SUR

APPLICATION DU CALCUL DES PROBABILITÉS

AUX OPÉRATIONS GÉODÉSIQUES

Pierre Simon Laplace*

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We determine the length of a great arc, on the surface of the Earth, by a chain of triangles which are supported on a base measured with exactitude. But whatever precision that we bring into the measure of the angles, their inevitable errors can, by accumulating, deviate sensibly from the truth, the value of the arc that we have concluded from a great number of triangles. We know therefore only imperfectly this value, if we are not able to assign the probability that its error is comprehended within some given limits. The desire to extend the application of the Calculus of Probabilities to natural Philosophy, has made me seek the formulas proper to this object. [422]

This application consists in deducing from the observations, the most probable results and to determine the probability of the errors of which they are always susceptible. When, these results being known very nearly, we wish to correct them with a great number of observations, the problem is reduced to determining the probability of one or many linear functions of the partial errors of the observations, the law of probability of these errors being supposed known. I have given, in my *Théorie analytique des Probabilités*, a method and some general formulas for this object, and I have applied them, to some interesting points of the System of the world, in the Connaissance des Temps of 1818, and in a supplement to the work that I just cited. In questions of Astronomy, each observation furnishes, in order to correct the elements, an equation of condition: when these equations are very manifold, my formulas give at the same time the most advantageous corrections, and the probability that the errors after these corrections, will be contained within some assigned limits, whatever be moreover the law of probability of the errors of each observation. It is so much more necessary to be rendered independent of this law, that the simplest laws are always infinitely less probable, seeing the infinite number of those which are able to exist in nature. But the unknown law which the observations follow of which we make use, introduces into the formulas an indeterminate which would permit not at all to reduce them to numbers, if we did not succeed to eliminate it. This is that which I have done by means [423]

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of the sum of the squares of the remainders, when we have substituted into each equation of condition, the most probable corrections. The geodesic questions offering not at all similar equations; it was necessary to seek another means to eliminate from the formulas of probability, the indeterminate dependent on the law of probability of the errors of each partial observation. The quantity by which the sum of the angles of each observed triangle surpasses two right angles plus the spherical excess, has furnished me this means; and I have replaced by the sum of the squares of these quantities, the sum of the squares of the remainders of the equations of condition. Thence, we are able to determine numerically the probability that the final result of a long sequence of geodesic operations, does not exceed a given quantity. It will be easy to apply these formulas, to the part of our meridian which extends from the base of Perpignan to the isle of Formentera; that which is so much more useful, that any base of verification having been measured toward the south part of this meridian, the exactitude of that part reposes entirely on the precision with which the angles of the triangles have been measured.

A perpendicular to the meridian of France, will soon be measured from Strasbourg to Brest. These formulas will make an estimate of the errors, not only of the total arc, but further the difference in longitude of its extreme points, concluded from a chain of the triangles which unite them, and of the azimuths of the first and of the last side of this chain. If we diminish, as much as it is possible, the number of triangles and if we give a great precision to the measure of their angles, two advantages that the use of the repetitive circle and of the reflectors procure, this way to have the difference in longitude of the extreme points of the perpendicular, will be one of the better of which we are able to make use.

In order to be assured of the exactitude of a great arc which is supported on a base measured toward one of its extremities, we measure a second base toward the other extremity, and we conclude from one of these bases the length of the other. If the length thus calculated deviates very little from observation, there is everywhere to believe that the chain of triangles is quite nearly exact, likewise the value of the great arc which results from it. We correct next this value, by modifying the angles of the triangles, in a manner that the bases calculated accord themselves with the measured bases, that which is able to be made in an infinity of ways. Those that we have until the present employed are based on some vague and uncertain considerations. The methods exposed [424]
in my *Théorie analytique des Probabilités* lead to some very simple formulas in order to have directly the correction of the total arc, which results from the measures of many bases. These measures have not only the advantage to correct the arc, but further to increase that which I have named the *weight* of the errors, that is to say to render the probability of the errors, more rapidly decreasing; so that the same errors become less probable with the multiplicity of the bases. I expose here the laws of probability of the errors of the total arc, that the addition of new bases gives birth to. After we brought in the observations and in the calculations, the exactitude that we require now; we considered the sides of the geodesic triangles, as rectilinear, and we supposed the sum of their angles, equal to two right angles. Next we corrected the observed angles, by subtracting from each of them, the third of the quantity of which the sum of the three observed angles, surpassed two right angles. Legendre has noted first, that the two errors that we commit thus, compensate themselves mutually, that is to say that by

subtracting from each angle of a triangle, the third of the spherical excess, we are able to neglect the curvature of its sides, and to regard them as rectilinear. But the excess of the three observed angles over two right angles, is composed of the spherical excess and the sum of the errors of the measure of each of the angles. The analysis of the probabilities shows that we must yet subtract from each angle, the third of this sum, in order to have the law of probability of the errors of the results, most rapidly decreasing. Thus, by the equal apportionment of the error of the sum observed of the three angles of the triangle considered as rectilinear, we correct at the same time the spherical excess, and the errors of the observations. The weight of the angles thus corrected, increases; so that the same errors become by this correction, less probable. There is therefore advantage to observe the three angles of each triangle, and to correct them as we have just said it. Simple good sense makes us recognize this advantage; but the Calculus of probabilities is able alone to estimate, and to show that by this correction it becomes the greatest possible.

In order to apply with success the formulas of probability, to the observations, it is necessary to return faithfully all those that we would admit if they were isolated, and to reject none of them by the sole consideration that it is extended a little from the others. Each angle must be uniquely determined by its measures, without regard to the [425] two other angles of the triangle in which it belongs; otherwise, the error of the sum of the three angles would not be the simple result of the observations, as the formulas of probability suppose it. This remark seems to me important in order to disentangle the truth in the middle from the slight uncertainties that the observations present.

