SUR

LE CALCUL DES PROBABILITÉS

APPLIQUÉ A LA PHILOSOPHIE NATURELLE

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I have given, in my *Théorie analytique des probabilités* and in that which preceded, some general formulas in order to have the probability that the errors of the results obtained by the whole of a great number of observations, and determined by the most advantageous method, are comprehended within some given limits. The advantage of these formulas, is to be independent of the law of probability of the errors of the observations, a law always unknown, and which does not permit to reduce into numbers, the expressions which contain it. I am happy to succeed to eliminate from my formulas, the factor $\frac{2k''}{k}a^2s$, which depends on this law, by observing that the number s of observations being quite great, this factor is very probably equal to the sum of the squares of the errors of the observations, and that this sum is very probably the sum of the squares of the residuals of the equations of condition, when we have substituted the elements determined by the most advantageous method. I suppose that we have before the eyes §§ 19, 20 and 21 of the second Book of my *Théorie analytique des probabilités*. The importance of these formulas in natural philosophy, requires that the uncertainty that they can leave is dissipated, and the only one which remains yet, is relative to the equalities of which I just spoke. I myself propose here, to clarify this delicate point of the theory of probabilities, and to show that these equalities can be employed without sensible errors.

The sum of the squares of the errors of the observations being supposed equal to $\frac{2k''}{k}a^2s + a^2r\sqrt{s}$, the probability that the value of r is comprehended within the given limits is, by the § 19 cited,

$$\frac{1}{\sqrt{2\pi}}\int \beta' dr\, c^{-\frac{\beta'^2 r^2}{2}},$$

the integral being taken within the given limits. Let us represent the general equation of condition of the elements $z,\ z'$, etc. by this one,

$$\epsilon^{(i)} = p^{(i)}z + q^{(i)}z' + \text{etc.} - \alpha^{(i)},$$

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$\epsilon^{(i)}$ being the error of the observation. The elements $z,\ z'$, etc. being determined by the most advantageous method, let us designate by $u,\ u'$, etc. their errors, we will have by naming $\epsilon'^{(i)}$ the rest of the function

$$p^{(i)}z + q^{(i)}z' + \text{etc.} - \alpha^{(i)},$$

when we have substituted for $z,\ z'$, etc. their values thus determined

$$\epsilon^{(i)} = \epsilon'^{(i)} + p^{(i)}u + q^{(i)}u' + \text{etc.},$$

that which gives

$$\mathrm{S}\epsilon^{(i)^2} = \mathrm{S}\epsilon'^{(i)^2} + 2\mathrm{S}\epsilon'^{(i)}(p^{(i)}u + q^{(i)}u' + \text{etc.}) + \mathrm{S}(p^{(i)}u + q^{(i)}u' + \text{etc.})^2,$$

the integral sign S extending to all the values of i, from $i = 0$ to $i = s - 1$. But by the conditions of the most advantageous method, we have $\mathrm{S}p^{(i)}\epsilon'^{(i)} = 0$, $\mathrm{S}q^{(i)}\epsilon'^{(i)} = 0$, etc.; we have therefore

$$\mathrm{S}\epsilon^{(i)^2} = \mathrm{S}\epsilon'^{(i)^2} + \mathrm{S}(p^{(i)}u + q^{(i)}u' + \text{etc.})^2.$$

By comparing this value of $\mathrm{S}\epsilon^{(i)^2}$ to its preceding value

$$\frac{2k''}{k}a^2s + a^2r\sqrt{s},$$

we will have

$$a^2r\sqrt{s} = \mathrm{S}\epsilon'^{(i)^2} - \frac{2k''}{k}a^2s + \mathrm{S}(p^{(i)}u + q^{(i)}u' + \text{etc.})^2.$$

Let us make

$$\mathrm{S}\epsilon'^{(i)^2} - \frac{2k''}{k}a^2s = t\sqrt{s},$$

$$u = \frac{\nu}{\sqrt{s}}, \quad u' = \frac{\nu'}{\sqrt{s}}, \quad \text{etc.},$$

we will have

$$a^2r = t + \frac{\mathrm{S}(p^{(i)}\nu + q^{(i)}\nu' + \text{etc.})^2}{s\sqrt{s}};$$

the exponential $c^{-\frac{\beta'^2r^2}{4}}$ becomes thus

$$c^{-\frac{\beta'^2}{4a^2}\left[t+\frac{\mathrm{S}(p^{(i)}\nu+q^{(i)}\nu'+\text{etc.})^2}{s\sqrt{s}}\right]^2};$$

thus the probability of t is proportional to this exponential.

