LEÇONS DE MATHÉMATIQUES DONNÉES À L'ÉCOLE NORMALE EN 1795

P.S. Laplace*

Oeuvres complètes **XIV**, pp. 146–177

DIXIÈME SÉANCE
SUR LES PROBABILITÉS[1]

In order to follow the plan which I have traced in the program of the course of Mathematics, I must talk to you again of the integral and differential calculus in the differences, either finite, or infinitely small; of Mechanics, of Astronomy and of the theory of probabilities. The short length of the École Normale does not permit it of me; but I myself propose to supply it, relatively to Mechanics and Astronomy, by the publication of a work which will have for title *Exposition du système du Monde*, and in which I have presented, independently of Analysis, the series of the discoveries which have been made, to this day, on the system of the World. I will speak to you, in this last Lesson, on the theory of probabilities, an interesting theory by itself and by its numerous relationships with the most useful objects of society.

All events, even those which, by their smallness, seem not to depend on the grand laws of the Universe, are a sequence as necessary as the revolutions of the Sun. In ignorance of the links which unify them to the entire system of nature, we have made them to depend on final causes or on chance, according as they happened and succeeded themselves with regularity or without apparent order; but these imaginary causes have been successively moved back with the boundaries of our knowledge, and disappear entirely before sound philosophy, which sees in them only the expression of the ignorance that we have of the true causes.

We will be convinced of this important result of the progress of enlightenment if we recall that once an extreme rain or drought, a comet dragging after it a quite extended tail, eclipses, the aurora borealis, and generally all the extraordinary phenomena were regarded as so many signs of celestial anger. We invoked the sky in order to turn away their fatal influence; we prayed it not suspend the course of the planets and of the sun: the observation had soon been made to sense the uselessness of these prayers; but, because these phenomena, happening and disappearing at some long intervals and without apparent causes, seemed opposing to the order of nature, we supposed that

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. August 30, 2010

[1]This lesson is reproduced, with some extended developments, in the Introduction to the *Théorie analytique des probabilités*.

the sky gave birth to them and modified them at its will to punish the crimes of Earth. Thus the long tail of the comet of 1456 spread the terror into Europe already dismayed by the rapid successes of the Turks who came to overturn the late Roman Empire; and Pope Callixte ordered public prayers in which we implored the comet and the Turks. This star, after four of its revolutions, has excited among us an interest quite different. The knowledge of the laws of the system of the World, acquired in this interval, has dissipated the childish fears by the ignorance of the true relationship of man with the universe; and Halley having recognized the identity of the comet with those of the years 1531, 1607 and 1682, announced its next return for the end of 1758 or the beginning of 1759. The scholarly world awaited with impatience this return which must confirm one of the greatest discoveries which we had made in the sciences, and to accomplish the prediction of Seneca when he has said, in speaking of the revolution of these stars which descend from an enormous distance: "The day will come that, by a study following many centuries, the things actually hidden will appear with evidence, and posterity will marvel that some truths so clear had escaped us." Clairaut undertook then to submit to analysis the perturbations that the comet had experienced by the action of the two largest planets, Jupiter and Saturn. After immense calculations, he fixed its next passage at perihelion, near the beginning of April 1759; that which observation did not delay to verify. The regularity that Astronomy shows us in the movement of the comets holds, without any doubt, in all phenomena; the curve described by the slightest atom is regulated in a manner as certain as the planetary orbits; there is a difference between them only that our ignorance sets.

Probability is relative in part to this ignorance, and in part to our knowledge. We know that out of three or a greater number of events one alone must exist; but nothing sustains to believe that one of them will happen rather than the others; in this state of indecision it is impossible for us to pronounce with certitude on their existence. It is however probable that one of these events, taken at will, will not exist, because we see many equally possible cases which exclude its existence, while one alone favors it.

The theory of chances consists in reducing all events of the same kind to a certain number of equally possible cases, that is to say such that we are equally indecisive on their existence; and to determine the number of cases favorable to the event of which we seek the probability. The ratio of this number to the one of all possible cases is the measure of this probability, which is thus only a fraction of which the numerator is the number of the favorable cases, and of which the denominator is the number of all possible cases.

The preceding notion of probability supposes that by having belief in the same ratio of the number of favorable cases to the one of possible cases, the probability remains the same. In order to convince ourselves of it, let us consider two urns A and B, of which the first contains four white balls and two black, and of which the second contains only two white balls and one black. We can imagine the two black balls of the first urn tied by a thread which is broken at the moment when we seize one of them, and the four white balls form two similar systems. All the chances which make to seize one of the balls of the black system will bring a black ball. If we imagine now that the threads which unify the balls are not able to break, it is clear that the number of possible chances will not change, no more than the one of the chances favorable to the extraction of the black balls, only we will draw from the urn two balls at a time. The probability of extracting a black ball from the urn will be therefore the same as before; but then we have evidently the case of urn B, with the sole difference that the three balls of this last urn are replaced by three systems of two balls invariably united. Here the equally possible cases are not the extraction of the balls; these are the chances which bring them forth and of which the sum, supposed the same for each urn, is apportioned on six balls in the first and on three in the second. The just estimation of the equally possible cases is one of the most delicate points in the analysis of chances.

When all possible cases are favorable to an event, its probability is changed to certitude, and its expression must equal unity. Under this relation, certitude and probability are comparables, whatever be an essential difference between the two states of the mind; when a truth is rigorously demonstrated to it, or when it realizes yet a small source of errors.

In the things which are only probables, the difference of the data that each man has over them is one of the principal causes of the diversity of opinions that we see reign on the same object. We suppose, for example, that we have three urns A, B, C, of which one contains only black balls, while the others contain only some white balls; we must draw a ball from urn C and we demand the probability that this ball will be black. If we ignore which is the one of the three urns which contains only the black balls, so that we have no reason to believe that it is rather C than B or A, these three hypotheses will seem equally possible; and, as a black ball can be extracted only from the first, the probability of the extract is equal to a third. If we know that urn A contains only some white balls, the indecision carries further then only over the urns B and C, and the probability that the ball extracted from urn C will be black is a half; finally this probability is changed to certitude if we are assured that the urns A and B contain only some white balls.