I dare to hope that these researches interest the Geometers at a time where we are occupied to measure the diverse countries of Europe, and where the King just ordered the execution of a new map of France, by competing for details, the operations of the cadastre which thence will become better and more useful yet. Thus the magnitude and the curve of the surface of Europe will be known in all the senses; and our meridian extends to the north to the parallel of the Shetland isles, by its junction with the geodesic operations made in England, and it being terminated to the south at the isle Formentera in the Mediterranean, will embrace near a quarter of the distance from the pole to the equator.

§ 1. Let us conceive, on a sphere, an arc of great circle A, A', A'', etc. and suppose that we have formed about the chain of triangles CAC', $CC'C''$, $C''C'C'''$, $C''C'''C^{iv}$, etc.; of which the sides CC', $C'C''$, $C''C'''$, etc. cut this arc at A', A'', A''', etc. I do not give at all the figure, because it is easy to trace it after these indications. Let A be the angle CAA', $A^{(1)}$ the angle $CA'A$, $A^{(2)}$ the angle $C'A''A'''$, etc. Let further C, be the angle ACC', $C^{(1)}$ the angle $CC'C''$, $C^{(2)}$ the angle $C'C''C'''$, etc.; we will have

$$A + A^{(1)} + C - \alpha = \pi + t,$$

α being the error of the observed angle C, t being the excess of the angles of the spherical triangle ACA' over π which expresses two right angles. We will have similarly

$$A^{(1)} + A^{(2)} + C^{(1)} - \alpha^{(1)} = \pi + t^{(1)},$$

$\alpha^{(1)}$ being the error of the observed angle $CC'C''$, and $t^{(1)}$ being the excess of the angles of the spherical triangle $A'C'A''$ over two right angles. We will form similarly the equations

$$A^{(2)}+A^{(3)}+C^{(2)}-\alpha^{(2)}=\pi+t^{(2)},$$
$$A^{(3)}+A^{(4)}+C^{(3)}-\alpha^{(3)}=\pi+t^{(3)},$$
etc.;

whence we deduce easily

$$\begin{aligned} A^{(2n)} = A + C - C^{(1)} + C^{(2)} - C^{(3)} \cdots + C^{(2n-2)} - C^{(2n-1)} \\ -\alpha + \alpha^{(1)} - \alpha^{(2)} \cdots + \alpha^{(2n-1)} \\ -t + t^{(1)} - t^{(2)} \cdots + t^{(2n-1)}, \end{aligned}$$

and [426]

$$\begin{aligned} A^{(2n-1)} = \pi - A - C + C^{(1)} - C^{(2)} + C^{(3)} \cdots - C^{(2n-2)} \\ +\alpha - \alpha^{(1)} + \alpha^{(2)} \cdots + \alpha^{(2n-2)} \\ +t - t^{(1)} + t^{(2)} \cdots + t^{(2n-2)}; \end{aligned}$$

by supposing therefore A well known, the error of the angle $A^{(2n)}$ is

$$\begin{aligned} -\alpha \quad -\alpha^{(2)} \ldots - \alpha^{(2n-2)}, \\ +\alpha^{(1)} + \alpha^{(3)} \ldots + \alpha^{(2n-1)}; \end{aligned}$$

because the values of t, $t^{(1)}$, etc., are quite small, and able to be determined with precision. The concern now is to have the probability that this error will be contained within some given limits.

For this, I will suppose that the probability of any error α is proportional to $c^{-h\alpha^2}$, c being the number of which the hyperbolic logarithm is unity. This supposition the most natural and the most simple of all, results from the use of the repeating circle in the measure of the angles of the triangles. In fact, let us name $\phi(q)$ the probability of an error q in the measure of a simple angle, and let s be the number of the simple angles observed in all the series that we have made, in order to determine the same angle. The probability that the error of the mean result, or of the angle concluded from this series, is $\pm\frac{r}{\sqrt{s}}$, will be by § 18 of the second book of my *Théorie analytique des Probabilités*, proportional to

$$c^{-\frac{kr^2}{2k''}}$$

$\pm a$ being the limits of the errors, k is, by the same section, equal to $2\int \frac{dq}{a}\,\phi\left(\frac{q}{a}\right)$, and k'' is equal to $\int \frac{q^2}{a^2}\frac{dq}{da}\,\phi\left(\frac{q}{a}\right)$, the integrals being taken from q null to $q=a$; and the negative errors being supposed as probable as the positive errors. By making therefore

$$r=\frac{\alpha\sqrt{s}}{a}, \qquad h=\frac{ks}{4k''a^2};$$

$c^{-h\alpha^2}$ will be the probability of the error α. But we will see, at the end of this Memoir, that the following results always hold, whatever be the probability of α.

Let β and γ be the errors of the two angles $AC'C$ and CAC' of the first triangle ACC'; the probability of the three errors α, β and γ will be proportional to $c^{-h\alpha^2-h\beta^2-h\gamma^2}$; but the observation of these angles give the sum $\alpha+\beta+\gamma$ of the

three errors; because the sum of the three angles having to be equal to two right angles [427]
plus the surface of the triangle ACC', if we name T the excess of the three angles observed on this quantity, we will have

$$\alpha + \beta + \gamma = T;$$

the preceding exponential becomes thus

$$c^{-\alpha^2 - h\beta^2 - h\gamma^2 - h(T-\alpha-\gamma)^2},$$

or

$$c^{-2h(\beta+\frac{1}{2}\alpha-\frac{1}{2}T)^2 - \frac{3h}{2}(\alpha-\frac{1}{3}T)^2 - \frac{h}{3}T^2},$$

β being susceptible to all the values from $-\infty$ to ∞; it is necessary to multiply this exponential by $d\beta$ and take the integral within these limits, that which gives an integral which has for factor

$$c^{-\frac{3h}{2}(\alpha-\frac{1}{3}T)^2 - \frac{h}{3}T^2};$$

the probability of α is therefore proportional to this factor. The most probable value of α is evidently that which renders null the quantity $\alpha - \frac{1}{3}T$; it is necessary therefore to correct the three angles of each triangle by the third of the excess T of their sum observed, over two right angles plus the spherical excess. This is that which we do commonly.

