RECHERCHÉS
Sur le calcul intégral aux différences infiniment petìtes, & aux différences finies.[*]

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I.

Among the great number of differential equations which we encounter in the resolution of the problems where the concern is to apply the calculus to nature, the most ordinary, & the most remarkable are comprehended under the general form.

$$X = y + H \cdot \frac{dy}{dx} + H' \cdot \frac{d^2y}{dx^2} + H'' \cdot \frac{d^3y}{dx^3} \cdots + H^{n-1} \cdot \frac{d^ny}{dx^n}$$

X, H, H', H'', &c. being some functions any whatever of the variable x, of which the difference is supposed constant. It would be therefore very important to have a general method to resolve them, & there is no doubt that mechanics, & more particularly yet physical astronomy may derive great advantages from it.

Messrs. d'Alembert, & Euler have resolved a long time ago the case where we have H, H', H'' &c. constants; the first, by his fine method of undetermined coefficients which is surely one of the most ingenious, & of the most fertile of analysis, the second, by a very fine consideration, & which is of the greatest usage in the integral calculus. In the third volume of his memoirs, we find some profound researches of Mr. de la Grange, in which this great Geometer integrates the case where we have,

$$H = A(x + h)$$
$$H' = A'(x + h)^2$$
$$H'' = A''(x + h)^3$$
$$\&c,$$

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[*]*Translator's note*: The notation used by Laplace is very awkward, but has been retained as far as possible. As is usual with papers of this period, the typesetting was poorly done. Characters and accents are omitted, put in the wrong place, or inverted. Consistent use of upper and lower case is lacking. A raised dot, often used to represent multiplication, also seems to be an artifact of setting space into formulas. Symbols of grouping are not matched. It is often not clear if Laplace intended to begin a new sentence or not. Therefore, in the interest of making the paper more readable, I have taken the liberty to display equations which were in-line, correct what appear to be "obvious" errors of language and notation, and break long sequences of material into sentences. With very few exceptions, there will be no indication that a correction has been made.

[†]Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. August 14, 2010

A, A' &c., & h, being some constants. He shows moreover that this equation

$$X = y + H\frac{dy}{dx} + H' \cdot \frac{d^2y}{dx^2} + \text{&c.}$$

is generally integrable in the same case as this

$$0 = y + H\frac{dy}{dx} + H' \cdot \frac{d^2y}{dx^2} + \text{&c.}$$

This beautiful theorem of which Mr. d'Alembert has given in the same volume a quite simple demonstration, is a very important step towards the general resolution of this kind of equations. Mr. Bezout has made long since on these equations some analogous remarks which he has since given in the fourth volume of his course of mathematics.

Here is presently a method which has led me not only to the demonstration of this theorem, but moreover to find immediately the integral of the first of these equations when we have that of the second. This method is not limited moreover to the infinitely small differences, we will see, hereafter that it is applicable equally well to the finite differences.

Remark.

By a particular integral of a differential equation between x, & y, we can understand, either a function of, x, which substituted for, y, in this equation makes vanish all the terms or else a parallel function of, x, which moreover is comprehended in the general integral of this equation by determining in a certain manner the arbitrary constants that the integration introduces there[1]; because Mr. Euler has shown that the first of these two properties very well subsist in the second. It is in the first sense that I will take for gold by having the particular integral of a differential equation.

PROBLEM I.

Let it be proposed to integrate the differential equation

$$X = y + H\frac{dy}{dx} + H'\frac{ddy}{dx^2} \cdots + H^{n-1} \cdot \frac{d^ny}{dx^n} \tag{A}$$

X, H, H' &c. being some functions any whatever of, x, &c. dx being supposed constant.

Solution.

Let there be,[2]

$$w\frac{dy}{dx} + y = T \tag{B}$$

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[1] *Translator's note*: "…que l'integration y introduit;" I have read this rather as "que l'integration y introduit."

[2] Equation (*B*) is actually written as œ$\frac{dy}{dx} + y = T$. But the symbol œ seems to have been used in error.

w, & T, being some functions of, x; this equation differentiated successively becomes

$$w \cdot \frac{ddy}{dx^2} + \left(\frac{dw}{dx} + 1\right) \cdot \frac{dy}{dx} = \frac{dT}{dx}$$

$$w \cdot \frac{d^3y}{dx^3} + \left(\frac{2dw}{dx} + 1\right) \cdot \frac{ddy}{dx^2} + \frac{ddw}{dx^2} \cdot \frac{dy}{dx} = \frac{ddT}{dx^2}$$

$$w \cdot \frac{d^4y}{dx^4} + \left(\frac{3dw}{dx} + 1\right) \cdot \frac{d^3y}{dx^3} + \frac{3ddw}{dx^2} \cdot \frac{ddy}{dx^2} + \frac{d^3w}{dx^3} \cdot \frac{dy}{dx} = \frac{d^3T}{dx^3}$$

$$w \cdot \frac{d^5y}{dx^5} + \left(\frac{4dw}{dx} + 1\right) \cdot \frac{d^4y}{dx^4} + \frac{6ddw}{dx^2} \cdot \frac{d^3y}{dx^3} + \frac{4d^3w}{dx^3} \cdot \frac{d^2y}{dx^2} + \frac{d^4w}{dx^4} \cdot \frac{dy}{dx} = \frac{d^4T}{dx^4}$$

