can always return to the integration of similar differentials, the formulas given by the theory of generating functions. We are going therefore to occupy ourselves first at length, with the approximation of this kind of integrals. [88]

If we designate by u, u', u'', etc. and ϕ arbitrary functions of x, and by s, s', s'', etc., very great numbers, each differential function which contains functions raised to some great powers, will be comprehended in the term $\phi\, dx\, u^s u'^{s'} u''^{s''}$ etc. In order to have in convergent series, its integral taken from $x = 0$ to $x = \theta$, we will make

$$\phi u^s u'^{s'} u''^{s''} \text{etc.} = y;$$

and, by designating by Y that which y becomes when we change x to θ there, we will suppose

$$y = Yc^{-t},$$

c always being the number of which the hyperbolic logarithm is unity. We will have thus [89]

$$t = \log\frac{Y}{y}.$$

If we consider x as a function of t given by this equation; we will have, by supposing dt constant,

$$x = \theta + t\frac{dx}{dt} + \frac{t^2}{1.2}\frac{ddx}{dt^2} + \frac{t^3}{1.2.3}\frac{d^3x}{dt^3} + \text{etc.},$$

t needing to be supposed null after the differentiations, in the values of $\frac{dx}{dt}$, $\frac{ddx}{dt^2}$, etc. Now we have generally

$$\frac{d^nx}{dt^n} = \frac{1}{dt}d\cdot\frac{1}{dt}d\cdot\frac{1}{dt}\cdots d\cdot\frac{dx}{dt};$$

the differential characteristic being related to all that which follows it, and dt being able to vary in any manner whatever in the second member of this equation; moreover, if we differentiate the preceding equation of t by y, and if we designate $-\frac{ydx}{dy}$ by v, we will have $dt = \frac{dx}{v}$; we will have therefore

$$\frac{d^nx}{dt^n} = \frac{vdv\,dv\ldots dv}{dx^{n-1}},$$

dx being supposed constant in the second member of this equation. Thus, by naming U that which v becomes when we change x into θ; the value of $\frac{d^nx}{dt^n}$ which corresponds to $x = \theta$, or, that which returns to the same, to $t = 0$, will be equal to

$$\frac{U\,d\,U\,dU\ldots dU}{d\theta^{n-1}};$$

we will have therefore

$$x = \theta + Ut + \frac{U\,dU}{1.2\,d\theta}t^2 + \frac{U\,dU\,dU}{1.2.3\,d\theta^2}t^3 + \text{etc.},$$

whence we deduce

$$dx = U\,dt\left(1 + \frac{dU}{d\theta}t + \frac{d\,U\,dU}{1.2\,d\theta^2}t^2 + \text{etc.}\right)$$

[90] consequently

$$\int y\,dx = UY\int dt\,c^{-t}\left(1 + \frac{dU}{d\theta}t + \frac{d\,U\,dU}{1.2\,d\theta^2}t^2 + \text{etc.}\right).$$

If we take the integral from $t = 0$ to t infinity, we will have generally

$$\int t^n dt\,c^{-t^n} = 1.2.3\ldots n;$$

hence

$$\int y\,dx = UY\left(1 + \frac{dU}{d\theta} + \frac{d\,U\,dU}{d\theta^2} + \frac{d.U\,d\,UdU}{d\theta^3} + \text{etc.}\right),$$

the integral relative to x being taken from $x = \theta$ to the value of x which corresponds to t infinite.

Let us name Y' and U' that which y and v become when we change x into θ'; we will have similarly

$$\int y\,dx = U'Y'\left(1 + \frac{dU'}{d\theta'} + \frac{d(U'\,dU')}{d\theta'^2} + \frac{d.U'\,d\,U'dU'}{d\theta'^3} + \text{etc.}\right);$$

the integral relative to x being taken from $x = \theta'$ to the value of x which corresponds to t infinite. By subtracting therefore these two equations from one another, we will have

$$\begin{aligned}\int y\,dx =& UY\left(1 + \frac{dU}{d\theta} + \frac{d\,U\,dU}{d\theta^2} + \frac{d.U\,d\,UdU}{d\theta^3} + \text{etc.}\right)\\ &- U'Y'\left(1 + \frac{dU'}{d\theta'} + \frac{d\,U'\,dU')}{d\theta'^2} + \frac{d\,U'\,d\,U'dU'}{d\theta'^3} + \text{etc.}\right);\end{aligned} \tag{A}$$

the integral relative to x being taken from $x = \theta$ to $x = \theta'$, so that the consideration of t disappears in this formula. If θ and θ' were originally contained in y, it would be necessary to vary only the quantities θ and θ' which introduce in U and U', the changes from x into θ and θ' in the function v.

Formula (A) will be very convergent, if v or $-\frac{y\,dx}{dy}$ is a very small quantity; now y being, by assumption, equal to $\phi u^s u'^{s'} u''^{s''}$. etc., we have

$$v = -\frac{1}{\frac{s\,du}{u\,dx} + \frac{s'du'}{u'dx} + \frac{s''du''}{u''dx} + \text{ etc. } + \frac{1}{\phi}\frac{d\phi}{dx}};$$

Thus in the case where s, s', s'', etc. are very great numbers, v will be very small; [91]
and if we make $\frac{1}{s} = \alpha$, α being a very small fraction, the function v will be of the order α, and the successive terms of formula (A) will be respectively of the order α, α^2, α^3, etc.

This formula would cease to be convergent, if the assumption of $x = \theta$ rendered very small the denominator of the expression of v. Let us suppose, for example, that $(x-a)^\mu$ is a factor of this denominator; it is clear that the successive terms of formula (A) are respectively divided by $(\theta-a)^\mu$, $(\theta-a)^{2\mu+1}$, $(\theta-a)^{3\mu+2}$, etc. and will become very large, if θ is little different from a; the convergence of this formula requires therefore that $(\theta-a)^\mu$, $(\theta'-a)^\mu$ be greater than α; it cannot consequently be used in the interval where $(x-a)^\mu$ is equal or less than α; but, in this case, we could make use of the following method.

§23. If we name Y that which y becomes when we change x into a; it is clear that $(x-a)^\mu$ being a factor of $-\frac{dy}{y\,dx}$, or, what returns to the same, of $\frac{d\log\frac{Y}{y}}{dx}$; $(x-a)^{\mu+1}$ will be a factor of $\log\frac{Y}{y}$. Let therefore

$$y = Yc^{-t^{\mu+1}}$$

$$v = \frac{x-a}{(\log Y - \log y)^{\frac{1}{\mu+1}}};$$

we will have

$$x = a + vt,$$

v at no point becoming infinite, by the assumption $x = a$. If we designate next by U, $\frac{dU^2}{dx}$, $\frac{d^2U^3}{dx^2}$, etc. that which v, $\frac{dv^2}{dx}$, $\frac{d^2v^3}{dx^2}$, etc., become when we change x into a after the differentiations; we will have, by formula (p) of §21 of Book II of the *Méchanique céleste*,

$$x = a + Ut + \frac{dU^2}{1.2\,dx}t^2 + \frac{d^2U^3}{1.2.3\,dx^2}t^3 + \text{etc.};$$

that which gives [92]

$$\int y\,dx = Y\int dt\,c^{-t^{\mu+1}}\left(U + \frac{dU^2}{dx}t + \frac{d^2U^3}{1.2\,dx^2}t^2 + \text{etc.}\right); \qquad \text{(B)}$$

this formula could be used in each interval where x differs very little from a; it can consequently serve as supplement to formula (A) of the preceding section; but instead of being ordered, as it, with respect to the powers of α, it is only with respect to the powers of $\alpha^{\frac{1}{\mu+1}}$; because it is clear that, in this last case, v is only of order $\alpha^{\frac{1}{\mu+1}}$.

In order to determine more easily the quantities U, $\frac{dU^2}{dx}$, etc., let us suppose

$$\log Y - \log y = (x-a)^{\mu+1}\left[A + B(x-a) + C(x-a)^2 + \text{etc.}\right].$$

We will have, by changing x into a after the differentiations,

$$A = -\frac{d^{\mu+1}\log y}{1.2.3\ldots(\mu+1)\,dx^{\mu+1}},$$
$$B = -\frac{d^{\mu+2}\log y}{1.2.3\ldots(\mu+2)\,dx^{\mu+2}},$$
etc.

We will have next, whatever be r,

$$v^r = [A + B(x-a) + C(x-a)^2 + \text{etc.}]^{-\frac{r}{\mu+1}};$$

whence it is easy to conclude by developing this expression of v^r, and naming $Q(x-a)^{r-1}$ the term of this development, which has for factor $(x-a)^{r-1}$,

$$\frac{d^{r-1}Ur}{1.2.3\ldots(r-1)dx^{r-1}} = Q.$$

Formula (B) presents thus no more difficulties other than those which result from
[93] the integration of the quantities of the form $\int t^n dt\, c^{-t^{\mu+1}}$; and we have generally,

$$\int t^n dt\, c^{-t^{\mu+1}} = -\frac{c^{t^{\mu+1}}}{\mu+1}\left\{\begin{matrix} t^{n-\mu} + \frac{n-\mu}{\mu+1}t^{n-2\mu-1} + \frac{(n-\mu)(n-2\mu-1)}{(\mu+1)^2}t^{n-3\mu-2} \\ \cdots + \frac{(n-\mu)(n-2\mu-1)\cdots(n-r\mu+\mu-r+2)t^{n-r\mu-r+1}}{(\mu+1)^{r-1}}\end{matrix}\right\}$$
$$+\frac{(n-\mu)(n-2\mu-1)\cdots(n-r\mu-r+1)}{(\mu+1)^r}\int t^{n-r\mu-r}dt\, c^{-t^{\mu+1}};$$

r being equal to the quotient of the division of n by $\mu+1$, if the division is possible, or to the number immediately inferior, if it is not. The determination of the integral $\int y\,dx$ depends therefore on the integrals of this form

$$\int dt\, c^{-t^{\mu+1}}, \quad \int t\,dt\, c^{-t^{\mu+1}}, \quad \ldots \quad \int t^{\mu-1}dt\, c^{-t^{\mu+1}}.$$

It is not possible to obtain exactly these integrals by known methods; but it will be easy in all cases, to have their approximate values.

§24. We will have principally need in the following, of the value of $\int y\,dx$, taken for the whole interval comprehended between two consecutive values of x, which render y null; we are going consequently to expose the simplifications of which this value is then susceptible. The variable y having been supposed, in the preceding section, equal to $Y c^{-t^{\mu+1}}$, it is clear that the two values of x which render y null, render equally null the quantity $c^{-t^{\mu+1}}$; that which requires that $\mu+1$ be an even number, and that one of these values of x corresponds to $t=-\infty$, and the other to $t=\infty$; Y is therefore then the *maximum* of y, comprehended between these values. Let $\mu+1=2i$; if we take the integral $\int t^{2n+1}dt\, c^{-t^{2i}}$, from $t=-\infty$ to $t=\infty$, its value will be null; because it is clear that the elements of this integral, which correspond to the negative values of t, are equal and of contrary sign to those which correspond to the same values taken positively. The integral $\int t^{2n}\,dt\,c^{-t^{2i}}$ is equal to $2\int t^{2n}\,dt\,c^{-t^{2i}}$, this last integral being taken from t null to t infinity; and in this case, we have by the preceding section, [94]

$$\int t^{2n}\,dt\,c^{-t^{2i}} = \frac{(2n-2i+1)(2n-4i+1)\cdots(2n-2ri+1)}{(2i)^r}\int t^{2n-2ri}dt\,c^{-t^{2i}}$$

r being equal to the whole number of the quotient of the division of n by i. Let therefore, by taking the integrals from t null to t infinity,

$$\begin{aligned} k &= \int dt\,c^{-t^{2i}}, \\ k^{(1)} &= \int t^2\,dt\,c^{-t^{2i}}, \\ k^{(2)} &= \int t^4\,dt\,c^{-t^{2i}}, \\ &\cdots\cdots\cdots\cdots\cdots, \\ k^{(i-1)} &= \int t^{2i-2}\,dt\,c^{-t^{2i}}; \end{aligned}$$

formula (B) of the preceding section will become

$$\begin{aligned} \int y\,dx = 2k\quad & Y\left\{U + \frac{1}{2i}\cdot\frac{d^{2i}U^{2i+1}}{1.2.3\ldots 2i\,dx^{2i}} + \frac{2i+1}{4i^2}\cdot\frac{d^{4i}U^{4i+1}}{1.2.3\ldots 4i\,dx^{4i}} + \text{etc.}\right\} \\ & + 2k^{(1)}Y\left\{\begin{aligned} &\frac{d^2U^3}{1.2\,dx^2} + \frac{3}{2i}\cdot\frac{d^{2i+2}U^{2i+3}}{1.2.3\cdots(2i+2)\,dx^{2i+2}} \\ &\qquad + \frac{3(2i+3)}{4i^2}\cdot\frac{d^{4i+2}U^{4i+3}}{1.2.3\cdots(4i+2)\,dx^{4i+2}} + \text{etc.}\end{aligned}\right\} \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & + 2k^{(i-1)}Y\left\{\begin{aligned} &\frac{d^{2i-2}U^{2i-1}}{1.2.3\cdots(2i-2)\,dx^{2i-2}} + \frac{2i-1}{2i}\cdot\frac{d^{4i-2}U^{4i-1}}{1.2.3\cdots(4i-2)\,dx^{4i-2}} \\ &\qquad + \frac{(2i-1)(4i-1)}{4i^2}\cdot\frac{d^{6i-2}U^{6i-1}}{1.2.3\cdots(6i-2)\,dx^{6i-2}} + \text{etc.}\end{aligned}\right\}. \end{aligned}$$

This formula is the sum of a number i of different series, decreasing as the powers of α, since U is of the order $\alpha^{\frac{1}{2i}}$, and multiplied respectively by the transcendentals k, $k^{(1)}$, etc., which it is, consequently, important to know; but it suffices for this to know a number equal to the greatest whole number comprehended within $\frac{i}{2}$.

Let us consider for this, the double integral

$$\iint ds\,dx\,c^{-s(1+x^n)},$$

the integrals being taken from s and x null to their infinite values. By integrating [95]
first with respect to s, it is reduced to

$$\int\frac{dx}{1+x^n};$$

but this last integral is $\frac{\pi}{n\sin\frac{\pi}{n}}$, n being any whole or fractional number; we have therefore

$$\iint ds\,dx\,c^{-s(1+x^n)} = \frac{\pi}{n\sin\frac{\pi}{n}}.$$

Let us integrate now this double integral, first with respect to x. By making $sx^n = t^n$, it becomes

$$\int \frac{dsc^{-s}}{s^{\frac{1}{n}}} \int dt\,c^{-t^n},$$

and if we make $s = t^n$, we will have

$$n\int dt\,c^{-t^n} \int t^{n-2}dt\,c^{-t^n} = \frac{\pi}{n\sin\frac{\pi}{n}},$$

the integrals being taken from t null to t infinity. If we change n into $\frac{n}{r-1}$, this equation becomes

$$n^2\int dt\,c^{-t^{\frac{n}{r-1}}} \int t^{\frac{n}{r-1}-2}dt\,c^{-t^{\frac{n}{r-1}}} = \frac{(r-1)^2\pi}{\sin\left(\frac{r-1}{n}\right)\pi},$$

and if in this new equation, we change t into t^{r-1}, we will have

$$n^2\int t^{r-2}dt\,c^{-t^n} \int t^{n-r}dt\,c^{-t^n} = \frac{\pi}{\sin\left(\frac{r-1}{n}\right)\pi}. \tag{T}$$

We will have, by means of this formula, by making $n = 2i$, all the values of k, $k^{(1)}$,... $k^{(i-1)}$, when we will know the half of it, if i is even, or the half less a half, if i is odd.

[96] By making $n = 2$ and $r = 2$, this formula gives this remarkable result

$$\int dt\,c^{-t^2} = \tfrac{1}{2}\sqrt{\pi}.$$

§25. We can by virtue of the generality of the analysis, extend the preceding results, to the case where t is imaginary. Let us consider the integral $\int dx\,\cos rx\,c^{-a^2x^2}$, taken from x null to x infinity. We can put it under this form

$$\tfrac{1}{2}\int dx\,c^{-a^2x^2+rx\sqrt{-1}} + \tfrac{1}{2}\int dx\,c^{-a^2x^2-rx\sqrt{-1}};$$

The integral $\int dx\,c^{-a^2x^2+rx\sqrt{-1}}$ is equal to

$$c^{-\frac{r^2}{4a^2}}\int dx\,c^{-\left(ax-\frac{r\sqrt{-1}}{2a}\right)^2}.$$

If we make

$$t = ax - \frac{r\sqrt{-1}}{2a},$$

it becomes

$$\frac{c^{-\frac{r^2}{4a^2}}}{a}\int dt\,c^{-t^2}:$$

here the integral relative to t must be taken from $t = -\frac{r\sqrt{-1}}{2a}$ to t infinity, because these two limits correspond to x null and to x infinity.

By making r negative in this formula, we will have the expression of the integral $\int dx\, c^{-a^2x^2-rx\sqrt{-1}}$; but in this case, the limits of the integral relative to t are $t = \frac{r\sqrt{-1}}{2a}$, and t infinity; the union of these two integrals is therefore equal to

$$\frac{c^{-\frac{r^2}{4a^2}}}{a}\int dt\, c^{-t^2},$$

the integral being taken from $t = -\infty$ to $t = \infty$; for the first integral adds to the second, that which is lacking to it in order to form the half of the integral taken between the two infinite limits; now this latter integral is $\frac{c^{-\frac{r^2}{4a^2}}\sqrt{\pi}}{a}$; we have therefore

$$\int dx\ \cos rx.c^{-a^2x^2} = \frac{\sqrt{\pi}}{2a}c^{-\frac{r^2}{4a^2}}.$$

The analysis which just led us to this result, is based on the passage from the real to the imaginary; for we treat the integrals relative to t and taken between two limits, of which one is imaginary and the other is infinite, as if these limits were each reals. But we can arrive to this result in the following manner. [97]

Let us name y the integral $\int x\, dx\ \cos rx.c^{-a^2x^2}$, taken from x null to infinity; we will have

$$\frac{dy}{dr} = -\int xdx\ \sin rx.c^{-a^2x^2}$$

$$= \frac{1}{2a^2}\sin rx.c^{-a^2x^2} - \frac{r}{2a^2}\int dx\ \cos rx.c^{-a^2x^2};$$

we will have therefore, by taking the integral from x null to x infinity,

$$\frac{dy}{dr} + \frac{r}{2a^2}y = 0.$$

The integral of this equation is

$$y = Bc^{-\frac{r^2}{4a^2}};$$

B being an arbitrary constant which we will determine by observing that r being null, we have

$$y = B = \int dx\, c^{-a^2x^2}.$$

This last integral is, by the preceding section, $\frac{\sqrt{\pi}}{2a}$; therefore B$= \frac{\sqrt{\pi}}{2a}$; consequently

$$\int dx\ \cos rx.c^{-a^2x^2} = \frac{\sqrt{\pi}}{2a}c^{-\frac{r^2}{4a^2}};$$

that which is conformed to the result found above by the passage from the real to the imaginary.

By differentiating $2n$ times with respect to r, we will have

$$\int x^{2n}dx\ \cos rx.c^{-a^2x^2} = \pm\frac{\sqrt{\pi}}{2a}\frac{d^{2n}}{dr^{2n}}c^{-\frac{r^2}{4a^2}},$$

[98] the + sign having place if n is even, and the − sign if n is odd. This last equation differentiated with respect to r, gives

$$\int x^{2n+1}dx\ \sin rx.c^{-a^2x^2} = \mp\frac{\sqrt{\pi}}{2a}\frac{d^{2n+1}}{dr^{2n+1}}c^{-\frac{r^2}{4a^2}}.$$

By integrating once with respect to r, the expression of $\int dx\ \cos rx.c^{-a^2x^2}$, we will have

$$\int \frac{dx\ \sin rx}{x}c^{-a^2x^2} = \frac{\sqrt{\pi}}{2a}\int dr\, c^{-\frac{r^2}{4a^2}}.$$

When a is null, $\frac{r}{a}$ becomes infinite, and the integral $\int \frac{dr}{2a}c^{-\frac{r^2}{4a^2}}$ taken from r null, becomes $\frac{1}{2}\sqrt{\pi}$; therefore

$$\int \frac{dx\ \sin rx}{x} = \frac{\pi}{2}.$$

§26. We can thence conclude the values of some singular definite integrals to which I have been led, as we will see in the sequel, by the passage from the real to the imaginary.

Let us consider the double integral

$$\iint 2dx\, y\, dy\, c^{-y^2(1+x^2)}\cos rx,$$

the integrals being taken from x and y nulls to x and y infinity. By integrating first with respect to y, it becomes

$$\int \frac{dx\ \cos rx}{1+x^2}.$$

Let us now integrate it with respect to x. We have by the preceding section,

$$\int dx\ \cos rx.c^{-y^2x^2} = \frac{\sqrt{\pi}}{2y}c^{-\frac{r^2}{4y^2}};$$

that which gives

$$\iint 2y\, dy\, dx\ \cos rx.c^{-y^2(1+x^2)} = \sqrt{\pi}\int dy\, c^{-y^2-\frac{r^2}{4y^2}}.$$

The concern now is to have this last integral taken from y null to y infinity.

For that, let us give to it this form

$$c^r\int dy\, c^{-\left(\frac{2y^2+r}{2y}\right)^2}.$$

[99] r being supposed positive, the quantity $\left(\frac{2y^2+r}{2y}\right)^2$ has a *minimum* which corresponds to $y = \sqrt{\frac{r}{2}}$; that which gives $2r$ for this *minimum*; let therefore

$$y = \tfrac{1}{2}z + \tfrac{1}{2}\sqrt{z^2+2r};$$

y needing to be extended from $y = 0$ to $y = \infty$, z must be extended from $z = -\infty$ to $z = \infty$. This value of y gives

$$dy = \tfrac{1}{2}dz + \tfrac{1}{2}\frac{zdz}{\sqrt{z^2 + 2r}}.$$

By taking the integrals from $z = -\infty$ to $z = \infty$, we have

$$\int dz\, c^{-z^2} = \sqrt{\pi}; \qquad \int \frac{zdz\, c^{-z^2}}{\sqrt{z^2 + 2r}} = 0;$$

we have therefore

$$\int dy\, c^{-\left(\frac{2y^2+r}{2y}\right)^2} = \int dy\, c^{-z^2-2r} = c^{-2r}\int \tfrac{1}{2}dz\, c^{-z^2} = \frac{c^{-2r}\sqrt{\pi}}{2};$$

hence

$$\int dy\, c^{-y^2-\frac{r^2}{4y^2}} = \frac{c^{-r}\sqrt{\pi}}{2}.$$

We will have generally by the same analysis, the integral

$$\int y^{\pm 2n} dy\, c^{-y^2-\frac{r2}{4y2}},$$

taken from y null to y infinity, and consequently also within the same limits, the integral

$$\int x^{\pm\frac{n}{2}}\, dx\, c^{-ax-\frac{b}{x}},$$

a and b being positives and n being odd. This premised, we will have

$$\iint 2y\, dy\, dx\, \cos rx . c^{-y^2(1+x^2)} = \frac{\pi}{2c^r};$$

we have therefore

$$\int \frac{dx\, \cos rx}{1 + x^2} = \frac{\pi}{2c^r}.$$

By differentiating with respect to r, we have [100]

$$\int \frac{x\, dx\, \sin rx}{1 + x^2} = \frac{\pi}{2c^r};$$

thence it is easy to conclude the value of the integral

$$\int \frac{(a + bx)dx\, \cos rx}{m + 2nx + x^2},$$

taken from $x = -\infty$ to x infinity, the denominator having no real factors in x of the first degree. If we make

$$x = -n + x'\sqrt{m - n^2},$$

this integral becomes, by supposing $\frac{a-bn}{\sqrt{m-n^2}} = a'$,

$$\int \frac{(a' + bx')dx'[\cos(rx'\sqrt{m - n^2})\cos rn + \sin(rx'\sqrt{m - n^2})\sin rn]}{1 + x'^2}.$$

This integral must be taken as that relative to x, from $x' = -\infty$ to $x' = \infty$; now the integral $\int \frac{x'\,dx'\ \cos(rx'\sqrt{m-n^2})}{1+x'^2}$, taken within these limits, is null; because its negative elements destroy its corresponding positive elements; it is likewise of the integral $\int \frac{dx'\ \sin(rx'\sqrt{m-n^2})}{1+x'^2}$; the preceding integral function is reduced therefore to

$$\int \frac{[a'\cos rn\cos(rx'\sqrt{m-n^2}) + b\sin rn\sin(rx'\sqrt{m-n^2})]x'\,dx'}{1+x'^2}.$$

We have by that which precedes,

$$\int \frac{dx'\ \cos(rx'\sqrt{m-n^2})}{1+x'^2} = \pi c^{-r\sqrt{m-n^2}}.$$

By differentiating this expression with respect to r, we have

$$\int \frac{x'\,dx'\ \sin(rx'\sqrt{m-n^2})}{1+x'^2} = \pi c^{-r\sqrt{m-n^2}};$$

we have therefore

$$\int \frac{(a+bx)dx\ \cos rx}{m+2nx+x^2} = (a'\cos rn + b\sin rn)\pi c^{-r\sqrt{m-n^2}}.$$

[101] We will find by the same analysis,

$$\int \frac{(a+bx)dx\ \sin rx}{m+2nx+x^2} = (b\cos rn - a'\sin rn)\pi c^{-r\sqrt{m-n^2}}.$$

If we differentiate the first of these two equations, $i-1$ times with respect to m, and next $2s$ times with respect to r, we will have the expression of the integral

$$\int \frac{x^{2s}dx(a+bx)\cos rx}{(m+2nx+x^2)^i}. \qquad (i)$$

Now M and N being rational and integer functions of x, the degree of the first being supposed smaller than the one of the second, and N being supposed to have no real factor of first degree; we will be able, as we know, to decompose the integral $\int \frac{M}{N}dx\ \cos rx$, into different terms of the form (i); we will have therefore generally the expression of this definite integral.

We will have in the same manner, the value of the integral

$$\int \frac{M}{N}dx\ \sin rx.$$

§27. Let us take now formula (B) of §23. The case of $\mu + 1 = 2$ being most ordinary, we are going to exhibit here the formulas which are relative there. Formula (B) becomes, in this case,

$$\int y\,dx = Y\int dt\,c^{-t^2}\left\{\begin{matrix} U + t\dfrac{dU^2}{dx} + \dfrac{t^2}{1.2}\dfrac{d^2U^3}{dx^2} \\ + \dfrac{t^3}{1.2.3}\dfrac{d^3U^4}{dx^3} + \text{etc.}\end{matrix}\right\}; \qquad (b)$$

here we have

$$t = \sqrt{\log Y - \log y}, \qquad v = \frac{x-a}{\sqrt{\log Y - \log y}},$$

Y being the *maximum* of y, and a being the value of x which corresponds to this *maximum*; U, $\frac{dU}{dx}$, … are that which v, $\frac{dv}{dx}$, etc. become, when we change x into a. This formula gives, by integrating from $t = T$ to $t = T'$, [102]

$$\begin{aligned}\int y\,dx = Y\left(U + \frac{1}{2}\cdot\frac{d^2U^3}{1.2\,dx^2} + \frac{1.3}{2^2}\cdot\frac{d^4U^5}{1.2.3.4\,dx^4} + \text{etc.}\right)\int dt\,c^{-t^2} \\ + \frac{Y}{2}c^{-T^2}\left(\frac{dU^2}{dx} + T\frac{d^2U^3}{1.2\,dx^2} + \frac{(T^2+1)d^3U^4}{1.2.3\,dx^3} + \text{etc.}\right); \\ - \frac{Y}{2}c^{-T'^2}\left(\frac{dU^2}{dx} + T'\frac{d^2U^3}{1.2\,dx^2} + \frac{(T'^2+1)d^3U^4}{1.2.3\,dx^3} + \text{etc.}\right)\end{aligned} \qquad (c)$$

the integral $\int dt\,c^{-t^2}$ being taken from $t = T$ to $t = T'$, and the integral $\int y\,dx$ being taken from the value of x which agrees with $t = T$, to that which agrees with $t = T'$.

If we suppose T$= -\infty$ and $T' = \infty$, we will have generally

$$T^n c^{-T^2} = 0, \quad T'^n c^{-T'^2} = 0.$$

We have besides by §24 $\int dt\,c^{-t^2} = \sqrt{\pi}$; the preceding formula becomes thus

$$\int y\,dx = Y\sqrt{\pi}\left(U + \frac{1}{2}\cdot\frac{d^2U^3}{1.2\,dx^2} + \frac{1.3}{2^2}\cdot\frac{d^4U^5}{1.2.3.4\,dx^4} + \text{etc.}\right), \qquad (d)$$

the integral $\int y\,dx$ being taken between the values of x which render y null, and Y being the *maximum* of y, comprehended between these values. The different terms of this formula will be determined easily by §23, and we will have

$$U = \frac{1}{\sqrt{-\frac{d^2\log y}{2dx^2}}};$$

x needing to be changed into a, after the differentiations. We have

$$d^2\log y = \frac{ddy}{y} - \frac{dy^2}{y^2};$$

the assumption of $x = a$ makes dy disappear; we will have therefore

$$\frac{d^2\log y}{dx^2} = \frac{d^2Y}{Y\,dx^2},$$

Y and $\frac{ddY}{dx^2}$ being that which y and $\frac{ddy}{dx^2}$, become when we change x into a. Thus, by considering in formula (d) only the first term of the series, we will have very nearly [103]

$$\int y\,dx = \frac{\sqrt{2\pi}Y^{\frac{3}{2}}}{\sqrt{-\frac{ddY}{dx^2}}}.$$

This expression of $\int y\,dx$ will be so much more near, as the factors of y will be raised to higher powers.