Let us name $\bar{\alpha}$ and $\bar{\beta}$, the quantities $\alpha - \frac{1}{3}T$ and $\beta - \frac{1}{3}T$; the probability of $\bar{\alpha}$ will be proportional therefore to

$$c^{-\frac{3}{2}h\alpha^2}.$$

If we diminish in the preceding expression of $A^{(2n)}$, the angle C by $\frac{1}{3}T$, that is to say if we employ the corrected angles of each triangle; this moment $\bar{C}$, $\bar{C}^{(1)}$, etc. that which the angles C, $C^{(1)}$, etc. become by these corrections; we will have

$$\begin{aligned} A^{(2n)} = & A + \bar{C} - \bar{C}^{(1)} + \bar{C}^{(2)} - \text{etc.} \\ & - \bar{\alpha} + \bar{\alpha}^{(1)} - \bar{\alpha}^{(2)} + \text{etc.} \\ & - t + t^{(1)} - t^{(2)} + \text{etc.} \end{aligned}$$

The probability that the quantity $-\bar{\alpha} + \bar{\alpha}^{(1)} - \text{etc.}$, where the error of the angle $A^{(2n)}$ will be comprehended within the limits $\pm r\sqrt{2n}$, will be, by § 18 cited,

$$\frac{2\sqrt{\frac{3}{2}h}}{\sqrt{\pi}} \int dr\, c^{-\frac{3}{2}hr^2}.$$

§ 2. The concern is no longer but to have the value of h. For this, I will take as [428]
given from the observations, the sums T, $T^{(1)}$, etc. of the errors of the angles of each triangle, and I will determine the value of h, which renders most probable, the observed value of the sum of their squares. By that which precedes, the probability of the values of $\bar{\alpha}$ and of T, is proportional to

$$c^{-\frac{3h}{2}\bar{\alpha}^2 - \frac{h}{3}T^2};$$

by multiplying this exponential by $d\bar{\alpha}$, and taking the integral from $\bar{\alpha} = -\infty$ to $\bar{\alpha} = \infty$, the integral will have for factor $c^{-\frac{h}{3}T^2}$, and this factor will be proportional to the probability of T. This probability will be therefore

$$\frac{dT c^{-\frac{h}{3}T^2}}{\int dT\, c^{-\frac{h}{3}T^2}},$$

the integral being taken from $T = -\infty$ to $T = \infty$; it will be thus

$$\frac{\sqrt{\frac{1}{3}h}}{\sqrt{\pi}}.dT.c^{-\frac{h}{3}T^2}.$$

If we multiply by T^2, this function; the integral taken from $T = -\infty$, to $T = \infty$, and multiplied by $2n$, will be the most probable value of the sum $T^2 + {T^{(1)}}^2 +$ etc. By naming θ^2 the observed sum, and by equating it to this product, we will have

$$h = \frac{3}{2}\frac{2n}{\theta^2};$$

the probability that the error of $A^{(2n)}$ is comprehended within the limits $\pm\frac{2}{3}r'\theta$ is thus

$$\frac{2\int dr.c^{-r'^2}}{\sqrt{\pi}},$$

the integral being taken from r' null.

Let us suppose the line AA' is perpendicular to the meridian of the point A, and that we have observed with exactitude, the angle that the last side $C^{(2n-1)}A^{(2n)}C^{(2n)}$ forms at the point $A^{(2n)}$ with the meridian of this point. By naming E this angle, $\pi - E - A^{(2n)}$ will be the angle that this meridian forms with that perpendicular. Let ϕ be the angle formed by the meridans of the points A and $A^{(2n)}$, or the difference in longitude of these points, and l the latitude of the point A; we will have [429]

$$\sin\phi = \frac{\cos(\pi - E - A^{(2n)})}{\sin l},$$

designating therefore by $\delta\phi$ and $\delta A^{(2n)}$ the errors of the angles ϕ and $A^{(2n)}$, we will have

$$\delta\phi = -\frac{\delta A^{(2n)}.\sin(E + A^{(2n)})}{\sin l.\cos\phi};$$

the preceding integral will give therefore the probability that the error respecting the longitude concluded from the observed azimuths observed at A and $A^{(2n)}$ will be comprehended within the limits

$$\pm\frac{2}{3}\theta r'\frac{\sin(E + A^{(2n)})}{\sin l.\cos\phi}.$$

§ 3. Let us determine presently the probability that the error of the measure of the line $AA'A''\ldots$ will be comprehended within some given limits. For this, let us suppose that in the triangles CAC', $C'CC''$, etc., we had corrected the angles as we do ordinarily, that is to say by subtracting from each, the third of the sum of which the three observed angles surpass two right angles plus the spherical excess; that we lower the vertices C, C', C'' etc. of the perpendiculars CI, $C'I'$, $C''I''$, etc., onto the line AA'; we will have, very nearly,

$$AI = AC\cos IAC;$$

we will have next quite nearly,

$$II' = CC'\sin A^{(1)}$$

and, generally,

$$I^{(i)}I^{(i+1)} = C^{(i)}C^{(i+1)}\sin A^{(i+1)}.$$

by supposing therefore that δ is the characteristic of the errors, we will have

$$\frac{\delta.I^{(i)}I^{(i+1)}}{I^{(i)}I^{(i+1)}} = \frac{\delta.C^{(i)}C^{(i+1)}}{C^{(i)}C^{(i+1)}} - \delta A^{(i+1)}\cot A^{(i+1)}.$$