It is thus that the same fact, recited before a numerous audience, obtains diverse degrees of belief, according to the extent of the knowledge of those who listen to it. If the man who reports it appears intimately persuaded and if his state and his virtues are proper to inspire a grand confidence, however extraordinary be his story, it will have, with respect to the audience devoid of knowledge, the same degree of probability as an ordinary fact reported by the same man, and they will credit to him an entire trust. However, if some one of them has had occasion to hear some contrary facts affirmed by some other equally respectable men, it will be in doubt; and the fact will be judged false by the enlightened audience who will find it opposed, either to some well sworn facts, or to the immutable laws of nature. What indulgence must we therefore not have for the different opinions of ours, since this difference often depends only on some diverse points of view where circumstances have placed us? We illuminate those who we judge insufficiently enlightened; but, previously, we examine severely our proper opinions, and we ponder with impartiality their respective probabilities.

The difference of opinions depends yet on the manner by which each determines the influence of the data which are known to him. The theory of probabilities is so difficult, it holds to some considerations so delicate, that it is not surprising that with the same data two persons find the results different, especially in the matters too complicated to be subjected to a rigorous calculus. The mind has its illusions as the sense of sight; and, in the same way touch rectifies this one, reflection and calculus correct equally the first. Probability founded on a daily experience, or exaggerated by fear or hope, strike us more than a superior probability which is only a simple analytic result; it will be therefore to desire that in all cases we can subject the probabilities to the calculus; but most often the thing is impossible, and we are forced to return ourselves to it by some perceived deceiver. Then, analogy, induction, a sane critique, a feeling given by nature and improved by some comparisons multiplied by its indications with experience, supply, as much as this is able, the applications of Analysis.

It is by analogy that we attribute similar effects to the same cause or to some similar causes, and reciprocally; thus we judge that of the beings provided with the same organs, performing the same things and communicating together, experience the same sensations. It is again thus that in seeing the Sun hatch, by the beneficent action of its light and its heat, the plants and the animals which cover the Earth, we judge that it produces some similar effects on the other planets; because it is not natural to think that the manner by which we observe the fecundity developed in such a fashion is sterile on a planet likewise as large as Jupiter which, as the terrestrial globe, has its days, its nights and its years, and on which the observations indicate some changes which suppose some very active forces. But it would be to give too great extension to the analogy to conclude the similitude of the inhabitants of the planets with those of the Earth. The man made for the temperature which he enjoys at its surface would not, according to all appearance, live on the other planets. But must he not have an infinity of organizations relative to the diverse temperatures of the worlds of this Universe? If the only difference of the elements and the climates puts so much variety in the terrestrial productions, how much more must differ those of the diverse planets and of their satellites? The most active imagination can form no idea of them; but their existence is at the least quite probable.

You have seen that often the laws of the analytic expressions manifest themselves in their first terms, and that those of nature are indicated by a small number of observations; the characteristic of the genius is to untangle them from the middle of the circumstances in which they are enveloped, and to expose them in a light such that it is impossible to disregard them. This way to arrive there is named *induction*; in order to increase the probability, we form new *terms*, or we make new observations, and, if the laws of which we have suspected the existence continue to satisfy them, they acquire a degree of probability which ends by confounding itself with certitude.

That which we observe in Analysis has place equally in nature, of which the phenomena are, in fact, only the mathematical results of a small number of invariable laws. In order to discover these laws, it is necessary to choose or give birth to phenomena more proper to this object, to multiply them in order to vary the circumstances, and to observe that which they have in common among them. Thus we are elevated to some relationships more and more extended, and we arrive finally to the general laws that we verify, either by some proofs or some direct experiences, when this is possible, or by examining if they satisfy all the known phenomena.

Such is the more certain method which can guide us in the search of the truth. We owe to it the most beautiful discoveries in the sciences; but its most sublime and most extended application is that which Newton has made to the system of the World, as you can see in the work which I have announced at the beginning of this lesson.

This system offers a remarkable example of a probability quite superior to that of a great number of historical facts on which we permit ourselves no doubt, but which, being no analogue at all to the probabilities of which we make habitual usage, is not generally sensed. Observation shows us the planets and their satellites describe some nearly circular orbits, and turning about themselves in the sense of the rotation of the Sun, and in some planes slightly inclined to its equator. If we apply the calculus to this extraordinary phenomenon, we find that there are some million of billions to wager against one that it is not at all due to chance, and that it depends on a general cause which, originally, embraces all the bodies of the planetary system, without exercising influence on the observed comets, because they move themselves in every sense and under every inclination to the solar equator. However, their orbits being quite eccentric, while those of the planets are nearly circular, it is natural to think that the same cause makes to vanish, at the origin, the orbits which introduced the nuances intermediary between a great and a small eccentricity. The cause which I have assigned elsewhere to these singular phenomena appear to me to be the sole which can satisfy to their entirety; but, this discussion being strange here, I limit myself to refer for this object to my *Exposition du système du Monde*.

One of the most delicate points of the theory of probabilities, and the one which furnishes most to the illusions, is the manner by which the probabilities increase or diminish by their mutual combinations. If the events are independent of one another, the probability of the existence of their entirety is the product of their particular probabilities. Thus the probability to bring forth one ace with a single die being one sixth, that to bring two aces by projecting two dice at once is a thirty-sixth. In fact, each of the faces of one being able to be combined with the six faces of the other, there are thirty-six possible cases among which one alone gives the two aces. Generally, the probability that a simple event under the same circumstances will happen in sequence a given number of times is equal to the probability of the simple event elevated to a power indicated by this number. Thus the successive powers of a fraction less than unity diminishing without ceasing, an event which depends on a sequence of probabilities quite great can become extremely small probably. We suppose that a fact is transmitted to us by twenty witnesses, in a manner that the first has transmitted to the second, the second to the third, and thus in sequence; we suppose further that the probability of each testimony is equal to nine-tenths; that of the fact will be less than one eighth, that is to say that there will be more than seven to one odds against that it is false. We can better compare this diminution of the probability only with the extinction of the clarity of objects by the interposition of many pieces of glass, a thickness not that considerable sufficing to steal the view of an object which a single piece permits perceiving in a distinct manner. The historians appear not to have paid enough attention to this degradation of the probability of the facts when they are seen to traverse a great number of successive generations; many historical events, reputed as certain, will be at least doubtful if we submitted them to this analysis.