Formula (c) contains the indefinite integral $\int dt\, c^{-t^2}$ taken from $t = T$ to $t = T'$; that which returns to taking it from $t = 0$ to the limits T and T', and by subtracting the first integral from the second. It is not possible to obtain in finite terms, the integral taken from t null; but we will obtain it in a manner quite near, if T is not very large, by the following series:

$$\int dt\, c^{-t^2} = T - \frac{T^3}{3} + \frac{1}{1.2}\cdot\frac{T^5}{5} - \frac{1}{1.2.3}\cdot\frac{T^7}{7} + \frac{1}{1.2.3.4}\cdot\frac{T^9}{9} - \text{etc.}$$

This series has the advantage of being alternately smaller or greater than the integral, according as we arrest ourselves at a positive or negative term. This kind of series that we can name *series-limits*, has thus the advantage to make known the limits of the errors of the approximations. We have thus

$$\int dt\, c^{-t^2} = Tc^{-T^2}\left(1 + \frac{2T^2}{1.3} + \frac{(2T^2)^2}{1.3.5} + \frac{(2T^2)^3}{1.3.5.7} + \text{etc.}\right).$$

These two series always terminate by being convergent, whatever be the value of T; but their convergence commences only at some terms distant from the first, if $2T^2$ has a large value; it is appropriate therefore to use them only for some values equal or less than four. For greater values, we will be able to make use of the following series, which gives the value of the integral $\int dt\, c^{-t^2}$ from $t = T$ to t infinity,

$$\int dt\, c^{-t^2} = \frac{c^{-T^2}}{2T}\left(1 - \frac{1}{2T^2} + \frac{1.3}{2^2T^4} - \frac{1.3.5}{2^3T^6} + \text{etc.}\right),$$

This series is again a series-limit. By subtracting it from $\frac{1}{2}\sqrt{\pi}$, the value of the integral $\int dt\, c^{-t^2}$ taken from t null to t infinity, we will have the value of the integral taken
[104] from t null to $t = T$. But the series has the inconvenience to end by being divergent: we obviate this inconvenience, by transforming it into a continued fraction, as I have done in Book X of the *Mécanique céleste*, where I have found that by making $q = \frac{1}{2T^2}$, we have, the integral being taken from $t = T$ to infinity.

$$\int dt\, c^{-t^2} = \frac{c^{-T^2}}{2T}\cfrac{1}{1 + \cfrac{q}{1 + \cfrac{2q}{1 + \cfrac{3q}{1 + \cfrac{4q}{1 + \cfrac{5q}{1 + \text{etc.}}}}}}}$$

In order to make use of this expression, it is necessary to reduce the continued fraction

$$\cfrac{1}{1 + \cfrac{q}{1 + \cfrac{2q}{1 + \text{etc.}}}}$$

into fractions alternately greater and lesser than the entire fraction. The first two fractions are $\frac{1}{1}$, $\frac{1}{1+q}$; the numerators of the following fractions are such, that the numerator of the i^{th} fraction is equal to the numerator of the $(i-1)^{\text{st}}$, plus to the numerator of the $(i-2)^{\text{nd}}$ fraction, multiplied by $(i-1)q$; the denominators are formed in the same manner. These successive fractions are

$$\frac{1}{1},\quad \frac{1}{1+q},\quad \frac{1+2q}{1+3q},\quad \frac{1+5q}{1+6q+3q^2},\quad \frac{1+9q+8q^2}{1+10q+15q^2},\quad \text{etc.}$$

When q or $\frac{1}{2T^2}$ will be equal or less than $\frac{1}{4}$, these fractions will give in a prompt and near manner the value of the entire fraction.

§28. We can easily extend the preceding analysis to double, triple, etc. integrals. For that, let us consider the double integral $\iint y\,dx\,dx'$, y being a function of x and of x', which contains factors raised to some great powers. Let us suppose that the [105] integral relative to x' must be taken from a function X of x to another function $X'+X$ of the same variable. By making $x' = X + tX'$, the integral $\iint y\,dx\,dx'$ will be changed into this here, $\iint y\,X'dx\,dt$; the integral relative to t needing to be taken from $t = 0$ to $t = 1$: we can thus therefore reduce the integral $\iint y\,dx\,dx'$ to some limits constant and independent of the variables which it contains. We will suppose that it has this form, and that the integral relative to x is taken from $x = \theta$ to $x = \varpi$, and that the integral relative to x' is taken from $x' = \theta'$ to $x' = \varpi'$. This premised, by naming Y that which y becomes when we change x and x' into θ and θ', we will make

$$y = Yc^{-t-t'};$$

by supposing next

$$x = \theta + u, \qquad x' = \theta' + u';$$

we will reduce $\log\frac{Y}{y}$ to a series ordered with respect to the powers of u and of u', and we will have an equation of this form

$$Mu + M'u' = t + t',$$

in which M is the part of the development into series, of $\log\frac{Y}{y}$ which contains all the terms multiplied by u, and M' is the other part which contains the terms multiplied by u', and which are independent of u. We will divide the preceding equation, into the two following

$$Mu = t, \quad M'u' = t';$$

whence we will deduce this, by the reversion of the series,

$$u = Nt, \quad u' = N't',$$

N being a series ordered with respect to the powers of t and of t', and N' being uniquely ordered with respect to the powers of t' and being independent of t; these two series are very convergent, if y contains very elevated factors. Now we have $dx\,dx' = du\,du'$; moreover we have

$$du = \left(\frac{dNt}{dt}\right) dt + \left(\frac{dNt}{dt'}\right) dt',$$

$$du' = \left(\frac{dN't'}{dt'}\right) dt';$$

[106] but in the product $du\,du'$, the differential du is taken by making u' constant, that which renders t' constant, or $dt' = 0$; we have therefore

$$du = \left(\frac{dNt}{dt}\right) dt;$$

consequently

$$du\,du' = \left(\frac{dNt}{dt}\right)\left(\frac{dN't'}{dt'}\right) dt\,dt';$$

that which gives

$$\int y\,dx\,dx' = Y \int \left(\frac{dNt}{dt}\right)\left(\frac{dN't'}{dt'}\right) dt\,dt' c^{-t-t'}.$$

It is easy to integrate the different terms of the second member of this equation, since the question is only of integrating the terms of the form $\int t^n dt\ c^{-t}$.

If we take the integral relative to t', from t' null to t' infinite, and if we name Q the result of the integration, we will have

$$\int y\,dx' = YQ,$$

the integral relative to x' being taken from $x' = \theta'$ to the value of x', which corresponds to t' infinite. If we change next in Y and Q, θ' into ϖ', and if we name Y' and Q', that which these quantities then become; we will have

$$\int y\,dx' = Y'Q',$$

the integral being taken from $x' = \varpi'$ to the value of x', which corresponds to t' infinite.

By naming R and R' the integrals $\int Q\,dt$ and $\int Q'dt$, taken from t null to t infinity; we will have

$$\int ydx\,dx' = YR - Y'R',$$

the integral relative to x' being taken from $x' = \theta'$ to $x' = \varpi'$, and the integral relative to x being taken from $x = \theta$ to the value of x which corresponds to t infinite. If in Y, R, Y', R', we change θ into ϖ, and if we name Y_1, R_1, Y_1', R_1', that which these quantities then become; we will have

$$\int y\,dx\,dx' = Y_1R_1 - Y_1'R_1',$$

the integral relative to x' being taken between the limits θ' and ϖ', and the integral [107]
relative to x being taken from $x = \varpi$ to the value x which corresponds to t infinite; we will have therefore

$$\int y\,dx\,dx' = YR - Y'R' - Y_1R_1 + Y_1'R_1',$$

the integral relative to x being taken between the limits θ and ϖ, and the integral relative to x' being taken between the limits θ' and ϖ'.

This formula corresponds to formula (A) of §22, which is relative only to a single variable; it has, like it, the inconvenience of not being able to be extended to the intervals near the *maximum* of y. It is necessary, for these intervals, to use a method analogous to that of §23. Thus, by supposing that, in the interval comprehended between θ and ϖ, y becomes a *maximum* relative to x, so that the condition of this *maximum* makes only the differential of y vanish, taken with respect to x, we will make

$$y = Yc^{-t-t'},$$

Y being the value of y which corresponds to this *maximum* and to $x' = \theta'$; and if, in the interval comprehended between the limits of the integrations relative to x and to x', y becomes a *maximum*, we will make

$$y = Yc^{-t^2-t'^2}.$$

As we will have need principally, in the following, of the integral $\int y\,dx\,dx'$ taken between the limits of x and x' which render y null, we are going to discuss this case.

Let us consider the integral $\int y\,dx\,dx'$, y being a function of x, x', which contains factors raised to some great powers. If we name a, a', the values of x, x' which correspond to the *maximum* of y, and if we name Y this *maximum*; we will make

$$y = Yc^{-t^2-t'^2};$$

by supposing next

$$x = a + \theta, \qquad x' = a' + \theta';$$

we will substitute these values into the function $\log\frac{Y}{y}$, and by developing it into a series ordered with respect to the powers and to the products of θ, θ', we will have [108]
an equation of this form

$$M\theta^2 + 2N\theta\theta' + P\theta'^2 = t^2 + t'^2.$$

This equation can be set under the form

$$M\left(\theta + \frac{N}{M}\theta'\right)^2 + \left(P - \frac{N^2}{M}\right)\theta'^2 = t^2 + t'^2;$$

we will make therefore

$$t = \theta\sqrt{M} + \frac{N\theta'}{\sqrt{M}}, \qquad t' = \theta'\sqrt{P - \frac{N^2}{M}}.$$

By differentiating these equations, we will have the differentials of this form

$$dt = L\,d\theta + I\,d\theta'$$
$$dt' = L'dt + I'd\theta'.$$

Now we have

$$\int y\,dx\,dx' = \int y\,d\theta\,d\theta';$$

in the product $d\theta\,d\theta'$, $d\theta$ is taken by supposing θ' constant, and then we have

$$dt = L\,d\theta;$$

next dt' must be taken by regarding t constant, in the product $dt\,dt'$; then we have

$$0 = L\,d\theta + I\,d\theta'$$
$$dt' = L'd\theta + I'd\theta';$$

that which gives

$$dt' = \frac{LI' - L'I}{L} d\theta';$$

we have therefore

$$dt\,dt' = d\theta\,d\theta'(LI' - L'I);$$

by this means, the integral $\int y\,d\theta\,d\theta'$ is transformed into this here:

$$Y \int \frac{dt\,dt'\,c^{-t^2-t'^2}}{LI' - L'I}.$$

The denominator $LI' - L'I$ is a function of θ and of θ' that we will reduce to a function of t and t', by means of the values of t and of t' in θ and θ'. We will obtain
[109] thus the preceding integral in a series of terms of the form $\int t^n t'^{n'} dt\,dt'\,c^{-t^2-t'^2}$, the integrals being taken from t and t' equal to $-\infty$, to their positive infinite values. These integrals are nulls, when one of the two numbers n and n' is odd; and in the case where they are both even, n being equal to $2i$, and n' to $2i'$, we have

$$\int t^{2i} t'^{2i'} dt\,dt'\,c^{-t^2-t'^2} = \frac{1.3.5\ldots(2i-1).1.3.5\ldots(2i'-1)}{2^i.2^{i'}}\sqrt{\pi}.$$

If the powers to which the factors of y are raised, are very great; then we will have, very nearly

$$M = -\frac{\left(\frac{ddY}{dx^2}\right)}{2Y}, \quad 2N = -\frac{\left(\frac{ddY}{dx\,dx'}\right)}{Y}, \quad P = -\frac{\left(\frac{ddY}{dx'^2}\right)}{2Y},$$

$\left(\frac{ddY}{dx^2}\right)$, $\left(\frac{ddY}{dxdx'}\right)$, $\left(\frac{ddY}{dx'^2}\right)$, being that which $\left(\frac{ddy}{dx^2}\right)$, $\left(\frac{ddy}{dx\,dx'}\right)$, and $\left(\frac{ddy}{dx'^2}\right)$ become when we change x and x' into a and a' there; the integral $\int y\,dx\,dx'$ become thus very nearly,

$$\frac{2\pi Y^2}{\sqrt{\left(\frac{ddY}{dx^2}\right)\left(\frac{ddY}{dx'^2}\right) - \left(\frac{ddY}{dx\,dx'}\right)^2}}.$$

CHAPTER 2

On integration by approximation, of linear equations in the finite and infinitely small differences

§29. We have seen in §21, that the integrals of equations linear in the differences among one variable s, of which the finite difference is supposed constant, and a function y_s of this variable, can be set under the form $y_s = \int x^s \phi\, dx$, ϕ being a function of x of the same nature as the generating function of the equation proposed in the differences, and the integral being taken within some determined limits of x. By supposing s a very great number, we will have by the preceding analysis, a very near value of this integral, and consequently of y_s. But this method of approximation being very important in the theory of probabilities, we are going to develop it at length. [110]

Let us consider the equation in finite differences

$$S = Ay_s + B\triangle y_s + C\triangle^2 y_s + \text{etc.}, \tag{1}$$

A, B, C being some rational and integral functions of s, to which we will give this form

$$\begin{aligned}
A &= a + a^{(1)}s + a^{(2)}s(s-1) + a^{(3)}s(s-1)(s-2) + \text{etc.},\\
B &= b + b^{(1)}s + b^{(2)}s(s-1) + b^{(3)}s(s-1)(s-2) + \text{etc.},\\
C &= e + e^{(1)}s + e^{(2)}s(s-1) + e^{(3)}s(s-1)(s-2) + \text{etc.},\\
&\text{etc.};
\end{aligned}$$

$\triangle y_s$ is the finite difference of y_s, s being supposed to vary by unity; $\triangle^2 y_s$, $\triangle^3 y_s$, etc. are the second, third, etc. differences of y_s; and S is a function of s. This premised, let us represent y_s by the integral $\int x^s \phi\, dx$, ϕ being a function of x which it is necessary to determine, as well as the limits of the integral. By designating x^s by δy, we will [111]
have

$$\triangle y_s = \int \delta y(x-1)\phi\, dx, \qquad \triangle^2 y_s = \int \delta y(x-1)^2\phi\, dx, \qquad \text{etc.};$$

we will have next

$$sx^s = x\frac{d\,\delta y}{dx}, \qquad s(s-1)x^s = x^2\frac{d^2\,\delta y}{dx^2}, \qquad \text{etc.};$$

equation (1) in the differences becomes thus

$$S = \int \phi\, dx \left\{ \begin{array}{l} \quad \delta y[a + b(x-1) + e(x-1)^2 + \text{etc.}] \\ + \dfrac{x\, d\, \delta y}{dx}[a^{(1)} + b^{(1)}(x-1) + e^{(1)}(x-1)^2 + \text{etc.}] \\ + \dfrac{x^2\, d^2\, \delta y}{dx^2}[a^{(2)} + b^{(2)}(x-1) + e^{(2)}(x-1)^2 + \text{etc.}] \\ + \text{etc.} \end{array} \right\}.$$

Instead of making y_s equal to $\int x^s\, \phi\, dx$, we can suppose it equal to $\int c^{-sx} \phi\, dx$; then we have

$$\triangle y_s = \int c^{-sx}(c^{-x} - 1)\phi\, dx, \qquad \triangle^2 y_s = \int c^{-sx}(c^{-x} - 1)^2 \phi\, dx, \qquad \text{etc.}$$

Moreover, if we designate c^{-sx} by δy, we will have

$$sc^{-sx} = -\frac{d\delta y}{dx}, \qquad s^2 c^{-sx} = \frac{d^2 \delta y}{dx^2}, \qquad \text{etc.;}$$

by setting therefore the coefficients of equation (1) under this form,

$$\begin{aligned} A &= a + a^{(1)} s + a^{(2)} s^2 + \text{etc.}, \\ B &= b + b^{(1)} s + b^{(2)} s^2 + \text{etc.}, \\ C &= e + e^{(1)} s + e^{(2)} s^2 + \text{etc.}, \\ &\text{etc.} \end{aligned}$$

this equation will take the form

$$S = \int \phi\, dx \left\{ \begin{array}{l} \quad \delta y[a + b(c^{-x}-1) + e(c^{-x}-1)^2 + \text{etc.}] \\ - \dfrac{d\, \delta y}{dx}[a^{(1)} + b^{(1)}(c^{-x}-1) + e^{(1)}(c^{-x}-1)^2 + \text{etc.}] \\ + \dfrac{d^2\, \delta y}{dx^2}[a^{(2)} + b^{(2)}(c^{-x}-1) + e^{(2)}(c^{-x}-1)^2 + \text{etc.}] \\ - \text{etc.} \end{array} \right\}.$$

[112] By representing generally y_s by $\int \delta y\, \phi\, dx$, the two forms that equation (1) takes under the assumption $\delta y = x^s$ and of $\delta y = c^{-sx}$ will be comprehended in the following

$$S = \int \phi\, dx \left(M\, \delta y + N \frac{d\, \delta y}{dx} + P \frac{d^2\, \delta y}{dx^2} + Q \frac{d^3\, \delta y}{dx^3} + \text{etc.} \right),$$

M, N, P, Q, etc. being functions of x independent of the variable s, which enters into the second member of this equation, only as far as δy and its differences are functions of it.

Now, in order to satisfy it, we will integrate by parts, its different terms; now we have

$$\int \frac{d\,\delta y}{dx} N\phi\,dx = \delta y N\phi - \int \delta y\, d(N\phi),$$
$$\int \frac{d^2\,\delta y}{dx^2} P\phi\,dx = \frac{d\,\delta y}{dx} P\phi - \delta y \frac{d(P\phi)}{dx} - \int \delta y\, \frac{d^2(P\phi)}{dx^2} dx,$$
etc.;

the preceding equation becomes thus

$$\begin{aligned} S = \int \delta y\, dx \left(M\phi - \frac{d(N\phi)}{dx} + \frac{d^2(P\phi)}{dx^2} + \frac{d^3(Q\phi)}{dx^3} + \text{etc.} \right) \\ + C + \delta y \left(N\phi - \frac{d(P\phi)}{dx} + \frac{d^2(Q\phi)}{dx^2} + \text{etc.} \right) \\ + \frac{d\,\delta y}{dx} \left(P\phi - \frac{d(Q\phi)}{dx} + \text{etc.} \right) \\ + \frac{d^2\,\delta y}{dx^2} (Q\phi - \text{etc.}) \\ + \text{etc.}, \end{aligned}$$

C being an arbitrary constant.

Since the function ϕ must be independent of s, and consequently of δy, we must separately equate to zero, the part of this equation, affected with the $\int$ sign; that which divides the preceding equation into the two following,

$$0 = M\phi - \frac{d(N\phi)}{dx} + \frac{d^2(P\phi)}{dx^2} + \frac{d^3(Q\phi)}{dx^3} + \text{etc.}, \tag{2}$$

[113]

$$\begin{aligned} S = C + \delta y \left(N\phi - \frac{d(P\phi)}{dx} + \frac{d^2(Q\phi)}{dx^2} - \text{etc.} \right) \\ + \frac{d\,\delta y}{dx} \left(P\phi - \frac{d(Q\phi)}{dx} + \text{etc.} \right); \\ + \frac{d^2\,\delta y}{dx^2} (Q\phi - \text{etc.}) \\ + \text{etc.} \end{aligned} \tag{3}$$

The first of these equations serves to determine the function ϕ; and the second determines the limits in which the integral $\int \delta y \phi\, dx$ is comprehended.

We can observe here that equation (2) is the equation of condition which must hold, in order that the differentiable function

$$\left(M\,\delta y + N \frac{d\,\delta y}{dx} + P \frac{d^2\,\delta y}{dx^2} + \text{etc.} \right) \phi\, dx$$

is an exact differential, whatever be δy; and in this case, the integral of this function is equal to the second member of equation (3); ϕ is therefore the factor in x alone which must multiply the equation

$$0 = M\,\delta y + N\frac{d\,\delta y}{dx} + P\frac{d^2\,\delta y}{dx^2} + \text{etc.},$$

in order to render it integrable. If ϕ were known, we could lower this equation by one degree; and, reciprocally, if this equation were lowered by a degree; the coefficient of δy, in its differential divided by $M dx$, would give a value of ϕ; this equation and equation (2) are consequently linked between them, in a manner that an integral of one gives an integral of the other.

The value of ϕ being supposed known, we will have that of y_s by means of a definite integral. The integration of equation (1) in the finite differences, is therefore thus brought back to the integration of equation (2) in the infinitely small differences, and to a definite integral.

Let us consider presently equation (3), and let us make first $S = 0$. If we suppose that δy, $\frac{d\,\delta y}{dx}$, $\frac{d^2\,\delta y}{dx^2}$, etc. become nulls, by means of one same value of x, which we will designate by h, and which is independent of s, it is clear that by supposing C
[114] null, this value will satisfy equation (3), and that thus it will be one of the limits between which we must take the integral $\int \delta y\phi\,dx$. The preceding supposition holds clearly, in the two cases of $\delta y = x^s$ and of $\delta y = e^{-sx}$; in the first case, the equation $x = 0$, and in the second case, the equation $x = \infty$, render null the quantities δy, $\frac{d\,\delta y}{dx}$, $\frac{d^2\,\delta y}{dx^2}$, etc. In order to have some other limits of the integral $\int \delta y\phi\,dx$, we will observe that these limits needing to be independent of s, it is necessary in equation (3), to equate separately to zero, the coefficients of δy, $\frac{d\,\delta y}{dx}$, etc.; that which gives the following equations:

$$\begin{aligned}
&0 = N\phi - \frac{d(P\phi)}{dx} + \frac{d^2(Q\phi)}{dx^2} - \text{etc.},\\
&0 = P\phi - \frac{d(Q\phi)}{dx} + \text{etc.},\\
&0 = Q\phi - \text{etc.},\\
&\text{etc.}
\end{aligned}$$

These equations are in number i, if i is the order of the differential equation (2); we will be able therefore to eliminate, by their means, all the arbitrary constants of the value of ϕ, less one; and we will have a final equation in x, of which the roots will be as many as limits of the integral $\int \delta y\phi\,dx$. We will seek by this equation, a number of different values of x, equal to the degree of the differential equation (1). Let q, $q^{(1)}$, $q^{(2)}$, etc. be these values; they will give as many different values of ϕ, since the arbitrary constants of ϕ, less one, are determined as functions of these values. We could thus represent the values of ϕ, corresponding to the limits q, $q^{(1)}$, $q^{(2)}$, etc., by $B\lambda$, $B^{(1)}\lambda^{(1)}$, $B^{(2)}\lambda^{(2)}$, etc., B, $B^{(1)}$, $B^{(2)}$, etc. being some arbitrary constants; and we will have for the complete value of y_s,

$$y_s = B\int \delta y\lambda\, dx + B^{(1)}\int \delta y\lambda^{(1)} dx + B^{(2)}\int \delta y\lambda^{(2)}\, dx + \text{etc.};$$

the integral of the first term being taken from $x = h$ to $x = q$, that of the second term being taken from $x = h$ to $x = q^{(1)}$, and thus of the rest. We will determine the constants B, $B^{(1)}$, etc., by means of so many particular values of y_s.

Let us suppose now that in equation (3), S is not null. If we take the integral $\int \delta y\phi\, dx$ from $x = h$ to x equal to any quantity p; it is clear that we will have $C = 0$, and that S will be that which the function [115]

$$\begin{aligned}&\delta y\left(N\phi - \frac{d(P\phi)}{dx} + \text{etc.}\right)\\ &+\frac{d\,\delta y}{dx}(P\phi - \text{etc.})\\ &+\text{etc.};\end{aligned}$$

becomes when we change x into p. Thus, for the success of the preceding method, it is necessary that S have the form of this function. Let us make, for example, $\delta y = x^s$, and

$$S = p^s[l + l^{(1)}s + l^{(2)}s(s-1) + l^{(3)}s(s-1)(s-2) + \text{etc.}];$$

by comparing this value of S to the preceding, we will have

$$\begin{aligned}l &= N\phi - \frac{d(P\phi)}{dx} + \text{etc.},\\ l^{(1)}p &= P\phi - \text{etc.},\\ &\text{etc.},\end{aligned}$$

x needing to be changed into p in the second members of these equations of which the number is equal to the degree of the differential equation (2). We could therefore, by their means, determine the arbitrary constants of the value of ϕ; and if we designate by ψ, that which ϕ becomes, when we have thus determined its arbitraries, we will have

$$y_s = \int x^s\psi\, dx.$$

Thence and from this that equation (1) is linear, it is easy to conclude that if S is equal to

$$\begin{aligned}&p^s[l + l^{(1)}s + l^{(2)}s(s-1) + \text{etc.}]\\ &+ p_1^s[l_1 + l_1^{(1)}s + l_1^{(2)}s(s-1) + \text{etc.}]\\ &+ \text{etc.}\end{aligned}$$

By naming ψ', etc., that which ψ becomes when we change successively p, l, $l^{(1)}$, etc., into p_1, l_1, $l_1^{(1)}$, etc., into p_2, etc.; we will have [116]

$$y_s = \int x^s\,\psi\, dx + \int x^s\,\psi'\, dx + \text{etc.};$$

the first integral being taken from $x = h$ to $x = p$, the second integral being taken from $x = h$ to $x = p_1$, etc. This value of y_s contains no arbitrary constant; but, by joining it to that which we have found previously for the case of S null, we will have the complete expression of y_s.

§30. Let us suppose now that we have any number of linear equations in the finite differences among a like number of variables y_s, y'_s, y''_s, etc., and of which the coefficients are rational and integral functions of s. Let us make then

$$y_s = \int x^s \phi\, dx, \qquad y'_s = \int x^s \phi'\, dx, \qquad y''_s = \int x^s \phi''\, dx, \qquad \text{etc.};$$

these different integrals being taken between the same limits determined and independent of s. We will have

$$\begin{aligned} \triangle y_s &= \int x^s(x-1)\phi\, dx, \quad \triangle^2 y_s = \int x^s(x-1)^2\phi\, dx, \quad \text{etc.};\\ \triangle y'_s &= \int x^s(x-1)\phi'\, dx, \quad \triangle^2 y'_s = \int x^s(x-1)^2\phi'\, dx, \quad \text{etc.};\\ &\text{etc.} \end{aligned}$$

The equations of which there is concern, will be able to be set under the following forms

$$S = \int x^s z\, dx, \qquad S' = \int x^s z'\, dx, \qquad S'' = \int x^s z''\, dx, \qquad \text{etc.},$$

S, S', S'', etc. being functions of s alone, and z, z', z'', etc., being rational and integral functions of the same variable, and of x, ϕ, ϕ', ϕ'', etc., in which ϕ, ϕ', etc., are under a linear form.

Let us consider first the equation

$$S = \int x^s z\, dx,$$

we have

$$z = \text{Z} + s\triangle\text{Z} + \frac{s(s-1)}{1.2}\triangle^2\text{Z} + \frac{s(s-1)(s-2)}{1.2.3}\triangle^3\text{Z} + \text{etc.};$$

[117] the characteristic $\triangle$ of the finite differences being relative to the variable s, and Z, $\triangle$Z, etc. being that which z, $\triangle z$, etc. become, when we suppose $s = 0$. We will have therefore

$$S = \int x^s dx \left(\text{Z} + s\triangle\text{Z} + \frac{s(s-1)}{1.2}\triangle^2\text{Z} + \text{etc.}\right).$$

If we make $x^s = \delta y$, we will have

$$sx^s = x\frac{d\,\delta y}{dx}, \qquad s(s-1)x^s = x^2\frac{d^2\,\delta y}{dx^2}, \qquad \text{etc.};$$

the preceding equation becomes thus

$$S = \int dx \left(\text{Z}\ \delta y + x\triangle\text{Z}\frac{d\,\delta y}{dx} + \frac{x^2\triangle^2\text{Z}}{1.2}\frac{d^2\,\delta y}{dx^2} + \text{etc.}\right),$$

whence we deduce by integrating by parts, as in the preceding section, the following two equations,

$$0 = \mathrm{Z} - \frac{d(x\,\triangle \mathrm{Z})}{dx} + \frac{d^2(x^2\,\triangle^2 \mathrm{Z})}{1.2\,dx^2} - \text{etc.}, \tag{a}$$

$$\begin{aligned} S = C + \delta y \left(x\triangle \mathrm{Z} - \frac{d(x^2\,\triangle^2 \mathrm{Z})}{1.2\,dx} + \text{etc.} \right) \\ + \frac{d\,\delta y}{dx} \left(\frac{x^2\,\triangle^2 \mathrm{Z}}{1.2} - \text{etc.} \right) \\ + \text{etc.}, \end{aligned} \tag{b}$$

C being an arbitrary constant. The equation

$$S' = \int x^s\, z'\, dx,$$

treated in the same manner, will give

$$0 = \mathrm{Z}' - \frac{d(x\,\triangle \mathrm{Z}')}{dx} + \frac{d^2(x^2\,\triangle^2 \mathrm{Z}')}{1.2\,dx^2} - \text{etc.}, \tag{a'}$$

$$\begin{aligned} S' = C' + \delta y \left(x\triangle \mathrm{Z}' - \frac{d(x^2\,\triangle^2 \mathrm{Z}')}{1.2\,dx} + \text{etc.} \right) \\ + \frac{d\,\delta y}{dx} \left(\frac{x^2\,\triangle^2 \mathrm{Z}'}{1.2} - \text{etc.} \right) \\ + \text{etc.} \end{aligned} \tag{b'}$$

the equations $S'' = \int x^s\, z''\, dx$, $S''' = \int x^s\, z'''\, dx$, etc., will produce some similar equations, which we will designate by (a''), (b''); (a'''), (b'''); etc.