We have, by that which precedes,

$$\delta A^{(i+1)} = \pm\{\bar{\alpha} - \bar{\alpha}^{(1)} + \bar{\alpha}^{(2)}\cdots \pm \bar{\alpha}^{(i)}\};$$

the $+$ sign having place if i is even, and the $-$ sign if it is odd; next we have, in the $(i+1)^{\text{st}}$ triangle, [430]

$$C^{(i+1)}C^{(i)} = \frac{C^{(i)}C^{(i-1)}\sin C^{(i+1)}C^{(i-1)}C^{(i)}}{\sin C^{(i-1)}C^{(i+1)}C^{(i)}},$$

that which gives

$$\frac{\delta.C^{(i)}C^{(i+1)}}{C^{(i)}C^{(i+1)}} = \frac{\delta.C^{(i)}C^{(i-1)}}{C^{(i)}C^{(i-1)}} + \delta C^{(i+1)}C^{(i-1)}C^{(i)}\cot C^{(i+1)}C^{(i-1)}C^{(i)} - \delta C^{(i-1)}C^{(i+1)}C^{(i)}\cot C^{(i-1)}C^{(i+1)}C^{(i)};$$

but $\bar{\alpha}^{(i)}$ is the error of the angle $C^{(i)}$ or $C^{(i-1)}C^{(i)}C^{(i+1)}$; let $\bar{\beta}^{(i)}$ be the error of the angle $C^{(i-1)}C^{(i+1)}C^{(i)}$; $-(\bar{\alpha}^{(i)}+\bar{\beta}^{(i)})$ will be the error of the angle ... $C^{(i+1)}C^{(i-1)}C^{(i)}$; we will have therefore

$$\frac{\delta.C^{(i)}C^{(i+1)}}{C^{(i)}C^{(i+1)}} = \frac{\delta.C^{(i)}C^{(i-1)}}{C^{(i)}C^{(i-1)}} + (\bar{\alpha}^{(i)} + \bar{\beta}^{(i)})\cot C^{(i+1)}C^{(i-1)}C^{(i)} - \bar{\beta}^{(i)}\cot C^{(i-1)}C^{(i+1)}C^{(i)};$$

that which gives, by observing that, in the first triangle, the side $C^{(i-1)}C$ is AC;

$$\frac{\delta.C^{(i)}C^{(i+1)}}{C^{(i)}C^{(i+1)}} = -\mathrm{S}\left\{\begin{array}{l}(\bar{\alpha}^{(i)} + \bar{\beta}^{(i)})\cot C^{(i+1)}C^{(i-1)}C^{(i)} \\ +\,\bar{\beta}^{(i)}\cot C^{(i-1)}C^{(i+1)}C^{(i)}]\end{array}\right\},$$

the finite integral S expressing the sum of all the quantities that this sign contains, from $i = 0$ inclusively to i inclusively. We will have therefore the value of $\delta . I^{(i)} I^{(i+1)}$.

By reuniting all these values, we will have for the entire error of the measured line, an expression of this form

$$p\bar{\alpha} + q\bar{\beta} + p^{(1)}\bar{\alpha}^{(1)} + q^{(1)}\bar{\beta}^{(1)} + \text{etc.} \qquad (o)$$

The probability of the simultaneous values of $\bar{\alpha}$ and of $\bar{\beta}$ is, by that which precedes, proportional to

$$c^{-2h\left(\bar{\beta}+\frac{1}{2}\bar{\alpha}\right)^2 - \frac{3}{2}h\bar{\alpha}^2};$$

By making

$$\bar{\beta} + \tfrac{1}{2}\bar{\alpha} = \tfrac{1}{2}\underline{\alpha}\sqrt{3},$$

the preceding exponential becomes

$$c^{-\frac{3}{2}h\underline{\alpha}^2 - \frac{3}{2}h\bar{\alpha}^2},$$

and the function (o) takes then this form [431]

$$r\underline{\alpha} + r^{(1)}\bar{\alpha} + r^{(2)}\underline{\alpha}^{(1)} + r^{(3)}\bar{\alpha}^{(1)} + \text{etc.}$$

The probability that the error of the function (o) is comprehended within the limits $\pm s$, is, by § 20 of the second book of my *Théorie analytique des Probabilités*,

$$\frac{2\int dt\, c^{-t^2}}{\sqrt{\pi}},$$

the integral being taken from t null, to t being

$$s\sqrt{\frac{\frac{3}{2}h}{r^2 + {r^{(1)}}^2 + {r^{(2)}}^2 + \text{etc.}}};$$

now we have evidently

$$r = \frac{1}{2}q\sqrt{3}, \quad r^{(1)} = p - \frac{1}{2}q;$$

the value of t will be therefore

$$s\sqrt{\frac{\frac{3}{2}h}{p^2 - pq + q^2 + {p^{(1)}}^2 - p^{(1)}q^{(1)} + {q^{(1)}}^2 + \text{etc.}}}$$

§ 4. Let us suppose that in order to verify the operations, we have measured the last part $I^{(2n)}I^{(2n+1)}$ of the line $AA'A''$, etc.; the expression of the error of this part, will be by that which precedes, of the form

$$l\bar{\alpha} + m\bar{\beta} + l^{(1)}\bar{\alpha}^{(1)} + m^{(1)}\bar{\beta}^{(1)} + \text{etc.} \qquad (p)$$