$$\cdots$$

$$w \cdot \frac{d^ny}{dx^n} + \left(\frac{n-1}{1} \cdot \frac{dw}{dx} + 1\right) \cdot \frac{d^{n-1}y}{dx^{n-1}} + \frac{\overline{n-1 \cdot n-2}}{1.2} \cdot \frac{ddw}{dx^2} \cdot \frac{d^{n-2}y}{dx^{n-2}} + \&c. \cdots$$

$$+ \frac{d^{n-1}w}{dx^{n-1}} \cdot \frac{dy}{dx} = \frac{d^{n-1}T}{dx^{n-1}}$$

I multiply the first of these equations by, w', the second by w''. the third by w''', &c. I add them with the equation (B), this which gives

$$T + w' \cdot \frac{dT}{dx} + w'' \cdot \frac{d^2T}{dx^2} + w''' \cdot \frac{d^3T}{dx^3} \cdots + w^{n-1} \cdots \frac{d^{n-1}T}{dx^{n-1}}$$

$$= y + \frac{dy}{dx} \cdot \left(w + w' + w'\frac{dw}{dx} + w'' \cdot \frac{ddw}{dx^2} + w''' \cdot \frac{d^3w}{dx^3}\right.$$

$$\cdots + w^{n-1} \cdot \frac{d^{n-1}w}{dx^{n-1}}\Big) + \frac{ddy}{dx^2}\left(ww' + w'' + 2w'' \cdot \frac{dw}{dx}\right.$$

$$+ 3w'''\frac{d^2w}{dx^2} + 4w^{iv} \cdot \frac{d^3w}{dx^3} \cdots \frac{n-1}{1}w^{n-1} \cdot \frac{d^{n-2}w}{dx^{n-2}}\Big) + \frac{d^3y}{dx^3} \cdot$$

$$\left(ww'' + w''' + 3w''' \cdot \frac{dw}{dx} + 6w^{iv} \cdot \frac{ddw}{dx^2} + 10w^v \cdot \frac{d^3w}{dx^3} \cdots \right. \qquad (C)$$

$$+ \frac{\overline{n-1 \cdot n-2}}{1.} \cdot w^{n-1}\frac{d^{n-3}w}{dx^{n-3}}\Big) \cdots$$

$$+ \frac{d^{n-2}y}{dx^{n-2}}\left(ww^{n-3} + w^{n-2} + \frac{n-2}{1} \cdot \frac{dw}{dx} + \frac{\overline{n-2 \cdot n-1}}{1.2} \cdot \right.$$

$$w^{n-1} \cdot \frac{ddw}{dx^2}\Big) + \frac{d^{n-1}y}{dx^{n-1}} \cdot \left(ww^{n-2} + w^{n-1} + \frac{n-1}{1} \cdot w^{n-1} \cdot \frac{dw}{dx}\right)$$

$$+ \frac{d^ny}{dx^n} \cdot ww^{n-1}.$$

by comparing this equation with equation (A), we will have $1°$ this.

$$X = T + w' \cdot \frac{dT}{dx} + w'' \cdot \frac{d^2T}{dx^2} \cdots + w^{n-1} \cdot \frac{d^{n-1}T}{dx^{n-1}}$$

2 ° the following.

$$ww^{n-1} = H^{n-1}$$

$$ww^{n-2} + w^{n-1} + \frac{n-1}{1} \cdot w^{n-1} \cdot \frac{dw}{dx} = H^{n-2}$$

$$ww^{n-3} + w^{n-2} + \frac{n-2}{1} \cdot w^{n-2} \cdot \frac{dw}{dx} + \frac{\overline{n-1 \cdot n - 2}}{1.2} \cdot w^{n-1} \cdot \frac{ddw}{dx^2} = H^{n-3} \text{ \&c.}$$

whence we will conclude the following

$$w^{n-1} = \frac{H^{n-1}}{w}$$

$$w^{n-2} = \frac{H^{n-2}}{w} - \left(1 + \frac{n-1}{1} \cdot \frac{dw}{dx}\right) \cdot \frac{H^{n-2}}{ww}$$

$$w^{n-3} = \frac{H^{n-3}}{w} - \left(1 + \frac{n-2}{1} \cdot \frac{dw}{dx}\right) \cdot \left(\frac{H^{n-2}}{ww} - \right.$$

$$\left[1 + \frac{n-1}{1} \frac{dw}{dx}\right] \cdot \frac{H^{n-1}}{w^3}\bigg) - \frac{\overline{n-1 \cdot n - 2}}{1 \cdot 2} \cdot \frac{ddw}{dx^2} \cdot \frac{H^{n-1}}{ww} \text{ \&c.}$$

hence

$$w' = \frac{H'}{w} - \left(1 + \frac{2dw}{dx}\right) \cdot \left(\frac{H''}{ww} - \left(1 + \frac{3dw}{dx}\right) \cdot \left[\frac{H'''}{w^3} - \text{\&c.}\right]\right)$$

$$- \frac{3ddw}{dx^2}\left(\frac{H'''}{w^3} - \text{\&c.}\right)$$

$$w'' = \frac{H''}{w} - \left(1 + \frac{3dw}{dx}\right) \cdot \left(\frac{H'''}{w^3} - \text{\&c.}\right)$$

$$w''' = \frac{H'''}{w} - \text{\&c.}$$

If we substitute these values into the equation

$$H = w + w' + w'\frac{dw}{dx} + w''\frac{ddw}{dx^2} + \text{\&c.}$$

& into this one

$$X = T + w' \cdot \frac{dT}{dx} + \text{\&c.}$$

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we will form the following two