In the purely mathematical sciences, the most remote consequences participate in the certitude of the principle from which they derive. In the application of Analysis to Physics, the consequences have all the certitude of facts or of experiences. But in the moral sciences, where each consequence is deduced from that which precedes only in a probable manner, these deductions are only probables, the chance of an error grows with their number and ends by surpassing the chance of the truth in the consequences very extended from the principle.

When the possibility of the simple events is known, the probability of the composite events can be determined by the theory of combinations; but the most direct and the most general method to arrive there consists in observing the law of the variation that it experiences by the addition of one or many simple events, and by making it depend on one equation in the ordinary or partial finite differences. The integral of this equation is the analytic expression of the sought probability. The theory of generating functions, which I have given besides in the *Mémoires de l'Académie des Sciences*, can be here of great use[2]. This theory has for object the relations of the coefficients of the powers of an indeterminate variable, in the development of a function of this variable, to the function itself. From the simple consideration of these relations there results, with an extreme facility, the integration of the equations in the ordinary or partial differences, the analogy of the powers and the differences, and generally the transport from the exponents of the powers to the characteristics which express the manner of being of the variables.

The theory of the generating functions is extended to the infinitely small differences; because, if we develop all the terms of one equation in the differences with respect to the powers of the supposed difference indeterminate, but infinitely small, and if we neglect the infinitely small of an order superior relatively to the those of an inferior order, we will have an equation in the infinitely small differences, of which the integral is that of the equation in the finite differences, in which we neglect likewise the infinitely small with respect to the finite quantities.

The quantities which we neglect in these passages from the finite to the infinitely small seem to remove from the infinitesimal calculus the rigor of the geometric results; but in order to render to it rigor it suffices to envision the quantities which we conserve in the development of an equation in the finite differences and of its integral, with respect to the powers of the indeterminate difference, as having all for factor the smallest power of which we compare the coefficients among them. This comparison being rigorous, the differential calculus, which is evidently only this same comparison, has all the rigor of the other algebraic operations. But the consideration of the infinitely small of different orders, the facility of recognizing them *a priori* by the inspection alone of the magnitudes, and the omission of the infinitely small of an order superior to the one which we conserve, in measure as they present themselves, simplifies extremely the calculus and are one of the principal advantages of the infinitesimal Analysis, which besides, in realizing the infinitely small and attributing to them very small values, gives, by a first approximation, the differences and the sums of the quantities.

The passage from the finite to the infinitely small has the advantage of illuminating many points of the infinitesimal Analysis which have been the object of great contests among the geometers. It is thus that, in the *Mémoires de l'Académie des Sciences* for the year 1779[3], I have shown that the arbitrary functions which the integration of partial differential equations introduce could be discontinuous, and I have determined the conditions in which this discontinuity must be subject. The transcendent results of the Analysis are, as all the abstractions of the understanding, some general signs of

[2] *Oeuvres de Laplace*, t. VIII to XII. These diverse Memoirs form the first part of the *Théorie analytique des probabilités*. t. VII.

[3] *Oeuvres de Laplace*, t. X, p. 59.

which we can determine the true extent only by ascending, by metaphysical Analysis, to the elementary ideas which have led there, this which presents often great difficulties; because the human mind tries less yet to carry itself into the future than to retire within itself.

It seems that Fermat, the true inventor of the differential Calculus, has considered this calculus as a derivation of the one of the finite differences, by neglecting the infinitely small of an order superior, with respect to those of an order inferior; it is, at least, that which he had made in his method *de Maximis* and in that of the tangents, which he has extended to the transcendent curves. We see still by his beautiful solution of the problem of the refraction of light, by supposing that it arrive from one point to another in the shortest time and by imagining that it is moved, in diverse diaphanous surroundings, with different speeds, we see, I say, that he knew to extend his calculus to the irrational functions, by being cleared of the irrationalities by the elevation of the radicals to the powers. Newton has since rendered this calculus more analytic in his *Method of Fluxions*, and he has simplified and generalized the process by the invention of his theorem of the binomial; finally, almost at the same time, Leibnitz has enriched the differential Calculus with a very fortunate notation and which is adapted of itself to the extension which the differential Calculus has received by the considerations of the partial differentials. The language of Analysis, the most perfect of all, being by itself a powerful instrument of discovery, its notations, when they are necessary and fortunately imagined, are the germs of new calculations. Thus the simple idea that Descartes had to indicate the powers of the quantities represented by some letters, by writing against the height of these letters the numbers which express the degree of these powers, has given birth to the exponential Calculus; and Leibnitz has been led by its notation to the singular analogy of the powers and the differences. The calculus of the generating functions, which give the true origin of this analogy, offers so many examples of this transport of the exponents from the powers to the characteristics, that it can yet be considered as the exponential calculus of the characteristics.

After this short digression which I have permitted myself to supplant, in some regards, the lessons that I should make on infinitesimal Analysis, I return to the probabilities: when the events that we consider are in very great number, the formulas to which we are led are composed of one so great multitude of terms and of factors that their numerical calculation becomes impractical. It is then indispensable to have a method which transforms these formulas into convergent series. I have given for this object, in the *Mémoires de l'Académie des Sciences*,[4] a method based on the transformation of the formulas of functions of very great numbers, into definite integrals which we integrate by some very convergent series; and there is this of the remarkable, namely, that the quantity under the integral sign is the generating function of the function expressed by the definite integral; so that the theories of the generating functions and of the approximations of the formulas of functions of very great numbers can be considered as the two branches of one same calculus which I designate under the name of *Calculus of the generating functions*.

By its means we can determine with facility the limits of the probability of the results and of the causes indicated by the events considered in great number, and the laws

[4]*Oeuvres de Laplace*, t. IX, X and XII.

according to which this probability approaches to its limits in measure as the events are multiplied. This research, the most delicate of the theory of chances, merits the attention of the geometers by the analysis which it requires, and that of the philosophers, by showing how the regularity ends by establishing itself in the same things which appear to us entirely delivered by chance, and by unveiling to us the causes hidden, but constant, on which this regularity depends. But I must here limit myself to present to you the principles and the general results of the theory of probabilities.

When two events depend on one another, the probability of the composite event is the product of the probability of the first of these events by the probability that, this event having arrived, the other will take place.