Let λ be this error, or the quantity of which the line $I^{(2n)}I^{(2n+1)}$, concluded from the value of the side AC measured with care, surpasses the direct measure of this line; we will equate the function (p) to λ. If in this function we make $\beta + \frac{1}{2}\bar{\alpha} = \frac{1}{2}\underline{\alpha}\sqrt{3}$; it will take the form

$$f\underline{\alpha} + f^{(1)}\bar{\alpha} + f^{(2)}\underline{\alpha}^{(1)} + f^{(3)}\bar{\alpha}^{(1)} + \text{etc.}$$

and the probability of that function, will be by that which precedes, proportional to

$$c^{-\frac{3}{2}h(\underline{\alpha}^2 + \bar{\alpha}^2 + \underline{\alpha}^{(1)^2} + \bar{\alpha}^{(1)^2} + \text{etc.})}.$$

by substituting for $\underline{\alpha}$, its value

$$\frac{\lambda - f^{(1)}\bar{\alpha} - f^{(2)}\underline{\alpha}^{(1)} - \text{etc.}}{f},$$

the preceding exponential becomes [432]

$$c^{-\frac{3}{2}h(\bar{\alpha}^2 + \underline{\alpha}^{(1)^2} + \bar{\alpha}^{(1)^2} + \text{etc.}) - \frac{3h}{2}\frac{(\lambda - f^{(1)}\bar{\alpha} - f^{(2)}\underline{\alpha}^{(1)} - \text{etc.})^2}{f^2}}.$$

the values of $\bar{\alpha}$, $\underline{\alpha}^{(1)}$, etc. the most probable, are those which render the exponent of c a *minimum*. We will differentiate therefore this exponent and we will equate to zero the coefficients of $d\bar{\alpha}$, $d\bar{\alpha}^{(1)}$, $d\underline{\alpha}^{(1)}$, etc.; that which gives

$$\begin{aligned} f^2\bar{\alpha} &= f^{(1)}\{\lambda - f^{(1)}\bar{\alpha} - f^{(2)}\underline{\alpha}^{(1)} - \text{etc.}\}, \\ f^2\bar{\alpha}^{(1)} &= f^{(2)}\{\lambda - f^{(1)}\bar{\alpha} - f^{(2)}\underline{\alpha}^{(1)} - \text{etc.}\}, \\ &\text{etc.} \end{aligned}$$

From these diverse equations, we deduce

$$\begin{aligned} \underline{\alpha} &= \frac{\lambda f}{f^2 + f^{(1)^2} + f^{(2)^2} + \text{etc.}}, \\ \bar{\alpha} &= \frac{\lambda f^{(1)}}{f^2 + f^{(1)^2} + f^{(2)^2} + \text{etc.}}, \\ \underline{\alpha}^{(1)} &= \frac{\lambda f^{(2)}}{f^2 + f^{(1)^2} + f^{(2)^2} + \text{etc.}}, \\ &\text{etc.}, \end{aligned}$$

that which gives, by observing that

$$f\underline{\alpha} + f^{(1)}\bar{\alpha} = l\bar{\alpha} + m\bar{\beta},$$

and that

$$\underline{\alpha} = \frac{2\bar{\beta} + \bar{\alpha}}{\sqrt{3}},$$

$$
\begin{aligned}
\bar{\alpha} &= \frac{\lambda(l - \frac{1}{2}m)}{l^2 - ml + m^2 + {l^{(1)}}^2 - m^{(1)}l^{(1)} + {m^{(1)}}^2 + \text{etc.}},\\
\bar{\beta} &= \frac{\lambda(m - \frac{1}{2}l)}{l^2 - ml + m^2 + {l^{(1)}}^2 + \text{etc.}},\\
\bar{\alpha}^{(1)} &= \frac{\lambda(l^{(1)} - \frac{1}{2}m^{(1)})}{l^2 - ml + m^2 + {l^{(1)}}^2 + \text{etc.}},\\
&\text{etc.}
\end{aligned}
$$

By substituting therefore these values into the function (o), we will have the correction resulting from the measure of the part $I^{(2n)}I^{(2n+1)}$.

But we can arrive by the following method, to this result, and have at the same time the new law of the errors of the measure of the entire arc, which results from the measure of a second base.

$c^{-\frac{3}{2}h.\underline{\alpha}^2}$ and $c^{-\frac{3}{2}h.\bar{\alpha}^2}$ being proportionals to the probabilities of $\underline{\alpha}$ and of $\bar{\alpha}$, it is easy to conclude from § 21 of the second book of my *Théorie analytique des Probabilités*, that by supposing the function (o) equal to e, the probability of e will be proportional to [433]

$$
c^{\frac{-\frac{3}{2}h\left(e-\lambda\frac{\text{S}r^{(i)}f^{(i)}}{\text{S}f^{(i)2}}\right)^2}{\text{S}r^{(i)2}-\frac{(\text{S}r^{(i)}f^{(i)})^2}{\text{S}f^{(i)2}}}},
$$

the sign S extending to all the values of i, from $i = 0$, inclusively; by making therefore

$$
e - \lambda\frac{\text{S}r^{(i)}f^{(i)}}{\text{S}f^{(i)^2}} \pm u,
$$

the probability that the function (o) will be comprehended within the limits

$$
\frac{\lambda\text{S}r^{(i)}f^{(i)}}{\text{S}f^{(i)^2}} \pm u,
$$

will be proportional to

$$
c^{\frac{-\frac{3}{2}hu^2}{\text{S}r^{(i)2}-\frac{(\text{S}r^{(i)}f^{(i)})^2}{\text{S}f^{(i)2}}}}.
$$

thus we see that it is necessary to diminish the arc measured $AI^{(2n+1)}$, by the quantity

$$
\frac{\lambda\text{S}r^{(i)}f^{(i)}}{\text{S}f^{(i)^2}}
$$

we see next by this correction, the weight of the error to fear is increased. Because before the measure of the second base, it was

$$
\frac{-\frac{3}{2}h}{\text{S}r^{(i)2}},
$$

and by this measure it becomes

$$\frac{-\frac{3}{2}h}{\mathrm{S}r^{(i)2} - \frac{(\mathrm{S}r^{(i)}f^{(i)})^2}{\mathrm{S}f^{(i)2}}}.$$

we are able to observe that

$$r^2 + r^{(1)} = p^2 - pq + q^2;$$
$$f^2 + {f^{(1)}}^2 = l^2 - ml + m^2;$$
$$rf + r^{(1)}f^{(1)} = l(p - \tfrac{1}{2}q) + m(q - \tfrac{1}{2}p);$$

we will be able therefore to form easily $\mathrm{S}r^{(i)2}$ and $\mathrm{S}r^{(i)}f^{(i)}$ by means of the coefficients of $\bar{\alpha}$, $\bar{\beta}$, $\bar{\alpha}^{(1)}, \ldots$ in the functions of (o) and (p). [434]

If we had measured some other bases, we would have, by the method of the section cited, the corrections which it would be necessary to make to the measured arc, and the law of its errors.