$$0 = w - H + \left(1 + \frac{dw}{dx}\right) \cdot \left\{ \begin{array}{c} \frac{H'}{w} - \left(1 + \frac{2dw}{dx}\right) \cdot \left[\frac{H''}{ww} - \text{\&c.}\right] \\ \left(1 + \frac{3dw}{dx}\right) \cdot \left[\frac{H'''}{w^3} - \text{\&c.}\right] \\ - \frac{3ddw}{dx^2} \cdot \left[\frac{H'''}{w^3} - \text{\&c.}\right] \end{array} \right\}$$

$$+ \frac{ddw}{dx^2} \cdot \left(\frac{H''}{w} - \left(1 + \frac{3dw}{dx}\right) \cdot \left[\frac{H'''}{w^3} - \text{\&c.}\right]\right) \qquad (D)$$

$$+ \frac{d^3w}{dx^3} \cdot \left(\frac{H'''}{w} - \text{\&c.}\right)$$

$$\ldots$$

$$+ \frac{d^{n-1}w}{dx^{n-1}} \cdot \frac{H^{n-1}}{w}$$

$$X = T + \frac{dT}{dx} \left\{ \begin{array}{c} \frac{H'}{w} - \left(1 + \frac{2dw}{dx}\right) \cdot \left[\frac{H''}{ww} - \left(1 + \frac{3dw}{dx}\right) \cdot \right. \\ \left. \left[\frac{H'''}{w^3} - \text{\&c.}\right] - \frac{3ddw}{dx^2} \cdot \left[\frac{H''}{w^2} - \text{\&c.}\right] \end{array} \right\}$$

$$+ \frac{ddT}{dx^2} \cdot \left(\frac{H''}{w} - \left(1 + \frac{3dw}{dx}\right) \cdot \left[\frac{H'''}{w^3} - \text{\&c.}\right]\right) \qquad (E)$$

$$+ \frac{d^3T}{dx^3} \left(\frac{H'''}{w} - \text{\&c.}\right)$$

$$\ldots$$

$$+ \frac{d^{n-1}T}{dx^{n-1}} \cdot \frac{H^{n-1}}{w}$$

III.

Equations (*D*) & (*E*) are of a degree inferior to the proposed, thus equation (*A*) can always be reduced to two others of a degree less by one unit. It is not even necessary to resolve generally these equations, it suffices to find in equation (*D*) a number, n, of particular values for, w, which satisfy this differential equation.

Let, β, β', β'' &c., be these values or these particular integrals, we will substitute them successively into equation (*E*), & we will form from them by this manner a number, n, of equations of which it will suffice to find a particular integral for each; because β, β', β'' &c., being the particular values of, w; let T, T', T'' &c., be the particular

and corresponding values of, T. We will have the number, n, of equations following,

$$\beta \frac{dy}{dx} + y = T$$

$$\beta' \frac{dy}{dx} + y = T'$$

$$\beta'' \frac{dy}{dx} + y = T''$$

$$\dots$$

$$\beta^{n-1} \cdot \frac{dy}{dx} + y = T^{n-1}$$

Nonetheless the complete integral of equation (A), will be

$$
\begin{aligned}
y = & e^{-\int \frac{dx}{\beta}} \left(C + \int \frac{T dx}{\beta} \cdot e^{\int \frac{dx}{\beta}} \right) \\
& + e^{-\int \frac{dx}{\beta'}} \left(C' + \int \frac{T' dx}{\beta'} \cdot e^{\int \frac{dx}{\beta'}} \right) \\
& + \&\text{c.} \\
& \dots \\
& + e^{-\int \frac{dx}{\beta^{n-1}}} \left(C^{n-1} + \int \frac{T^{n-1} dx}{\beta^{n-1}} \cdot e^{\int \frac{dx}{\beta^{n-1}}} \right),
\end{aligned}
\tag{$\widetilde{\omega}$}
$$

e being the number of which the hyperbolic logarithm, is unity.

IV.

If we substitute into equation (E), β, in the place of, w, it will become

$$
\begin{aligned}
X = T & + \frac{dT}{dx} \left[\frac{H'}{\beta} - \left(1 + \frac{2d\beta}{dx} \right) \cdot \left(\frac{H''}{\beta\beta} - \&\text{c.} \right) \right. \\
& + \frac{ddT}{dx^2} \cdot \left(\frac{H''}{\beta} - \&\text{c.} \right) \\
& \dots \\
& + \frac{d^{n-1}T}{dx^{n-1}} \cdot \frac{H^{n-1}}{\beta}
\end{aligned}
$$

If we suppose that $T = Z$, is the complete integral of this equation, Z, then it will contain a number, $n-1$, of arbitrary constants. Hence the complete integral of equation (A) will be[3]

$$y = e^{-\int \frac{dx}{\beta}} \cdot \left(C + \int \frac{Z\, dx}{\beta} \cdot e^{\int \frac{dx}{\beta}} \right) \tag{$\widetilde{\omega}$}$$

since this integral contains a number, n, of arbitrary constants.

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[3] *Translator's note*: The equation label is lacking and has been added.