Equations (a), (a'), (a''), etc. will determine the variables ϕ, ϕ', ϕ'', etc. as function [118]
of x; and the equations (b), (b'), (b''), etc. will determine the limits within which we must take the integrals $\int x^s z\, dx$, $\int x^s z'\, dx$, etc. One of these limits is $x = 0$. In order to have the others, we will suppose first S, S', S'', etc. nulls; the constants C, C', C''', etc. will be consequently nulls in the equations (b), (b'), etc., since the supposition of $x = 0$ renders null the other terms of these equations. By equating next separately to zero, the coefficients of δy, $\frac{d\,\delta y}{dx}$, etc. in these same equations, we will have the

following,

$$
\begin{aligned}
&0 = x\,\triangle \mathrm{Z} - \frac{d(x^2\triangle^2\mathrm{Z})}{1.2\,dx} + \text{etc.},\\
&0 = \frac{x^2\triangle^2\mathrm{Z}}{1.2} - \text{etc.},\\
&\quad\text{etc.};\\
&0 = x\,\triangle \mathrm{Z}' - \frac{d(x^2\triangle^2\mathrm{Z}')}{1.2\,dx} + \text{etc.},\\
&0 = \frac{x^2\triangle^2\mathrm{Z}'}{1.2} - \text{etc.},\\
&\quad\text{etc.};\\
&\quad\text{etc.}
\end{aligned}
$$

We will eliminate, by means of these equations, all the arbitrary constants, less one, of the values of ϕ, ϕ', ϕ'', etc., and we will arrive to one final equation in x, of which the roots are the limits of the integrals $\int x^s\phi\,dx$, $\int x^s\phi'\,dx$, etc. We will determine as many of these limits as it is necessary, in order that the values of y_s, y'_s, etc. are complete.

Let us suppose now that S is not null, and that it is equal to

$$p^s[l + l^{(1)}s + l^{(2)}s(s-1) + \text{etc.}].$$

By making $C = 0$ in equation (b) and by putting x^s in the place of δy, we will have

$$
\begin{aligned}
p^s[l + l^{(1)}s + l^{(2)}s(s-1) + \text{etc.}] = {} & x^s\left(x\,\triangle\mathrm{Z} - \frac{d(x^2\triangle^2\mathrm{Z})}{1.2\,dx} + \text{etc.}\right)\\
& + sx^s\left(\frac{x\triangle^2\mathrm{Z}}{1.2} - \text{etc.}\right)\\
& + \text{etc.}
\end{aligned}
$$

[119] whence we conclude first $x = p$, so that the integrals $\int x^s\phi\,dx$, $\int x^s\phi'\,dx$, etc., must be taken from $x = 0$ to $x = p$. The comparison of the coefficients of s, $s(s-1)$, etc., will give next as many equations between l, $l^{1)}$, etc. and the arbitrary constants of the expressions of ϕ, ϕ', etc. The equating to zero of these same coefficients, in equations (b'), (b''), etc., will give some new equations among these arbitraries that we could thus determine by means of all these equations. We will have, by this process, the particular values of y_s, which satisfy in the case where S', S'', etc. being nulls, S has the form that we just supposed to it, or, more generally, is equal to any number of functions of the same form.

Similarly, if we suppose that S, S'', etc. being nulls, S' is the sum of any number of similar functions, we will determine the particular values of y_s, y'_s, etc., which satisfy this case, and thus of the rest. By joining next all these values, to those which we will have determined in the case where S, S', etc. are nulls, we will have the complete expressions of y_s, y'_s, etc. corresponding to the case where S, S', … have the preceding forms.

It is easy to extend this method to the equations in infinitely small differences, or in finite parts, and in infinitely small parts, and in which the coefficients of the principal variables and of their differences, are rational functions of s, that we can always render integral, by making the denominators vanish. If we designate, as above, by y_s, y'_s, etc., the principal values of these equations, and if we make

$$y_s = \int x^s \phi \, dx, \qquad y'_s = \int x^s \phi' \, dx, \qquad \text{etc.};$$

we will have

$$\frac{dy_s}{ds} = \int x^s \phi \, dx \log x, \qquad \frac{d^2 y_s}{ds^2} = \int x^s \phi \, dx \, (\log x)^2, \quad \text{etc.};$$

$$\triangle y_s = \int x^s (x-1) \phi \, dx, \quad \triangle^2 y_s = \int x^s (x-1)^2 \phi \, dx, \quad \text{etc.},$$

etc.;

$$\frac{dy'_s}{ds} = \int x^s \phi' \, dx \log x, \qquad \text{etc.},$$

etc.;

The proposed equations will take thus the following forms, [120]

$$S = \int x^s z \, dx, \qquad S' = \int x^s z' \, dx, \qquad \text{etc.}$$

In treating them by the preceding method, we will determine the values of ϕ, ϕ', etc. as functions of x, and the limits of the integrals $\int x^s \phi \, dx$, $\int x^s \phi' \, dx$, etc.

By making

$$y_s = \int c^{-sx} \phi \, dx, \qquad y'_s = \int c^{-sx} \phi' \, dx, \qquad \text{etc.};$$

we would arrive to some similar equations. In many circumstances, these forms of y_s, y'_s, etc. will be more suitable than the preceding.

§31. The principal difficulty that the application of the preceding method presents, consists in the integration of the linear differential equations which determine ϕ, ϕ', ϕ'', etc. in x. The degrees of these equations depend not at all on those of the equations in the differences in y_s, y'_s, etc.; they depend uniquely on the highest powers of s, in their coefficients. By considering therefore only a single variable y_s, the differential equation in ϕ will be of a degree equal to the highest exponent of s, in the coefficients of the equation in the differences in y_s. The differential equation in ϕ will be thus resolvable generally only in the case where the highest exponent is unity. Let us develop this case quite at length.

Let us represent the differential equation in y_s by the following,

$$0 = V + sT,$$

V and T being linear functions of the principal variable y_s and of its differences, either finite, or infinitely small. If we make

$$y_s = \int \delta y\, \phi\, dx,$$

δy being equal to x^s, or to c^{-sx}, it will become

$$0 = \int \phi\, dx \left(M\delta y + N\frac{d\,\delta y}{dx} \right);$$

[121] M and N being functions of x; we will have therefore, by integrating by parts as in the preceding section, the following two equations,

$$0 = M\phi - \frac{d(N\phi)}{dx},$$
$$0 = C + N\phi\,\delta y.$$

The first gives by integrating it,

$$\phi = \frac{H}{N} c^{\int \frac{M}{N} dx},$$

H being an arbitrary constant. Let us suppose C null in the second equation; $x = 0$ or $x = \infty$ will be one of the limits of the integral $\int \delta y \phi\, dx$, according as we take x^s or c^{-sx} for δy. We will determine the other limits, by resolving the equation $0 = N\phi\,\delta y$.

Let us apply to this integral, the method of approximation of §23. If we designate by a, the value of x, given by the equation

$$0 = d(N\,\phi\,\delta y),$$

and by Q that which the function $N\phi\,\delta y$ becomes, when we change x into a in it, we will make

$$N\,\phi\,\delta y = Qc^{-t^2},$$

that which gives

$$t = \sqrt{\log Q - \log(N\phi) - \log \delta y}.$$

$\log \delta y$ being of order s; if we suppose s very great, and if we make $\frac{1}{s} = \alpha$, α will be a very small coefficient. The quantity under the radical will take this form $\frac{(x-a)^2}{\alpha} X$, X being a function of $x - a$ and of α; we will have therefore, by the reversion of the series, the value of x in t, by a series of this form

$$x = a + \alpha^{\frac{1}{2}} ht + \alpha h^{(1)} t^2 + \alpha^{\frac{3}{2}} h^{(2)} t^3 + \text{etc.}$$

Now, y_s being equal to $\int \delta y\, \phi\, dx$, if we substitute into this integral, in the place of $\phi\,\delta y$, its value $\frac{Qc^{-t^2}}{N}$, it will become Q$\int \frac{dx}{N} c^{-t^2}$; and if in $\frac{dx}{N}$, we substitute for x, its
[122] preceding value in t, we will have y_s by a series of this form,

$$y_s = \alpha^{\frac{1}{2}} Q \int dt\, c^{-t^2} [l + \alpha^{\frac{1}{2}} l^{(1)} t + \alpha l^{(2)} t^2 + \alpha^{\frac{3}{2}} l^{(3)} t^3 + \text{etc.}],$$

the limits of the integral relative to t needing to be determined by the condition that at these limits, the quantity $N\phi\,\delta y$, or its equivalent Qc^{-t^2}, be null; whence it follows that these limits are $t = -\infty$ and $t = \infty$; we will have therefore, by §24,

$$y_s = \alpha^{\frac{1}{2}} Q\sqrt{\pi}\left(l + \frac{1}{2}\alpha l^{(2)} + \frac{1.3}{2^2}\alpha^2 l^{(4)} + \frac{1.3.5}{2^3}\alpha^3 l^{(6)} + \text{etc.}\right).$$

This expression has the advantage of being independent of the determination of the limits in x, which render null the function $N\phi\,\delta y$; so that it subsists in the case even where this function, equated to zero, has no real roots; it subsists further in the case of s negative. This remark analogous to that which we have made in §25, and which holds, as it, to the generality of analysis, is very remarkable in this that it gives the means to extend the preceding formula, to a great number of cases to which the method has led us, seems first to be refused.

This formula contains only the arbitrary constant H, and consequently, it is only a particular integral of the differential equation proposed in y_s, if this equation is of an order superior to unity. In order to have the complete integral in this case, it will be necessary to seek in the equation $0 = d(N\phi\,\delta y)$, as many different values of x, as it has units in this order. Let a, a', a'', etc. be these values; we will change successively in the preceding expression of y_s, a into a', a'', etc., and H into H',H'', etc.; we will have as many particular values which will each contain one arbitrary, and of which the sum will be the complete expression of y_s.

When the coefficients of the proposed in y_s contain powers of s superior to unity; we can sometimes decompose this equation into many others which contain only that first power. If we have, for example, the equation

$$y_{s+1} = My_s,$$

M being a rational and integral function of s; we will set this function under the form [123]

$$\frac{q(s+b)(s+b')(s+b'').\text{etc.}}{(s+f)(s+f')(s+f'').\text{etc.}};$$

we will make next

$$z_{s+1} = q(s+b)z_s,\quad z'_{s+1} = (s+b')z'_s,\qquad \text{etc.};$$
$$t_{s+1} = q(s+f)t_s,\quad t'_{s+1} = (s+f')t'_s,\qquad \text{etc.}$$

It is easy, by that which precedes, to determine z_s, t_s, etc. as definite integrals, and to reduce these integrals into convergent series, when s is a great number. We will have next

$$y_s = \frac{z_s z'_s.\text{etc.}}{t_s t'_s.\text{etc.}}.$$

In many cases where the differential equation in ϕ being of an order superior to the first, cannot be integrated rigorously, we can determine ϕ by a very convergent approximation; by substituting next this value of ϕ into the integral $\int x^s\phi\,dx$, we can obtain in a manner quite close the value of this integral.

§32. The analysis exposed in the preceding sections, is extended further to the equations in partial differences, finite and infinitely small. For this, let us consider first the equation linear in the partial differences of which the coefficients are constants. In designating by $y_{s,s'}$ the principal variable, s and s' being the two variables of which it is a function, and representing this equation by this one, $V = 0$, V being a linear function of $y_{s,s'}$ and of its partial differences, we will suppose

$$y_{s,s'} = \int x^s x'^{s'} \phi\, dx,$$

ϕ being a function of x; then the equation $V = 0$ takes this form

$$0 = \int M x^s u^{s'} \phi\, dx,$$

M being a function of x and of x', with neither s nor s'. In equating therefore M
[124] to zero, we will have the value of x' in x, and this value substituted into the integral $\int x^s x'^{s'} \phi\, dx$, will give the general expression of $y_{s,s'}$, in which ϕ is an arbitrary function of x; the limits of the integral being independent of x, but moreover arbitraries. If the proposed equation $0 = V$, is of order n, it will be necessary, by means of the equation $M = 0$, to determine a number n of values of x' in x. The sum of the n values of $\int x^s x'^{s'} \phi\, dx$ which will result from it, and in which we could set for ϕ the different arbitrary functions of x, will be the expression of $y_{s,s'}$.

There results from that which we have said in the first part of this Book, that the equation $M = 0$ is the generating equation of the proposed equation $V = 0$.

Let us consider presently the equation in the partial differences

$$0 = V + sT + s'R,$$

in which V, T, and R are any linear functions of $y_{s,s'}$ and of its partial differences, either finite or infinitely small. If we suppose, as above

$$y_{s,s'} = \int x^s x'^{s'} \phi dx,$$

x' being a function of x which the concern is to determine. we will have an equation in this form

$$0 = \int x^s x'^{s'} \phi\, dx (M + Ns + Ps'),$$

M, N, and P being functions of x and x', with neither s nor s'; now we have

$$\frac{d(x^s x'^{s'})}{dx} = x^s x'^{s'} \left(\frac{s}{x} + \frac{s' dx'}{x' dx} \right);$$

therefore if we determine x' by this equation

$$\frac{dx'}{x'} = \frac{P\, dx}{Nx},$$

we will have

$$x^s x'^{s'} (Ns + Ps') = Nx \frac{d(x^s x'^{s'})}{dx};$$

consequently, if we designate $x^s x'^{s'}$ by δy, and if we suppose that we have substituted into M and N for x' it value in x, we will have [125]

$$0 = \int \phi\, dx \left(M\,\delta y + Nx \frac{d\,\delta y}{dx} \right).$$

This equation integrated by parts, as in the preceding sections, gives the following two,

$$0 = M\phi - \frac{d(Nx\phi)}{dx};$$
$$0 = Nx\phi\,\delta y.$$

The first determines ϕ in x, and the second gives the limits of the integral $\int \delta y\,\phi\,dx$.

This value of $y_{s,s'}$ containing no arbitrary function at all, is only a particular integral of the proposed equation in the partial differences. In order to render it complete, we will observe that the integral of the equation

$$\frac{dx'}{x'} = \frac{Pdx}{Nx},$$

which determines x' in x, is $x' = Q$, Q being a function of x, and of one arbitrary constant that we will designate by u; in representing therefore by ψ, an arbitrary function of u, the proposed equation in the partial differences will be satisfied by this value of $y_{s,s'}$,

$$y_{s,s'} = \iint x^s Q^{s'} \psi\, dx\, du;$$

the integral relative to x being taken between the limits determined by the equation $0 = N\phi\,\delta y$, and the integral relative to u being taken between any limits whatsoever. This value of $y_{s,s'}$ will be therefore the complete integral of the proposed equation in the partial differences, if this here is of first order; but, if it is of a superior order, it will be necessary, by means of the equation $0 = N\phi\,\delta y$, to determine as many values of x in u as there are units in that order. The union of the values of $y_{s,s'}$ to which we will arrive, will be the complete expression of $y_{s,s'}$.

CHAPTER 3

Application of the preceding method, to the approximation of diverse functions of very great numbers.

Among the diverse functions to which these methods can be applied, I am going to consider the products of numbers, the developments of the polynomials, and the infinitely small and finite differences of functions, these diverse quantities being those which are present most often in the analysis of hazards. [126]

Concerning the approximation of the products composed of a great number of factors, and the terms of the polynomials raised to great powers.

§33. Let us propose to integrate the equation in finite differences

$$0 = (s+1)y_s - y_{s+1}.$$

If we suppose

$$y_s = \int x^s \phi dx;$$

we will have, by designating x^s by δy,

$$0 = \int \phi\, dx \left[(1-x)\delta y + x\frac{d\,\delta y}{dx} \right];$$

whence we deduce by integrating by parts, following the previous method, the following two equations,

$$0 = \phi(1-x) - \frac{d(x\phi)}{dx},$$
$$0 = x^{s+1}\phi.$$

The first equation gives, by integrating it,

$$\phi = Ac^{-x};$$

and the second gives, in order to determine the limits of the integral $\int x^s\phi\, dx$, [127]

$$0 = x^{s+1}c^{-x};$$

these limits are consequently $x = 0$ and $x = \infty$. Thus we have

$$y_s = A\int x^s dx\, c^{-x},$$

the integral being taken from $x = 0$ to x infinity, and A being an arbitrary constant.

In order to have this integral in a series, we will determine, conformably to the method exposed in §23, the value of x, which renders $x^s c^{-x}$ a *maximum*; this value is s. We will make therefore, according to the method cited,

$$x^s c^{-x} = s^s c^{-s} c^{-t^2}.$$

In supposing $x = s + \theta$, this equation becomes

$$\left(1 + \frac{\theta}{s}\right)^s c^{-\theta} = c^{-t^2};$$

hence

$$t^2 = -s \log\left(1 + \frac{\theta}{s}\right) + \theta = \frac{\theta^2}{2s} - \frac{\theta^3}{3s^2} + \frac{\theta^4}{4s^3} - \text{etc.};$$

that which gives by the reversion of the series

$$\theta = t\sqrt{2s} + \frac{2}{3}t^2 + \frac{t^3}{9\sqrt{2s}} + \text{etc.};$$

consequently

$$dx = d\theta = dt\sqrt{2s}\left(1 + \frac{4t}{3\sqrt{2s}} + \frac{t^2}{6s} + \text{etc.}\right);$$

the function $\int x^s \, dx \, c^{-x}$ will become therefore

$$s^s c^{-s} \int dt \, c^{-t^2} \sqrt{2s}\left(1 + \frac{4t}{3\sqrt{2s}} + \frac{t^2}{6s} + \text{etc.}\right).$$

The integral relative to x needing to be taken from x null to x infinity, the integral relative to t must be taken from $t = -\infty$ to $t = \infty$. By integrating as in §31, we will have

$$y_s = A s^{s+\frac{1}{2}} c^{-s} \sqrt{2\pi}\left(1 + \frac{1}{12s} + \text{etc.}\right).$$

[128] We can determine quite simply the factor $1 + \frac{1}{12s} + \text{etc.}$ in this manner. Let us designate by

$$1 + \frac{B}{s} + \frac{C}{s^2} + \text{etc.};$$

that which gives

$$y_s = A s^{s+\frac{1}{2}} c^{-s} \sqrt{2\pi}\left(1 + \frac{B}{s} + \frac{C}{s^2} + \text{etc.}\right).$$

By substituting this value of y_s into the proposed equation

$$y_{s+1} = (s+1) y_s;$$

we will have

$$\left(1 + \frac{1}{s}\right)^{s+\frac{1}{2}} c^{-1}\left(1 + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \text{etc.}\right) = 1 + \frac{B}{s} + \frac{C}{s^2} + \text{etc.},$$

or

$$\left(1+\frac{B}{s}+\frac{C}{s^2}+\text{etc.}\right)\left[c^{1-\left(s+\frac{1}{2}\right)\log\left[1+\frac{1}{s}\right)}-1\right]$$
$$=-\frac{B}{s^2}+\frac{B-2C}{s^3}-\text{etc.};$$

now we have

$$1-\left(s+\frac{1}{2}\right)\log\left(1+\frac{1}{s}\right)=1-\left(s+\frac{1}{2}\right)\left(\frac{1}{s}-\frac{1}{2s^2}+\frac{1}{3s^3}-\frac{1}{4s^4}+\text{etc.}\right)$$
$$=-\frac{1}{12s^2}+\frac{1}{12s^3}-\text{etc.}.$$

We will have therefore, by observing that $c^{-\frac{1}{12s^2}+\text{etc.}}=1-\frac{1}{12s^2}+\text{etc.}$,

$$\left(1+\frac{B}{s}+\frac{C}{s^2}+\text{etc.}\right)\left(-\frac{1}{12s^2}+\frac{1}{12s^3}-\text{etc.}\right)=-\frac{B}{s^2}+\frac{B-2C}{s^3}-\text{etc.};$$

that which gives, by comparing the similar powers of $\frac{1}{s}$,

$$B=\frac{1}{12},\qquad C=\frac{1}{288},\qquad \text{etc.};$$

therefore

$$y_s=As^{s+\frac{1}{2}}c^{-s}\sqrt{2\pi}\left(1+\frac{1}{12s}+\frac{1}{288s^2}+\text{etc.}\right).$$

We will determine the arbitrary constant A, by means of a particular value of y_s; by [129]
supposing, for example, that, s being equal to μ, we have $y_s=Y$; we will have

$$Y=A\int x^\mu\,dx\,c^{-x},$$

that which gives

$$A=\frac{Y}{\int x^\mu\,dx\,c^{-x}};$$

consequently,

$$y_s=\frac{Y s^{s+\frac{1}{2}}c^{-s}\sqrt{2\pi}}{\int x^\mu\,dx\,c^{-x}}\left(1+\frac{1}{12s}+\frac{1}{288s^2}+\text{etc.}\right).\qquad (q)$$

We see now of what nature is the function y_s. For this, it is necessary to integrate the equation in the finite differences

$$y_{s+1}=(s+1)y_s.$$

We find easily that its integral is

$$y_s=Y(\mu+1)(\mu+2)(\mu+3)\dots s;$$

we will have therefore, by comparing this expression to formula (q),

$$(\mu+1)(\mu+2)(\mu+3)\ldots s = \frac{s^{s+\frac{1}{2}}c^{-s}\sqrt{2\pi}\left(1+\frac{1}{12s}+\frac{1}{288s^2}+\text{etc.}\right)}{\int x^{\mu}\,dx\,c^{-x}}. \qquad (q')$$

If we make $\mu=0$, we will have $\int x^{\mu}dx\,c^{-x}=1$; hence

$$1.2.3\ldots s = s^{s+\frac{1}{2}}c^{-s}\sqrt{2\pi}\left(1+\frac{1}{12s}+\frac{1}{288s^2}+\text{etc.}\right).$$

If we make $\mu=\frac{m}{n}$, m being less than n; we will have

$$s = s'+\frac{m}{n},$$

s' being a whole number; thus

$$s^{s+\frac{1}{2}} = \left(s'+\frac{m}{n}\right)^{s'+\frac{m}{n}+\frac{1}{2}} = s'^{s'+\frac{m}{n}+\frac{1}{2}}c^{\left(s'+\frac{m}{n}+\frac{1}{2}\right)\log\left(1+\frac{m}{ns'}\right)};$$

[130] now we have

$$\left(s'+\frac{m}{n}+\frac{1}{2}\right)\log\left(1+\frac{m}{ns'}\right) = \left(s'+\frac{m}{n}+\frac{1}{2}\right)\left(\frac{m}{ns'}-\frac{m^2}{2n^2s'^2}+\text{etc.}\right)$$
$$= \frac{m}{n}+\frac{m^2+mn}{2n^2s'}+\text{etc.}$$

We have moreover, by making $x=t^n$,

$$\int x^{\frac{m}{n}}\,dx\,c^{-x} = \frac{m}{n}\int x^{\frac{m}{n}-1}dx\,c^{-x} = m\int t^{m-1}dt\,c^{-t^n},$$

the integral relative to t being taken from $t=0$ to t infinity. In substituting these values into formula (q'), it will give

$$m(m+n)(m+2n)(m+3n)\cdots(m+s'n) = \frac{n^{s'}s'^{s'+\frac{m}{n}+\frac{1}{2}}c^{-s'}\sqrt{2\pi}\left(1+\frac{n^2+6mn+6m^2}{12n^2s'}+\text{etc.}\right)}{\int t^{m-1}dt\,c^{-t^n}}; \qquad (q'')$$

so that the approximate value of the product of all the terms of the arithmetic progression m, $m+n$, $m+2n$, etc. depend on the three transcendentals c, π and $\int t^{m-1}dt\,c^{-t^n}$.

If in this equation we make for greater simplicity, $n=1$, that which changes m into μ, and if we observe that $\int t^{\mu-1}dt\,c^{-t}=\frac{1}{\mu}\int t^{\mu}dt\,c^{-t}$; we will have

$$(1+\mu)(2+\mu)\ldots(s'+\mu) = s'^{s'+\mu+\frac{1}{2}}c^{-s'}\sqrt{2\pi}\,\frac{1+\frac{1+6\mu+6\mu^2}{12s'}+\text{etc.}}{\int t^{\mu}dt\,c^{-t}}.$$

By changing μ into $-\mu$, we will have

$$(1-\mu)(2-\mu)\ldots(s'-\mu) = s'^{s'-\mu+\frac{1}{2}}c^{-s'}\sqrt{2\pi}\,\frac{1+\frac{1-6\mu+6\mu^2}{12s'}+\text{etc.}}{\int t^{-\mu}dt\,c^{-t}}.$$

By multiplying these two equations by one another, we will have

$$(1-\mu^2)(4-\mu^2)\dots(s'^2-\mu^2)=\frac{s'^{2s'+1}c^{-2s'}.2\pi\left(1+\frac{1+6\mu^2}{6s'}+\text{etc.}\right)}{\int t^{-\mu}dt\,c^{-t}\int t^{\mu}dt\,c^{-t}}.$$

Equation (T) of §24 gives [131]

$$n^3\int t^{n+r-2}dt\,c^{-t^n}\int t^{n-r}dt\,c^{-t^n}=-\frac{(r-1)\pi}{\sin\frac{r-1}{n}\pi}.$$

By making $n=1$ and $\mu=r-1$, we have

$$\int t^{\mu}dt\,c^{-t}\int t^{-\mu}dt\,c^{-t}=\frac{\mu\pi}{\sin\mu\pi};$$

we have therefore

$$\sin\mu\pi=\tfrac{1}{2}\mu(1-\mu^2)(4-\mu^2)\cdots(s'^2-\mu^2)\left(1-\frac{1+6\mu^2}{6s'}+\text{etc.}\right)s'^{-2s'-1}c^{2s'}.$$

If we make μ infinitely small, this equation gives

$$2\pi=1^2.2^2.3^2\dots s'^2\left(1-\frac{1}{6s'}+\text{etc.}\right)s'^{-2s'-1}c^{2s'};$$

dividing therefore the preceding equations by this here, we will have

$$\sin\mu\pi=\mu\pi(1-\mu^2)\left(1-\frac{\mu^2}{4}\right)\left(1-\frac{\mu^2}{9}\right)\cdots\left(1-\frac{\mu^2}{s'^2}\right)\left(1-\frac{\mu^2}{s'}+\text{etc.}\right).$$

If we make s' infinity, we have for the expression of $\sin\phi$, ϕ being equal to $\mu\pi$, the infinite product

$$\phi\left(1-\frac{\phi^2}{\pi^2}\right)\left(1-\frac{\phi^2}{2^2\pi^2}\right)\left(1-\frac{\phi^2}{3^2\pi^2}\right)\left(1-\frac{\phi^2}{4^2\pi^2}\right)\text{etc.};$$

the expression of $\sin\phi$ is thus decomposable into an infinity of factors; that which we know besides.

By supposing ϕ imaginary and equal to $\phi'\sqrt{-1}$, $\sin\phi$ becomes $\frac{c^{-\phi'}-c^{\phi'}}{2\sqrt{-1}}$; we have therefore

$$c^{\phi'}-c^{-\phi'}=2\phi'\left(1+\frac{\phi'^2}{\pi^2}\right)\left(1+\frac{\phi'^2}{2^2\pi^2}\right)\left(1+\frac{\phi'^2}{3^2\pi^2}\right)\cdots$$
$$\cdots\left(1+\frac{\phi'^2}{s'^2\pi^2}\right)\left(1+\frac{\phi'^2}{s'\pi^2}+\text{etc.}\right);$$

and by making s' infinite, we see that $c^{\phi'}-c^{-\phi'}$ is equal to the infinite product

$$2\phi'\left(1-\frac{\phi'^2}{\pi^2}\right)\left(1-\frac{\phi'^2}{2^2\pi^2}\right)\text{etc.}$$

We will have, by a similar process, the continued product of factors of which the [132]
general term is an integral or fractional rational function of s. But the expression

to which we will arrive, will be able to contain other transcendentals dependent on definite integrals of the form $\int x^\mu dx\, c^{-x}$.

We can observe here that these products being set under the form $\int x^s \phi\, dx$, their differentiation with respect to the variable s, presents a clear idea, and then we have for this differential, $\int x^s \phi\, dx\, \log x$.