We will have similarly the correction that we must make to the angle $A^{(2n)}$ which gives the difference in longitude of the extreme points from a perpendicular to the meridian; because the correction of $A^{(2n)}$, or that which it is necessary to remove from it, being by that which precedes

$$-\bar{\alpha} + \bar{\alpha}^{(1)} - \bar{\alpha}^{(2)} + \text{etc.},$$

it will suffice to substitute, instead of the function (o), the function

$$-\bar{\alpha} + \bar{\alpha}^{(1)} - \bar{\alpha}^{(2)} + \text{etc.},$$

that which gives

$$p = -1, \quad q = 0, \quad p^{(1)} = 1, \quad q^{(1)} = 0, \quad p^{(2)} = -1, \quad \text{etc.},$$

thence it is easy to conclude that we must in order to correct the value of $A^{(2n)}$, add the quantity to it

$$\frac{+\lambda \left\{ \begin{array}{c} l - l^{(1)} + l^{(2)} - \text{etc.} \\ -\frac{1}{2}m + \frac{1}{2}m^{(1)} - \text{etc.} \end{array} \right\}}{l^2 - ml + m^2 + {l^{(1)}}^2 - \text{etc.}}.$$

the probability that the error of this value of $A^{(2n)}$, thus corrected, is within the limits $\pm u$ will be

$$\frac{2\int dt\, c^{-t^2}}{\sqrt{\pi}},$$

the integral being taken from t null to

$$t = \frac{u\sqrt{\frac{3}{2}h}}{\sqrt{2n - \frac{(l - l^{(1)} + l^{(2)} - \text{etc.} - \frac{1}{2}m + \frac{1}{2}m^{(1)} - \text{etc.})^2}{l^2 - ml + m^2 + {l^{(1)}}^2 - \text{etc.}}}}$$

§ 5. We are arrived to the preceding results, by supposing that the law of probability of the error α in the measure of the angle is proportional to $c^{-h\alpha^2}$, and we have proved that this law of probability is able to be admitted in regard to the angles measured with the repeating circle. We will show here that these results hold generally whatever be the law of probability of error α. Let $\phi(\alpha)$ be this law, the positive errors being supposed [435]
to have the same probability as the negative errors; let us make $\alpha - qT = \bar{\alpha}$, and let us seek the probability of the errors of the function

$$-\bar{\alpha} + \bar{\alpha}^{(1)} - \bar{\alpha}^{(2)} + \text{etc.} + \bar{\alpha}^{(2u-1)}; \qquad (i)$$

If we name α, and β the errors of the two angles of a triangle, and T the excess of their sum over two angles plus the spherical excess; $T - \alpha - \beta$ will be the error of the third angle; and the probability of the simultaneous existence of these three errors, will be

$$\phi(\alpha).\phi(\beta).\phi(T - \alpha - \beta).$$

The probability of $\bar{\alpha}$, will be therefore

$$\phi(\bar{\alpha} + qT).\phi(\beta).\phi\left(\begin{array}{c} (1-q).T - \beta \\ -\bar{\alpha} \end{array}\right),$$

by multiplying this product by $d\beta.dT$, and by taking the integrals from β and T equal to $-\infty$, to β and T equal to $+\infty$; we will have a function which will be proportional to the probability of $\bar{\alpha}$. We will suppose here, that which we are able to make, that the function $\phi(\alpha)$ is able to be extended to these infinite limits. Let us designate by $\psi(\bar{\alpha})$, the function resulting from these integrations. The probability that the error of the function (i) is $\pm r\sqrt{2n}$, will be by § 18 of my *Théorie analytique des Probabilités*, proportional to

$$c^{\frac{-H}{4H''}.r^r},$$

by making

$$H = 2\int d\bar{\alpha}.\psi(\bar{\alpha}); \qquad H'' = 2\int \bar{\alpha}^2 d\bar{\alpha}^2.\psi(\bar{\alpha}).\psi(\bar{\alpha}),$$

the integrals being taken within the positive and negative infinite limits. Now we have by integrating within these limits,

$$H = 2\int d\bar{\alpha}.d\beta.dT.\phi(\bar{\alpha} + qT).\phi(\beta).\phi\left(\begin{array}{c} T - \bar{\alpha} - \beta \\ -qT \end{array}\right),$$

and by the theory of double integrals, this second member is equal to

$$2\int d\alpha.d\beta.dT'.\phi(\alpha).\phi(\beta).\phi(T'),$$

by making

$$T' = T - qT - \bar{\alpha} - \beta;$$

by designating therefore by K the integral $\int d\alpha.\phi(\alpha)$ taken between the infinite limits, we will have evidently

$$H = 2K^3.$$

We will have next [436]

$$H'' = \int \bar{\alpha}^2.d\bar{\alpha}.d\beta.dT.\phi(\bar{\alpha} + qT).\phi(\beta).\phi(T - qT - \bar{\alpha} - \beta),$$

now we have

$$\bar{\alpha} = \alpha - qT = \alpha.(1 - q) - q\beta - qT';$$

we will have therefore

$$H'' = \int [(1 - q).\alpha - q\beta - qT']^2.d\alpha.d\beta.dT'.\phi(\alpha).\phi(\beta).\phi(T').$$