If in equation (A), $X = 0$, we can suppose then in the expression $(\widetilde{\omega})$ of, y, of article III., $T = 0$, $T' = 0$, $T'' = 0$ &c., & we will have

$$y = Ce^{-\int \frac{dx}{\beta}} + C'e^{-\int \frac{dx}{\beta'}} \cdots + C^{n-1}e^{-\int \frac{dx}{\beta^{n-1}}}$$

& if we suppose that in the equation

$$0 = T + \frac{dT}{dx}\left(\frac{H'}{\beta} - \left(1 + \frac{2d\beta}{dx}\right) \cdot \left[\frac{H'}{\beta\beta} - \&\text{c.}\right]\right)$$
$$+ \frac{ddT}{dx^2} \cdot \left(\frac{H''}{\beta} - \&\text{c.}\right)$$
$$\cdots$$
$$+ \frac{d^{n-1}T}{dx^{n-1}} \cdot \frac{H^{n-1}}{\beta}$$

we have for the complete integral

$$T = AR + A'R' + A''R'' \cdots + A^{n-2}R^{n-2}$$

A, A', A'' being some arbitrary constants, & R, R', R'' &c. being some particular integrals of the preceding equation, the expression $\overset{(\mathfrak{m})}{\sim}$ of, y, of the preceding article gives

$$y = Ce^{-\int \frac{dx}{\beta}} + e^{-\int \frac{dx}{\beta}} \int \frac{AR}{\beta} dx\, e^{\int \frac{dx}{\beta}}$$
$$+ e^{-\int \frac{dx}{\beta}} \int \frac{A'R'}{\beta} dx\, e^{\int \frac{dx}{\beta}}$$
$$+ \&\text{c.}$$
$$\cdots$$
$$+ e^{-\int \frac{dx}{\beta}} \int \frac{A^{n-2}R^{n-2}}{\beta} dx\, e^{\int \frac{dx}{\beta}}$$

By comparing this expression of, y, with this one

$$y = Ce^{-\int \frac{dx}{\beta}} + C'e^{-\int \frac{dx}{\beta'}} + C''e^{-\int \frac{dx}{\beta''}} \cdots + C^{n-1}e^{-\int \frac{dx}{\beta^{n-1}}}$$

we will form the following equations

$$e^{-\int \frac{dx}{\beta}} \int \frac{AR}{\beta} dx \cdot e^{\int \frac{dx}{\beta}} = C'e^{-\int \frac{dx}{\beta'}}$$
$$e^{-\int \frac{dx}{\beta}} \int \frac{A'R'}{\beta} dx \cdot e^{\int \frac{dx}{\beta}} = C''e^{-\int \frac{dx}{\beta''}}$$
$$\&\text{c.}$$

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whence we will have

$$AR = C' \cdot \frac{\beta' - \beta}{\beta'} \cdot e^{-\int \frac{dx}{\beta'}}$$

$$A'R' = C'' \cdot \frac{\beta'' - \beta}{\beta''} \cdot e^{-\int \frac{dx}{\beta''}}$$

$$\cdots$$

$$A^{n-2}R^{n-2} = C^{n-1} \cdot \frac{\beta^{n-1} - \beta}{\beta^{n-1}} \cdot e^{-\int \frac{dx}{\beta^{n-1}}}$$

We suppose now that we know how to integrate equation (*A*) in y by making $X = 0$, & let, Cu, $C'u'$, $C''u''$ &c. be the particular values of y, which satisfy this equation, so that its complete integral is

$$y = Cu + C'u' + C''u'' \cdots + C^{n-1}u^{n-1}$$

By comparing it with this one

$$y = Ce^{-\int \frac{dx}{\beta}} + C'e^{-\int \frac{dx}{\beta'}} + C''e^{-\int \frac{dx}{\beta''}} \cdots + C^{n-1}e^{-\int \frac{dx}{\beta^{n-1}}}$$

we will have
$$Cu = Ce^{-\int \frac{dx}{\beta}}, \quad C'u' = e^{-\int \frac{dx}{\beta'}}, \quad \&c.$$

Whence we will conclude

$$\beta = -\frac{u\,dx}{du}, \quad \beta' = -\frac{u'\,dx}{d\,u'}, \quad \beta'' = -\frac{u''\,dx}{du''} \quad \&c.$$

these values of β, β', β'' &c. will satisfy consequently for, w, in equation (*D*), & we will conclude from it, AR, $A'R'$, $A''R''$ &c., & consequently if we know how to integrate the equation

$$0 = y + H\frac{dy}{dx} + H'\frac{ddy}{dx^2} \cdots + H^{n-1} \cdot \frac{d^n y}{dx^n}$$

we will know 1° a number, n, of values which satisfy for, w, in equation (*D*), 2° a number, $n-1$, of particular values for, T, in the equation

$$0 = T + \frac{dT}{dx}\left(\frac{H'}{\beta} - \&c.\right)$$

$$\cdots$$

$$+ \frac{d^{n-1}T}{dx^{n-1}} \cdot \frac{H^{n-1}}{\beta}$$

& hence we will know the integral completely.

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Now if we know how to integrate the equation

$$X = T + \frac{dT}{dx}\left(\frac{H'}{\beta} - \&\text{c.}\right)$$

$$\cdots$$

$$+ \frac{d^{n-1}T}{dx^{n-1}} \cdot \frac{H^{n-1}}{\beta}$$

by supposing, Z, its complete integral, we will have by article IV.

$$y = e^{-\int \frac{dx}{\beta}}\left(C + \int \frac{Z\,dx}{\beta} \cdot e^{\int \frac{dx}{\beta}}\right)$$