The expressions of y_s given by formulas (q) and (q') of the preceding section, yet hold according to the remark of §30, in the case where s and μ are negatives, although in this case, the equation

$$0 = x^{s+1} c^{-x},$$

which determines the limits of the definite integral which represents the value of y_s, does not have many real roots. If in formula (q) of the preceding section, we change s into $-s$, and μ into $-\mu$, it becomes

$$y_{-s} = \frac{Y\sqrt{-1} c^s \sqrt{2\pi}\left(1 - \frac{1}{12s} + \frac{1}{288s^2} - \text{etc.}\right)}{(-1)^s s^{s-\frac{1}{2}} \int \frac{dx\, c^{-x}}{x^\mu}},$$

Y being the value of y_s which corresponds to $s = -\mu$. All difficulty is reduced to integrating $\int \frac{dx\, c^{-x}}{x^\mu}$. In order to arrive there, it is necessary to follow the same process of which we have made use in order to reduce into series, the integral $\int e^{-x} x^s\, dx$. We will make therefore

$$x = -\mu + \varpi\sqrt{-1},$$

$-\mu$ being the value of x given by the equation

$$0 = d\frac{c^{-x}}{x^\mu};$$

we will have thus

$$\int \frac{dx\, c^{-x}}{x^\mu} = \frac{c^\mu \sqrt{-1}}{(-1)^\mu} \int \frac{d\varpi c^{-\varpi\sqrt{-1}}}{(\mu - \varpi\sqrt{-1})^\mu}.$$

[133] The integral relative to x needing to be extended between the two limits which render null the quantity $\frac{c^{-x}}{x^\mu}$, it is clear that the integral relative to ϖ must be extended from $\varpi = -\infty$ to $\varpi = \infty$; by joining therefore the two quantities $\frac{c^{-\varpi\sqrt{-1}}}{(\mu - \varpi\sqrt{-1})^\mu}$ and $\frac{c^{\varpi\sqrt{-1}}}{(\mu + \varpi\sqrt{-1})^\mu}$, which correspond to the same values of ϖ, affected with contrary signs, we will have

$$\int \frac{dx\, c^{-x}}{x^\mu} = \frac{c^\mu \sqrt{-1}}{(-1)^\mu} \int d\varpi \left\{ \begin{array}{l} \cos\varpi \dfrac{(\mu + \varpi\sqrt{-1})^\mu + (\mu - \varpi\sqrt{-1})^\mu}{(\mu^2 + \varpi^2)^\mu} \\ \qquad + \sqrt{-1}\sin\varpi \dfrac{(\mu - \varpi\sqrt{-1})^\mu - (\mu + \varpi\sqrt{-1})^\mu}{(\mu^2 + \varpi^2)^\mu} \end{array} \right\},$$

the integral relative to ϖ being taken from $\varpi = 0$ to $\varpi = \infty$. If we develop the quantities under the $\int$ sign, the imaginaries disappear, and there remains only a real function which we will designate by $Q\, d\varpi$; we will have thus

$$\int \frac{dx\, c^{-x}}{x^\mu} = \frac{c^\mu \sqrt{-1}}{(-1)^\mu} \int Q d\varpi;$$

hence

$$y_{-s} = \frac{Yc^{s-\mu}\sqrt{2\pi}\left(1 - \frac{1}{12s} + \frac{1}{288s^2} - \text{etc.}\right)}{(-1)^{s-\mu}s^{s-\frac{1}{2}}\int Q d\varpi}.$$

Let us see presently what function of s is y_{-s}. For this, let us take the original equation

$$0 = (s+1)y_s - y_{s+1};$$

by changing s into $-s$, and making $y_{-s} = u_s$, it becomes

$$0 = (s-1)u_s - u_{s-1};$$

an equation of which the integral

$$u_s = \frac{(-1)^{s-\mu}Y}{\mu(1+\mu)(2+\mu)\cdots(s-1)} = y_{-s},$$

Y being as above, equal to $y_{-\mu}$. If we compare this expression of y_{-s} to the preceding, [134]
and if we observe that $s-\mu$ is a whole number, and that thus we have $(-1)^{2s-2\mu} = 1$; we will have

$$\frac{1}{(\mu+1)(\mu+2)\cdots(s-1)} = \frac{\mu\sqrt{2\pi}c^{s-\mu}\left(1 - \frac{1}{12s} + \frac{1}{288s^2} - \text{etc.}\right)}{s^{s-\frac{1}{2}}\int Q\, d\varpi}.$$

By dividing the two members of this equation by s, and by reciprocating them, we will have

$$(\mu+1)(\mu+2)(\mu+3)\cdots s = \frac{s^{s+\frac{1}{2}}c^{\mu-s}}{\mu\sqrt{2\pi}}\left(1 + \frac{1}{12s} + \text{etc.}\right)\int Q\, d\varpi.$$

If we compare this equation to formula (q') of the preceding section, we have this remarkable result

$$\int Q\, d\varpi = \frac{2\mu\pi c^{-\mu}}{\int x^\mu\, dx\, c^{-x}}; \qquad (O)$$

I am arrived to this general equation, in the *Mémoire de l'Académie des Sciences* for the year 1782,[1] by the preceding analysis, based, as we see, on the passage from the real to the imaginary. By making successively in Q, $\mu = 1$, $\mu = 2$, $\mu = 3$, etc., we will have the values of an infinite number of definite integrals; thus, in the case of $\mu = 1$, equation (O) gives

$$\int \frac{d\varpi(\cos\varpi + \varpi\sin\varpi)}{1+\varpi^2} = \frac{\pi}{c},$$

a formula that I have given similarly in the Memoirs cited. This formula and all those of the same kind, can be verified by the formulas of §26; for we have by this section,

$$\int \frac{d\varpi\cos\varpi}{1+\varpi^2} = \frac{\pi}{2c} = \int \frac{\varpi\, d\varpi\sin\varpi}{1+\varpi^2}.$$

[1] This is the paper "Mémoire sur les Approximations des Formules qui sont fonctions de très grands nombres", [**11**].

We will observe here, as in the Memoirs cited, that $\int \frac{dx\,c^{-x}}{x^\mu}$ being equal to $\frac{c^\mu\sqrt{-1}}{(-1)^\mu}\int Q\,d\varpi$; we have, by substituting instead of $\int Q\,d\varpi$, its value given by equation (O),

$$\int \frac{dx\,c^{-x}}{x^\mu} = \frac{2\mu\pi(-1)^{-\mu+\frac{1}{2}}}{\int x^\mu dx\,c^{-x}} = \frac{2\pi(-1)^{-\mu+\frac{1}{2}}}{\int x^{\mu-1}dx\,c^{-x}},$$

[135] the first integral being taken between the two imaginary values of x which render null the quantity $\frac{c^{-x}}{x^\mu}$, and the two other integrals being taken from x null to x infinity; that which gives an easy way to transform into these here, the integrals $\int \frac{dx\,\sin x}{x^\mu}$ and $\int \frac{dx\,\cos x}{x^\mu}$.

§34. Let us consider now the general equation

$$0 = (a' + b's)y_{s+1} - (a+bs)y_s.$$

If we make

$$\frac{a}{b} = n, \qquad \frac{a'}{b'} = n' + 1, \qquad \frac{b}{b'} = p;$$

it takes this form

$$0 = (n' + s + 1)y_{s+1} - (n+s)py_s.$$

Let us suppose

$$y_s = \int x^{s-1}\phi\,dx;$$

we will have, by integrating by parts,

$$\begin{aligned} 0 &= x^s\phi(x-p) \\ &\quad + \int x^{s-1}[\phi\,dx(n'x - np) + (p-x)x\,d\phi]. \end{aligned}$$

This first equation gives in order to determine ϕ, the following

$$0 = (n'x - np)\phi\,dx + (p-x)x\,d\phi,$$

whence we deduce by integrating

$$\phi = Ax^n(p-x)^{n'-n},$$

A being an arbitrary constant. We will have next in order to determine the limits of the integral, the equation

$$0 = x^s\phi(p-x)$$

or

$$0 = x^{n+s}(p-x)^{n'+1-n}.$$

These limits are therefore $x = 0$ and $x = p$, if $n+s$ and $n'+1-n$ are positive quantities. Thus we will have, by taking the integral within these limits,

$$y_s = A\int x^{n+s-1}dx\,(p-x)^{n'-n}.$$

We will determine the constant A, by means of a particular value of y_s. Let y_μ be this value; we will have [136]

$$A = \frac{y_\mu}{\int x^{n+\mu-1}dx\,(p-x)^{n'-n}};$$

consequently,

$$y_s = \frac{y_\mu \int x^{n+s-1}dx\,(p-x)^{n'-n}}{\int x^{n+\mu-1}dx\,(p-x)^{n'-n}}.$$

Let us integrate presently the proposed equation in the differences in y_s. Its integral is

$$y_s = \frac{(n+\mu)(n+\mu+1)\dots(n+s-1)}{(n'+\mu+1)(n+\mu+2)\dots(n'+s)} y_\mu p^{s-\mu}.$$

In this expression, as in all those formed from products, the factors of the numerator commence only for the value of s which renders the last factor equal to the first, that which holds here when s is equal to $\mu+1$; it is likewise for it of the factors of the denominator. For the value of s equal to μ, the numerator and the denominator are reduced to unity which is counted to multiply them both. If we compare the two preceding expressions of y_s, we will have

$$\frac{(n+\mu)(n+\mu+1)\dots(n+s-1)}{(n'+\mu+1)(n+\mu+2)\dots(n+s)} p^{s-\mu} = \frac{\int x^{n+s-1}dx\,(p-x)^{n'-n}}{\int x^{n+\mu-1}dx\,(p-x)^{n'-n}}.$$

Let us make $p-x=pu^2$; the second member of this equation will become

$$p^{s-\mu}\frac{\int u^{2n'-2n+1}du\,(1-u^2)^{n+s-1}}{\int u^{2n'-2n+1}du\,(1-u^2)^{n+\mu-1}},$$

the integrals being taken from $u=0$ to $u=1$, because these limits correspond to the limits $x=p$ and $x=0$. We have therefore

$$\frac{(n+\mu)(n+\mu+1)\dots(n+s-1)}{(n'+\mu+1)(n'+\mu+2)\dots(n'+s)} = \frac{\int u^{2n'-2n+1}du\,(1-u^2)^{n+s-1}}{\int u^{2n'-2n+1}du\,(1-u^2)^{n+\mu-1}}.$$

Let us suppose $n=\frac{1}{2}$, $n'=0$ and $\mu=1$; if we observe that

$$\int du\,\sqrt{1-u^2} = \tfrac{1}{4}\pi;$$

we will have [137]

$$\frac{(s+1)(s+2)\dots 2s}{1.2.3\dots s} = \frac{2^{2s+1}}{\pi}\int du\,(1-u^2)^{s-\frac{1}{2}}.$$

The first member of this equation is the coefficient of the middle term or term independent of a, of the binomial $\left(\frac{1}{a}+a\right)^{2s}$; we will have therefore by means of the preceding methods, this coefficient, by a rapid approximation, when s is a great number. For this, we will make

$$\frac{1}{s-\frac{1}{2}} = \alpha, \qquad 1-u^2 = c^{-\alpha t^2};$$

that which gives

$$u = \sqrt{1-c^{-\alpha t^2}}$$

and

$$\int du\,(1-u^2)^{s-\frac{1}{2}} = \int du\, c^{-t^2}.$$

Let us suppose

$$\sqrt{1-c^{-\alpha t^2}} = \alpha^{\frac{1}{2}} t(1+\alpha q^{(1)}t^2 + \alpha^2 q^{(2)} t^4 + \alpha^3 q^{(3)} t^6 + \text{etc.}).$$

By taking the logarithmic differences of the two members of this equation, we will have

$$\frac{1+3\alpha q^{(1)}t^2 + 5\alpha^2 q^{(2)}t^4 + 7\alpha^3 q^{(3)} t^6 + \text{etc.}}{t+\alpha q^{(1)}t^3 + \alpha^2 q^{(2)} t^5 + \alpha^3 q^{(3)} t^7 + \text{etc.}} = \frac{\alpha t c^{-\alpha t^2}}{1-c^{-\alpha t^2}};$$

and this last member is equal to

$$\frac{1-\alpha t^2 + \frac{\alpha^2}{1.2}t^4 - \frac{\alpha^3}{1.2.3}t^6 + \text{etc.}}{t\left(1-\frac{\alpha t^2}{1.2} + \frac{\alpha^2 t^4}{1.2.3} - \frac{\alpha^3 t^6}{1.2.3.4} + \text{etc.}\right)}.$$

We will have therefore by comparing this quantity to the first member, and reducing to the same denominator; the general equation

$$0 = 2iq^{(i)} - \frac{2i-3}{1.2}q^{(i-1)} + \frac{2i-6}{1.2.3}q^{(i-2)} - \frac{2i-9}{1.2.3.4}q^{(i-3)} + \frac{2i-12}{1.2.3.4.5}q^{(i-4)} - \text{etc.},$$

$q^{(0)}$ being equal to unity. If we make successively in this equation $i = 1$, $i = 2$, $i = 3$,
[138] etc.; we will have the successive values of $q^{(1)}$, $q^{(2)}$, $q^{(3)}$, etc.; and we will find

$$q^{(1)} = -\frac{1}{4}, \qquad q^{(2)} = \frac{5}{96}, \qquad \text{etc.}$$

We will have next

$$\int du(1-u^2)^{s-\frac{1}{2}} = \alpha^{\frac{1}{2}} \int dt\, c^{-t^2}[1+3\alpha q^{(1)}t^2 + 5\alpha^2 q^{(2)} t^4 + 7\alpha^3 q^{(3)} t^6 + \text{etc.}].$$

The integral relative to u needing to be taken from $u = 0$ to $u = 1$, the integral relative to t must be taken from t null to t infinity; we will have therefore, by §24,

$$\int du(1-u^2)^{s-\frac{1}{2}} = \frac{1}{2}\sqrt{\alpha\pi}\left\{1+\frac{1.3}{2}\alpha q^{(1)} + \frac{1.3.5}{2^2}\alpha^2 q^{(2)} + \frac{1.3.5.7}{2^3}\alpha^3 q^{(3)} + \text{etc.}\right\};$$

hence,

$$\frac{(s+1)(s+2)(s+3)\dots 2s}{1.2.3\dots s} = \frac{2^{2s}}{\sqrt{(s-\frac{1}{2})\pi}}\left\{1+\frac{1.3}{2}\alpha q^{(1)} + \frac{1.3.5}{2^2}\alpha^2 q^{(2)} + \frac{1.3.5.7}{2^3}\alpha^3 q^{(3)} + \text{etc.}\right\}.$$

Thus we will have by a very convergent series, the middle term or term independent of a, of the binomial $\left(\frac{1}{a}+a\right)^{2s}$.

We will arrive more simply to this result, by the following method, which can be extended to any polynomial.

§35. Let us name y_s, the middle term or term independent of a, of the binomial $\left(\frac{1}{a}+a\right)^{2s}$, or, that which reverts to the same, the term independent of $c^{\pm\varpi\sqrt{-1}}$, in the development of the binomial $(c^{\varpi\sqrt{-1}}+c^{-\varpi\sqrt{-1}})^{2s}$. If we multiply this development by $d\varpi$, and if we integrate from ϖ null to $\varpi=\frac{1}{2}\pi$; it is easy to see that this integral will be $\frac{1}{2}\pi y_s$, and that thus we have

$$y_s=\frac{2}{\pi}\int d\varpi\,(c^{\varpi\sqrt{-1}}+c^{-\varpi\sqrt{-1}})^{2s}.$$

Indeed, by developing the binomial contained under the $\int$ sign, and substituting instead of $c^{\pm 2r\varpi\sqrt{-1}}$, its value $\cos 2r\varpi \pm \sqrt{-1}\sin 2r\varpi$, we will have the middle term [139] of the binomial, plus a series of cosines of the angle 2ϖ and of its multiples; by multiplying them by $d\varpi$, and integrating them, this series will be transformed into a series of sines of the angle 2ϖ and of its multiples, sines which are null at the two limits $\varpi=0$ and $\varpi=\frac{1}{2}\pi$. There will remain thus in the integral only the middle term of the binomial, multiplied by $\frac{1}{2}\pi$. This premised, if we substitute instead of the binomial $c^{\varpi\sqrt{-1}}+c^{-\varpi\sqrt{-1}}$, its value $2\cos\varpi$, we will have

$$y_s=\frac{2^{2s+1}}{\pi}\int d\varpi\cos^{2s}\varpi;$$

by supposing $\sin\varpi=u$, we will have

$$y_s=\frac{2^{2s+1}}{\pi}\int du(1-u^2)^{s-\frac{1}{2}},$$

the integral being taken from $u=0$ to $u=1$; that which coincides with that which we have found in the preceding section.

Let us consider now the trinomial $\left(\frac{1}{a}+1+a\right)^{s}$, and let us name y_s the middle term or term independent of a, in the development of this trinomial. This term will be the term independent of $c^{\pm\varpi\sqrt{-1}}$, in the development of the trinomial $\left(c^{\varpi\sqrt{-1}}+1+c^{-\varpi\sqrt{-1}}\right)^{s}$; we will have consequently, by applying here the reasoning which precedes,

$$y_s=\frac{1}{\pi}\int d\varpi(1+2\cos\varpi)^s;$$

the integral being taken from $\varpi=0$ to $\varpi=\pi$. The condition of the *maximum* of the function $(1+2\cos\varpi)^s$ gives $\sin\varpi=0$, so that the two limits of the integral, $\varpi=0$ and $\varpi=\pi$, correspond to some *maxima* of this function; we will partition therefore the preceding integral into the two following,

$$\int d\varpi(1+2\cos\varpi)^s \qquad (-1)^s\int d\varpi(2\cos\varpi-1)^s;$$

the first of these two integrals being taken from ϖ null to the value of ϖ, which renders null the quantity $2\cos\varpi + 1$; and the second integral being taken from $\varpi = 0$, to its value which renders null the quantity $2\cos\varpi - 1$.

[140] In order to obtain the first integral in a convergent series, we will make

$$(1 + 2\cos\varpi)^s = 3^s c^{-t^2};$$

by supposing $\alpha = \frac{1}{s}$, extracting the root s of each member, and developing $\cos\varpi$ and $c^{-\alpha t^2}$, we will have

$$3 - \varpi^2 + \frac{\varpi^4}{12} - \text{etc.} = 3 - 3\alpha t^2 + \frac{3\alpha^2 t^4}{2} - \text{etc.};$$

whence we deduce by the reversion of the series,

$$\varpi = \alpha^{\frac{1}{2}} t\sqrt{3}\left(1 - \frac{\alpha t^2}{8} + \text{etc.}\right);$$

hence,

$$\int d\varpi (1 + 2\cos\varpi)^s = \frac{3^{s+\frac{1}{2}}}{\sqrt{s}} \int dt\, c^{-t^2}\left(1 - \frac{3t^2}{8s} + \text{etc.}\right).$$

The integral relative to t must be taken from t null to t infinity; we will have therefore

$$\int d\varpi (1 + 2\cos\varpi)^s = \frac{3^{s+\frac{1}{2}}\sqrt{\pi}}{2\sqrt{s}}\left(1 - \frac{3}{16s} + \text{etc.}\right).$$

We will find in the same manner

$$\int d\varpi (2\cos\varpi - 1)^s = \frac{\sqrt{\pi}}{2\sqrt{s}}\left(1 - \frac{5}{16s} + \text{etc.}\right);$$

we will have therefore

$$y_s = \frac{3^{s+\frac{1}{2}}}{2\sqrt{s\pi}}\left(1 - \frac{3}{16s} + \text{etc.}\right) + \frac{(-1)^s}{2\sqrt{s\pi}}\left(1 - \frac{5}{16s} + \text{etc.}\right);$$

s being a very large number, this quantity is reduced very nearly to $\frac{3^{s+\frac{1}{2}}}{2\sqrt{s\pi}}$. This is the rough approximation of the middle term or term independent of a, of the binomial $\left(\frac{1}{a} + 1 + a\right)^s$.

We will determine in the same manner, the middle term of any polynomial whatsoever, raised to a very high power. Let us suppose first the number of terms of the
[141] polynomial, odd and equal to $2n + 1$; and let us represent this polynomial by

$$\frac{1}{a^n} + \frac{1}{a^{n-1}} \cdots + \frac{1}{a} + 1 + a \cdots + a^{n-1} + a^n.$$

By substituting $c^{\varpi\sqrt{-1}}$ for a, this polynomial becomes

$$1 + 2\cos\varpi + 2\cos 2\varpi \cdots + 2\cos n\varpi;$$

now this function is equal to $\frac{\sin\left(\frac{2n+1}{2}\right)\varpi}{\sin\frac{1}{2}\varpi}$; the power s of the polynomial is therefore

$$\left(\frac{\sin\left(\frac{2n+1}{2}\right)\varpi}{\sin\frac{1}{2}\varpi}\right)^s.$$

The middle term of this power, is the term independent of ϖ, in its development in cosines of the angle ϖ and of its multiples. We will have evidently this term, by multiplying the power by $d\varpi$; by taking next the integral from $\varpi = 0$ to $\varpi = \pi$, and by dividing it by π. This term is therefore equal to

$$\frac{1}{\pi}\int d\varpi\left(\frac{\sin\left(\frac{2n+1}{2}\right)\varpi}{\sin\frac{1}{2}\varpi}\right)^s.$$

The condition of the *maximum* of $\frac{\sin\left(\frac{2n+1}{2}\right)\varpi}{\sin\frac{1}{2}\varpi}$ gives the equation

$$\tan\left(\frac{2n+1}{2}\right)\varpi = (2n+1)\tan\frac{1}{2}\varpi.$$

There is from ϖ null to $\varpi = \pi$, many *maxima*, alternatively positive and negative. The first corresponds to ϖ null and gives

$$\left(\frac{\sin\left(\frac{2n+1}{2}\right)\varpi}{\sin\frac{1}{2}\varpi}\right)^s = (2n+1)^s.$$

In order to have the preceding integral, from this *maximum* to the point where $\frac{\sin\left(\frac{2n+1}{2}\right)\varpi}{\sin\frac{1}{2}\varpi}$ is null, that which holds first when $\varpi = \frac{2\pi}{2n+1}$, we will make [142]

$$\left(\frac{\sin\left(\frac{2n+1}{2}\right)\varpi}{\sin\frac{1}{2}\varpi}\right)^s = (2n+1)^s c^{-t^2}.$$

By taking logarithms, and reducing into series relative to the powers of ϖ, the function

$$s\log\frac{\sin\left(\frac{2n+1}{2}\right)\varpi}{\sin\frac{1}{2}\varpi};$$

we will have

$$\frac{n(n+1)}{6}s\varpi^2 + \text{etc.} = t^2;$$

that which gives

$$d\varpi = \frac{dt\sqrt{6}}{\sqrt{n(n+1)s}} + \text{etc.};$$

the preceding integral becomes thus

$$\frac{(2n+1)^s}{\pi}\int\frac{dt\sqrt{6}}{\sqrt{n(n+1)s}}e^{-t^2} + \text{etc.}$$

It must be taken from t null to t infinity; for at the origin, or when ϖ is null, t is null; and at the limit, where $\varpi = \frac{2\pi}{2n+1}$, t is infinite; this integral becomes therefore, by considering only the first term, and neglecting the following which are, with respect to it, of order $\frac{1}{s}$,

$$\frac{(2n+1)^s\sqrt{3}}{\sqrt{n(n+1)2s\pi}}.$$

The second *maximum* is negative, and corresponds to a value of $\left(\frac{2n+1}{2}\right)\varpi$ comprehended between $\frac{5}{4}\pi$ and $\frac{3}{2}\pi$. Indeed, the equation of the *maximum*

$$\tan\left(\frac{2n+1}{2}\right)\varpi = (2n+1)\tan\frac{1}{2}\varpi,$$

gives

$$\tan\left(\frac{2n+1}{2}\right)\varpi > \left(\frac{2n+1}{2}\right)\varpi.$$

[143] Thus, $\left(\frac{2n+1}{2}\right)\varpi$ being comprehended within the second *maximum* between π and 2π, $\tan\left(\frac{2n+1}{2}\right)\varpi$ surpasses π; consequently $\left(\frac{2n+1}{2}\right)\varpi$ surpasses $\pi+\frac{1}{4}\pi$; it is therefore comprehended between $\frac{5}{4}\pi$ and $\frac{3}{2}\pi$. The preceding equation of the *maximum* gives

$$\frac{-\sin\left(\frac{2n+1}{2}\right)\varpi}{\sin\frac{1}{2}\varpi} = \frac{2n+1}{\sqrt{\cos^2\frac{1}{2}\varpi + (2n+1)^2\sin^2\frac{1}{2}\varpi}}.$$

This last member is smaller than

$$\frac{2n+1}{\left(\frac{2n+1}{2}\right)\varpi\frac{\sin\frac{1}{2}\varpi}{\frac{1}{2}\varpi}};$$

$\frac{1}{2}\varpi$ not surpassing $\frac{1}{2}\pi$, it is easy to be assured that $\frac{\sin\frac{1}{2}\varpi}{\frac{1}{2}\varpi}$ is never less that its value which corresponds to $\varpi=\pi$, and which is equal to $\frac{2}{\pi}$; the second member of which there is question, is therefore generally smaller than

$$\frac{2n+1}{\left(\frac{2n+1}{2}\right)\varpi}\cdot\frac{\pi}{2}.$$

Relative to the second *maximum*, $\left(\frac{2n+1}{2}\right)\varpi$ being comprehended between $\frac{5}{4}\pi$ and $\frac{3}{2}\pi$, this member will be smaller than $(2n+1)\frac{2}{5}$; thus the power of s of $\frac{\sin\left(\frac{2n+1}{2}\right)\varpi}{\sin\frac{1}{2}\varpi}$, will not surpass at all $(2n+1)^s\left(\frac{2}{5}\right)^s$; it will be therefore, when s is a very great number, incomparably smaller than the same power corresponding to the first *maximum*, and which is equal to $(2n+1)^s$.

We will see in the same manner, that the third *maximum* is comprehended between $\left(\frac{2n+1}{2}\right)\varpi = \frac{9}{4}\varpi$, and $\left(\frac{2n+1}{2}\right)\varpi = \frac{5}{2}\varpi$, and that at this *maximum*, the power s of $\frac{\sin\left(\frac{2n+1}{2}\right)\varpi}{\sin\frac{1}{2}\varpi}$ does not surpass $(2n+1)^s\left(\frac{2}{9}\right)^s$; that the fourth *maximum* is comprehended
[144] between $\left(\frac{2n+1}{2}\right)\varpi = \frac{13}{4}\varpi$, and $\left(\frac{2n+1}{2}\right)\varpi = \frac{7}{2}\varpi$, and that at this *maximum*, the power s of $\frac{\sin\left(\frac{2n+1}{2}\right)\varpi}{\sin\frac{1}{2}\varpi}$ does not surpass $(2n+1)^s\left(\frac{2}{13}\right)^s$ at all, and so forth.

Now, if, departing from any one of these *maxima*, we make

$$\left(\frac{\sin\left(\frac{2n+1}{2}\right)\varpi}{\sin\frac{1}{2}\varpi}\right)^s = \left(\frac{\sin\left(\frac{2n+1}{2}\right)\Pi}{\sin\frac{1}{2}\Pi}\right)^s c^{-t^2},$$

Π being the value of ϖ which corresponds to this *maximum*; and if we make

$$\varpi = \Pi + \varpi',$$

we will have by taking the logarithms of the two members of the preceding equation between ϖ and t,

$$s\log\sin\left(\frac{2n+1}{2}\right)(\Pi+\varpi') - s\log\sin\tfrac{1}{2}(\Pi+\varpi')$$

$$= s\left[\log\sin\left(\frac{2n+1}{2}\right)\Pi - \log\sin\tfrac{1}{2}\Pi\right] - t^2.$$

By developing the first member of this equation according to the powers of ϖ', the comparison of the first power will give first the equation of the *maximum*

$$\tan\left(\frac{2n+1}{2}\right)\Pi = (2n+1)\tan\tfrac{1}{2}\Pi.$$

By considering next only the second power of ϖ', we will have

$$\tfrac{1}{2}n(n+1)s\varpi'^2 = t^2.$$

that which gives

$$d\varpi' = \frac{2dt}{\sqrt{n(n+1)2s}};$$

the integral

$$\frac{1}{\pi}\int d\varpi\left(\frac{\sin\left(\frac{2n+1}{2}\right)\varpi}{\sin\frac{1}{2}\varpi}\right)^s,$$

taken between the two limits between which $\frac{\sin\left(\frac{2n+1}{2}\right)\varpi}{\sin\frac{1}{2}\varpi}$ is null on both sides of the [145]
maximum of this function, is therefore very nearly

$$\frac{2}{\sqrt{2n(n+1)s\pi}}\left(\frac{\sin\left(\frac{2n+1}{2}\right)\Pi}{\sin\frac{1}{2}\Pi}\right)^s.$$

This expression holds generally for the integrals relative to all the *maxima* which follow the first; only it is necessary to take only the half of it relative to the last which corresponds to $\Pi = \pi$. There results from that which precedes, that this expression, with respect to the second *maximum*, is less, setting aside the sign, than

$$\frac{2}{\sqrt{2n(n+1)s\pi}}\left(\frac{2}{5}\right)^s;$$

that, relative to the third *maximum*, it is less than

$$\frac{2}{\sqrt{2n(n+1)s\pi}}\left(\frac{2}{9}\right)^s;$$

and so forth. When s is a very great number, these quantities decrease with an extreme rapidity, and they are incomparably smaller than the quantity relative to the first *maximum*, and which, as we have seen, is

$$\frac{(2n+1)^s\sqrt{3}}{\sqrt{2n(n+1)s\pi}};$$

we can therefore have regard only for this last integral, and we see that it is rigorous in the case of n infinite; for the equation of condition of the *maximum* gives then $\left(\frac{2n+1}{2}\right)\Pi = \left(\frac{2r+1}{2}\right)\pi$, r being a whole number, that which renders $\frac{\sin\left(\frac{2n+1}{2}\right)\Pi}{\sin\frac{1}{2}\Pi}$ finite, when Π is zero excepted, that which corresponds to the first *maximum*.