Thus in the preceding case of the three urns A, B, C, of which two contain only some white balls and of which one contains only some black balls, the probability of drawing a white ball from urn C is $\frac{2}{3}$, since two of the three urns contain only some balls of this color; but, when we have extracted a white ball from urn C, the indecision relative to that of the urns which contain only some black balls carries further only onto urns A and B, the probability of extracting a white ball from urn B becomes $\frac{1}{2}$; the product of $\frac{2}{3}$ by $\frac{1}{2}$ or $\frac{1}{3}$ is therefore the probability of extracting from urns B and C two white balls.

We see, by that which precedes, the influence of the past events on the probability of future events. Because the probability to extract a white ball from urn B, which originally is $\frac{2}{3}$, is reduced to $\frac{1}{2}$ when we have extracted a white ball from urn C; it would be changed into certitude if we have extracted a black ball from the same urn. We will determine this influence of the past events by means of the following principle:

If we calculate a priori *the probabilities of the arrived event and of an event composed of this one and of another which we await, the second probability divided by the first will be the probability of the awaited event deduced from the observed event*.

When the possibilities of the simple events are totally unknown, we determine *a priori* the probability of a composite event by giving successively to these possibilities all the values of which they are susceptible, and by taking a mean among the probabilities relative to each of these values. We find thus, for example, that by going go back five thousand years to the most ancient period of history, the Sun having risen constantly in this interval at each revolution of twenty-four hours, this is odds of eighteen hundred twenty-six thousand against one that it will rise in the following revolution. But this number is incomparably more strong for the one who, knowing through the assembly of the celestial phenomena the principle regulator of the days and of the seasons, sees that nothing can, in the actual moment, arrest the course of it.

Here is presented the question discussed by some philosophers touching the influence of the past on the probability of the future.

Suppose that in a game of *heads* and *tails* we have brought forth *heads* more often than *tails*; by this alone we will be led to believe that, in the constitution of the piece, there exists a constant cause which favors it, the past throws influence therefore then on the probability of the future events. Thus, in the conduct of life, good luck is often a proof of ability which lucky persons must make use preferably. But if, by the instability of circumstances, we are restored without ceasing to the state of an absolute indecision on that which must happen; if, for example, we change the coin at each throw in the game of heads and tails, the past can cast no light on the future, and it will be absurd to take account of it. We see thence that which it is necessary to think in these veins of good luck or bad luck that men imagine in order to explicate the constancy of some events which are favorable or contrary to them. They fall even in this regard into an evident contradiction, since in many cases, and especially in the lotteries, they judge that an event which for a long time has not arrived becomes more probable. This quite common error appears to me to hold to an illusion by which we report back involuntarily to the origin of events. It is, for example, very improbable that in the game of heads and tails we will bring forth heads ten times in sequence; this improbability, which strikes us still when it has arrived nine times, leads us to believe that at the tenth cast it will not happen. But, far from us to make judgment thus, the past, in appearing to indicate in the coin more tendency for heads than for tails, renders the first of these events more probable than the other; it increases consequently the probability to arrive to heads in the next cast. By extending generally this remark to the unknown, but constant, causes which favor the events, we find this remarkable result, namely, *that they increase always the probability of the events composed of the repetition of one same simple event without increasing however the probability of its first arrival, since we are supposed to ignore first the events which favor these causes*. Thus, in the game of heads and tails, the unknown inequality which, according to all probability, exists among the facilities of the two faces, increases not at all the probability to bring forth heads or tails at the first cast; but it increases the probability to bring forth one or the other two times in sequence, a probability which will be $\frac{1}{4}$ if the facilities of the two faces were perfectly equal.

In a great number of cases, and these are the most interesting in the analysis of chances, the possibilities of the simple events are unknown, and we are reduced to seek in the past events the indices which can guide us in our conjectures on the causes on which they depend. But in what manner do these events reveal to us, in developing themselves, their causes and their respective possibilities? This is a problem of which the solution requires a very delicate analysis. This analysis leads to the following theorem:

When an event, composed of many simple events, such as a part of a game, has been repeated a great number of times, the possibilities of the simple events which render that which we have observed the most probable are those that the observation indicates with the most probability; in measure as the composite event is repeated, this probability increases without ceasing and ends by being confounded with certainty, under the assumption of an infinite number of repetitions.

There are here two sorts of approximations; the one of them is relative to the limits taken on both sides of the possibilities which give to the past the most probability; the other approximation is related to the probability that these possibilities fall within these limits. The repetition of the composite event increases more and more this probability, the limits remaining the same; it tightens more and more the interval of these limits, the probability remaining the same; at infinity this interval becomes null, and the probability is changed to certitude. The same analysis leads yet to this other theorem:

If we multiply indefinitely the observations or the experiences, their mean result converges towards a fixed term, in a manner that by taking on both sides of this term an interval as small as we will wish, the probability that the mean result will fall in this interval will end by differing from certitude only by a quantity less than every assignable magnitude. This term is the very truth if the positive and negative errors are equally facile; and, generally, it is the abscissa of the curve of facility of the errors corresponding to the center of gravity of the area of this curve, the origin of the abscissas being that of the errors.

Thus, the mean result of a great number of future observations will be very nearly the same as the one of a great number of similar observations already made.

The events which depend on chance offer in their assembly a regularity which seems to hold to a design, but which is at base only the development of their respective possibilities. The ratio of the annual births of boys to those of girls, in the great cities such as Paris and London, is an example of it. This ratio is not very variable; we have belief to see in this constancy a proof of the Providence which governs the world; but it is only a result of the first of the preceding theorems, according to which this ratio must always coincide very nearly with the one of the facilities of birth of the two sexes. We can likewise conclude from it, as general law, that the ratios of the effects of nature, such as the one of the births in the population, or of the marriages to the births, are quite nearly constant when these effects are considered in great number. Thus, despite the great variety of the years, the sum of the productions, during a considerable number of years, is sensibly the same; so that a man can, by a useful foresight, be set sheltered from the irregularity of the seasons by spreading equally on all the times the benefits that nature distributes to him in an unequal manner. I do not except even from the preceding law the effects of moral causes: in Paris, the number of annual births, since a great number of years, has differed little from nineteen thousand; and I have hearsay that in the post, the number of letters sent to the junk, by defects of the addresses, was very nearly the same each year.