Therefore the difficulty of integrating the equation

$$X = y + H\frac{dy}{dx} \cdots + H^{n-1} \cdot \frac{d^n y}{dx^n}$$

when we know how to integrate

$$0 = y + H\frac{dy}{dx} \cdots + H^{n-1} \cdot \frac{d^n y}{dx^n}$$

is reduced to integrating this one

$$X = T + \frac{dT}{dx}\left(\frac{H'}{\beta} - \&\text{c.}\right) \qquad (\triangle)$$

$$\cdots$$

$$+ \frac{d^{n-1}T}{dx^{n-1}} \cdot \frac{H^{n-1}}{\beta}$$

of degree, $n - 1$, & if we know how to integrate, when we suppose $X = 0$, we will make in the same manner, & by the same method the resolution of that one to depend on another of degree, $n-2$, & thus in sequence until we arrive to an equation of degree, $n - n$, or purely algebraic; whence there results that the equation

$$X = y + H\frac{dy}{dx} \cdots + H^{n-1} \cdot \frac{d^n y}{dx^n}$$

is integrable in the same case as this one

$$0 = y + H\frac{dy}{dx} \cdots + H^{n-1} \cdot \frac{d^n y}{dx^n}$$

this which is the beautiful theorem of Mr. de la Grange.

If we knew only a number, $n - 1$, of particular values for, y, in this last equation, or that which comes to the same of a particular value for, w, in equation (D), the integration would have no more difficulty, because instead of arriving to a purely algebraic equation, we would arrive to an equation of the first degree of this form

$$X = S + Q\frac{dS}{dx}$$

an equation that we know how to resolve generally, X, & Q, being some functions of x.

<div align="center">VII.</div>

The preceding method furnishes us not only the demonstration of the theorem of Mr. de la Grange, it leads us yet to find immediately the expression of, y, in the equation

$$X = y + H\frac{dy}{dx}\cdots + H^{n-1}\cdot\frac{d^n y}{dx^n} \tag{A}$$

when we know how to resolve this one

$$0 = y + H\frac{dy}{dx}\cdots + H^{n-1}\cdot\frac{d^n y}{dx^n}$$

because let, as previously, u, u', u'', &c., be the particular values of, y, in this same equation, so that its complete integral is

$$y = Cu + C'u' + C''u''\cdots + C^{n-1}u^{n-1}$$

The complete integral of equation (A) will be by the preceding article

$$y = u\left(C - \int\frac{Z\,du}{uu}\right)$$

Z, being the complete integral of, T, in equation $(\triangle)$. If we name, $\overline{u}$, $\overline{u'}$, $\overline{u''}$, &c. the particular values of , T, in this equation $(\triangle)$ by supposing in it, $X = 0$, we will have in the same manner

$$Z = \overline{u}\left(C' - \int\frac{Z'\,d\overline{u}}{\overline{u}u}\right)$$

We will form likewise

$$Z' = \overline{\overline{u}}\left(C'' - \int\frac{Z''\,d\overline{\overline{u}}}{\overline{\overline{u}}\,\overline{u}}\right)$$

$$Z'' = \overline{\overline{\overline{u}}}\left(C''' - \int\frac{Z'''\,d\overline{\overline{\overline{u}}}}{\overline{\overline{\overline{u}}}\,\overline{\overline{u}}}\right)$$

until finally we arrive to this equation,

$$Z^{n-1} = X.$$

We seek now the values of $\overline{u}$, $\overline{\overline{u}}$, $\overline{\overline{\overline{u}}}$, &c. now we have by article V.

$$R = \overline{u} = \frac{\beta' - \beta}{\beta'}\cdot e^{-\int\frac{dx}{\beta'}}$$

(by substituting in place of, β, & β' their values,) $= u' - \frac{u\,du'}{du}$, where, $\overline{u}, = \frac{u'\,du - u\,du'}{du}$. in the same manner

$$\overline{u'} = \frac{u''\,du - u\,du''}{du}$$

$$\overline{u''} = \frac{u'''\,du - u\,du'''}{du}$$

&c.

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we will form likewise

$$\overline{\overline{u}} = \frac{\overline{u'}\,d\overline{u} - \overline{u}d\overline{u'}}{d\overline{u}} \qquad \overline{\overline{\overline{u}}} = \frac{\overline{\overline{u'}}\,d\overline{\overline{u}} - \overline{\overline{u}}\,d\overline{\overline{u'}}}{d\overline{\overline{u}}}$$

$$\overline{\overline{u'}} = \frac{\overline{u''}\,d\overline{u} - \overline{u}d\overline{u''}}{d\overline{u}} \qquad \overline{\overline{\overline{u'}}} = \frac{\overline{\overline{u''}}\,d\overline{\overline{u}} - \overline{\overline{u}}\,d\overline{\overline{u''}}}{d\overline{\overline{u}}}$$

$$\&\text{c.} \qquad\qquad \&\text{c.}$$

This put by substituting, we will have

$$y = u\left(C - \int\left[\frac{\overline{u}\,du}{uu}\cdot\left[C' - \int\left[\frac{\overline{\overline{u}}\,du}{\overline{u}\,\overline{u}}\cdot\left[C'' - \int\frac{\overline{\overline{\overline{u}}}\,d\overline{\overline{u}}}{\overline{\overline{u}}\,\overline{\overline{u}}}\right.\right.\right.\right.$$

$$\left[C''' \cdots - \int\left[\frac{\overset{(n-1)}{\overline{u}}\,d\overset{(n-2)}{\overline{u}}}{\overset{(n-1)}{\overline{u}}\,\overset{(n-2)}{\overline{u}}}\left[C^{n-1} - \int\frac{X d\overset{(n-1)}{\overline{u}}}{\overset{(n-1)}{\overline{u}}\,\overset{(n-1)}{\overline{u}}}\right.\right. \tag{Z}$$