The probability of the simultaneous existence of the quantities $u,\ u'$, etc. is, by § 21 of the second Book of the *Théorie analytique des probabilités*, proportional to the exponential

$$c^{-\frac{k}{4k''a^2s}\mathrm{S}(p^{(i)}\nu+q^{(i)}\nu'+\text{etc.})^2},$$

the probability of the simultaneous existence of t, ν, ν', etc. is therefore proportional to

$$c^{-\frac{\beta'^2}{4a^2}\left[t+\frac{\mathrm{S}(p^{(i)}\nu+q^{(i)}\nu'+\text{etc.})^2}{s\sqrt{s}}\right]^2-\frac{k}{4k''a^2s}\mathrm{S}(p^{(i)}\nu+q^{(i)}\nu'+\text{etc.})^2}.$$

By substituting for $\frac{4k''a^2s}{k}$ its value $2\mathrm{S}\epsilon'^{(i)^2} - 2t\sqrt{s}$, this exponential is reduced, by neglecting the terms of order $\frac{1}{s}$, to the following function:

$$\left[1-\frac{t\sqrt{s}}{2(\mathrm{S}\epsilon'^{(i)2})^2}\mathrm{S}(p^{(i)}\nu+\text{etc.})^2\right].$$
$$c^{-\frac{\beta'^2}{4a^4}\left[t+\frac{\mathrm{S}(p^{(i)}\nu+q^{(i)}\nu'+\text{etc.})^2}{s\sqrt{s}}\right]^2-\frac{\mathrm{S}(p^{(i)}\nu+q^{(i)}\nu'+\text{etc.})^2}{2\mathrm{S}\epsilon'^{(i)2}}}$$

Now, in order to have the probability that the value of ν is comprehended within some given limits, it is necessary: 1° to multiply this function by $dt\ d\nu\ d\nu'$etc.; 2° to take the integral of the product, for all the possible values of t, ν', etc. and, with respect to ν, to integrate only within the given limits; 3° to divide the whole by this same integral, taken with respect to all the possible values of t, ν, ν', etc. The unknown value of $\frac{2k''a^2s}{k}$ being able to vary from zero to infinity, the value of t is able to vary from $\frac{\mathrm{S}\epsilon'^{(i)^2}}{\sqrt{s}}$ to negative infinity, and as $\mathrm{S}\epsilon'^{(i)2}$ is of the order of s, t is able to vary from negative infinity to a positive value of order $\sqrt{s}$; the preceding exponential will become therefore, at the extremity of the integral taken with respect to t, of the form c^{-Q^2s}, and will be able to be neglected, because of the magnitude supposed to s; thus we can take the integral relative to t, from $t=-\infty$ to $t=\infty$. Similarly the integrals relative to ν, ν', etc. can be taken within the same limits. If we make

$$t+\frac{\mathrm{S}(p^{(i)}\nu+q^{(i)}\nu'+\text{etc.})^2}{s\sqrt{s}}=z;$$

the integral relative to z will be able to be taken with respect to z, from $z=-\infty$ to $z=\infty$.

Thence it is easy to conclude that the probability that ν is contained within the given limits, is equal to the integral

$$\int d\nu d\nu'\text{etc.}c^{-\frac{\mathrm{S}(p^{(i)}\nu+q^{(i)}\nu'+\text{etc.})^2}{2\mathrm{S}\epsilon'^{(i)2}}}\left\{1+\frac{\left[\mathrm{S}(p^{(i)}\nu+q^{(i)}\nu'+\text{etc.})^2\right]^2}{(2\mathrm{S}\epsilon'^{(i)2})^2s}\right\},$$

the integral being taken from ν', ν'', etc. equal to $-\infty$, to their infinite values and, with respect to ν, within the given limits, and being divided by the same integral extended to the positive and negative infinite values of ν, ν', ν'', etc.

The consideration of the difference between $\frac{2k''}{k}a^2s$ and $\mathrm{S}\epsilon'^{(i)^2}$introduces therefore, into the expression of the probability of which there is concern, only a term of order $\frac{1}{s}$, an order that I myself am permitted to neglect in the work cited, seeing the magnitude supposed to s.