In the midst of the inconstancy of the phenomena which seem to depend the most on chance, there exists therefore some fixed ratios towards which they tend without ceasing, but which they can reach only at infinity. The research on these ratios and on the laws according to which the results of the phenomena approach themselves is one of the most interesting points of the theory of probabilities.

Each of the causes to which an observed event can be attributed is indicated with so much more probability as it is more probable that, this cause being supposed to exist, the event will take place; the probability of the existence of any of these causes is therefore a fraction of which the numerator is the probability of the event resulting from this cause, and of which the denominator is the sum of the same probabilities relative to all the causes.

This is the fundamental principle of this branch of Analysis of chances, which consists to carry up from the events to the causes.

This principle gives the reason for which we attribute the regular events to a particular cause. Some philosophers have believed that these events are less possible than the others and that in the game of heads and tails, for example, the combination in which heads arrives twenty times in sequence is less facile to nature than that where heads and tails are mixed in an irregular fashion. But this opinion supposes that the past events influence on the possibility of the future events, that which is not at all admissible. The regular combinations arrive rarely only because they are less numerous. If we seek a cause there where we perceive from the symmetry, it is not that we regard a symmetrical event as being less possible than the others; but, this event must be the effect of a regular cause or the one of chance, the first of these suppositions is more probable than the second. We see on a table some characters of print disposed in this order, *Constantinople*, and we judge that this arrangement is not the effect of chance, not because it is less possible than the others, since, if this word were not employed in any language, this arrangement will be neither more nor less possible in itself, and however we would suspect to it then no other particular cause; but, this word being in use among us, it is incomparably more probable that a person will have thus disposed the preceding characters than it is that this arrangement is due to chance.

Thence we must generally conclude that, the more a fact is extraordinary, the more there it has need to be supported by strong proofs; because, those who attest to it being able either to deceive or to have been deceived, these two causes are so much more probable than the reality of the fact is less in itself. There are some things so extraordinary that nothing can, in the eyes of enlightened men, balance the improbability of it. But this, by the effect of a dominant opinion, can be reduced to the point of appearing inferior to the probability of the witnesses; and when this opinion just changed, an absurd story, admitted generally in the century which has given birth to it, offers to the centuries following only a new proof of the great influence of opinion on the better minds.

Having to pronounce on the existence of a cause which seems indicated by the observed events, it is necessary to determine its probability resulting from these events; alternately, one would be exposed with respect to a constant cause this regularity which affects sometimes the events due to chance, and which no longer continues when they are very multiplied; but this distinction requires an entirely peculiar analysis. By applying it with respect to the births of boys to those of girls observed in the diverse parts of Europe, we find that this ratio, everywhere nearly the one of 22 to 21, indicates with an extreme probability a greater facility in the births of boys. If we consider next that it is the same in Naples as in Petersburg, we will see that in this regard the influence of the climate is insensible. We could therefore suspect, contrary to common opinion, that this superiority of masculine births subsists in the Orient even. I had, in consequence, invited the French scholars sent into Egypt to make some researches on this interesting question; but the difficulty to obtain some precise information on the births has not permitted their resolution of it.

The registrations of births can serve to determine the population without recourse to the denumeration of the inhabitants; but it is necessary, for this, to know the ratio of the population to the births. The method to arrive to it most exactly consists: 1° to choose many townships in each department in order to have a mean among the small differences which the local causes bring to the results; 2° to make the denumeration of the inhabitants of these townships at a given period; 3° to determine, by the summary of the births during many years which precede or follow this period, the corresponding number of the annual births. This number, divided by the one of the inhabitants, will give the ratio of the births to the population, in a manner so much more precise as the denumeration will be more considerable. We find, by the analysis of the chances, that this denumeration must be elevated to twelve or fifteen hundred thousand inhabitants, in order to have a great probability that the errors of the entire population of France, determined by the births, will be contained in some narrow limits. The Government, convinced of the utility of a similar denumeration, has well wanted to order the execution of it, at my request. In thirty departments distributed over the surface of France, we have chosen some townships which can give the most precise information. They have furnished, for the 1st vendémiaire[5] year XI, some denumerations of which the sum is elevated to 2037615 individuals. The summary of the births, of the marriages and of the deaths, during the years VIII, IX and X, has given for the three years:

Births	Marriages	Deaths
110 312 boys	46 037	103 659 males
105 287 girls		99 443 females

The ratio of the population to the annual births is therefore $28\frac{3528}{10000}$; it is therefore greater than we have estimated it until here. The ratio of the births of the boys to those of the girls, which this summary presents, is the one of 22 to 21; and the marriages are to the births as 3 to 14.

In Paris, the baptisms of the infants of the two sexes deviate a little from the ratio of 22 to 21. Since the beginning of 1745, a period in which we have commenced to distinguish the sexes in the registers of the births, to the end of 1784, we have baptized in this great city 393 386 boys and 377 555 girls. The ratio of the two numbers is very nearly the one of 25 to 24; it appears therefore that in Paris a particular cause brings nearer to equality the baptisms of the two sexes; and, if we apply to this object the Calculus of probabilities, we find that there are odds about 238 against 1 in favor of its existence, this which suffices to authorize the research of it. Now I have suspected that the difference observed in this regard between Paris and the rest of France can be held to this that, in the country and in the provinces, the parents, finding some advantage to retain near them the boys, had sent to the hospice[6] of the found infants in Paris in a ratio less than the one of the births of the two sexes. It is that which the summary of the registers of this hospice has shown with evidence. Since the beginning of 1745 to the end of 1809, there are entered 159 405 girls and 163 499 boys; and this last number exceeds only by $\frac{1}{38}$ the preceding, which it would have ought to surpass by $\frac{1}{21}$, according to the ratio observed of the births. This which achieved to confirm the assigned cause, it is that, if we have no regard at all to the found infants, the ratio of the two sexes, in Paris, is the one of 22 to 21, as in the departments.

The probability of the events serves to determine the expectation[7] and the fear of the persons interested in their existence. The word *expectation* has diverse meanings:

[5] *Translator's note*: Year I of the first French republic extended from 22 September 1792 to 21 September 1793. Year XI thus spanned 22 September 1803 to 21 September 1804. This is the first month of the calendar.