If we observe now that $\alpha\, d\alpha.\phi(\alpha)$ is null, because $\phi(\alpha)$ being supposed the same for the two errors α and $-\alpha$, the two elements $-\alpha\, d\alpha.\phi(-\alpha)$, and $\alpha\, d\alpha\, \phi(\alpha)$ is destroyed in the preceding integral taken between the infinite limits; if we designate next by K'', the integral $\int \alpha^2.d\alpha.\phi(\alpha)$; we will have evidently

$$H'' = K''.K^2.(\overline{1 - q}^2 + 2q^2);$$

thus the probability of r will be proportional to

$$c^{\frac{-Kr^2}{2K''.(\overline{1-q}^2+2q^2)}}.$$

The value of q which renders this probability most rapidly decreasing is that which renders $(1 - q)^2 + 2q^2$, a *minimum*, and that value is $\frac{1}{3}$; it is necessary therefore to have the probability of error most rapidly decreasing, to diminish by a third of T, each angle of the triangle; and then the probability of r becomes

$$c^{\frac{3K.r^2}{4K''}}.$$

It is necessary now to determine by the observations, the value of $\frac{-3K}{4K''}$. For this, we will observe that the probability of T, will be proportional to the integral

$$\int d\alpha.d\beta.\phi(\alpha).\phi(\beta).\phi(T - \alpha - \beta),$$

taken within the infinite limits. Let $\Pi(T)$ be this integral. The most probable sum of the values of T^2 in the $2n$ observed triangles, will be by § 19 of the second book of the work cited,

$$\frac{Q''}{Q}.2n,$$

by supposing

$$Q = \int dT.\Pi(T); \qquad Q'' = \int T^2 dt.\Pi(T),$$

the integrals being taken between the infinite limits. Now we have [437]

$$\int dT.\Pi(T) = \int d\alpha.d\beta.dT.\phi(\alpha).\phi(\beta).\phi(T-\alpha-\beta);$$

and by that which precedes, this second member is equal to K^3; next we have

$$\begin{aligned}\int T^2 dT.\Pi(T) &= \int (T'+\alpha+\beta)^2.d\omega.d\beta.dT'.\phi(\alpha).\phi(\beta).\phi(T') \\ &= \int (T'^2+\alpha^2+\beta^2).d\alpha.d\beta.dT'.\phi(\alpha).\phi(\beta).\phi(T').\end{aligned}$$

This second member is evidently $3K2.K''$; we will have therefore

$$\frac{K}{K''} = \frac{6n}{\theta^2}.$$

and then the probability of r become proportional to

$$c^{\frac{-9}{4}.2n.\frac{r^2}{\theta^2}};$$

thus the probability of the error $\pm\frac{2}{3}r'\theta$ of the function (i) will be as previously

$$\frac{2\int dr'\, c^{-r'^2}}{\sqrt{\pi}}.$$

It is easy to see that we are able to extend the same reasoning, to all the results to which we are arrived, by departing from the law of probability of the error α, proportional to $c^{-h\alpha 2}$. Thence these results become independent of this law, and are extended to all the laws that are able to exist in nature.

Let us consider in fact, the function

$$p\bar{\alpha} + q\bar{\beta} + p^{(1)}\alpha^{(1)} + q^{(1)}\bar{\beta}^{(1)} + \text{etc.}, \qquad (o)$$

and let us seek the probability that the value of this function will be s. In designating by $\psi(\bar{\alpha},\bar{\beta})$ the probability of the coexistence of the values of $\bar{\alpha}$ and of $\bar{\beta}$; we will have

$$\psi(\bar{\alpha},\bar{\beta}) = \int dT.\phi(\bar{\alpha}+\tfrac{1}{3}T).\phi(\bar{\beta}+\tfrac{1}{3}T).\phi(\tfrac{1}{3}T-\bar{\alpha}-\bar{\beta}),$$

the integral being taken within the infinite limits, $T=-\infty$, and $T=\infty$. Next we see by § 20 of the second book of my *Théorie analytique des Probabilités*, that the sought probability will be

$$\frac{1}{\pi}\int d\omega.c^{-s\omega\sqrt{-1}}\left\{\begin{array}{l}\int d\bar{\alpha}.d\bar{\beta}.\psi(\bar{\alpha},\bar{\beta}).(\bar{\omega})\cos(p\bar{\alpha}+q\bar{\beta}).\omega \\ \times\int d\bar{\alpha}^{(1)}.d\bar{\beta}^{(1)}.\psi(\bar{\alpha}^{(1)},\bar{\beta}^{(1)})\cos(p^{(1)}\bar{\alpha}^{(1)}+q^{(1)}\bar{\beta}^{(1)}).\omega \\ \times\text{etc.}\end{array}\right.$$

The integral relative to ω being taken from $\omega = -\pi$ to $\omega = \pi$ and the integrals relative to $\bar{\alpha}$, and $\bar{\beta}$, being taken within their infinite limits. Let us develop into a series ordered with respect to the powers of ω, the logarithm of the factor of $d\omega.c^{-s\omega\sqrt{-1}}$, under the $\int$ sign. We have [438]

$$\begin{aligned} &\log \int d\bar{\alpha}.d\bar{\beta}.\psi(\bar{\alpha},\bar{\beta}) \cos(p\bar{\alpha} + q\bar{\beta}).\omega \\ &= \log \int d\bar{\alpha}.d\bar{\beta}.\psi(\bar{\alpha},\bar{\beta}) \\ &- \frac{\omega^2}{2} \cdot \frac{\int d\bar{\alpha}.d\bar{\beta}.\psi(\bar{\alpha},\bar{\beta}) \cos(p\bar{\alpha} + q\bar{\beta})}{\int d\bar{\alpha}.d\bar{\beta}.\psi(\bar{\alpha},\bar{\beta})} - \text{etc.}, \end{aligned}$$

the integral $\int d\bar{\alpha}.d\bar{\beta}.\psi(\bar{\alpha},\bar{\beta})$ is equal to