If the polynomial is composed of any number of terms, even and equal to $2n$, such that

$$\frac{1}{a^{n-\frac{1}{2}}} + \frac{1}{a^{n-\frac{3}{2}}}\cdots + \frac{1}{a^{\frac{1}{2}}} + a^{\frac{1}{2}}\cdots + a^{n-\frac{3}{2}} + a^{n-\frac{1}{2}},$$

[146] by substituting $c^{\varpi\sqrt{-1}}$ in place of a, it becomes

$$2\cos\frac{1}{2}\varpi + 2\cos\frac{3}{2}\varpi\cdots + 2\cos\frac{2n-1}{2}\varpi,$$

or $\frac{\sin n\varpi}{\sin\frac{1}{2}\varpi}$. This polynomial raised to an integral and positive power, can have a middle term or term independent of the cosines of $\frac{1}{2}\varpi$ and of its multiples, only so much as that power is even; let us represent it by $2s$: then the middle term will be

$$\frac{1}{\pi}\int d\varpi\left(\frac{\sin n\varpi}{\sin\frac{1}{2}\varpi}\right)^{2s},$$

the integral being taken from ϖ null to $\varpi = \pi$. This integral is composed of diverse partial integrals, relative to the diverse *maxima* of the function $\frac{\sin n\varpi}{\sin\frac{1}{2}\varpi}$; but we will be assured easily, by the preceding analysis, that all these integrals, when $2s$ is a very great number, and when n is greater than unity, are incomparably smaller than the one which is relative to the first *maximum* which corresponds to ϖ null; and then we find very nearly the middle term of the $2s$ power of the polynomial equal to

$$y_s = \frac{(2n)^{2^s}\sqrt{3}}{\sqrt{(2n+1)(2n+1)s\pi}}.$$

In bringing closer this result, of the preceding, we see that if we name generally n' the number of terms of the polynomial, and s' the power to which it is raised; the middle term of the development will be, when there is one of them,

$$\frac{n'^{s'}\sqrt{3}}{\sqrt{\frac{n'^2-1}{2}s'\pi}};$$

and provided that it has a middle term, $(n'-1)s'$ must be an even number; that is that one or the other at least, of the numbers $n'-1$ and s', must be even.

§36. The preceding analysis gives further the coefficient of $a^{\pm l}$ in the development of the polynomial

$$(a^{-n}+a^{-n+1}\cdots+a^{-1}+1+a\cdots+a^{n-1}+a^n)^s;$$

In order to obtain it, we will observe that the coefficient of a^r in the development of this polynomial, is the same as the one of a^{-r}; by naming therefore A_r this coefficient, by making $a=c^{\varpi\sqrt{-1}}$, and uniting the two terms of the development, relative to a^r and a^{-r}, we will have $2A_r\cos r\varpi$ for their sum. Now, if we multiply this polynomial, or its value $\left(\frac{\sin\left(\frac{2n+1}{2}\right)\varpi}{\sin\frac{1}{2}\varpi}\right)^s$ by $d\varpi\,\cos l\varpi$, and if we integrate the product from $\varpi=0$ to $\varpi=\pi$; it is clear that all the terms vanish, except the one where r is equal to l; the integral will be reduced therefore to $2A_l\int d\varpi\cos^2 l\varpi$; that which gives [147]

$$A_l=\frac{1}{\pi}\int d\varpi\,\cos l\varpi\left(\frac{\sin\left(\frac{2n+1}{2}\right)\varpi}{\sin\frac{1}{2}\varpi}\right)^s.$$

In order to integrate this function, we will make as above,

$$\left(\frac{\sin\left(\frac{2n+1}{2}\right)\varpi}{\sin\frac{1}{2}\varpi}\right)^s=(2n+1)^s c^{-t^2}.$$

By taking the logarithms and developing with respect to the powers of ϖ, we will have by the inversion of the series, for ϖ, an expression of this form,

$$\varpi=\frac{t\sqrt{6}}{\sqrt{n(n+1)s}}(1+At^2+\text{etc.});$$

that which transforms the integral into this here,

$$\frac{(2n+1)^s}{\pi}\frac{\sqrt{6}}{\sqrt{n(n+1)s}}\int dl\,\cos\left[\frac{lt\sqrt{6}}{\sqrt{n(n+1)s}}\right]c^{-t^2}(1+3At^2+\text{etc.}),$$

the integral being taken from t null to t infinity. We can easily obtain it by §26, and we find, by having regard only to its first term, for its value,

$$\frac{(2n+1)^s\sqrt{3}}{\sqrt{n(n+1).2s\pi}}c^{-\frac{\frac{3}{2}l^2}{n(n+1)s}}.$$

This is the value sought of the coefficient of $a^{\pm l}$ in the development of the polynomial, when its power s is very elevated. [148]

Let us seek now the sum of all these coefficients, from the one of a^{-l} inclusively, to the one of a^l inclusively, l being a great number, but of an order inferior to s. For this, we will observe that we have, by §10,

$$\Sigma y_l = \frac{1}{c^{\frac{dy_l}{dl}} - 1} = \frac{1}{\frac{dy_l}{dl}\left\{1 + \frac{1}{2}\frac{dy_l}{dl} + \frac{1}{6}\left(\frac{dy_l}{dl}\right)^2 + \text{etc.}\right\}}$$
$$= \left(\frac{dy_l}{dl}\right)^{-1} - \frac{1}{2}\left(\frac{dy_l}{dl}\right)^0 + \frac{1}{12}\frac{dy_l}{dl} + \text{etc.};$$

whence we deduce by the section cited,

$$\Sigma y_l = \int y_l\,dt - \tfrac{1}{2}y_l + \tfrac{1}{12}\frac{dy_l}{dl} + \text{etc.} + \text{constant}.$$

By taking the integral from the term corresponding to l null inclusively, we will have the sum of the values of y_l, from this origin to the term y_l exclusively. The arbitrary constant will be equal then to $\frac{1}{2}y_0 - \frac{1}{12}\frac{dy_0}{dl} - \text{etc.}$; thus the sum of the values of y_l, from l null inclusively to y_l inclusively, will be

$$\int y_l\,dl + \tfrac{1}{2}y_0 + \tfrac{1}{2}y_l + \tfrac{1}{12}y\frac{dy_l}{dl} - \tfrac{1}{12}\frac{dy_0}{dl} + \text{etc.}$$

Let us suppose now

$$y_l = \frac{(2n+1)^s\sqrt{3}}{\sqrt{n(n+1)2s\pi}}c^{-\frac{\frac{3}{2}l^2}{n(n+1)s}};$$

then the differences of y_l will be successively of an order inferior the ones to the others; by considering therefore only the first three terms of the preceding series, we will have

$$\int y_l\,dl + \tfrac{1}{2}y_0 + \tfrac{1}{2}y_l$$

for the sum of the coefficients of the terms of the development of the s power of the polynomial, from l null inclusively to y_l inclusively. By doubling this sum, and by subtracting from this double, the term y_0, we will have for the sum of the coefficients,
[149] from the one of the term corresponding to a^{-l} inclusively, to the one of the term corresponding to a^l inclusively,

$$\frac{(2n+1)^s\sqrt{6}}{\sqrt{n(n+1)s\pi}}\left(\int dl\,c^{-\frac{\frac{3}{2}l^2}{n(n+1)s}} + \frac{1}{2}c^{-\frac{\frac{3}{2}l^2}{n(n+1)s}}\right).$$

§37. We have supposed in the preceding examples, that the equations in the differences in y_s, had no last term at all; let us give an example of an equation enjoying a last term, and for this, let us consider the equation in the differences

$$p^s = sy_s + (s-i)y_{s+1}.$$

By making

$$y_s = \int x^{s-1}\phi\,dx,$$

we will have

$$p^s = x^s\phi(1+x) - \int x^s[(x+1)d\phi + (i+1)\phi\,dx];$$

that which gives first in order to determine ϕ, the equation

$$(1+x)d\phi + (i+1)\phi\,dx = 0;$$

whence we deduce by integrating,

$$\phi = \frac{A}{(1+x)^{i+1}},$$

A being an arbitrary constant. Next we have

$$p^s = x^s\phi(1+x),$$

or

$$p^s = \frac{Ax^s}{(1+x)^i};$$

whence we deduce

$$x = p, \qquad A = (1+p)^i;$$

so that

$$y_s = (1+p)^i \int \frac{x^{s-1}dx}{(1+x)^{i+1}},$$

the integral being taken from $x = 0$ to $x = p$. By adding to that value of y, this here

$$B(1+p)^i \int \frac{x^{s-1}dx}{(1+x)^{i+1}},$$

the integral being taken from x null to x infinite, and B being an arbitrary; we will [150]
have for the complete integral of the proposed

$$y_s = B\int \frac{x^{s-1}dx}{(1+x)^{i+1}} + (1+p)^i \int \frac{x^{s-1}dx}{(1+x)^{i+1}},$$

an expression that we can set under this form

$$y_s = B'\int \frac{x^{s-1}dx}{(1+x)^{i+1}} - (1+p)^i \int \frac{x^{s-1}dx}{(1+x)^{i+1}},$$

the first integral being taken from x null to x infinity, and the second being taken from $x = p$ to x infinity.

Now, the integral of the proposed

$$p^s = sy_s + (s-i)y_{s+1}$$

is

$$y_s = \frac{1.2.3\dots(s-1)}{i(i-1)(i-2)\dots(i-s+1)}\left(Q - \sum \frac{i(i-1)(i-2)\dots(i-s+1)}{1.2.3\dots s}p^s\right),$$

Q being an arbitrary, and $\sum$ being the characteristic of the finite differences; so that the function $\sum \frac{i(i-1)(i-2)\dots(i-s+1)}{1.2.3\dots s}p^s$ is equal to

$$1 + ip + \frac{i(i-1)}{1.2}p^2 \dots + \frac{i(i-1)(i-2)\dots(i-s+2)}{1.2.3\dots(s-1)}p^{s-1},$$

that is, the sum of s first terms of the binomial $(1+p)^i$. If we compare this expression of y_s to the preceding, we will have

$$B'\int\frac{x^{s-1}dx}{(1+x)^{i+1}}-(1+p)^i\int\frac{x^{s-1}dx}{(1+x)^{i+1}}$$
$$=\frac{1.2.3\ldots(s-1)}{i(i-1)(i-2)\ldots(i-s+1)}\left(Q-\sum\frac{i(i-1)(i-2)\ldots(i-s+1)}{1.2.3\ldots s}p^s\right).$$

If we make $s=1$ in this equation, and if we observe that the product $1.2.3\ldots(s-1)$ is reduced then to unity, as we have seen in §34; we find, after the integrations, $B'=Q$: thus, B' being an arbitrary, this equation is partitioned into the two following,

$$\frac{1.2.3\ldots(s-1)}{i(i-1)(i-2)\ldots(i-s+1)}=\int\frac{x^{s-1}dx}{(1+x)^{i+1}},$$
$$\frac{1.2.3\ldots(s-1)}{i(i-1)(i-2)\ldots(i-s+1)}\sum\frac{i(i-1)(i-2)\ldots(i-s+1)}{1.2.3\ldots s}p^s=(1+p)^i\int\frac{x^{s-1}dx}{(1+x)^{i+1}};$$

[151] whence we deduce

$$1+ip+\frac{i(i-1)}{1.2}p^2\cdots+\frac{i(i-1)(i-2)\ldots(i-s+2)}{1.2.3\ldots(s-1)}p^{s-1}=(1+p)^i\frac{\int\frac{x^{s-1}dx}{(1+x)^{i+1}}}{\int\frac{x^{s-1}dx}{(1+x)^{i+1}}},$$

the integral of the numerator being taken from $x=p$ to x infinity; and that of the denominator being taken from x null to x infinity. When s and i are large numbers, it will be easy to reduce these two integrals to convergent series, by the formulas of §22 and §23. We will have thus the sum of the first s terms of the binomial raised to a great power, by an approximation so much more rapid, as this power will be higher.

If we effect the integrations, the preceding equation becomes

$$1+ip+\frac{i(i-1)}{1.2}p^2\cdots+\frac{i(i-1)(i-2)\ldots(i-s+2)}{1.2.3\ldots(s-1)}p^{s-1}$$
$$=(1+p)^{s-1}\left\{\begin{array}{l}1+\frac{i-s+1}{1}\frac{p}{1+p}+\frac{(i-s+1)(i-s+2)}{1.2}\frac{p^2}{(1+p)^2}\\ \cdots+\frac{(i-s+1)\ldots(i-1)}{1.2.3\ldots(s-1)}\frac{p^{s-1}}{(1+p)^{s-1}}\end{array}\right\}$$

The second member of this equation is a transformation of the partial sum of the terms of the binomial $(1+p)^i$, a transformation which is able to be useful.

Concerning the approximation of the infinitely small and finite differences, very elevated, of functions.

§38. Let us consider any function of z, that we will represent by $\phi(z)$. By changing z into $z+t$, let us designate by y_s the coefficient of t^s in the development of this function; we will have

$$\frac{d^s\phi(z+t)}{dt^s}=1.2.3\ldots s.y_s,$$

t being supposed null after the differentiations, and, as we have $\frac{d\phi(z+t)}{dt} = \frac{d\phi(z)}{dz}$, by supposing t null, we will have

$$\frac{d^s\phi(z)}{dz^s} = 1.2.3\ldots s.y_s.$$

Thus the pursuit of the s^{th} difference of $\phi(z)$, is reduced to developing the function $\phi(z+t)$ into series. [152]

Let us suppose that this function of t is a power of a polynomial in t, that we will represent by

$$(a + bt + ct^2 + \text{etc.})^\mu.$$

In expressing by

$$y_0 + y_1 t + y_2 t^2 \cdots y_s t^3 + \text{etc.}$$

its development into series, we will have, by taking the logarithmic differences,

$$\frac{\mu(b + 2ct + \text{etc.})}{a + bt + ct^2 + \text{etc.}} = \frac{y_1 + 2y_2 t + \cdots + sy_s t^{s-1} + \text{etc.}}{y_0 + y_1 t + y_2 t^2 + \cdots + y_s t^s + \text{etc.}}.$$

Cross multiplying, and comparing the terms multiplied by t^{s-1}, we will have

$$asy_s + b(s-1)y_{s-1} + c(s-2)y_{s-2} + \text{etc.} = \mu b y_{s-1} + 2\mu c y_{s-2} + \text{etc.}$$

Let us represent by $\int x^{s-1}\phi\, dx$, the expression of y_s; this equation becomes

$$0 = x^s\left(a + \frac{b}{x} + \frac{c}{x^2} + \text{etc.}\right)\phi - \int x^s \left\{ \begin{array}{l} d\phi\left(a + \dfrac{b}{x} + \dfrac{c}{x^2} + \text{etc.}\right) \\ + \mu\phi\, dx\left(\dfrac{b}{x^2} + \dfrac{2c}{x^3} + \text{etc.}\right) \end{array} \right\}.$$

By equating separately to zero, the part of this equation, affected of the integral sign, we have

$$0 = d\phi\left(a + \frac{b}{x} + \frac{c}{x^2} + \text{etc.}\right) + \mu\phi\, dx\left(\frac{b}{x^2} + \frac{2c}{x^3} + \text{etc.}\right);$$

that which gives by integrating,

$$\phi = A\left(a + \frac{b}{x} + \frac{c}{x^2} + \text{etc.}\right)^\mu,$$

A being an arbitrary constant. The part of the preceding equation, beyond the integral sign, will give next in order to determine the limits of the integral,

$$0 = x^s\left(a + \frac{b}{s} + \frac{c}{x^2} + \text{etc.}\right)^{\mu+1};$$

these limits are therefore $x = 0$, and x equal to the diverse roots of the equation [153]

$$0 = a + \frac{b}{x} + \frac{c}{x^2} + \text{etc.}$$

We will have therefore by the preceding methods, and by a very prompt approximation, the coefficients of the very elevated powers of t, in the development into series of the power

$$(a+bt+ct^2+\text{etc.})^\mu,$$

and consequently we will have the very elevated differentials of the power

$$(a'+b'z+c'z^2+\text{etc.})^\mu,$$

which is changed into the preceding, by changing z into $z+t$ and making

$$\begin{aligned} a &= a' + \ \ b'z + c'z^2 + \text{etc.},\\ b &= b' + 2c'z + \text{etc.},\\ c &= c' + \text{etc.},\\ &\text{etc.} \end{aligned}$$

Let us apply this analysis to an example.

z being the sine of an angle θ, we will have

$$\frac{d^{s+1}\theta}{dz^{s+1}} = \frac{d^s}{dz^s}\frac{1}{\sqrt{1-z^2}}.$$

In order to have the expression of the second member of this equation, we will observe that we have, by that which we just saw,

$$\frac{d^s}{dz^s}\frac{1}{\sqrt{1-z^2}} = 1.2.3\dots s.y_s,$$

y_s being the coefficient of t^s in the development of $[1-(z+t)^2]^{-\frac{1}{2}}$. We have next

$$y_s = A\int x^{s-1}dx\left[1-\left(z+\frac{1}{x}\right)^2\right]^{-\frac{1}{2}},$$

the limits of the integral being given by the equation

$$0 = x^s\left[1-\left(z+\frac{1}{x}\right)^2\right]^{-\frac{1}{2}}.$$

[154] These limits are

$$x = -\frac{1}{1+z}, \qquad x = 0, \qquad x = \frac{1}{1-z}.$$

As x has three values, the expression of y_s takes this form, by §29,

$$y_s = A\int x^{s-1}dx\left[1-\left(z+\frac{1}{x}\right)^2\right]^{-\frac{1}{2}} + A'\int x^{s-1}dx\left[1-\left(z+\frac{1}{x}\right)^2\right]^{-\frac{1}{2}},$$

A and A' being arbitrary constants, and the first integral being taken from $x=-\frac{1}{1+z}$ to $x=0$, and the second being taken from $x=0$ to $x=\frac{1}{1-z}$. If we make

$$x = \frac{z+\cos\varpi}{1-z^2}.$$

The preceding expression of y_s becomes

$$y_s = B\int\frac{d\varpi(z+\cos\varpi)^s}{(1-z^2)^{s+\frac{1}{2}}} + B'\int\frac{d\varpi(z+\cos\varpi)^s}{(1-z^2)^{s+\frac{1}{2}}},$$

the first integral being taken from $\varpi = 0$ to ϖ equal to the angle of which the cosine is $-z$, and the second being taken from this last angle to $\varpi = \pi$. In order to determine the arbitraries B and B', we will observe that

$$y_0 = \frac{1}{\sqrt{1-z^2}}, \qquad y_1 = \frac{1}{(1-z^2)^{\frac{3}{2}}};$$

whence it is easy to conclude

$$B = B' = \frac{1}{\pi};$$

hence

$$y_s = \frac{1}{\pi(1-z^2)^{s+\frac{1}{2}}}\int d\varpi(z+\cos\varpi)^s,$$

the integral being taken from $\varpi = 0$ to $\varpi = \pi$. By taking this integral, and observing that

$$\int d\varpi\cos^{2r}\varpi = \frac{1}{2^{2r}}\int d\varpi(c^{\varpi\sqrt{-1}} + c^{-\varpi\sqrt{-1}})^{2r}$$
$$= \frac{1.2.3\ldots 2r}{2^{2r}(1.2.3\ldots r)^2}\pi = \frac{1.3.5\ldots(2r-1)}{2.4.6\ldots 2r}\pi;$$

we will have [155]

$$y_s = \frac{1}{(1-z^2)^{s+\frac{1}{2}}}\left\{\begin{array}{l} z^s + \frac{1}{2}\frac{s(s-1)}{1.2}z^{s-2} + \frac{1.3}{2.4}\frac{s(s-1)(s-2)(s-3)}{1.2.3.4}z^{s-4} \\ \quad + \frac{1.3.5}{2.4.6}\frac{s(s-1)(s-2)(s-3)(s-4)(s-5)}{1.2.3.4.5.6}z^{s-6} \\ \quad + \text{etc.}\end{array}\right\}; \qquad (a)$$

this expression is quite compound, when s is a large number; but then we can obtain its value in a very close manner, by applying to the expression of y_s under the form of definite integral, the methods exposed above. The function under the integral sign having two *maxima*, one at the origin of the integral, and the other at its extremity, we will decompose it into the following two

$$y_s = \frac{1}{\pi(1-z^2)^{s+\frac{1}{2}}}\left[\int d\varpi(z+\cos\varpi)^s + (-1)^s\int d\varpi(\cos\varpi - z)^s\right];$$

the first integral being taken from ϖ null to ϖ equal to the angle of which the cosine is $-z$, and the second integral being taken from ϖ null to ϖ equal to the angle of which z is the cosine. Let $\frac{1}{s} = \alpha$, and let us make

$$(z+\cos\varpi)^s = (1+z)^s c^{-t^2};$$

we will have by taking the logarithms and reducing $\cos\varpi$ into series,

$$\log\left(1-\frac{\varpi^2}{2(1+z)}+\frac{\varpi^4}{24(1+z)}-\text{etc.}\right)=-\alpha t^2;$$

whence it is easy to conclude

$$\varpi=\alpha^{\frac{1}{2}}t\sqrt{2(1+z)}\left(1-\frac{\alpha(2-z)}{12}t^2+\text{etc.}\right);$$

we will have thus, by observing that the integral must be taken from t null to t infinity,

$$\int d\varpi(z+\cos\varpi)^s=\frac{\alpha^{\frac{1}{2}}\sqrt{2\pi}}{2}(1+z)^{s+\frac{1}{2}}\left[1-\frac{\alpha(2-z)}{8}+\text{etc.}\right].$$

By changing z into $-z$, we will have

$$\int d\varpi(\cos\varpi-z)^s=\frac{\alpha^{\frac{1}{2}}\sqrt{2\pi}}{2}(1-z)^{s+\frac{1}{2}}\left[1-\frac{\alpha(2+z)}{8}+\text{etc.}\right];$$

[156] hence

$$\begin{aligned}y_s=&\frac{1}{(1-z)^{s+\frac{1}{2}}\sqrt{2s\pi}}\left(1-\frac{\alpha(2-z)}{8}+\text{etc.}\right)\\&+\frac{(-1)^s}{(1+z)^{s+\frac{1}{2}}\sqrt{2s\pi}}\left(1-\frac{\alpha(2+z)}{8}+\text{etc.}\right);\end{aligned}\qquad(b)$$

in the case of s very great, this expression is reduced to very nearly this very simple term,

$$\frac{1}{(1-z)^{s+\frac{1}{2}}\sqrt{2s\pi}}.$$

If we multiply the expression (b) of y_s by the product $1.2.3\ldots s$, a product which by §33, is equal to

$$s^{s+\frac{1}{2}}c^{-s}\sqrt{2\pi}\left(1+\frac{\alpha}{12}+\text{etc.}\right);$$

we will have very nearly

$$\frac{d^{s+1}\theta}{dz^{s+1}}=\frac{d^s\frac{1}{\sqrt{1-z^2}}}{dz^s}=\frac{s^sc^{-s}}{(1-z)^{s+\frac{1}{2}}}.$$

§39. When a function y_s of s can be expressed by a definite integral of the form $\int x^s\phi\,dx$, the infinitely small and finite differences of any order n, will be by §21,

$$\frac{d^ny_s}{ds^n}=\int x^s\phi\,dx(\log x)^n,$$

$$\triangle^ny_s=\int x^s\phi\,dx(x-1)^n.$$

If instead of expressing the function of s, by the integral $\int x^s\phi\,dx$, we express it by the integral $\int c^{-sx}\phi\,dx$, then we have

$$\frac{d^n y_s}{ds^n} = (-1)^n \int x^n \phi\,dx\,c^{-sx},$$
$$\triangle^n y_s = \int \phi\,dx\,c^{-sx}(c^{-x}-1)^n.$$

In order to have the n^{th} integrals, either finite, or infinitely small, it will suffice to make n negative in these expressions. We can observe that they are generally true, whatever be n, by supposing it even fractional; that which gives the way to [157]
have the differences and the integrals corresponding to some fractional indices. All the difficulty is reduced to putting under the form of definite integrals, a function of s; that which we can make by §§29 and 30, when this function is given by an equation linear in the infinitely small or finite differences. As we are principally led in the analysis of hazards, to some expressions which are only the finite differences of functions, or a part of these differences; we are going to apply the preceding methods, and to determine their values in convergent series.

§40. Let us consider first the function $\frac{1}{s^i}$. In designating it by y_s, it will be determined by the equation in the infinitely small differences

$$0 = s\frac{dy_s}{ds} + iy_s.$$

If we suppose in this equation,

$$y_s = \int c^{-sx}\phi\,dx, \qquad c^{-sx} = \delta y,$$

it will become

$$0 = \int \phi\,dx\left(i\delta y + x\frac{d\,\delta y}{dx}\right);$$

whence we deduce by integrating by parts, conformably to the method of §29, the two equations

$$0 = i\phi - \frac{d(x\phi)}{dx},$$
$$0 = x\phi\,\delta y.$$

The first gives by integrating it,

$$\phi = Ax^{i-1},$$

A being an arbitrary. The second equation gives for the limits of the integral $\int c^{-sx}\phi\,dx$, $x=0$ and $x=\infty$. We will have therefore within these limits,

$$\frac{1}{s^i} = A\int x^{i-1}dx\,c^{-sx}.$$

[158] In order to determine the constant A, we will observe that s being 1, the first member of this equation is reduced to unity; that which gives

$$A = \frac{1}{\int x^{i-1} dx\, c^{-x}};$$

hence

$$\frac{1}{s^i} = \frac{\int x^{i-1} dx\, c^{-sx}}{\int x^{i-1} dx\, c^{-x}}:$$

we will have therefore by the preceding section

$$\triangle^n \frac{1}{s^i} = \frac{\int x^{i-1} dx\, c^{-sx}(c^{-x}-1)^n}{\int x^{i-1} dx\, c^{-x}}, \qquad (\mu)$$

the integrals of the numerator and of the denominator being taken from x null to x infinity.

In order to develop this expression into series, let us suppose

$$x^{i-1} c^{-sx}(c^{-x}-1)^n = a^{i-1} c^{-sa}(c^{-a}-1)^n c^{-t^2},$$

a being the value of x which corresponds to the *maximum* of the first member of this equation. If we make $x = a+\theta$, we will have, by taking the logarithm of each member, and by developing the logarithm of the first, into a series ordered with respect to the powers of θ,

$$h\theta^2 + h'\theta^3 + h''\theta^4 + \text{etc.} = t^2;$$

the quantities a, h, h', h'', etc. being given by the following equations:

$$0 = \frac{i-1}{a} - s - \frac{nc^{-a}}{c^{-a}-1},$$

$$h = \frac{i-1}{2a^2} - \frac{n}{2}\frac{c^{-a}}{c^{-a}-1} + \frac{n}{2}\left(\frac{c^{-a}}{c^{-a}-1}\right)^2,$$

$$h' = -\frac{i-1}{3a^3} - \frac{n}{6}\frac{c^{-a}}{c^{-a}-1} - \frac{n}{2}\left(\frac{c^{-a}}{c^{-a}-1}\right)^2 + \frac{n}{3}\left(\frac{c^{-a}}{c^{-a}-1}\right)^3,$$

$$h'' = -\frac{i-1}{4a^4} - \frac{n}{24}\frac{c^{-a}}{c^{-a}-1} + \frac{7n}{24}\left(\frac{c^{-a}}{c^{-a}-1}\right)^2 - \frac{n}{2}\left(\frac{c^{-a}}{c^{-a}-1}\right)^3 + \frac{n}{4}\left(\frac{c^{-a}}{c^{-a}-1}\right)^4,$$

etc.;

we will have therefore by the reversion of series,

$$\theta = \frac{t}{\sqrt{h}}\left(1 - \frac{h't}{2h\sqrt{h}} + \frac{5h'^2 - 4hh''}{8h^3}t^2 + \text{etc.}\right);$$

[159] and this series will be so much more convergent, as the number n will be greater. By substituting this value of θ into the function $\int d\theta\, c^{-t^2}$, and taking the integral

within the limits $t=-\infty$ and $t=\infty$, limits which correspond to the limits $x=0$ and $x=\infty$, we will have

$$\int x^{i-1}dx\,c^{-sx}(c^{-x}-1)^n = a^{i-1}c^{-sa}(c^{-a}-1)^n\frac{\sqrt{\pi}}{\sqrt{h}}\left(1+\frac{15h'^2-12hh''}{16h^3}+\text{etc.}\right).$$

We have besides

$$\int x^{i-1}dx\,c^{-x} = \frac{1}{i}\int x^i dx\,c^{-x};$$

and when i is very great, we have, by §32,

$$\int x^i dx\,c^{-x} = i^{i+\frac{1}{2}}c^{-i}\sqrt{2\pi}\left(1+\frac{1}{12i}+\text{etc.}\right);$$

by dividing therefore the one by the other, the two values of

$$\int x^{i-1}dx\,c^{-sx}(c^{-x}-1)^n \quad\text{and}\quad \int x^i dx\,c^{-x},$$

we will have

$$\triangle^n\frac{1}{s^i} = \frac{\left(\frac{a}{i}\right)^{i-1}c^{i-sa}(c^{-a}-1)^n}{\sqrt{2hi}}\left\{\begin{array}{l}1+\dfrac{15h'^2-12hh''}{16h^3}+\text{etc.}\\ -\dfrac{1}{12i}-\text{etc.}\end{array}\right\}.$$

In order to have the n^{th} finite difference of the positive power s^i; it suffices, by §30, to change in this equation i into $-i$, and we will have

$$\begin{aligned}\triangle^n s^i &= (s+n)^i - n(s+n-1)^i + \frac{n(n-1)}{1.2}(s+n-2)^i - \text{etc.}\\ &= \frac{\left(\frac{i}{a}\right)^{i+1}c^{sa-i}(c^a-1)^n}{\sqrt{\frac{i(i+1)}{a^2}-ni\frac{c^a}{(c^a-1)^2}}}\left(1+\frac{15l'^2-12ll''}{16l^3}+\frac{1}{12i}+\text{etc.}\right);\end{aligned} \qquad (\mu')$$

a, l, l', l'', etc. being given by the equations [160]

$$\begin{aligned}0 &= \frac{i+1}{a}-s-\frac{nc^a}{c^a-1},\\ l &= -\frac{i+1}{2a^2}-\frac{n}{2}\frac{c^a}{c^a-1}+\frac{n}{2}\left(\frac{c^a}{c^a-1}\right)^2,\\ l' &= -\frac{i+1}{3a^3}-\frac{n}{6}\frac{c^a}{c^a-1}-\frac{n}{2}\left(\frac{c^a}{c^a-1}\right)^2+\frac{n}{3}\left(\frac{c^a}{c^a-1}\right)^3,\\ l'' &= -\frac{i+1}{4a^4}-\frac{n}{24}\frac{c^a}{c^a-1}+\frac{7n}{24}\left(\frac{c^a}{c^a-1}\right)^2-\frac{n}{2}\left(\frac{c^a}{c^a-1}\right)^3+\frac{n}{4}\left(\frac{c^a}{c^a-1}\right)^4,\end{aligned}$$

etc.