[6] *Translator's note*: Foundling hospital.

[7] *Translator's note*: I have rendered *expérance* as expectation rather than hope.

it expresses generally the advantage of the one who awaits some benefit, under an assumption which is only probable. In the theory of chances, this advantage is the product of the expected sum by the probability to obtain it; this is the partial sum which must arrive, when we wish not at all to incur the risks of the event, by supposing that the distribution of the entire sum is made proportionally to the probabilities. This manner of distributing it is the sole equitable one, when we set aside all the strange circumstances, because with an equal degree of probability we have an equal right to the expected sum. We will name this advantage *mathematical expectation*, in order to distinguish it from *moral expectation*, which depends, as it, on the expected sum and on the probability to obtain it, but which is regulated still on a thousand variable circumstances which it is nearly impossible to define and furthermore to subject to the Calculus. These circumstances, it is true, make only to increase or diminish the value of the expected benefit; then we can consider the moral expectation itself as the product of this value by the probability to obtain it; but we must distinguish, in the expected benefit, its value relative to its absolute value. This is independent of the motives which make it desired, instead that the first increases with these motives.

We cannot give some general principle in order to estimate this relative value. There is however one proposed by Daniel Bernoulli, and which can serve in many cases. *The relative value of an infinitely small sum is equal to its absolute value divided by the total wealth of the interested person.* In fact, it is clear that 1 franc having little of value for the one who possessed a great number of them, the most natural manner to estimate its relative value is to suppose it in ratio inverse to this number.

By applying Analysis to this principle, we arrive to diverse results corresponding to the indications of common sense, but that we can estimate by this means with some exactitude. Such is this rule dictated by prudence, and which consists in exposing his fortune by parts to some dangers independent from one another, rather than to expose the entire whole to the same danger. There results next from the same principle that in the most equal game the loss is always relatively greater than the gain. Thus, we find that by supposing the fortune of the players of 100 francs and their stake in the game of 50 francs, their fortune is found reduced to 87 francs; the game is therefore disadvantageous in the same case when the loss is equal to the product of the expected sum by the probability to obtain it. We can judge thence the immorality of the games in which the promised sum is below this product: they subsist only by the false reasonings and the cupidity which they foment and which, leading the people to sacrifice its necessity to some chimerical expectations of which it is out of someone's power to estimate the improbability, are the source of an infinity of evils.

There exists, in the repetition of an advantageous event, a fixed term towards which the mean benefit converges in measure as the event is multiplied. The real benefit is more and more probable and increases without ceasing; it becomes certain under the hypothesis of an infinite number of repetitions and, by dividing it by their number, the quotient is the mathematical expectation itself or the relative advantage to each event. It is likewise of the loss, which becomes certain at length, if the event is disadvantageous in the least.

This theorem on the benefits or the losses is analogous to those that we have given previously on the ratios which indicate the indefinite repetitions of the simple or composite events; and, as them, they prove that the regularity ends by being established in the things most subordinated to that which we name *chance*.

We have constructed some Tables of mortality which present thoroughly this troubling result, namely: that the half of the human race perishes before having terminated its twentieth year. The manner of forming these Tables is very simple. We take from the registers of the births and of the deaths a great number of infants who we follow during the course of their life, by determining how many there remain of them at the end of each year of their age, and we write this number opposite each concluding year. But, as in the two or three first years of life the mortality is very rapid, it is necessary, for more exactitude, to indicate in this first age the number of survivors at the end of each half-year The diverse states of life offer, in regard to mortality, some very sensible differences, relative to the hardships and to the inseparable dangers of each state, and of which it is indispensable to take account in the calculus based on the duration of life. But these differences have not yet been sufficiently determined. They will be one day: then we will know what sacrifice of life each profession requires, and we will profit from this knowledge in order to diminish the dangers of them.

If we divide the sum of the years of life of all the individuals considered in a Table of mortality by the number of these individuals, we have the mean duration of life, which we find thus of twenty-eight years and a half. The mean duration of that which remains yet to live, when we are arrived to any age, is determined by making a sum of the years who have lived to the end of this age all the individuals who have attained it, and by dividing it by the number of these individuals. It is not at all at the moment of birth that this duration is greatest; it is when one has escaped from the dangers of the first childhood, and then it is around forty-three years. The probability to arrive to any age, by departing from a given age, is equal to the ratio of the two numbers of individuals indicated in the Table at these two ages.

We imagine that the precision of these results require that we consider a very great number of births; but the analysis of the probabilities shows us that they approach without ceasing to the truth, with which they finish by coinciding, when the number of the births considered becomes infinity.

We have observed that there exist more women than men, while there are born more boys than girls. Now, in the countries where the population is constant, the ratio of the population to the annual births is equal to the number of years of the mean duration of life; this duration is therefore greater for the women than for the men, either by virtue of their constitution, or because they are exposed less to dangers.

It is clear that the mean duration of life will be increased if the wars became more rare, if ease were greater and more general and if, by any means, men happened to render more salubrious the soil which he inhabits and to diminish the number and the dangers of the maladies. This is that which he has done in regard to the smallpox, one of the most destructive curses of the human species. Daniel Bernoulli has found, by an ingenious application of the Calculus of probabilities, that inoculation increases sensibly the mean life, by supposing even that an inoculation perishes one out of two hundred; it is therefore not doubtful that it is advantageous to the State. But the one who wishes to be inoculated must compare the very small danger; but near, to die from it to the much greater danger, but more extended, to die of the natural smallpox; and, while the consideration of the proximity of the danger is null for the State, which envisions only the mass of citizens, it is not for individuals. However, the well conducted inoculation causes to die so small a number of persons, and the ravages of the natural smallpox are so considerable, that the particular interest joins itself to the one of the State in order to adopt this method. The father of a family, of whom the attachment for his infants increases with them, must not at all hold undecided to submit them to an operation which delivers them from the anxiety and from the dangers of a too cruel malady, and which assures to him the fruit of his care and of their education. I hesitate not at all therefore to recommend the salutary practice of inoculation and to regard it as one of the most advantageous results Medicine has drawn from experience.[8]

We have founded, on the Tables of mortality, divers establishments, such as the life annuities and the tontines; but the most useful of these establishments are those in which, by means of a slight sacrifice of his revenue, one assures the existence of his family for a time when he must fear of no longer being able to be sufficient to their needs. As much as the game is immoral, so much these establishments are advantageous to the morals by favoring the softest tendencies of nature. Besides, of the funds which, by their smallness, would be sterile between the hands of each particular, they become productive and nourish the commerce in the great establishments which receive them and which, by the multitude of these capitals, produce a certain benefit when they are well conceived and sagely administered. They offer not at all the inconvenience which we have remarked in even the most equitable games, the one to render the loss more sensible than the gain, since to the contrary they give the means to exchange the superfluous against some assured resources in the future. The Government must therefore encourage these establishments and to respect them in their vicissitudes; because the expectations which they present carry on an extended future, they can prosper only sheltered from all anxiety on their duration.