$$\int d\bar{\alpha}.d\bar{\beta}.dT.\phi(\bar{\alpha} + \tfrac{1}{3}T).\phi(\bar{\beta} + \tfrac{1}{3}T).\phi(\tfrac{1}{3}T - \bar{\alpha} - \bar{\beta}),$$

or to

$$\int d\alpha.d\beta.dT'.\phi(\alpha).\phi(\beta).\phi(T'),$$

all these integrals being taken within the infinite limits. This last integral is, by that which precedes, equal to K^3. the integral

$$\int d\bar{\alpha}.d\bar{\beta}.\psi(\bar{\alpha},\bar{\beta})(p\bar{\alpha} + q\bar{\beta})^2,$$

is equal to

$$\int d\alpha.d\beta.dT'.\phi(\alpha).\phi(\beta).\phi(T'). \begin{bmatrix} p.(\frac{9}{3}\alpha - \frac{1}{3}\beta + \frac{1}{3}T') \\ +q.(\frac{2}{3}\beta - \frac{1}{3}\alpha + \frac{1}{3}T') \end{bmatrix}^2,$$

In the squared factor under the $\int$ sign, we are able to neglect the products of α, β, and T', because they produce nothing, as we have seen, in the integral; then, it is easy to see that this integral is reduced to $\frac{2}{3}K^2K''.(p^2 - pa + q^2)$. Thence, it is easy to conclude that the sought probability of the value s, is proportional to

$$\int d\omega.c^{-s\omega\sqrt{-1}} - \frac{K''.\omega^2}{3K}.\text{S}.(p^2 - pq + q^2) - \text{etc.},$$

by designating by $\text{S}(p^2 - pq + q^2)$, the sum $p^2 - pq + q^2 + {p^{(1)}}^2 - p^{(1)}q^{(1)} + {q^{(1)}}^2 + \text{etc.}$ We are able by § 20 cited, to consider only the square of ω and to neglect its superior powers; by setting next the preceding integral under this form

$$\int d\omega.c^{\frac{-K''}{3K}.\text{S}(p^2-pq+q^2)\cdot\left[\omega - s\sqrt{-1}.\frac{3K}{2K''.\int(p^2-pq+q^2)}\right]^2 - \frac{3}{4}\cdot\frac{Ks^2}{K''.\text{S}(p^2-pq+q^2)}},$$

the integral is able, by this same section, to be taken from $\omega = -\infty$, to $\omega = \infty$; the probability of the value of s, is thus proportional to

$$c^{-\frac{3}{4}\cdot\frac{Ks^2}{K''.\text{S}(p^2-pq+q^2)}},$$

and consequently to [439]

$$c^{-\frac{9}{4}\cdot\frac{s^2.2n}{\theta^2.\mathrm{S}(p^2-pq+q^2)}};$$

thus the probability that the value of s is comprehended within the limits

$$\pm\frac{3}{2}t\theta\cdot\sqrt{\frac{\mathrm{S}(p^2-pq+q^2)}{2n}},$$

will be

$$\frac{2\int dt.c^{-t2}}{\sqrt{\pi}},$$

the integral being taken from t null; that which is conformed to that which precedes.

We have supposed in that which precedes,

$$\bar{\alpha}=\alpha-\tfrac{1}{3}T; \qquad \bar{\beta}=\beta-\tfrac{1}{3}T; \quad \text{etc.};$$

that is, that we correct each of the three angles of each triangle, by one third of the observed error of the sum of its three angles. But is this correction here the most advantageous? This is that which we will examine. Let us suppose generally in the function (o)

$$\bar{\alpha}=\underline{\alpha}+iT; \quad \bar{\beta}=\underline{\beta}+lT; \quad \bar{\alpha}^{(1)}=\underline{\alpha}^{(1)}+i^{(1)}T; \quad \text{etc.}$$

Then $\bar{\alpha}$ being $\alpha-\frac{1}{3}T$; $\bar{\beta}$ being $\beta-\frac{1}{3}T$; we will have

$$\alpha=\underline{\alpha}+(1+\tfrac{1}{3}).T; \quad \beta=\underline{\beta}+(1+\tfrac{1}{3}).T; \quad \text{etc.},$$

and the function (o) will become

$$p\underline{\alpha}+p\underline{\beta}+p^{(1)}\underline{\alpha}^{(1)}+\text{etc.}+\mathrm{S}.(pi+ql).T;$$

and the correction of the calculated arc $AI^{(2n)}$ will be $-\mathrm{S}.(pi+ql)T$; and its error will become

$$p\underline{\alpha}+p\underline{\beta}+p^{(1)}\underline{\alpha}^{(1)}+\text{etc.}$$

By applying to this function, the preceding analysis; we will find easily that the probability of the value s of this function, is proportional to

$$c^{-\frac{-\frac{9}{4}.2n.\frac{s^2}{\theta^2}}{\mathrm{S}.(p^2-pq+q^2+\frac{9}{2}(pi+ql)^2}},$$

$2n$ being the number of triangles employed.

It is clear that the values of i and of l which render the coefficient of s^2 a *maximum*, [440] are those which render $pi+ql$, null. Then the preceding correction of the arc $AI^{(2n)}$ is null, and the law of probability of its errors is the same as in the case of i and of l nulls. This case gives therefore the law of probability of the errors, the most rapidly decreasing, a law which must be evidently adopted.

We will note here that in the calculation of the function (o), we are able to apply at will, the errors α and β to two of the angles of the triangle. Similarly in the calculation of the arc $AI^{(2n)}$, we are able at will, in the series of the triangles which serve to this calculation, to name them first, second, etc., triangles; only we will observe that α and β belong to the first triangle, $\alpha^{(1)}$ and $\beta^{(1)}$ to the second and thus of the rest. But it will be simpler to enumerate them according to the order in which we use them.