we can give to this expression a simpler form by considering that

$$\frac{\overline{u}\,du}{uu} = -\frac{u'du - u\,du'}{du} = -d\left(\frac{u'}{u}\right);$$

in the same manner

$$\frac{\overline{\overline{u}}\,d\overline{u}}{\overline{u}\,\overline{u}} = -d\left(\frac{\overline{u'}}{\overline{u}}\right), \ \&\text{c.}$$

the formula (Z) will become consequently

$$y = u\left[C + \int\left[d\left(\frac{u'}{u}\right)\cdot\left[C' + \int\left[d\left(\frac{\overline{u'}}{\overline{u}}\right)\cdot\left[C'' \cdots\right.\right.\right.\right.\right.$$

$$+ \int\left[d\left(\frac{\overset{(n-2)}{\overline{u}}}{\overset{(n-2)}{\overline{u}}}\right)\cdot\left[C^{n-1} - \int\frac{X d\overset{(n-1)}{\overline{u}}}{\overset{(n-1)}{\overline{u}}\,\overset{(n-1)}{\overline{u}}}\right.\right. \tag{Y}$$

& we will have

$$d\left(\frac{\overline{u'}}{\overline{u}}\right) = d\left(\frac{d\left(\frac{u''}{u}\right)}{d\left(\frac{u'}{u}\right)}\right)$$

$$d\left(\frac{\overline{\overline{u'}}}{\overline{\overline{u}}}\right) = d\left\{\frac{d\left(\frac{d\left(\frac{u'''}{u}\right)}{d\left(\frac{u''}{u}\right)}\right)}{d\left(\frac{d\left(\frac{u''}{u}\right)}{d\left(\frac{u'}{u}\right)}\right)}\right\}$$

$$\&\text{c.}$$

If we would know only a number, $n - 1$, of particular values for, y, in the equation

$$0 = y + H\frac{dy}{dx}\cdots + H^{n-1}\cdot\frac{d^n y}{dx^n}$$

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or that which reverts to the same, a number, $n-1$, of values for, w, in equation (D), then we could arrive, as we can be assured by the preceding article to the following equation

$$X = Z^{n-2} + dZ^{n-2} \cdot \cfrac{H^{n-1}}{-\cfrac{udx}{du} \cdot -\cfrac{\overline{u}\,dx}{d\overline{u}} \cdots - \cfrac{\overset{(n-2)}{\overline{u}}\,dx}{\overset{(n-2)}{d}\,\overline{u}}}$$

by means of which we will have easily, Z^{n-2}, & we will have

$$y = u\left[C - \int\left(\frac{\overline{u}\,du}{uu} \cdot \left[C' \cdots \int \frac{\overset{(n-2)}{\overline{u}}\,\overset{(n-3)}{d}\,\overline{u}}{\overset{(n-2)}{\overline{u}}\,\overset{(n-2)}{\overline{u}}}\right.\right.\right.$$

$$\left.\left.\left[C^{n-2} - \int \frac{Z^{n-2}\,\overset{(n-2)}{d}\,\overline{u}}{\overset{(n-2)}{\overline{u}}\,\overset{(n-2)}{\overline{u}}}\right.\right.\right. \qquad (X)$$

Remark.

We will observe here that if we know a number, $n-1$, of particular values for, w, in equation (D), we will know how to integrate it completely; because by means of these values we will have by the preceding articles the complete integral of this one

$$0 = y' + H\frac{dy}{dx} \cdots + H^{n-1} \cdot \frac{d^n y}{dx^n}$$

We suppose that this complete integral of, y, is as that above

$$y = Cu + C'u' \cdots + C^{n-1} \cdot u^{n-1}$$

& that the complete value of, w, is γ; γ, will contain consequently a number, $n-1$, of arbitrary constants; but we will have $\gamma\frac{dy}{dx} + y = 0$ hence $y = Ae^{-\int \frac{dx}{\gamma}}$ & this expression will be the complete integral of, y, since it contains a number, n, of arbitrary constants; therefore

$$Ae^{-\int \frac{dx}{\gamma}} = Cu + C'u' \cdots + C^{n-1} \cdot u^{n-1}$$

Whence we will conclude

$$\gamma = -\left(\frac{u + \frac{C'}{C}u' + \frac{C''}{C}u'' \cdots + \frac{C^{n-1}}{C} \cdot u^{n-1}}{\frac{du}{dx} + \frac{C'}{C} \cdot \frac{du'}{dx} \cdots + \frac{C^{n-1}}{C}\frac{du^{n-1}}{dx}}\right)$$

an expression which contains, $n-1$, arbitrary constants.

VIII.

We will resume now the formula

$$y = u \left[C + \int \left[d \left(\frac{u'}{u} \right) \cdot \left[C' + \int \left[d \left(\frac{\overline{u'}}{\overline{u}} \right) \cdot \left[C'' \cdots \right. \right. \right. \right. \right.$$

$$+ \int \left[d \left(\frac{\overset{(n-2)}{\overline{u}}}{\overset{(n-2)}{\overline{u}}} \right) \cdot \left[C^{n-1} - \int X d \left(\frac{1}{\overset{(n-1)}{\overline{u}}} \right) \right. \right.$$

if we divide by, u, & if we differentiate we will have

$$d \left(\frac{y}{u} \right) = d \left(\frac{u'}{u} \right) \cdot \left[C' + \int \left[d \left(\frac{\overline{u'}}{\overline{u}} \right) \cdot \left[C'' \cdots \right. \right. \right.$$