The most general and the simplest method to calculate the benefits and the charges of these establishments consists in reducing them to actual capitals to the mean by this principle: *The actual capital equivalent to a sum, which must be probably paid only after a certain number of years, is equal to that sum multiplied by the probability that it will be paid in this period, and divided by unity increased by the rate of interest raised to a power equal to the number of these years.* The annual interest of the unit is that which we name *rate of interest*.

It is easy to apply this principle to the life annuities on one or many heads, and to the savings banks and to assurance companies, of any nature. We suppose that we propose to form a Table of life annuities according to a given Table of mortality. A life annuity payable, for example, at the end of five years, and reduced to actual capital, will be, by this principle, equal to the product of the following two quantities, namely: the pension divided by the fifth power of unity increased by the rate of interest; and the probability of paying it: this probability is the inverse ratio of the number of persons at the age of the one who constitutes the pension to the number of persons living at this age increased by five years. By forming therefore a sequence of fractions of which the denominators are the products of the numbers of the persons indicated in the Table of

[8]Since the publication of his lessons, all the fears of inoculation that the smallpox left yet have been dissipated by the inestimable discovery of the vaccine, to which we are indebted to Jenner, who, thence, has rendered himself one of the greatest benefactors of the human species.

mortality, as living to the age of the one who constitutes the pension, by the successive powers of unity increased by the rate of interest, and of which the numerators are the products of the pension, by the number of persons living to the same age, increased successively by one year, two years, . . ., the sum of these fractions will be the capital required for the life annuity to this age.

We suppose now that a person wishes, by means of a life pension, to assure to her heirs a capital payable to the end of the year of her death. In order to determine the value of this pension, we can imagine that the person borrows in life, in a case of assurance, this capital divided by unity increased by the rate of interest, and that she places it at perpetual interest in the same fund. It is clear that this capital will be owed by the fund to her heirs at the end of the year of her death; but she will have paid each year only the excess of the life interest on the perpetual interest; the Table of life-annuities makes therefore known this that the person must pay annually to the fund in order to assure this capital after her death.

The maritime assurances are calculated on the same principles. A merchant has some vessels at sea; he wishes to assure their value and that of their cargo against the dangers which they can incur; for this, he gives a sum to a company which responds to him with the estimated value of his cargos and of his vessels. The ratio of this value to the sum which must be given for the price of the assurance depends on the dangers to which the vessels are exposed, and can be estimated only by some numerous observations on the sort of vessels departed from the port for the same destination. But these establishments and all those of the same genre, such as the assurances against fires and storms, can succeed only so much as they have a proper advantage to provide to the expenses which they carry away. It is necessary moreover that they have very numerous relations, so that this advantage developing from it produces a certain benefit and makes coincide their mathematical and moral expectation.

There remains to me to speak to you of the means that it is necessary to choose among results of observations and of the probability of the decisions of the assemblies.

When we wish to correct by the assembly of a great number of observations one or many elements already quite nearly known, we form in the following manner some equations which we name *equations of condition*.

The analytic expression of each observation being a function of the elements, we substitute the approximate value of each of them, plus its correction; by developing next the expression into series and by neglecting, because of their smallness, the squares and the products of the corrections, we equate the series to the observation which it represents; we have thus an equation of condition among the corrections of the elements. Each observation furnishes a similar equation of condition. If the observations were exact, it would suffice to have from them a number equal to the one of the elements; but, seeing the errors of which they are always susceptible, we consider a great number of them, so that the errors compensate themselves quite nearly in the mean result. The observer must choose the most favorable circumstances to the determination of the elements; the art of the calculator consists in combining in the most advantageous manner the equations of condition, furnished by the observations, in order to reduce them to a number equal to the one of the elements. All the combinations which we can make come back to multiplying respectively each equation by a particular factor and to make a sum of all these products; this which gives a first final equation relative to the system of factors employed. A second system of factors will give a second final equation, and thus in sequence, until we have as many final equations as there are elements. It is clear that we will have the most precise corrections if we choose the systems of factors such that the mean error to fear more or less on each element is a minimum; by *mean error* we must understand the sum of the products of each error to fear by its probability. The research on this minimum, one of the most useful of the theory of probabilities, requires some singular artifices of analysis. We will limit ourselves here to say that we are led to this remarkable result, namely, that the most advantageous manner to combine the equations of condition consists in rendering minimum the sum of the squares of the errors of the observations; this which furnishes as many final equations as there are corrections to determine.

The probability of the decisions of an assembly depend on the plurality of the votes, on the enlightenment and on the impartiality of the members who compose it. So much of passions and of particular interests mix often their influence in it, that it is impossible to submit this probability to the Calculus. Here is however a general result to which we are led by Analysis. If the assembly is not very enlightened on the object submitted to its decision, if this object requires some delicate considerations and in the range of the smallest number; or if the truth on this point is contrary to some received prejudices, so that it had greater odds than one against one that each voter will deviate from it, it will be probable that the reason will be on the side of the minority; and the more the assembly will be numerous, the more it will take place to fear that the decision of the majority is wrong. It will be the contrary if the assembly is composed of instructed men. We imagine, for example, one hundred persons assembled indistinctly, and we propose to them to resolve on this question: *Does the Sun revolve, each day, about the Earth?* There is every place to believe that the decision of the majority will be for the affirmative, and this one will become the more probable again if, instead of one hundred persons, you suppose one thousand or ten thousand of them reunited. Thence we can deduce this consequence, dictated by simple good sense: it is that it signifies extremely to the state that the instruction is quite widespread and that the national representation is the elite of the just and enlightened men. *Vérité, justice, humanité*, here are the eternal laws of the social order which must rest uniquely on the true relationships of man with his assemblies and with nature; they are as necessary to his maintenance as universal gravitation to the existence of physical order; the most dangerous of the errors is to believe that we can sometimes be separated from them and to deceive or subdue the men for their proper happiness; fatal experiences have proved, in every time, that these sacred laws are never violated with impunity.