The series (μ') ceases to be convergent, when a is a very small fraction of order $\frac{1}{n}$; because it is clear that the quantities l, l', l'',etc., forming then an increasing progression, each term of the series is of the same order as the one which precedes it. In order to determine in what case a is very small, let us resume again the equation

$$0 = \frac{i+1}{a} - s - \frac{nc^a}{c^a - 1}.$$

We can transform it into the following, when a is very small,

$$0 = \frac{i+1}{a} - s - \frac{n}{a}\left(1 + \frac{a}{2} + \text{etc.}\right);$$

whence we deduce very nearly, under the assumption of a very small,

$$a = \frac{i+1-n}{s+\frac{n}{2}};$$

thus a will be quite small all the time that $i-n$ will be not very considerable relative to $s+\frac{n}{2}$. In this case, we will determine $\triangle^n s^i$ by the following method.

Let us resume again the equation

$$\triangle^n s^i = \frac{\int \frac{dx}{x^{i+1}} c^{-sx}(c^{-x}-1)^n}{\int \frac{dx}{x^{i+1}} c^{-x}},$$

[161] in which formula (μ) is changed, when we make i negative and equal to $-i$. We can put the function $(c^{-x}-1)^n$ under this form

$$\begin{aligned} c^{-\frac{nx}{2}}\left(c^{-\frac{x}{2}} - c^{\frac{x}{2}}\right)^n &= (-1)^n c^{-\frac{nx}{2}} x^n \left(1 + \frac{1}{1.2.3}\frac{x^2}{2^2} + \frac{1}{1.2.3.4.5}\frac{x^4}{2^4} + \text{etc.}\right)^n \\ &= (-1)^n c^{-\frac{nx}{2}} x^n \left(1 + \frac{nx^2}{24} + \frac{n(5n-2)}{15.16.24}x^4 + \text{etc.}\right); \end{aligned}$$

we will have therefore

$$\int \frac{dx}{x^{i+1}} c^{-sx}(c^{-x}-1)^n = (-1)^n \int \frac{dx}{x^{i+1-n}} c^{-\left(s+\frac{n}{2}\right)x}\left(1 + \frac{nx^2}{24} + \text{etc.}\right).$$

If we make

$$\left(s+\frac{n}{2}\right)x = x',$$

we will have generally

$$\frac{dx}{x^r} c^{-\left(s+\frac{n}{2}\right)x} = \left(s+\frac{n}{2}\right)^{r-1} \int \frac{dx' c^{-x'}}{x'^r};$$

now we have found in §33, by passage from the real to the imaginary,

$$\int \frac{dx'\, c^{-x'}}{x'^r} = \frac{2\pi(-1)^{r-\frac{1}{2}}}{\int x'^{r-1} dx'\, c^{-x'}} = \frac{2\pi(-1)^{r-\frac{1}{2}}}{(r-1)(r-2)(r-3).\text{etc.}};$$

hence we will have

$$\triangle^n s^i = (i-n+1)(i-n+2)\dots\left(s+\frac{n}{2}\right)^{i-n}$$
$$\times\left\{\begin{array}{l} 1+(i-n)(i-n-1)\dfrac{n}{24\left(s+\frac{n}{2}\right)^2} \\ \quad +(i-n)(i-n-1)(i-n-2)(i-n-3)\dfrac{n(5n-2)}{15.16.24\left(s+\frac{n}{2}\right)^4} \\ \quad +\text{etc.} \end{array}\right\}. \qquad (\mu'')$$

This series will be very convergent, if $i-n$ is not very considerable relative to $s+\frac{n}{2}$; it can moreover be employed in the case where i is fractional, as it is easy to be convinced. As for the product $(i-n+1)(i-n+2)\dots i$, it is easy to obtain it in convergent series, by §33.

The preceding formula is a very simple application of the equation [162]

$$\triangle^n y_s = \left(c^{\frac{dy_{s+\frac{n}{2}}}{dx}} - c^{-\frac{dy_{s+\frac{n}{2}}}{dx}}\right)^n$$

that we have given in §10; because by developing the second member of this equation, and making $y_s = s^i$, we obtain directly that formula that we have concluded from the passages from the real to the imaginary; that which confirms the justice of these passages.

§41. Formulas (μ') and (μ'') of the previous sections, suppose n equal or less than i. Indeed, if we consider the expression

$$\triangle^n s^i = \frac{\int \frac{dx\, c^{-sx}}{x^{i+1}}(c^{-x}-1)^n}{\int \frac{dx\, c^{-x}}{x^{i+1}}},$$

of which the development has produced these formulas; we see that the limits of the integrals of the numerator and of the denominator being determined by the preceding section, by equating to zero the product of the quantities under the integral sign, by x; these limits will be totally imaginaries, when i will be greater than n; instead that in the case where i will be less than n, the limits of the integral of the numerator will be reals, while those of the denominator will be imaginaries; it is necessary therefore then to bring back these last limits to the real state. In order to arrive there, we will observe that we have generally

$$\int x^{i-1} dx\, c^{-x} = \frac{\int x^{i+r} dx\, c^{-x}}{i(i+1)(i+2)\dots(i+r)}.$$

If we make in this expression, i negative and equal to $-r-\frac{m}{n}$, m being less than n; we will have

$$\int \frac{dx\, c^{-x}}{x^{i+1}} = \frac{(-1)^{r+1}\int x^{-\frac{m}{n}} dx\, c^{-x}}{\frac{m}{n}\left(1+\frac{m}{n}\right)\left(2+\frac{m}{n}\right)\cdots i};$$

[163] now we have by §33, the integrals being taken from x null to x infinity,

$$\left(1+\frac{m}{n}\right)\left(2+\frac{m}{n}\right)\ldots i=\frac{\int x^i dx\, c^{-x}}{\int x^{\frac{m}{m}} dx\, c^{-x}},$$

i being here positive: this is the expression of $\int \frac{dx\, c^{-x}}{x^{i+1}}$ of which we must make use in the case that we examine here. If we make $x=t^n$, we will have

$$\frac{m}{n}\int x^{-\frac{m}{n}} dx\, c^{-x}\int x^{\frac{m}{n}} dx\, c^{-x}=n^2\int t^{n-m-1}dt\, c^{-t^n}\int t^{m-1}dt\, c^{-t^n};$$

and equation (T) of §24 gives, by changing r into $m+1$,

$$n^2\int t^{n-m-1}dt\, c^{-t^n}\int t^{m-1}dt\, c^{-t^n}=\frac{\pi}{\sin\frac{m}{n}\pi};$$

we will have therefore

$$\int\frac{dx\, c^{-x}}{x^{i+1}}=\frac{(-1)^{r+1}\pi}{\sin\frac{m}{n}\pi\int x^i dx\, c^{-x}},$$

whence we deduce, by substituting this value into the preceding expression of $\triangle^n s^i$,

$$\triangle^n s^i=\frac{(-1)^{r+1}\sin\frac{m\pi}{n}}{\pi}\int x^i dx\, c^{-x}\int\frac{dx}{x^{i+1}}c^{-sx}(c^{-x}-1)^n, \qquad (\mu''')$$

the integrals being taken from x null to x infinity.

The process which just led us to this equation, is based on the reciprocal passages from the real to the imaginary; but we can arrive there directly by the following analysis which will confirm thus the justice of these passages.

If we take the integral $\int\frac{dx\, c^{-sx}}{x^{i+1}}$ from $x=\alpha$ to x infinity; we will have, by making
[164] $i=r+\frac{m}{n}$, the function

$$\frac{(-1)^r c^{-s\alpha}}{\frac{m}{n}\left(1+\frac{m}{n}\right)\left(2+\frac{m}{n}\right)\cdots i\alpha^{\frac{m}{n}}}\left\{\begin{aligned}&s^r-\frac{m}{n}\frac{s^{r-1}}{\alpha}+\frac{m}{n}\left(1+\frac{m}{n}\right)\frac{s^{r-2}}{\alpha^2}-\text{etc.}\\&+(-1)^r\frac{m}{n}\left(1+\frac{m}{n}\right)\left(2+\frac{m}{n}\right)\cdots i\frac{1}{\alpha^r}\end{aligned}\right\}$$
$$+\frac{(-1)^{r+1}s^{r+1}}{\frac{m}{n}\left(1+\frac{m}{n}\right)\left(2+\frac{m}{n}\right)\cdots i}\int\frac{dx\, c^{-sx}}{x^{\frac{m}{n}}};$$

now we have generally, when α is infinitely small,

$$\frac{\triangle^n c^{-s\alpha}s^{r-f}}{\alpha^f}=0,$$

f being zero or a positive whole number; for, if we develop $c^{-s\alpha}$ in series, and if we designate by $k\alpha^q s^q$ any term of this series, we will have

$$k\alpha^{q-f}\triangle^n s^{q+r-f}=0.$$

Indeed, if q surpasses f, this term becomes null by the assumption of α infinitely small. If q is equal or less than f, $q+r-f$ will be equal or less than r, and consequently, it will be smaller than n; and then, by the known property of finite differences, $\triangle^n s^{q+r-f}$ will be null. It follows thence that $\triangle^n \int \frac{dx\, c^{-sx}}{x^{i+1}}$, or $\int \frac{dx\, c^{-sx}(c^{-x}-1)^n}{x^{i+1}}$ is reduced to

$$\frac{(-1)^{r+1} \triangle^n s^{r+1} \int \frac{dx\, c^{-sx}}{x^{\frac{m}{n}}}}{\frac{m}{n}\left(1+\frac{m}{n}\right)\left(2+\frac{m}{n}\right)\cdots i},$$

the integral being taken from x null to x infinity. If we make $x = \frac{x'}{s}$, we will have

$$\int \frac{dx\, c^{-sx}}{x^{\frac{m}{n}}} = s^{\frac{m}{n}-1} \int \frac{dx'\, c^{-x'}}{x'^{\frac{m}{n}}};$$

the integrals being taken from x and x' nulls to x and x' infinities; we will have therefore [165]

$$\int \frac{dx\, c^{-sx}(c^{-x}-1)^n}{x^{i+1}} = \frac{(-1)^{r+1} \int \frac{dx'\, c^{-x'}}{x'^{\frac{m}{n}}} \triangle^n s^i}{\frac{m}{n}\left(1+\frac{m}{n}\right)\left(2+\frac{m}{n}\right)\cdots i}.$$

By substituting for $\left(1+\frac{m}{n}\right)\left(2+\frac{m}{n}\right)\cdots i$ its value $\frac{\int x^i dx\, c^{-x}}{\int x^{\frac{m}{n}}\, dx\, c^{-x}}$, and observing that we have by that which precedes,

$$\frac{m}{n} \int x'^{-\frac{m}{n}}\, dx'\, c^{-x'} \int x^{\frac{m}{n}}\, dx\, c^{-x} = \frac{\pi}{\sin \frac{m}{n}\pi},$$

we will have formula (μ''').

If i is a very large number, we will have by §33, the integral $\int x^i dx\, c^{-x}$; we will have next by that which precedes, the integral $\int \frac{dx\, c^{-sx}(c^{-x}-1)^n}{x^{i+1}}$; thus we will obtain, by a very convergent series, the value of the second member of the formula cited.

Let us suppose i infinitely small, r will be null, and $\frac{m}{n}$ will be an infinitely small fraction; we will have therefore

$$\sin \frac{m}{n}\pi = \frac{m}{n}\pi = i\pi,$$

$$\triangle^n \left(\frac{s^i - 1}{i}\right) = \triangle^n \log s;$$

formula (μ''') will give thus

$$\triangle^n \log s = -\int \frac{c^{-sx} dx}{x}(c^{-x}-1)^n,$$

an expression that we will reduce easily into convergent series, when n is a great number.

§42. We have need often, in the analysis of hazards, to consider in the expression of $\triangle^n s^i$, only the part in which the quantities raised to the power i are positive. We are going to determine the sum of all these terms. For that, let us resume the formula [166] (μ''') of the preceding section. If we substitute instead of $\triangle^n s^i$ its value

$$(s+n)^i - n(s+n-1)^i + \frac{n(n-1)}{1.2}(s+n-2)^i - \text{etc.};$$

and if we change next s into $-s$, we will have, in continuing the two series of the first member of the following equation, only to the terms in which the quantity raised to the power i, become negative, and observing that the $+$ sign holds, if n is even, and the $-$ sign, if n is odd,

$$\begin{aligned}&(1)^i\left[(n-s)^i - n(n-s-1)^i + \frac{n(n-1)}{1.2}(n-s-2)^i - \text{etc.}\right]\\ &\pm(-1)^i\left[s^i - n(s-1)^i + \frac{n(n-1)}{1.2}(s-2)^i - \text{etc.}\right]\\ &=\frac{(-1)^{r+1}}{\pi}\sin\frac{m\pi}{n}\cdot\int x^i dx\, c^{-x}\int\frac{dx}{x^{i+1}}c^{sx}(c^{-x}-1)^n.\end{aligned}$$

If we change in the last integral, x into $-2x'\sqrt{-1}$, it becomes, after all the reductions,

$$2^{n-i}(-1)^{\frac{n+i}{2}}\int x'^{n-i-1}dx'[\cos(2s-n)x' - \sqrt{-1}\sin(2s-n)x']\left(\frac{\sin x'}{x'}\right)^n;$$

the integral relative to x' being taken from x' null to x' infinity. We will have therefore

$$\begin{aligned}&(1)^i\left[(n-s)^i - n(n-s-1)^i + \frac{n(n-1)}{1.2}(n-s-2)^i - \text{etc.}\right]\\ &\pm(-1)^i\left[s^i - n(s-1)^i + \frac{n(n-1)}{1.2}(s-2)^i - \text{etc.}\right]\\ &=\frac{(-1)^{r+1}}{\pi}2^{n-i}(-1)^{\frac{n+i}{2}}\sin\frac{m\pi}{n}\int x\,dx\,c^{-x}\int x'^{n-i-1}dx'\\ &\times[\cos(2s-n)x' - \sqrt{-1}\sin(2s-n)x']\left(\frac{\sin x'}{x'}\right)^n.\end{aligned}\qquad (o)$$

Let us suppose $r = n-1$, that which gives $i = n-1+\frac{m}{n}$, and let us compare separately the real parts and the imaginary parts of the preceding equation. We have

$$(1)^i = (1)^{n-1}(1)^{\frac{m}{n}} = 1^{\frac{m}{n}};$$

[167] now we have

$$1 = \cos 2l\pi + \sqrt{-1}\sin 2l\pi,$$

l being a whole number; we will have therefore

$$(1)^{\frac{m}{n}} = \cos\frac{2lm\pi}{n} + \sqrt{-1}\sin\frac{2lm\pi}{n}.$$

The corresponding values of $(-1)^{\frac{m}{n}}$ are

$$\cos(2l+1)\frac{m\pi}{n} + \sqrt{-1}\sin(2l+1)\frac{m\pi}{n}.$$

Now $(1)^i$ needing to be supposed equal to unity, in equation (o), it is necessary to choose l in a manner that $\cos\frac{2lm\pi}{n} + \sqrt{-1}\sin\frac{2lm\pi}{n}$ be 1, that which requires that we have

$$\frac{2lm\pi}{n} = 2f\pi,$$

f being a whole number that we are able to suppose null; then we have

$$(-1)^{\frac{m}{n}} = \cos\frac{m\pi}{n} + \sqrt{-1}\sin\frac{m\pi}{n};$$

but we have

$$\pm(-1)^i = \pm(-1)^{n-1+\frac{m}{n}} = -(-1)^{\frac{m}{n}};$$

the imaginary part of the first member of equation (o) is therefore

$$-\sqrt{-1}\sin\frac{m\pi}{n}\left[s^i - n(s-1)^i + \frac{n(n-1)}{1.2}(s-2)^i - \text{etc.}\right].$$

Let us determine the imaginary part of the second member of equation (o). We have

$$(-1)^{r+n-1} = (-1)^{2n-2} = 1;$$

we have next

$$(-1)^{\frac{n+i}{2}+r+1} = -\sqrt{-1}(-1)^{\frac{m}{2n}}$$

because of $r = n-1$ and of $i = n-1+\frac{m}{n}$; now we have by that which precedes, [168]

$$(-1)^{\frac{m}{2n}} = \cos\frac{m\pi}{2n} + \sqrt{-1}\sin\frac{m\pi}{2n};$$

we will have therefore, for the imaginary part of the second member of equation (o),

$$-2^{n-1}\sqrt{-1}\frac{\sin\frac{m\pi}{n}}{\pi}\int dx'\,x'^{-\frac{m}{n}}\cos\left[(2s-n)x' - \frac{m\pi}{2n}\right]\left(\frac{\sin x'}{x'}\right)^n \int x^i dx\,c^{-x}.$$

If we equate this function to the imaginary part of the first member of this equation; if we observe moreover that

$$\begin{aligned}\int x^i dx\,c^{-x} &= \left(1+\frac{m}{n}\right)\left(2+\frac{m}{n}\right)\dots i\int x^{\frac{m}{n}}dx\,c^{-x}\\ &= \left(1+\frac{m}{n}\right)\left(2+\frac{m}{n}\right)\dots ink,\end{aligned}$$

by making $k = \int t^{n+m-1}dt\,c^{-t^n}$, the integral being taken from t null to t infinity; finally, if we suppose $2s - n = z$; we will have

$$\begin{aligned}&\frac{(n+z)^{n-1+\frac{m}{n}} - n(n+z-2)^{n-1+\frac{m}{n}} + \frac{n(n-1)}{1.2}(n+z-4)^{n-1+\frac{m}{n}} - \text{etc.}}{\left(1+\frac{m}{n}\right)\left(2+\frac{m}{n}\right)\cdots\left(n-1+\frac{m}{n}\right)}\\ &\qquad = \frac{nk2^n}{\pi}\int x'^{-\frac{m}{n}}dx'\cos\left(zx' - \frac{m\pi}{2n}\right)\left(\frac{\sin x'}{x'}\right)^n.\end{aligned} \tag{p}$$

In the first member of this formula, the series must be continued until we arrive to a negative quantity raised to the power $n-1+\frac{m}{n}$, z not surpassing n; in the second member, the integral must be taken from x' null to x' infinity.

The comparison of the real parts of the two members of equation (o) leads to the same result; and besides, it proves that for the coincidence of the two results deduced from the comparison of the real quantities between them and of the imaginary quantities between them, it is necessary to suppose, as we have done, $f = 0$.

We can further arrive to formula (p), by means of the following equation: [169]

$$i[\phi(z+2,n) - \phi(z,n)] = (n+z+2)\phi'(z+2,n) + (n-z)\phi'(z,n),$$

$\phi'(z,n)$ being the coefficient of dz in the differential of $\phi(z,n)$, and $\phi(z,n)$ being equal to

$$(n+z)^i - n(n+z-2)^i + \frac{n(n-1)}{1.2}(n+z-4)^i - \text{etc.};$$

all the terms in which the quantity raised to the power i is negative, needing to be rejected, and z not surpassing n, so that the quantity raised to the power i, never surpasses $2n$. In resolving this equation in the infinitely small and finite differences, by the method of §30, and determining conveniently the arbitrary constants, we arrive to the form (p).

We are going now to give some applications of this formula, which will lead us to many curious theorems of analysis.

Let us suppose m null; then we have

$$k = \int t^{n+m-1} dt\, c^{-t^n} = \frac{1}{n};$$

formula (p) thus becomes

$$\frac{(n+z)^{n-1} - n(n+z-2)^{n-1} + \frac{n(n-1)}{1.2}(n+z-4)^{n-1} - \text{etc.}}{1.2.3\ldots(n-1)2^n} = \frac{\int dx' \cos zx' \left(\frac{\sin x'}{x'}\right)^n}{\pi},$$

we have

$$\log\left(\frac{\sin x'}{x'}\right)^n = n\log\left(1 - \frac{1}{6}x'^2 + \frac{1}{120}x'^4 - \text{etc.}\right);$$

that which gives

$$\left(\frac{\sin x'}{x'}\right)^n = c^{-\frac{n}{6}x'^2}\left(1 - \frac{nx'^4}{180} + \text{etc.}\right);$$

[170] we will have therefore, by §26, by making $z = r\sqrt{n}$,

$$\begin{aligned}\frac{\int dx' \cos sx' \left(\frac{\sin x'}{x'}\right)^n}{\pi} &= \sqrt{\frac{3}{2n\pi}}c^{-\frac{3}{2}r^2}\left[1 - \frac{3}{20n}(1-6r^2+3r^4) + \text{etc.}\right] \\ &= \frac{(n+r\sqrt{n})^{n-1} - n(n+r\sqrt{n}-2)^{n-1} + \frac{n(n-1)}{1.2}(n+r\sqrt{n}-4)^{n-1} - \text{etc.}}{1.2.3\ldots(n-1)2^n}\end{aligned} \qquad (q)$$

the series of this last member needing to be arrested at the powers of the negative quantities.

By differentiating this equation with respect to r, we will have, with the condition of the exclusion of the powers of the negative quantities,

$$\begin{aligned}&\frac{n}{1.2.3\ldots(n-2)2^n}\left[(n+r\sqrt{n})^{n-2} - n(n+r\sqrt{n}-2)^{n-2} + \frac{n(n-1)}{1.2}(n+r\sqrt{n}-4)^{n-2} - \text{etc.}\right] \\ &\qquad = -3r\sqrt{\frac{3}{2\pi}}c^{-\frac{3}{2}r^2}\left[1 - \frac{3}{20n}(5-10r^2+3r^4) + \text{etc.}\right].\end{aligned}$$

By continuing to differentiate thus, we will have the values of the inferior differences, provided however that the number of these differentiations be quite small relative to the number n. We can observe that these equations subsist, by making r negative; for $\cos zx'$ or $\cos x'r\sqrt{n}$ is the same in the two cases of r positive and of r negative.

We can, by integrating successively equation (q), obtain analogous theorems on the finite differences of the powers superior to n, by excluding always the powers of the negative quantities. Thus we have, by a first integration,

$$\frac{(n+r\sqrt{n})^n - n(n+r\sqrt{n}-2)^n + \frac{n(n-1)}{1.2}(n+r\sqrt{n}-4)^n - \text{etc.}}{1.2.3\ldots n2^n}$$
$$=\sqrt{\frac{3}{2\pi}}\int dr\, c^{-\frac{3}{2}r}\left[1-\frac{3}{20n}(1-6r^2+3r^4)\right]+\text{etc.}$$
$$=C+\sqrt{\frac{3}{2\pi}}\left[\int dr\, c^{-\frac{3}{2}r^2}-\frac{3}{20n}r(1-r^2)c^{-\frac{3}{2}r^2}+\text{etc.}\right].$$

We will determine the arbitrary constant C, by starting with r, the integral $\int dr\, c^{-\frac{3}{2}r^2}$, and by observing that then r remaining null, the last member of the equation is [171] reduced to this constant. In this case, the first becomes

$$n^n - n(n-2)^2 + \frac{n(n-1)}{1.2}(n-4)^n - \text{etc.}.$$

But we have, as we know, without the exclusion of the power of the negative quantities,

$$n^n - n(n-2)^n + \text{etc.} \mp n(2-n)^n \mp (-n)^n = 1.2.3\ldots n.2^n,$$

the superior sign holding if n is even, and the inferior sign if n is odd. In the two cases, we see that the sum of the terms in which the quantities raised to the power n are negatives, is equal to the sum of the other terms; we have therefore, with the exclusion of the powers of the negative quantities,

$$n^n - n(n-2)^n + \frac{n(n-1)}{1.2}(n-4)^n - \text{etc.} = 1.2.3\ldots n.2^{n-1};$$

that which gives $C=\frac{1}{2}$; consequently,

$$\frac{(n+r\sqrt{n})^n - n(n+r\sqrt{n}-2)^n + \frac{n(n-1)}{1.2}(n+r\sqrt{n}-4)^n - \text{etc.}}{1.2.3\ldots n.2^n}$$
$$=\frac{1}{2}+\sqrt{\frac{3}{2\pi}}\left[\int dr\, c^{-\frac{3}{2}r^2}-\frac{3}{20n}r(1-r^2)c^{-\frac{3}{2}r2}+\text{etc.}\right].$$

By integrating anew this expression, and determining conveniently the arbitrary constant, we find

$$\frac{(n+r\sqrt{n})^{n+1} - n(n+r\sqrt{n}-2)^{n+1} + \frac{n(n-1)}{1.2}(n+r\sqrt{n}-4)^{n+1} - \text{etc.}}{1.2.3\ldots(n+1)2^n\sqrt{n}}$$
$$=\sqrt{\frac{3}{2\pi}}\left\{r\int dr\, c^{-\frac{3}{2}r^2}+c^{-\frac{3}{2}r^2}\left[\frac{1}{3}+\frac{1}{60n}(1-3r^2)\right]\right\}+\frac{1}{2}r.$$

§43. We can extend the preceding methods to the determination of the n^{th} difference of any power of a rational function of s. It suffices for that to reduce, by the method of §29, this function to the form $\int x^s \phi\, dx$. But we have seen then that
[172] we arrive in order to determine ϕ, to a differential equation of a degree equal to the highest exponent of s in this function, and which most often is not integrable. We can obviate this disadvantage, by means of multiple integrals, in the following manner.

Let us consider generally the function

$$\frac{1}{(s+p)^i (s+p')^{i'} (s+p'')^{i''}.\text{etc.}}.$$

If in the integral $\int x^{i-1} dx\, c^{-(s+p)x}$, taken from x null to x infinity, we change $(s+p)x$ into x', it becomes $\frac{1}{(s+p)^i} \int x'^{i-1} dx'\, c^{-x'}$, the new integral being taken within the preceding limits. The comparison of the two integrals will give

$$\frac{1}{(s+p)^i} = \frac{\int x^{i-1} dx\, c^{-(s+p)x}}{\int x^{i-1} dx\, c^{-x}}.$$

It follows thence that

$$\frac{1}{(s+p)^i (s+p')^{i'} (s+p'')^{i''}.\text{etc.}}$$
$$= \frac{\int x^{i-1} x'^{i'-1} x''^{i''-1}.\text{etc.} dx\, dx'\, dx''.\text{etc.} c^{-px-p'x'-p''x''-\text{etc.}-s(x+x'+x''+\text{etc.})}}{\int x^{i-1} dx\, c^{-x}\, x'^{i'-1} dx'\, c^{-x'} \int x''^{i''-1} dx''\, c^{-x''}.\text{etc.}}$$

all the integrals being taken from x, x', x'', etc. null to their infinite values; we will have therefore

$$\triangle^n \frac{1}{(s+p)^i (s+p')^{i'}.\text{etc.}}$$
$$= \frac{\int x^{i-1} x'^{i'-1}.\text{etc.} dx\, dx'.\text{etc.}.c^{-px-p'x'-\text{etc.}-s(x+x'+\text{etc.})} (c^{-x-x'-\text{etc.}} - 1)^n}{\int x^{i-1} dx\, c^{-x}\, x'^{i'-1} dx'\, c^{-x'}.\text{etc.}}$$

We will reduce easily into convergent series, by the method of §40, the numerator and the denominator of this expression; and if we change in this series, the signs of i, i', etc.; we will have the very near value of

$$\triangle^n (s+p)^i (s+p')^{i'}.\text{etc.},$$

[173] n, i, i', etc. being supposed very large numbers. We will find by the section cited,

$$\triangle^n (s+p)^i (s+p')^{i'} \text{etc.}$$
$$= \frac{\left(\frac{i}{a}\right)^{i+1} \left(\frac{i'}{a'}\right)^{i'+1} .\text{etc.} c^{(s+p)a+(s+p')a'+\text{etc.}-i-i'-\text{etc.}} (c^{a+a'+\text{etc.}} - 1)^n}{\sqrt{\left(\frac{i(i+1)}{a^2} - \frac{nic^{a+a'+\text{etc.}}}{(c^{a+a'+\text{etc.}}-1)^2}\right)\left(\frac{i'(i'+1)}{a'^2} - \frac{ni'c^{a+a'+\text{etc.}}}{(c^{a+a'+\text{etc.}}-1)^2}\right).\text{etc.}}}$$

a, a', etc. being determined by the equations

$$\begin{aligned}
0 &= \frac{i+1}{a} + s - p - \frac{nc^{a+a'+\text{etc.}}}{c^{a+a'+\text{etc.}} - 1}, \\
\frac{i'+1}{a'} &= \frac{i+1}{a} + p' - p, \\
\frac{i''+1}{a''} &= \frac{i+1}{a} + p'' - p, \\
&\text{etc.}
\end{aligned}$$

The most ordinary case is the one in which the exponents i, i', i'', etc. are equal, and $s+p$, $s+p'$, etc. form an arithmetic progression. We can obtain then, by the following method, the finite difference of their product elevated to a high power.