$$+ \int \left[d \left(\frac{\overset{(n-2)}{\overline{u}}}{\overset{(n-2)}{\overline{u}}} \right) \cdot \left[C^{n-1} - \int X d \left(\frac{1}{\overset{(n-1)}{\overline{u}}} \right) \right. \right.$$

by dividing by $d \left(\frac{u'}{u} \right)$, & differentiating again we will have

$$d \left\{ \frac{d \left(\frac{y}{u} \right)}{d \left(\frac{u'}{u} \right)} \right\} = d \left(\frac{\overline{u'}}{\overline{u}} \right) \cdot [C'' \cdots + \&c.]$$

by continuing to operate thus, we will arrive to a differential equation of this form

$$C^{n-1} + \int X d \left(\frac{1}{\overset{(n-1)}{\overline{u}}} \right) = \gamma y + \gamma' \frac{dy}{dx} \cdots + \gamma^{n-1} \cdot \frac{d^{n-1} y}{dx^{n-1}} \qquad (1)$$

γ, γ' &c., being some functions of u, u', u'' &c., & of their differences. We will form, $\overset{(n-1)}{\overline{u}}$, by article VII., & for this we have considered in this article the values of, u, u' &c., in this order, u, u', u'', $u''' \ldots u^{n-1}$; but if instead of giving to, u', the second rank, we had put it in the first, & u in the second in the following order, u', u, u'', $u''' \ldots u^{n-1}$, then we will be arrived to this equation

$$C^{n-1} + \int X d \left(\frac{1}{\boxed{\overset{(n-1)}{\overline{u}}}} \right) = \boxed{\gamma} y + \boxed{\gamma'} \frac{dy}{dx} \cdots + \boxed{\gamma^{n-1}} \cdot \frac{d^{n-1} y}{dx^{n-1}} \quad (2)$$

$\boxed{\overset{(n-1)}{\overline{u}}}$, $\boxed{\gamma}$, &c. being that which, $\overset{(n-1)}{\overline{u}}$, γ, &c. become when we change, u, into u', & reciprocally, u', into, u; now if we suppose, $X = 0$, the two equations (1)

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& (2) become

$$C^{n-1} = \gamma y + \gamma' \frac{dy}{dx} \cdots + \gamma^{n-1} \cdot \frac{d^{n-1}y}{dx^{n-1}}$$

$$C^{n-1} = \boxed{\gamma} y + \boxed{\gamma'} \frac{dy}{dx} \cdots + \boxed{\gamma^{n-1}} \cdot \frac{d^{n-1}y}{dx^{n-1}}$$

hence,

$$\gamma y + \gamma' \frac{dy}{dx} \cdots + \gamma^{n-1} \cdot \frac{d^{n-1}y}{dx^{n-1}} = \boxed{\gamma} y + \boxed{\gamma'} \frac{dy}{dx} \cdots + \boxed{\gamma^{n-1}} \cdot \frac{d^{n-1}y}{dx^{n-1}} \quad (3)$$

an equation which must be identical, because without this, as we have

$$y = Cu + C'u' \cdots + C^{n-1} \cdot u^{n-1}$$

the integral of equation (3), whatever it be of order, $n-1$, would contain a number, n, of arbitrary constants, that which is absurd. We will have therefore by comparing the equations (1) & (2)

$$C^{n-1} + \int Xd\left(\frac{1}{(n-1)\atop\overline{u}}\right) = C^{n-1} + \int Xd\left(\boxed{\frac{1}{(n-1)\atop\overline{u}}}\right)$$

hence we will have, $\dfrac{(n-1)}{\overline{u}} = \boxed{\dfrac{(n-1)}{\overline{u}}}$. Thus the expression $\dfrac{(n-1)}{\overline{u}}$, will remain the same, whether we change or not, u', into u, & u, into u'. We will prove in the same manner that it will remain constantly the same, be it that we change, u'', into, u', & u', into u''; u''', into u', & u', into u''', &c., & that in general, by forming $\dfrac{(n-1)}{\overline{u}}$, we can without changing its value give to, u, u', u'' &c., such order as we will wish, provided that we consider, $\dfrac{(n-1)}{\overline{u}}$, as the last of these quantities.

Let now $\dfrac{d\left(\frac{1}{(n-1)\atop\overline{u}}\right)}{dx} =, Z^{n-1}$; let, Z^{n-2}, that which Z^{n-1} becomes, when we change, u^{n-1}, into, u^{n-2}, &, u^{n-2}, into, u^{n-1}; we will have by the same method which has made us arrive to the equation (1),

$$C^{n-2} + \int X Z^{n-2} dx = \underline{\gamma} y + \underline{\gamma'} \frac{dy}{dx} \cdots + \underline{\gamma}^{n-1} \cdot \frac{d^{n-1}y}{dx^{n-1}}$$

$\underline{\gamma}$, $\underline{\gamma'}$, $\underline{\gamma''}$ &c. being that which γ, γ' &c. become when we change, u^{n-1}, into u^{n-2}, & reciprocally; we will have likewise by treating successively, u^{n-3}, u^{n-4}, &c., as the last of the quantities, u, u', &c.

$$C^{n-3} + \int X z^{n-3} dx = \underline{\underline{\gamma}} y + \underline{\underline{\gamma'}} \frac{dy}{dx} \cdots + \underline{\underline{\gamma}}^{n-1} \cdot \frac{d^{n-1}y}{dx^{n-1}} \text{ &c.}$$