It is often difficult to know and even to define the view of an assembly, in the middle of the variety of the opinions of its members. We try to give on this some rules, and we consider the two most ordinary cases, the election among many candidates and that among many propositions relative to the same object.

When an assembly must choose among diverse candidates who present themselves for one or many places of the same genre, that which appears the simplest consists in making each voter write, on a ticket, the names of all the candidates in the order of merit which he attributes to them. By supposing that he classifies them in good faith, the inspection of the tickets will make known the results of the elections, in some manner that the candidates are compared among them, so that anew elections can be taught nothing more in this regard. The question is presently to conclude from them the order of preference that they establish among the candidates. We imagine that we give to each elector an urn which contains an infinity of balls by means of which he may nuance all the degrees of merit of the candidates; we imagine again that he draws from his urn a number of balls proportional to the merit of each candidate, and we suppose this number written on his ticket beside the name of the candidate. It is clear that by making a sum of all the numbers relative to each candidate, on each ticket, the one of all the candidates who will have the greatest sum will be the candidate who the assembly prefers, and that in general the order of preference of the candidates will be the one of the sums relative to each of them. But the tickets indicate not at all the number of balls that each elector gives to the candidates; they indicate only that the first of them has more of them than the second, the second more than the third, and thus in sequence. By supposing therefore to the first, on a given ticket, any number of balls, all the combinations of the inferior numbers, which fill the preceding conditions, are equally admissible, and we will have the number of balls relative to each candidate, by making a sum of all the numbers which give him each combination and by dividing it by the entire number of combinations. If these numbers are very considerable, as we must suppose in order that they can express all the nuances of merit, a quite simple analysis shows that the numbers which it is necessary to write on each ticket beside the first name, the second name, are among them as the following: 1 ° the number of the candidates; 2 ° this number diminished by one unit; 3 ° this number diminished by two units, etc. It suffices therefore to write on each ticket these last numbers and to add the numbers relative to each candidate on all the tickets; these diverse sums will indicate, by their magnitude, the order of preference which must be established among the candidates. We will simplify the calculus by writing on each ticket, zero, beside the last candidate and the numbers 1, 2, 3, . . . respectively beside the superior candidates. Such is the mode of election that the theory of probabilities indicates. It will be without doubt the better, if each elector wrote down on his list the names of the candidates according to the order of merit which he supposes them: but the passions, the particular interests and many strange considerations to the merit must often trouble this order and make placement at the last rank the rival most to fear for the one who we prefer; this which, by giving a great advantage to the rivals of a mediocre merit, renders this mode of election inferior to those which we employ commonly.

The choice among many propositions relative to the same object seems to must be subject to the same rules as the election among many candidates; however there exists between these two cases this essential difference, that the merit of a candidate excludes not at all that of his rivals; instead that, if the propositions among which it is necessary to choose are contraries, the truth of the one excludes the truth of the others. Here is how we can then envision the question.

We give again to each voter an urn which contains a very great number of balls, and we imagine that he distributes them on each proposition by reason of the probability that he supposes to it. It is clear that the total number of the balls expressing the certitude, and the voter being, by hypothesis, assured that one of the propositions is true, he must apportion the number of the balls from the urn on these diverse propositions; the problem is reduced therefore to determine the comparisons in which all the balls are apportioned on the propositions, in a manner that there are more on the first than on the second, more on the second than on the third, ...; to make the sums of all the numbers of balls, relative to each proposition in the diverse combinations and to divide these sums by the number of combinations; the quotients will be the number of balls which we must attribute to the propositions on any ticket. We find thus, by analysis, that these quotients, by departing from the last proposition in order to go up to the first, are among them as the following quantities: 1° unity divided by the number of propositions; 2° unity divided by the number of propositions, plus unity divided by this number diminished by one; 3° unity divided by the number of propositions, plus unity divided by this number diminished by one; plus, unity divided by the same number diminished by two, and thus of the rest; we will write therefore these quantities on each ticket, beside the corresponding propositions and, by adding the quantities relative to each proposition on the diverse tickets, the sums will indicate by their magnitude the order of preference that the assembly gives to these propositions.

I just examined most of the objects to which we have until now applied the calculus of probabilities. We can, by taking account of all the results of observation and of experience, extend these applications and thus improve the political economy. The questions that this science presents are so complicated; they hold to so much of inestimable or unknown elements, that it is impossible to resolve them *a priori*. We can have in their regard only of the perceived, and the calculus, in the matters which are susceptible of them, shows us how they are deceivers. We treat the economy, as we have treated physics, by the way of experience and of analysis. We consider, besides, the great number of truths that this method has made discovered on nature and, of the other, the multitude of errors that the mania of the systems has produced; you know then the necessity to consult experience in all. It is a slow guide, but always sure; by abandoning it, we expose ourselves to the most dangerous gaps.

If we consider the analytic methods to which the theory of probabilities has already given birth and those which it can yet made be born, the justice of the principles which serve it at base, the rigorous logic that requires their use in the solution of problems, the great number and importance of the objects which it embraces, the establishment of public utility which is supported on it; if we observe next that, in the same things which can not be submitted to the calculus, this theory gives the surest appearances which can guide us in our judgments and which it teaches to steer clear of some illusions which often lead us astray, we will see that there is no science at all more worthy of our meditations and of which the results are more useful. It owes the birth to two French geometers of the XVII$^{\text{th}}$ century, so fecund in great men and in great discoveries, and perhaps the one of all the centuries which gives the most honor to the human spirit. Pascal and Fermat proposed and resolved themselves some problems on the probabilities. Huygens united these solutions and extended them in a small Tract on this matter, which next had been considered in a more general manner by Bernoulli, Montmort, Moivre and by many celebrated geometers of these last times.