Let us consider the difference $\triangle^n[s(s-1)]^i$. If we make $s = s' + \frac{1}{2}$, it becomes

$$\triangle^n s'^{2i}\left(1 - \frac{1}{4s'^2}\right)^i.$$

By developing this function in series, we have

$$\triangle^n s'^{2i} - \frac{i}{4}\triangle^n s'^{2i-2} + \frac{i(i-1)}{1.2.4^2}\triangle^n s'^{2i-4} - \text{etc.}$$

The formulas of §40 will give the near value of each of the terms of this series, and we see, by these formulas, that, n and i being very great numbers, $\triangle^n s^{2i-2}$ is of an order less by two units than $\triangle^n s^{2i}$; whence it follows that each term of the preceding series is of an order inferior by one unit, to the one which precedes it; that which shows the convergence of the series.

We would arrive to the same result by resolving by approximation, the differential equation of the second order in ϕ, to which the method of §29 leads. When we suppose

$$\left(s'^2 - \frac{1}{4}\right)^{-i} = \int c^{-s'x}\phi\,dx;$$

we have [174]

$$2is'\int c^{-s'x}\phi\,dx = \left(s'^2 - \frac{1}{4}\right)\int c^{-s'x}\phi\,dx.$$

By making s' of the coefficients of this equation vanish, by the method cited, in the terms affected with the integral sign; equating next to zero, the sum of these terms, and supposing next, in the differential equation that we obtain thus, ϕ equal to an ascending series with respect to the powers of x, we will have a convergent series. We will have next

$$\triangle^n\left(s'^2 - \frac{1}{4}\right)^{-i} = \int c^{-s'x}(c^{-x} - 1)^n\phi\,dx;$$

whence we will deduce a value in series of $\triangle^n\left(s'^2 - \frac{1}{4}\right)^{-i}$, and in which it will suffice to change the sign of i, in order to have the value of $\triangle^n\left(s'^2 - \frac{1}{4}\right)^{i}$.

This manner to resolve by approximation, the differential equation in ϕ, and that we have indicated at the end of §30, can serve in a great number of cases where this equation is not integrable exactly.

General remark on the convergence of series.

§44. We will terminate this Introduction, with an important observation on the convergence of the series of which we have made a so frequent use. These series converge very rapidly in their first terms; but often this convergence diminishes and ends by being changed into divergence. It must not prevent the use of these series, by employing only their first terms, in which the convergence is rapid; because the rest of the series, which we neglect, is the development of an algebraic function or integral, very small with respect to that which precedes. In order to render this sensible by an example, let us consider the development into series, of the integral $\int dt\, c^{-t^2}$, taken
[175] from $t = T$ to t infinity. We have, by §27,

$$\int dt\, c^{-t^2} = \frac{c^{-T^2}}{2T}\left(1 - \frac{1}{2T^2} + \frac{1.3}{2^2T^4} - \frac{1.3.5}{2^3T^6} + \text{etc.}\right).$$

This series ends by being divergent, however great that the value be that we suppose to T; but then we can employ without sensible error, its first terms. Indeed, if we consider, for example, its first four terms, the rest of the series will be $\frac{1.3.5.7}{2^4}\int\frac{dt\, c^{-t^2}}{t^8}$; now this quantity, setting aside the sign, is smaller than the term $-\frac{1.3.5.c^{-T^2}}{2^4T^7}$ which precedes, that is that we have

$$\frac{c^{-T^2}}{T^7} > 7\int\frac{dt\, c^{-t^2}}{t^8};$$

for we have

$$7\int\frac{dt\, c^{-t^2}}{t^8} = \text{constant} - \frac{c^{-t^2}}{t^7} - 2\int\frac{dt\, c^{-t^2}}{t^6}.$$

In determining the constant, in a manner that the integral be null, when $t = T$, we will have $\frac{c^{-T^2}}{T^7}$ for this constant; we will have therefore, by taking the integral from $t = T$ to t infinity,

$$7\int\frac{dt\, c^{-t^2}}{t^8} = \frac{c^{-T^2}}{T^7} - 2\int\frac{dt\, c^{-t^2}}{t^6}.$$

The preceding series therefore can be employed, as long as it is convergent; since we are sure that that which we neglect, is below the term at which we arrest ourselves.

This series enjoys further this property, namely, that it is alternatively greater and smaller than its entire value, according as we are arrested at a positive term, or at a negative term. We can name for this reason, this kind of series, *limited-series.* Besides, we have seen in §27, that in the case where they are divergent; we can, in reducing them to continued fractions, obtain always convergent approximations.

That which we just said on the preceding series, can be extended to all those that we have considered, and must remove all disquiet on the uses that we have made of

them. Indeed, we can always stop these series at the point where they cease to be convergent, and represent the rest by an integral. This is that which we are going to [176]
demonstrate on the most general formula of the development of functions into series.

We have, in taking the integral from $z = 0$,

$$\int dz\, \phi'(x-z) = \phi(x) - \phi(x-z),$$

$\phi'(x)$ being the differential of $\phi(x)$ divided by dx. If we designate similarly by $\phi''(x)$ the differential of $\phi'(x)$ divided by dx; by $\phi'''(x)$ the differential of $\phi''(x)$ divided by dx, and so forth, we will have

$$\int dz\phi'(x-z) = z\phi'(x-z) + \int z dz\, \phi''(x-z),$$

$$\int dz\phi''(x-z) = \tfrac{1}{2}z^2\phi''(x-z) + \int \tfrac{1}{2}z^2 dz\, \phi'''(x-z),$$

etc.

By continuing thus, we will find generally

$$\int dz\phi'(x-z) = z\phi'(x-z) + \frac{z^2}{1.2}\phi''(x-z)\cdots + \frac{z^n}{1.2.3\ldots n}\phi^{(n)}(x-z)$$
$$= \int \frac{z^n dz}{1.2.3\ldots n}\phi^{(n+1)}(x-z).$$

By comparing this expression to the preceding, we will have

$$\phi(x) = \phi(x-z) + z\phi'(x-z) + \frac{z^2}{1.2}\phi''(x-z)\cdots + \frac{z^n}{1.2.3\ldots n}\phi^{(n)}(x-z)$$
$$= \frac{1}{1.2.3\ldots n}\int z^n dz\, \phi^{(n+1)}(x-z).$$

Let us make $x - z = t$, the preceding equation will take this form

$$\phi(t+z) = \phi(t) + z\phi'(t) + \frac{z^2}{1.2}\phi''(t)\cdots + \frac{z^n}{1.2.3\ldots n}\phi^{(n)}(t)$$
$$= \frac{1}{1.2.3\ldots n}\int z'^n dz'\, \phi^{(n+1)}(t+z-z'),$$

the integral being taken from $z' = 0$ to $z' = z$. It is clear that if we made in this integral $\phi^{(n+1)}(t+z-z')$ constant, we could have a too great result, if we took the greatest value of this quantity; and a too small result, by taking its least value. It has therefore in the interval of $z' = 0$ to $z' = z$, a value of z' such that in supposing this quantity constant, we will have an exact result. Let u be this value; the preceding [177]
integral becomes thus

$$\frac{z^{n+1}}{1.2.3\ldots(n+1)}\phi^{(n+1)}(t+z-u),$$

that which gives

$$\phi(t+z) = \phi(t) + z\phi'(t) \cdots + \frac{z^n}{1.2.3\ldots n}\phi^{(n)}(t) + \frac{z^{n+1}}{1.2.3\ldots(n+1)}\phi^{(n+1)}(t+z-u),$$

$z-u$ being comprehended between zero and z. We could thus judge from the convergence of the series and from the degree of approximation, when we stop ourselves at one of its terms.

END OF THE FIRST PART

Additions

I.

We have integrated by a very convergent approximation, in §34 of Book I, the equation in the finite differences, [462]

$$0 = (n' + s + 1)y_{s+1} - (n + s)y_s.$$

It is easy to conclude from our analysis, the expression of the ratio of the circumference to the radius, in infinite products, given by Wallis. In fact, this analysis has led us in the section cited, to the general expression

$$\frac{(n+\mu)(n+\mu+1)\cdots(n+s+1)}{(n'+\mu+1)(n'+\mu+2)\cdots(n'+s)} = \frac{\int u^{2n'-2n+1}du(1-u^2)^{n+s-1}}{\int u^{2n'-2n+1}du(1-u^2)^{n+\mu-1}}, \qquad (a)$$

the integrals being taken from $u = 0$ to $u = 1$. By making first $n' = 0$, $n = \frac{1}{2}$, $\mu = 1$ and observing that $\int du(1-u^2)^{\frac{1}{2}} = \frac{1}{4}\pi$, π being the ratio of the semi-circumference to the radius, we will have

$$\frac{4}{\pi} = \frac{3.5\ldots(2s-1)}{4.6\ldots 2s\int du(1-u^2)^{s-\frac{1}{2}}}.$$

By supposing therefore generally

$$\frac{1}{\int du(1-u^2)^s} = y_s;$$

one will have

$$\frac{4}{\pi} = \frac{3.5\ldots(2s-1)}{4.6\ldots 2s}y_{s-\frac{1}{2}} = \frac{3.5\ldots(2s+1)}{4.6\ldots(2s+2)}y_{s+\frac{1}{2}} = \text{etc.};$$

that which gives

$$y_{s-\frac{1}{2}} = \frac{2s+1}{2s+2}y_{s+\frac{1}{2}}.$$

If we make next in formula (a), $n' = -\frac{1}{2}$, $n = 0$ and $\mu = 1$, it gives [463]

$$\frac{3.5\ldots(2s-1)}{2.4\ldots(2s-2)} = y_{s-1};$$

whence we deduce

$$y_{s-1} = \frac{2s}{2s+1}y_s;$$

an equation which coincides with the preceding between $y_{s-\frac{1}{2}}$ and $y_{s+\frac{1}{2}}$, by changing s into $s + \frac{1}{2}$; so that this equation holds, s being whole, or equal to a whole plus $\frac{1}{2}$.

The two expressions of y_{s-1} and of $\frac{4}{\pi}$ give

$$\frac{4}{\pi} = \frac{3.3}{2.4} \cdot \frac{5.5}{4.6} \cdots \frac{(2s-1)(2s-1)}{(2s-2)2s} \frac{y_{s-\frac{1}{2}}}{y_{s-1}};$$

the equations in the differences in y_s and $y_{s-\frac{1}{2}}$ give

$$\frac{y_{s-\frac{1}{2}}}{y_{s-1}} = \frac{(2s+1)^2}{2s(2s+2)} \cdot \frac{y_{s+\frac{1}{2}}}{y_s} = \frac{(2s+1)^2}{2s(2s+2)} \cdot \frac{(2s+3)^2}{(2s+2)(2s+4)} \cdot \frac{y_{s+\frac{3}{2}}}{y_{s+1}} = \text{etc.}$$

The ratio $\frac{y_{s-\frac{1}{2}}}{y_{s-1}}$ is greater than unity: it diminishes without ceasing, in measure as s increases; and, in the case of s infinite, it becomes unity. In fact, this ratio is equal to

$$\frac{\int du(1-u^2)^{s-1}}{\int du(1-u^2)^{s-\frac{1}{2}}}.$$

Now the element $du(1-u^2)^{s-1}$ is greater than the element $du(1-u^2)^{s-\frac{1}{2}}$, or $du(1-u^2)^{s-1}(1-u^2)^{\frac{1}{2}}$; the integral of the numerator of the preceding fraction surpasses therefore that of the denominator; this fraction is therefore greater than unity. When s is infinite, these integrals have a sensible value only when u is infinitely small; because u being finite, the factor $(1-u^2)^{s-1}$ becomes a fraction having an infinitely great exponent; one can therefore then suppose $(1-u^2)^{\frac{1}{2}} = 1$, this which renders the ratio $\frac{s-\frac{1}{2}}{y_{s-1}}$ equal to unity.

[464] This ratio is equal to the product of an infinite sequence of fractions of which the first is $\frac{(2s+1)^2}{2s(2s+2)}$, and of which the others are deduced from it, by increasing successively s by one unit; it becomes $\frac{y_s}{y_{s-\frac{1}{2}}}$, by changing s into $s+\frac{1}{2}$, and the fraction $\frac{(2s+1)^2}{2s(2s+2)}$ becomes $\frac{(2s+2)^2}{(2s+1)(2s+3)}$; now we have, whatever be s,

$$\frac{(2s+1)^2}{2s(2s+2)} > \frac{(2s+2)^2}{(2s+1)(2s+3)};$$

we have therefore this inequality

$$\frac{y_{s-\frac{1}{2}}}{y_{s-1}} > \frac{y_s}{y_{s-\frac{1}{2}}}.$$

By changing s into $s-\frac{1}{2}$, we will have

$$\frac{y_{s-1}}{y_{s-\frac{3}{2}}} > \frac{y_{s-\frac{1}{2}}}{y_{s-1}}.$$

These two inequalities give

$$\frac{y_{s-\frac{1}{2}}}{y_{s-1}} > \sqrt{\frac{y_s}{y_{s-1}}} < \sqrt{\frac{y_{s-\frac{1}{2}}}{y_{s-\frac{3}{2}}}}.$$

Substituting in place of the ratios $\frac{y_s}{y_{s-1}}$ and $\frac{y_{s-\frac{1}{2}}}{y_{s-\frac{3}{2}}}$, their values given by the equations in the differences in y_s, we will have

$$\frac{y_{s-\frac{1}{2}}}{y_{s-1}} > \sqrt{1+\frac{1}{2s}} < \sqrt{1+\frac{1}{2s-1}};$$

we will have therefore

$$\left.\begin{aligned} \tfrac{4}{\pi} &> \tfrac{3.3}{2.4}\cdot\tfrac{5.5}{4.6}\cdots\tfrac{(2s-1)(2s-1)}{(2s-2)(2s)}\sqrt{1+\tfrac{1}{2s}},\\ \tfrac{4}{\pi} &< \tfrac{3.3}{2.4}\cdot\tfrac{5.5}{4.6}\cdots\tfrac{(2s-1)(2s-1)}{(2s-2)(2s)}\sqrt{1+\tfrac{1}{2s-1}}.\end{aligned}\right\} \qquad \text{(A)}$$

Wallis published in 1657, in his *Arithmetica infinitorum*[2], this beautiful theorem, one of the most curious in analysis, by itself, and by the manner in which the inventor is arrived there. His method containing the principles of the theory of definite integrals, that the geometers have specially cultivated in these recent times; I think that they will see with pleasure, a succinct exposition, in the actual language of Analysis. [465]

Wallis considers the series of fractions of which the general term is $\frac{1}{\int dx\left(1-x^{\frac{1}{n}}\right)^s}$, n and s being whole numbers, by commencing with zero. By developing the binomial contained under the integral sign, and integrating each term of the expansion, he obtains for one same value of n, the numerical values of the preceding fraction, corresponding to $s = 0$, $s = 1$, $s = 2$, etc.; that which gives to him a horizontal series, of which s is the index. By supposing successively $n = 0$, $n = 1$, $n = 2$, etc., he has so many horizontal series. Thence he forms a table in double entry, of which s is the horizontal index, and n the vertical index.

In this table, the horizontal and vertical series are the same, so that by designating by $y_{n,s}$ the term corresponding to the indices n and s, we have this fundamental equation,

$$y_{n,s} = y_{s,n}.$$

Wallis observes next that the first series is unity; that the second is formed of the natural numbers; that the third is formed of the triangular numbers, and so forth; in a manner that the general term $y_{n,s}$ of the horizontal series corresponding to n is

$$\frac{(s+1)(s+2)\cdots(s+n)}{1.2.3\ldots n};$$

this fraction being equal to

$$\frac{(n+1)(n+2)\cdots(s+n)}{1.2.3\ldots s},$$

we see clearly that $y_{n,s}$ is equal to $y_{s,n}$.

Now if we arrive to interpolate in the preceding table, the term corresponding to n and s equal to $\frac{1}{2}$, we would have the ratio of the square of the diameter to the surface of the circle; because the term of which there is concern is $\frac{1}{\int dx(1-x^2)^{\frac{1}{2}}}$, or $\frac{4}{\pi}$. Wallis seeks therefore to make this interpolation. It is easy in the case where one of [466]

[2]This work has been translated into English. See [**13, 14**].

the two numbers n and s is a whole number. Thus, by making successively s equal to a whole number less $\frac{1}{2}$, in the function $\frac{(s+1)(s+2)\cdots(s+n)}{1.2.3\ldots n}$, he obtains all the terms of the horizontal series, corresponding to the values of s, $-\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, etc.; and by making n equal to a whole number less $\frac{1}{2}$, in the function $\frac{(n+1)(n+2)\cdots(n+s)}{1.2.3\ldots s}$, he obtains all the terms of the vertical series, corresponding to the values of n, $-\frac{1}{2}$, $\frac{3}{2}$, etc. But the difficulty consists in finding the terms corresponding to n and s both equal, to some whole numbers less $\frac{1}{2}$.

Wallis observes for this that the equation

$$y_{n,s} = \frac{(s+1)(s+2)\cdots(s+n)}{1.2.3\ldots n}$$

gives

$$y_{n,s-1} = \frac{s(s+1)\cdots(s+n-1)}{1.2.3\ldots n},$$

and that thus we have

$$y_{n,s} = \frac{s+n}{s} y_{n,s-1}; \tag{a}$$

so that each term of a horizontal series is equal to the preceding, multiplied by the fraction $\frac{s+n}{s}$; whence it follows that all the terms of a horizontal series, departing from $s = -\frac{1}{2}$, s increasing successively by unity, are the products of $y_{n,-\frac{1}{2}}$, by the fractions $\frac{2n+1}{1}$, $\frac{2n+3}{3}$, $\frac{2n+5}{5}$, etc.; and, departing from $s = 1$, these terms are the products of $y_{n,0}$, by the fractions $\frac{n+1}{1}$, $\frac{n+2}{2}$, $\frac{n+3}{3}$, etc. He supposes that the same laws subsist in
[467] the case of n fractional and equal to $\frac{1}{2}$, so that we have all the terms, departing from $s = -\frac{1}{2}$, by multiplying $y_{\frac{1}{2},-\frac{1}{2}}$ by the series of fractions $\frac{2}{1}$, $\frac{4}{3}$, $\frac{6}{5}$, etc. By designating therefore by $\square$ the term corresponding to $n = \frac{1}{2}$ and $s = \frac{1}{2}$, a term which, as we have seen, is equal to $\frac{4}{\pi}$, we have

$$\square = \tfrac{2}{1} y_{\frac{1}{2},-\frac{1}{2}},$$

that which gives

$$y_{\frac{1}{2},-\frac{1}{2}} = \tfrac{1}{2}\square.$$

Departing from $y_{\frac{1}{2},0}$ or from unity, he obtains the successive terms of the series, corresponding to s whole, by multiplying successively unity, by the fractions $\frac{3}{2}$, $\frac{5}{4}$, $\frac{7}{6}$, etc. He forms thus the following horizontal series which corresponds to $n = \frac{1}{2}$, and to s successively equal to $-\frac{1}{2}$, 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, etc.

$$\frac{1}{2}\square, \quad 1, \quad \square, \quad \frac{3}{2}, \quad \frac{4}{3}\square, \quad \frac{3}{2}\cdot\frac{5}{4}, \quad \frac{4}{3}\cdot\frac{6}{5}\square, \quad \text{etc.}; \tag{i}$$

a series which represents this here,

$$\frac{1}{\int dx(1-x^2)^{-\frac{1}{2}}}, \quad \frac{1}{\int dx(1-x^2)^{0}}, \quad \frac{1}{\int dx(1-x^2)^{\frac{1}{2}}}, \quad \text{etc.}$$

Series (i) gives generally, s being a whole number,

$$y_{\frac{1}{2},s-\frac{1}{2}} = \frac{4}{3}\cdot\frac{6}{5}\cdots\frac{2s}{2s-1}\square,$$
$$y_{\frac{1}{2},s-1} = \frac{3}{2}\cdot\frac{5}{4}\cdots\frac{2s-1}{2s-2};$$

whence we deduce

$$\square = \frac{3.3}{2.4}\cdot\frac{5.5}{4.6}\cdots\frac{(2s-1)(2s-1)}{(2s-2)2s}\frac{y_{\frac{1}{2},s-\frac{1}{2}}}{y_{\frac{1}{2},s-1}}. \tag{B}$$

Wallis considers next that in the series (i), the ratio of each term to the one which precedes it by one unit, is greater than unity, and diminished without ceasing, so that we have

$$\frac{y_{\frac{1}{2},s}}{y_{\frac{1}{2},s-1}} > \frac{y_{\frac{1}{2},s+1}}{y_{\frac{1}{2},s}}.$$

This results in fact from the equation [468]

$$y_{\frac{1}{2},s} = \frac{2s+1}{2s}y_{\frac{1}{2},s-1}.$$

He supposes that this holds equally for all the consecutive terms of the series; so that we have the two inequalities

$$\frac{y_{\frac{1}{2},s-\frac{1}{2}}}{y_{\frac{1}{2},s-1}} > \frac{y_{\frac{1}{2},s}}{y_{\frac{1}{2},s-\frac{1}{2}}} < \frac{y_{\frac{1}{2},s-1}}{y_{\frac{1}{2},s-\frac{3}{2}}};$$

whence he deduces, as we have done above,

$$\frac{y_{\frac{1}{2},s-\frac{1}{2}}}{y_{\frac{1}{2},s-1}} > \sqrt{1+\frac{1}{2s}} < \sqrt{1+\frac{1}{2s-1}};$$

thence, it changes formula (B) into formula (A).

This manner to proceed by way of induction, must appear, and appeared in fact, extraordinary to the geometers accustomed to the rigor of the ancients. Thus we see that some great contemporary geometers of Wallis, were not very satisfied with it, and Fermat, in his correspondence with Digby,[3] made some objections not very worthy of him, against this method which he had not studied sufficiently deeply. It must be, without doubt, employed with an extreme circumspection: Wallis himself said, in responding to Fermat, that it is thus that he is served by it, and in order to confirm the exactitude, he supported it on a calculation by which lord Brouncker[4] had found, by means of formula (A), the ratio of the circumference to the diameter, comprehended between the limits

$$3.14159\,26535\,69,$$
$$3.14159\,26536\,96,$$

[3] Sir Kenelm Digby (1603–1665).
[4] William Brouncker (1620–1684)

limits which coincide in the first ten digits, with this ratio that we have carried beyond one hundred decimals. Notwithstanding these confirmations, it is always useful to demonstrate in rigor, that which one obtains by these means of invention. Wallis observes that the ancients had, without doubt, similar ones that they had not made known at all, being content to give their results supported on synthetic
[469] demonstrations. He regrets with reason, that they had concealed from us their ways to arrive there, and he said to Fermat, that one must be thankful to him not to have imitated them, and to not have *destroyed the bridge after the flood having passed.*[5] It is worthy to note that Newton who had profited from this method of induction of Wallis and of his results, in order to discover his theorem on the binomial, had merited the reproaches that Wallis made to the ancients geometers, in concealing the means which had led them to their discoveries.

Let us resume formula (B) of Wallis. If we suppose

$$\frac{y_{\frac{1}{2},s-\frac{1}{2}}}{y_{\frac{1}{2},s-1}} = u_s,$$

this formula will give

$$u_{s-1} = \frac{(2s-1)^2}{(2s-2)2s} u_s,$$

or

$$0 = 2s(2s-2)(u_s - u_{s-1}) + u_s. \qquad (l)$$

Let there be

$$u_s = A^{(0)} + \frac{A^{(1)}}{s+1} + \frac{A^{(2)}}{(s+1)(s+2)} + \frac{A^{(3)}}{(s+1)(s+2)(s+3)} + \text{etc.};$$

and let us consider that which produces in the second member of equation (l), the term

$$\frac{A^{(r)}}{(s+1)\cdots(s+r)}.$$

By having regard only to this term in u_s, we will have

$$u_s - u_{s-1} = \frac{-rA^{(r)}}{s(s+1)(s+2)\cdots(s+r)};$$

the term $2s(2s-2)(u_s - u_{s-1})$ of the equation (l) becomes thus

$$\frac{-4rA^{(r)}(s-1)}{(s+1)\cdots(s+r)},$$

or

$$\frac{-4rA^{(r)}}{(s+1)\cdots(s+r-1)} + \frac{4r(r+1)A^{(r)}}{(s+1)\cdots(s+r)}.$$

[5] Text: *détruit le pont après avoir passé le fleuve*, i.e., not conceal the means to safety.

The term of u_s depending on $A^{(r+1)}$, will produce similar terms, and thus of the others. [470]
By comparing therefore in equation (l) the terms which have the same denominator $(s+1)\cdots(s+r)$, we will have

$$0 = 4r(r+1)A^{(r)} - 4(r+1)A^{(r+1)} + A^{(r)},$$

that which gives

$$A^{(r+1)} = \frac{(2r+1)^2 A^{(r)}}{4(r+1)}.$$

It is clear, by that which precedes, that u_s is reduced to unity, when s is infinite, that which gives $A^{(0)} = 1$. Thence, we deduce

$$u_s = 1 + \frac{1^2}{4(s+1)} + \frac{1^2.3^2}{4^2.1.2(s+1)(s+2)} + \frac{1^2.3^2.5^2}{4^3.1.2.3(s+1)(s+2)(s+3)} + \text{etc.} = \frac{y_{s-\frac{1}{2}}}{y_{s-1}}.$$

The ratio of the mean term of the binomial $(1+1)^{2s}$ to the entire binomial, is

$$\frac{(s+1)(s+2)\cdots 2s}{2^{2s}.1.2.3\ldots s}$$

or

$$\frac{1.3.5\ldots(2s-1)}{2.4.6\ldots 2s}.$$

By naming therefore T this mean term, formula (B) will give

$$\mathrm{T}^2 = \frac{1}{s\pi u_s}.$$

This theorem and the preceding expression of u_s in series, are due to Stirling; and we see how they are attached to the theorem and to the analysis of Wallis. This value of T^2 is able to serve to determine by approximation, the ratio of the circumference to the diameter, that which was the object of Wallis; or this ratio being supposed known, it gives the mean term of the binomial, that which was the object of Stirling.

II.

The expression of $\triangle^n s^i$ given by formula (μ') of §40 of the first book, has been
[471] concluded from the expression of $\triangle^n \frac{1}{s^i}$, by changing in that one, i into $-i$. This passage from the positive to the negative, is analogous to the inductions that Wallis and other geometers have so happily employed. All these means of invention, which hold in the generality of analysis, require in their usage, a great circumspection, and it is always good to demonstrate the results directly. This is that which we are going to do relative to formula (μ').

Let us consider the integral

$$\int \frac{d\varpi\, c^{-as\varpi\sqrt{-1}}}{(1-\varpi)\sqrt{-1})^{i+1}},$$

taken from $\varpi = -\infty$, to $\varpi = \infty$. This integral is equal to

$$-\frac{\sqrt{-1}}{i}\frac{c^{-as\varpi\sqrt{-1}}}{(1-\varpi)\sqrt{-1})^{i}} + \frac{as}{i}\int \frac{d\varpi\, c^{-as\varpi\sqrt{-1}}}{(1-\varpi)\sqrt{-1})^{i+1}} + \text{constant}.$$

This constant is

$$\frac{\sqrt{-1}}{i}\frac{c^{as\varpi\sqrt{-1}}}{(1-\varpi)\sqrt{-1})^{i}},$$

ϖ being supposed infinite. By uniting it with the term

$$\frac{-\sqrt{-1}}{i}\frac{c^{-as\varpi\sqrt{-1}}}{(1-\varpi)\sqrt{-1})^{i}},$$

in which we must similarly suppose ϖ infinite, we will have

$$\frac{\frac{\sqrt{-1}}{i}\begin{cases}\cos(as\varpi)[(1-\varpi\sqrt{-1})^i - (1+\varpi\sqrt{-1})^i]\\ +\sqrt{-1}\sin(as\varpi)[(1-\varpi\sqrt{-1})^i + (1+\varpi\sqrt{-1})^i]\end{cases}}{(1+\varpi^2)^i}:$$

the numerator of this fraction is real, as also its denominator; and it is clear that it becomes null, in making ϖ infinity; we have therefore

$$\int \frac{d\varpi\, c^{-as\varpi\sqrt{-1}}}{(1-\varpi\sqrt{-1})^{i+1}} = \frac{as}{i}\int \frac{d\varpi\, c^{-as\varpi\sqrt{-1}}}{(1-\varpi\sqrt{-1})^{i}}.$$

[472] Thence it is easy to conclude that in making $i = r - \frac{m}{n}$, r being a positive whole number, we will have

$$\int \frac{d\varpi\, c^{-as\varpi\sqrt{-1}}}{(1-\varpi\sqrt{-1})^{i+1}} = \frac{a^r s^r}{i(i-1)\cdots\left(1-\frac{m}{n}\right)}\int \frac{d\varpi\, c^{-as\varpi\sqrt{-1}}}{(1-\varpi\sqrt{-1})^{1-\frac{m}{n}}}$$

Let $as\varpi = \varpi'$, and let us make $as = q$; we will have

$$a^r s^r \int \frac{d\varpi\, c^{-as\varpi\sqrt{-1}}}{(1-\varpi\sqrt{-1})^{1-\frac{m}{n}}} = q^i \int \frac{d\varpi'\, c^{-\varpi'\sqrt{-1}}}{(q-\varpi'\sqrt{-1})^{1-\frac{m}{n}}},$$

the integrals being taken from ϖ and ϖ' equal to $-\infty$, to ϖ and ϖ equal to $+\infty$. Let us designate by k the integral

$$\int \frac{d\varpi'\, c^{-as\varpi'\sqrt{-1}}}{(q-\varpi'\sqrt{-1})^{1-\frac{m}{n}}};$$

we will have

$$\begin{aligned}\frac{dk}{dq} &= -\left(1-\frac{m}{n}\right)\int \frac{d\varpi'\, c^{-\varpi'\sqrt{-1}}}{(q-\varpi'\sqrt{-1})^{2-\frac{m}{n}}}\\ &= \frac{\sqrt{-1}c^{-\varpi'\sqrt{-1}}}{(q-\varpi'\sqrt{-1})^{1-\frac{m}{n}}} - \int \frac{d\varpi'\, c^{-\varpi'\sqrt{-1}}}{(q-\varpi'\sqrt{-1})^{1-\frac{m}{n}}} + \text{constant.}\end{aligned}$$

We will see, as above, that this last member is reduced to the term affected with the integral sign, a term which is equal to $-k$; we have therefore

$$\frac{dk}{dq} = -k;$$

that which gives, by integrating,

$$k = Ac^{-q},$$

A being an arbitrary constant independent of q. It is clear that this equation supposes q positive; for by making q positive or negative infinity, k is infinitely small. We have therefore

$$\int \frac{d\varpi\, c^{as(1-\varpi\sqrt{-1})}}{(1-\varpi\sqrt{-1})^{i+1}} = \frac{Aa^i s^i}{i(i-1)\cdots\left(1-\frac{m}{n}\right)}.$$

This equation holds whatever be the value of a, provided that as is positive. By [473]
making $s=1$, and changing a into another constant a', we will have

$$\int \frac{d\varpi\, c^{-a'(1-\varpi\sqrt{-1})}}{(1-\varpi\sqrt{-1})^{i+1}} = \frac{Aa'^i}{i(i-1)\cdots\left(1-\frac{m}{n}\right)};$$

we will have therefore

$$s^i = \frac{a'^i}{a^i}\frac{\int \frac{d\varpi\, c^{as(1-\varpi\sqrt{-1})}}{(1-\varpi\sqrt{-1})^{i+1}}}{\int \frac{d\varpi\, c^{a'(1-\varpi\sqrt{-1})}}{(1-\varpi)\sqrt{-1})^{i+1}}};$$

that which gives

$$\triangle^n s^i = \frac{a'^i}{a^i}\frac{\int \frac{d\varpi\, c^{as(1-\varpi\sqrt{-1})}(c^{a(1-\varpi\sqrt{-1})}-1)^n}{(1-\varpi\sqrt{-1})^{i+1}}}{\int \frac{d\varpi\, c^{a'(1-\varpi\sqrt{-1})}}{(1-\varpi\sqrt{-1})^{i+1}}}.$$

In order to have the integrals in series; we will suppose

$$\frac{c^{as(1-\varpi\sqrt{-1})}(c^{a(1-\varpi\sqrt{-1})}-1)^n}{(1-\varpi\sqrt{-1})^{i+1}} = c^{as}(c^a-1)^n c^{-t^2};$$

we will have by taking logarithms,

$$-as\varpi\sqrt{-1} + n\log\left[1+\frac{c^a}{c^a-1}(c^{-a\varpi\sqrt{-1}-1)}\right] - (i+1)\log(1-\varpi\sqrt{-1}) = -t^2.$$

Let us determine a, in a manner that, in the development of the first member of this equation, the first power of ϖ vanishes, and let us suppose this development equal to

$$-fa^2\varpi^2 - f'a^3\varpi^3 - f''a^4\varpi^4 - \text{etc.} = -t^2;$$

we will have first

$$0 = \frac{i+1}{a} - s - \frac{nc^a}{c^a-1};$$

[474] next

$$f = \frac{i+1}{2a^2} + \frac{n}{2}\frac{c^a}{c^a-1} - \frac{n}{2}\left(\frac{c^a}{c^a-1}\right)^2,$$

$$f' = \sqrt{-1}\left[\frac{i+1}{3a^3} - \frac{n}{6}\frac{c^a}{c^a-1} + \frac{n}{2}\left(\frac{c^a}{c^a-1}\right)^2 - \frac{n}{3}\left(\frac{c^a}{c^a-1}\right)^3\right],$$

$$f'' = -\frac{i+1}{4a^4} - \frac{n}{24}\frac{c^a}{c^a-1} + \frac{7n}{24}\left(\frac{c^a}{c^a-1}\right)^2 - \frac{n}{2}\left(\frac{c^a}{c^a-1}\right)^3 + \frac{n}{24}\left(\frac{c^a}{c^a-1}\right)^4,$$

etc.

We have next, by the reversion of series,

$$a\varpi = \frac{t}{\sqrt{f}}\left(1 - \frac{f't}{2f\sqrt{f}} + \frac{5f'^2-4ff''}{8f^3}t^2 + \text{etc.}\right):$$

we have therefore, by taking the integrals from ϖ and t equal to $-\infty$, to t and ϖ equal to $+\infty$,

$$\begin{aligned}&\int \frac{d\varpi\, c^{as(1-\varpi\sqrt{-1})}(c^{a(1-\varpi\sqrt{-1})}-1)^n}{(1-\varpi\sqrt{-1})^n}\\ &= \frac{c^{as}(c^a-1)^n}{a}\int\frac{dt}{\sqrt{f}}\left(1 - \frac{f't}{f\sqrt{f}} + 3\frac{5f'^2-4ff''}{8f^3}t^2 + \text{etc.}\right)\\ &= \frac{\sqrt{\pi}}{\sqrt{f}}\left(1 + \frac{15f'^2-12ff''}{16f^3} + \text{etc.}\right)\frac{c^{as}(c^a-1)^n}{a}.\end{aligned}$$

If we suppose $s=1$, $n=0$ and if we change a into a', we will have

$$a' = i+1,\quad f = \frac{1}{2(i+1)},\quad f' = \frac{\sqrt{-1}}{3(i+1)^2},\quad f'' = \frac{1}{4(i+1)^3},\quad \text{etc.}$$

we will have therefore

$$\int \frac{d\varpi\, c^{a'(1-\varpi\sqrt{-1})}}{(1-\varpi\sqrt{-1})^{i+1}} = \frac{c^{i+1}}{i+1}\left(1 - \frac{1}{12i} + \text{etc.}\right)\sqrt{2(i+1)\pi}.$$

Thence it is easy to conclude

$$\triangle^n s^i = \frac{\left(\frac{i}{a}\right)^{i+1} c^{as-i}(c^a-1)^n}{\sqrt{\frac{i(i+1)}{a^2} - in\frac{c^a}{(c^a-1)^2}}}\left(1 + \frac{15f'^2-12ff''}{16f^3} + \frac{1}{12i} + \text{etc.}\right),$$

[475] a formula which coincides with formula (μ') of §40 of the first book.

This formula supposes a positive, and this is that which holds, when $i+1$ surpasses n. In fact, if, in the equation

$$0 = \frac{i+1}{a} - s - \frac{nc^a}{c^a - 1},$$

we suppose a infinitely small, the second member is positive and equal to $\frac{i+1-n}{a}$; next, a being positive and infinite, this second member becomes negative and equal to $-s-n$; there is therefore a positive value of a which satisfies this equation. But there is only one; because if there were two, the function $\frac{i+1}{a} - s - \frac{nc^a}{c^a-1}$ would have a *maximum* between these two values; we would have therefore at this *maximum*,

$$0 = -\frac{i+1}{a^2} + \frac{nc^a}{(c^a - 1)^2},$$

that which cannot be, a being positive. In fact, $(c^a-1)^2$ is greater than a^2c^a, or $c^a - 1 > ac^{\frac{a}{2}}$, that which is clear; because we have

$$c^{\frac{a}{2}} - c^{-\frac{a}{2}} = a + \frac{a^3}{4.1.2.3} + \text{etc.} > a;$$

we have therefore

$$\frac{nc^a}{(c^a-1)^2} < \frac{n}{a^2} < \frac{i+1}{a^2}.$$

Thus formula (μ') can be used, while $i+1$ surpasses n; that which is conformed to that which we have said in §41 of the first book, according to the consideration of the passages from the real to the imaginary, passages that the preceding analysis confirms.

III.

Formula (p) of §42 of the first book, is quite remarkable: it can be demonstrated in the following manner, which shows distinctly the reason for which the series of differences must be arrested, when the quantity under the exponent of the power, becomes negative.

Let us consider the integral [476]

$$\int x^{-\frac{m}{n}} dx \cos\left(zx - \frac{m\pi}{2n}\right)\left(\frac{\sin x}{x}\right)^n,$$

and let us give to it this form

$$\cos\frac{m\pi}{2n}\int x^{-\frac{m}{n}} dx \cos zx \left(\frac{\sin x}{x}\right)^n$$
$$+\sin\frac{m\pi}{2n}\int x^{-\frac{m}{n}} dx \sin zx \left(\frac{\sin x}{x}\right)^n,$$

the integrals being taken from x null to x infinity. Let us suppose first n even and equal to $2i$; we will have, by the known formulas,

$$\left(\frac{\sin x}{x}\right)^{2i} = \frac{(-1)^i}{2^{2i-1}x^{2i}}\left\{\begin{array}{r}\cos nx - n\cos(n-2)x + \dfrac{n(n-1)}{1.2}\cos(n-4)x - \text{etc.}\\ \pm\dfrac{1}{2}\cdot\dfrac{n(n-1)(n-2)\dots(n-i+1)}{1.2.3\dots i}\end{array}\right\},$$

the $+$ sign holding, if i is even, and the $-$ sign, if i is odd. By multiplying this equation by $\cos zx$, we will have

$$\left(\frac{\sin x}{x}\right)^{2i}\cos zx = \frac{(-1)^i}{2^{2i}x^{2i}}\left\{\begin{array}{r}\cos(n\pm z)x - n\cos(n-2\pm z)x \pm \text{etc.}\\ \pm\dfrac{1}{2}\dfrac{n(n-1)(n-2)\dots(n-i+1)}{1.2.3\dots i}\cos(\tfrac{1}{2}zx)\end{array}\right\},$$

where we must observe that by $\cos(n-2r\pm z)x$, I understand the sum of the cosines $\cos(n-2r+z)x$ and $\cos(n-2r-z)x$, $2r$ being here at most equal to n or $2i$. Let us multiply the second member of this equation by $x^{-\frac{m}{n}}dx$; we have generally

[477]
$$\begin{aligned}&\int x^{-n-\frac{m}{n}}dx\,\cos(n-2r\pm z)x\\ &= -\frac{\cos(n-2r\pm z)x}{\left(n+\frac{m}{n}-1\right)x^{n+\frac{m}{n}-1}}\\ &\quad+\frac{(n-2r\pm z)\sin(n-2r\pm z)x}{\left(n+\frac{m}{n}-1\right)\left(n+\frac{m}{n}-2\right)x^{n+\frac{m}{n}-2}}\\ &\quad+\frac{(n-2r\pm z)^n\cos(n-2r\pm z)x}{\left(n+\frac{m}{n}-1\right)\left(n+\frac{m}{n}-2\right)\left(n+\frac{m}{n}-3\right)x^{n+\frac{m}{n}-3}}\\ &\quad-\text{etc.}\\ &\quad\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ &\quad+\frac{(-1)^i(n-2r\pm z)^n}{\left(n+\frac{m}{n}-1\right)\cdots\frac{m}{n}}\int dx\,x^{-\frac{m}{n}}\cos(n-2r\pm z)x.\end{aligned}$$

We have therefore

$$\int x^{-\frac{m}{n}}dx\,\cos zx\left(\frac{\sin x}{x}\right)^n = \frac{(-1)^i}{2^{2i}x^{n+\frac{m}{n}}}$$

$$\times\left\{\begin{array}{l} -\dfrac{x}{n+\frac{m}{n}-1}\left\{\begin{array}{l} \quad\cos(n\pm z)x - n\cos(n-2\pm z)x \\ +\dfrac{n(n-1)}{1.2}\cos(n-4\pm z)x \\ -\text{etc.} \\ \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \pm\dfrac{1}{2}\dfrac{n(n-1)\cdots(n-i+1)}{1.2.3\ldots i}\cos(\pm zx) \end{array}\right\} \\ +\dfrac{x^2}{\left(n+\frac{m}{n}-1\right)\left(n+\frac{m}{n}-2\right)}\left\{\begin{array}{l} \quad(n\pm z)\sin(n\pm z)x \\ -n(n-2\pm z)\sin(n-2\pm z)x \\ +\text{etc.} \end{array}\right\} \\ +\dfrac{x^3}{\left(n+\frac{m}{n}-1\right)\left(n+\frac{m}{n}-2\right)\left(n+\frac{m}{n}-3\right)}\left\{\begin{array}{l} \quad(n\pm z)^2\cos(n\pm z)x \\ -\text{etc.} \end{array}\right\} \\ -\text{etc.} \end{array}\right\}$$

$$+\frac{1}{2^{2i}\left(n+\frac{m}{n}-1\right)\cdots\frac{m}{n}}\int dx\, x^{-\frac{m}{n}}\left\{\begin{array}{l} \quad(n\pm z)^n\cos(n\pm z)x \\ -n(n-2\pm z)^n\cos(n-2\pm z)x \\ +\dfrac{n(n-1)}{1.2}(n-4\pm z)^n\cos(n-4\pm z)x \\ -\text{etc.} \\ +\pm\dfrac{1}{2}\dfrac{n(n-1)\cdots(n+i-1)}{1.2.3\ldots i}z^n\cos(\pm zx) \end{array}\right.$$

$$+\text{ constant.}$$

This constant must be determined in a manner that the second member of this equation be null: when x is null: now we have, by that which precedes, [478]

$$\cos(n\pm z)x - n\cos(n-2\pm z)x + \text{etc.} = (-1)^i 2^{2i}(\sin x)^n\cos zx.$$

By differentiating this equation with respect to x, we have

$$\begin{aligned} &-[(n\pm z)\sin(n\pm z)x - n(n-2\pm z)\sin(n-2\pm z)x + \text{etc.}] \\ &\quad = (-1)^i 2^{2i}\frac{d[(\sin x)^n\cos zx]}{dx}; \end{aligned}$$

differentiating again, we have

$$\begin{aligned} &-[(n\pm z)^2\cos(n\pm z)x - n(n-2\pm z)^2\cos(n-2\pm z)x\text{etc.}] \\ &\quad = (-1)^i 2^{2i}\frac{d^2[(\sin x)^n\cos zx]}{dx^2}; \end{aligned}$$

and so forth: now we have to the two limits $x = 0$ and x infinity,

$$x^{-n-\frac{m}{n}+1} \quad (\sin x)^n \cos zx = 0,$$
$$x^{-n-\frac{m}{n}+2} \frac{d[(\sin x)^n \cos zx]}{dx} = 0,$$
etc.

We have therefore, by integrating next x null to x infinity,

$$\int x^{-\frac{m}{n}} dx \, \cos zx \left(\frac{\sin x}{x}\right)^n = \frac{1}{2^{2i}\left(n + \frac{m}{n} - 1\right) \cdots \frac{m}{n}}$$
$$\times \int x^{-\frac{m}{n}} dx \begin{cases} \quad (n \pm z)^n \cos(n \pm z)x \\ -n(n-2 \pm z)^n \cos(n-2 \pm z)x \\ +\text{etc.} \\ \pm \frac{1}{2} \frac{n(n-1) \cdots (n-i+1)}{1.2.3 \ldots i} z^n \cos(\pm zx) \end{cases}$$

Now we have, by making $(n - 2r \pm z)x = x'$,

$$\int x^{-\frac{m}{n}} dx \, (n - 2r \pm z)^n \cos(n - 2r \pm z)x$$
$$= (n - 2r \pm z)^{n-1+\frac{m}{m}} \int dx' \, x'^{-\frac{m}{n}} \cos x'.$$

[479] We have moreover, as we will demonstrate it hereafter,

$$\int x'^{-\frac{m}{n}} dx' \cos x' = k' \sin \frac{m\pi}{2n},$$
$$\int x'^{-\frac{m}{n}} dx' \sin x' = k' \cos \frac{m\pi}{2n},$$

k' being equal to $\int t^{-\frac{m}{n}} dt \, c^{-t}$, the integral being taken from t null to t infinity. This premised, we will have

$$\int x^{-\frac{m}{n}} dx \, \cos zx \left(\frac{\sin x}{x}\right)^n = \frac{k' \sin \frac{m\pi}{2n}}{2^n \left(n + \frac{m}{n} - 1\right)\left(n + \frac{m}{n} - 2\right) \cdots \frac{m}{n}}$$

$$\times\begin{cases} \quad (n+z)^{n-1+\frac{m}{n}} - n(n+z-2)^{n-1+\frac{m}{n}} \\ +\dfrac{n(n-1)}{1.2}(n+z-4)^{n-1+\frac{m}{n}} \\ \quad \dots\dots\dots\dots\dots\dots\dots\dots \\ \pm\dfrac{1}{2}\dfrac{n(n-1)\cdots(n-i+1)}{1.2.3\dots i}z^n \\ +(n-z)^{n-1+\frac{m}{n}} - n(n-z-2)^{n-1+\frac{m}{n}} + \text{etc.} \\ \quad \dots\dots\dots\dots\dots\dots\dots\dots \\ \pm\dfrac{1}{2}\dfrac{n(n-1)\cdots(n-i+1)}{1.2.3\dots i}(-z)^{n-1+\frac{m}{n}} \end{cases}$$

It is easy to see by the preceding analysis, that if $n-z-2r$ is negative, it is necessary to change the power $(n-z-2r)^{n-1+\frac{m}{n}}$ into $(2r+z-n)^{n-1+\frac{m}{n}}$, because we have

$$\cos(n-z-2r)x = \cos(2r+z-n)x.$$

We will find by the same analysis, [480]

$$\int x^{-\frac{m}{n}} dx \,\sin zx \left(\frac{\sin x}{x}\right)^n = \frac{1}{2^n\left(n+\frac{m}{n}-1\right)\cdots\frac{m}{n}}$$

$$\times\int x^{-\frac{m}{n}} dx \begin{cases} \quad (n+z)^n \sin(n+z)x \\ -n(n+z-2)^n \sin(n+z-2)x \\ +\text{etc.} \\ -(n-z)^n \sin(n-z)x \\ +n(n-z-2)^n \sin(n-z-2)x \\ +\text{etc.} \end{cases}$$

Now we have

$$\int x^{-\frac{m}{n}}(n\pm z-2r)^n dx \,\sin(n\pm z-2r)x$$
$$= (n\pm z-2r)^{n-1+\frac{m}{n}} k' \cos\frac{m\pi}{2n}.$$

If $(n-z-2r)$ is negative, we have

$$\int x^{-\frac{m}{n}}(n-z-2r)^n dx \,\sin(n-z-2r)x$$
$$= -\int x^{-\frac{m}{n}} dx\,(2r+z-r)^n \sin(2+z-n)x$$
$$= -(2r+z-n)^{n-1+\frac{m}{n}} k' \cos\frac{m\pi}{2n}.$$

Thence we deduce

$$\cos\frac{m\pi}{2n}\int x^{-\frac{m}{n}}dx\,\cos zx\left(\frac{\sin x}{x}\right)^n$$
$$+\sin\frac{m\pi}{2n}\int x^{-\frac{m}{n}}dx\,\sin zx\left(\frac{\sin x}{x}\right)^n$$
$$=\frac{k'\sin\frac{m\pi}{2n}\left[(n+x)^{n-1+\frac{m}{n}}-n(n+z-2)^{n-1+\frac{m}{n}}+\frac{n(n-1)}{1.2}(n+z-4)^{n-1+\frac{m}{n}}-\text{etc.}\right]}{2^n\left(n+\frac{m}{n}-1\right)\left(n+\frac{m}{n}-2\right)\cdots\frac{m}{n}};\qquad (i)$$

the series being continued to that which in the power $(n+z-2r')^{n-1+\frac{m}{n}}$, the quantity $n+z-2r'$ becomes negative, $2r'$ being able here to be extended to $2n$. In fact, it is clear that in the expressions of the two terms of the first member of equation (i),
[481] the terms relative to the power $(n+z-2r)^{n-1+\frac{m}{n}}$, are the same and are added. The terms relative to the power $(n-z-2r)^{n-1+\frac{m}{n}}$, are the same and of contrary signs, as long as $n-z-2r$ is positive; but they have the same sign, when $n-z-2r$ is negative; and the preceding power must, by that which precedes, be changed into $(2r+z-n)^{n-1+\frac{m}{n}}$. The sum of the terms relative to this power is

$$\frac{(-1)^r\frac{n(n-1)\cdots(n-r+1)}{1.2.3\ldots r}(z+2r-n)^{n-1+\frac{m}{n}}}{2^n\left(n+\frac{m}{n}-1\right)\cdots\frac{m}{n}}k'\sin\frac{m\pi}{n};$$

now this term is encountered in the series of the second member of equation (i). This series contains the term

$$\frac{(-1)^{r'}\frac{n(n-1)\cdots(n-r'+1)}{1.2.3\ldots r'}(n+z-2r')^{n-1+\frac{m}{n}}}{2^n\left(n+\frac{m}{n}-1\right)\cdots\frac{m}{n}}k'\sin\frac{m\pi}{n};$$

$n+z-2r'$ being supposed positive. If we make $n-2r'=2r-n$, that which gives $r'=n-r$, this term becomes equal to the preceding; because then we have $(-1)^{r'}=(-1)^r$, and

$$\frac{n(n-1)\cdots(n-r'+1)}{1.2.3\ldots r'}=\frac{n(n-1)\cdots(n-r+1)}{1.2.3\ldots r}.$$

Formula (T) of §24 of the first book, gives

$$\frac{1}{r-1}\int t^{r-1}dt\,c^{-t}\int t^{1-r}dt\,c^{-t}=\frac{\pi}{\sin(r-1)\pi},$$

the integrals being taken from t null to t infinity. If we suppose $r-1=\frac{m}{n}$, we will have

$$\int t^{\frac{m}{n}}dt\,c^{-t}\int t^{-\frac{m}{n}}dt\,c^{-t}=\frac{\frac{m}{n}\pi}{\sin\frac{m\pi}{n}}.$$

[482] That which we have named k in formula (p) from §42 of the first book, is equal to $\int t^{m+n-1}dt\,e^{-t^n}$, and it is easy to see that the integrals being taken from t null to t infinity, we have

$$\int t^{n-1+m}dt\,e^{-t^n}=\frac{1}{n}\int t^{\frac{m}{n}}dt\,c^{-t};$$

we have therefore

$$nkk' = \frac{\frac{m}{n}\pi}{\sin\frac{m}{n}}.$$

By multiplying the two members of equation (i) by $\frac{nk2^n}{\pi}$, and substituting into the second member thus multiplied, instead of nkk', its value $\frac{\frac{m}{n}\pi}{\sin\frac{m\pi}{n}}$, we will have the formula (p) cited.

The same analysis is applied to the case where n is an odd number. It shows distinctly the reason for which the series of the differences must be arrested, when the quantity raised to the power $n-1+\frac{m}{n}$ becomes negative.

There remains for us now to demonstrate the formulas

$$\int x'^{-\frac{m}{n}}dx'\cos x' = k'\sin\frac{m\pi}{n},$$
$$\int x'^{-\frac{m}{n}}dx'\sin x' = k'\cos\frac{m\pi}{n},$$

For this, let us consider the definite integral

$$\int \frac{dx\,c^{-ax}}{x^{\omega}}(\cos rx - \sqrt{-1}\sin rx),$$

this integral being taken from x null to x infinity; ω being less than unity. In developing it by the known expressions of $\cos rx$ and of $\sin rx$, into series, it becomes

$$\int \frac{dx\,c^{-ax}}{x^{\omega}}\left\{\begin{matrix} 1-\frac{r^2x^2}{1.2}+\frac{r^4x^4}{1.2.3.4} \\ -\,rx\sqrt{-1}\left(1-\frac{r^2x^2}{1.2.3}+\frac{r^4x^4}{1.2.3.4.5}-\text{etc.}\right)\end{matrix}\right\}$$

Now we have generally, by taking the integral from x null to x infinity, [483]

$$\int x^{i-\omega}dx\,c^{-ax} = \frac{(1-\omega)(2-\omega)\cdots(i-\omega)}{a^i}\int\frac{dx\,c^{-ax}}{x^{\omega}}.$$

By making next $ax=t$, we have

$$\int\frac{dx\,c^{-ax}}{x^{\omega}} = \frac{1}{a^{1-\omega}}\int t^{-\omega}dt\,c^{-t} = \frac{k'}{a^{1-\omega}},$$

the integral relative to t being taken from t null to t infinity, and k' being supposed to express the integral $\int t^{-\omega}dt\,c^{-t}$, taken within these limits. We will have thus

$$\int x^{i-\omega}dx\,c^{-ax} = \frac{(1-\omega)(2-\omega)\cdots(i-\omega)k'}{a^{i+1-\omega}};$$

whence we deduce

$$\int \frac{dx\, c^{-ax}}{x^\omega}(\cos rx - \sqrt{-1}\sin rx)$$
$$= \frac{k'}{a^{i-\omega}}\begin{cases} 1 - \dfrac{(1-\omega)(2-\omega)}{1.2}\dfrac{r^2}{a^2} + \dfrac{(1-\omega)(2-\omega)(3-\omega)(4-\omega)}{1.2.3.4}\dfrac{r^4}{a^4} - \text{etc.} \\ -\sqrt{-1}\left[(1-\omega)\dfrac{r}{a} - \dfrac{(1-\omega)(2-\omega)(3-\omega)}{1.2.3}\dfrac{r^3}{a^3} + \text{etc.}\right] \end{cases}$$

If we make $\frac{r}{a} = s$, the second member of this equation becomes

$$\frac{k'}{a^{1-\omega}(1+s\sqrt{-1})^{1-\omega}}.$$

Let A be an angle of which s is the tangent, we will have

$$\sin A = \frac{s}{\sqrt{1+s^2}}, \qquad \cos A = \frac{1}{\sqrt{1+s^2}},$$

that which gives

$$\cos A - \sqrt{-1}\sin A = \frac{\sqrt{1+s^2}}{1+s\sqrt{-1}},$$

whence we deduce, by the known theorem,

$$\cos(1-\omega)A - \sqrt{-1}\sin(1-\omega)A = \frac{(1+s^2)^{\frac{1-\omega}{2}}}{(1+s\sqrt{-1})^{1-\omega}}.$$

[484] The tangent s is not only the tangent of angle A, but further that of the same angle increased by any multiple of the semi-circumference; but the first member of this equation needing to be reduced to unity, when s is null, it is clear that we must take for A, the smallest of the angles which have s for tangent.

Now, this equation gives, by substituting $\frac{r}{a}$ in place of s,

$$\frac{k'}{a^{1-\omega}(1+s\sqrt{-1})^{1-\omega}} = \frac{k'}{(a^2+r^2)^{\frac{1-\omega}{2}}} \times [\cos(1-\omega)A - \sqrt{-1}\sin(1-\omega)A] :$$

we have therefore

$$\int \frac{dx\, c^{-ax}}{x^\omega}(\cos rx - \sqrt{-1}\sin rx)$$
$$= \frac{k'}{(a^2+r^2)^{\frac{1-\omega}{2}}}[\cos(1-\omega)A - \sqrt{-1}\sin(1-\omega)A].$$

In comparing separately the real and the imaginary quantities, we have

$$\int \frac{dx\, \cos rx\, c^{-ax}}{x^\omega} = \frac{k'}{(a^2+r^2)^{\frac{1-\omega}{2}}}\cos(1-\omega)A,$$
$$\int \frac{dx\, \sin rx\, c^{-ax}}{x^\omega} = \frac{k'}{(a^2+r^2)^{\frac{1-\omega}{2}}}\sin(1-\omega)A.$$

If a is null, $\frac{r}{a}$ is infinite, and the smallest angle, of which it is the tangent, is $\frac{\pi}{2}$; we have therefore

$$\int \frac{dx\ \cos rx}{x^{\omega}} = \frac{k'}{r^{1-\omega}} \sin \frac{\omega\pi}{2},$$
$$\int \frac{dx\ \sin rx}{x^{\omega}} = \frac{k'}{r^{1-\omega}} \cos \frac{\omega\pi}{2}.$$

By supposing $r = 1$ and $\omega = \frac{m}{n}$, we will have the equations that there was concern to demonstrate.

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