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by arranging all these equations in the following order

$$C^{n-1} + \int z^{n-1} X dx = \gamma y + \gamma' \frac{dy}{dx} \cdots + \gamma^{n-1} \cdot \frac{d^{n-1}y}{dx^{n-1}}$$

$$C^{n-2} + \int z^{n-2} X dx = \underline{\gamma} y + \underline{\gamma}' \frac{dy}{dx} \cdots + \underline{\gamma}^{n-1} \cdot \frac{d^{n-1}y}{dx^{n-1}}$$

$$C^{n-3} + \int z^{n-3} X dx = \underset{=}{\gamma} y + \underset{=}{\gamma}' \frac{dy}{dx} \cdots + \underset{=}{\gamma}^{n-1} \cdot \frac{d^{n-1}y}{dx^{n-1}}$$

$$C + \int z X dx = \underset{(n-1)}{\gamma} y + \underset{(n-1)}{\gamma}' \frac{dy}{dx} \cdots + \underset{(n-1)}{\gamma}^{n-1} \cdot \frac{d^{n-1}y}{dx^{n-1}}$$

& adding them together after having multiplied the first by, u^{n-1}, the second by, u^{n-2}, & thus in sequence. We will have an equation of this form

$$\lambda y + \lambda' \frac{dy}{dx} \cdots + \lambda^{n-1} \cdot \frac{d^{n-1}y}{dx^{n-1}} = u \left(C + \int z X dx \right)$$
$$+ u' \left(C' + \int z' X dx \right)$$
$$+ u'' \left(C'' + \int z'' X dx \right)$$
$$\cdots$$
$$+ u^{n-1} \left(C^{n-1} + \int z^{n-1} X dx \right)$$

If we suppose, $X = 0$, we will have

$$\lambda y + \lambda' \frac{dy}{dx} \cdots + \lambda^{n-1} \cdot \frac{d^{n-1}y}{dx^{n-1}} = Cu + C'u' + C''u'' \cdots + C^{n-1}u^{n-1}$$

but we have,

$$y = Cu + c'u' \cdots + c^{n-1}u^{n-1}.$$

therefore

$$y = \lambda y + \lambda' \frac{dy}{dx} \cdots + \lambda^{n-1} \cdot \frac{d^{n-1}y}{dx^{n-1}}$$

an equation which must be identical, because without this, whatever be the order, $n-1$, its integral would contain a number, n, of arbitrary constants, that which is absurd. We will have therefore

$$y = u \left(C + \int Z X dx \right)$$
$$+ u' \left(C' + \int Z' X dx \right)$$
$$\cdots$$
$$+ u^{n-1} \left(C^{n-1} + \int z^{n-1} X dx \right)$$

thence results this very simple rule in order to have the complete integral of the equation

$$X = y + H\frac{dy}{dx} + H'\frac{ddy}{dx^2} \cdots + H^{n-1} \cdot \frac{d^n y}{dx^n}$$

when we know how to integrate this one

$$0 = y + H\frac{dy}{dx} + H'\frac{ddy}{dx^2} \cdots + H^{n-1} \cdot \frac{d^n y}{dx^n}$$

let

$$y = Cu + C'u' + C''u'' \cdots + C^{n-1}u^{n-1}$$

be the integral of this last, & let us make

$$\overline{u} = \frac{u'du - udu'}{du} \quad \overline{\overline{u}} = \frac{\overline{u'}\,d\overline{u'} - \overline{u}d\overline{u'}}{d\overline{u}} \quad \overline{\overline{\overline{u}}} = \frac{\overline{\overline{u'}}\,d\overline{\overline{u}} - \overline{\overline{u}}d\overline{\overline{u'}}}{d\overline{\overline{u}}}$$

$$\overline{u'} = \frac{u''du - udu''}{du} \quad \overline{\overline{u'}} = \frac{\overline{u''}d\overline{u} - \overline{u}d\overline{u''}}{d\overline{u}} \quad \text{\&c.}$$

$$\overline{u''} = \frac{u'''du - udu'''}{du} \quad \text{\&c.}$$

&c.

& let us arrive to form thus, $\overset{(n-1)}{\overline{u}}$, let then $\dfrac{d\left(\frac{1}{\overset{(n-1)}{\overline{u}}}\right)}{dx} = Z^{n-1}$. If in the expression of, Z^{n-1}, we change, u^{n-1}, into, u^{n-2}, & reciprocally, we will form, Z^{n-2}; if in the same expression of, Z^{n-1}, we change, u^{n-1}, into u^{n-3}, & reciprocally we will form, Z^{n-3} &c., & thus in sequence, I say that the complete integral of the equation

$$X = y + H\frac{dy}{dx} \cdots + H^{n-1} \cdot \frac{d^n y}{dx^n} \tag{A}$$

will be

$$
\begin{aligned}
y =\; & u\left(C + \int ZX\,dx\right) \\
& + u'\left(C' + \int Z'X\,dx\right) \\
& \cdots \\
& + u^{n-1}\left(C^{n-1} + \int Z^{n-1}X\,dx\right)
\end{aligned}
$$

IX.

If we suppose now in equation (A), H, H', H'' &c. constants, we see easily that in order to have a number, n, of particular values which satisfy for, w, in the equation (D) it suffices to suppose, w, constant, & then this equation will be changed into this one

$$0 = w - H + \frac{H'}{w} - \frac{H''}{w^2} + \frac{H'''}{w^3} \cdots \pm \frac{H^{n-1}}{w^{n-1}}$$

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By resolving this last equation we will have a number, n, of values for, w, by means of which we will integrate easily equation (A). We apply to this case the rule given in the preceding article.

Let for this, $-\frac{1}{p} - \frac{1}{p'} - \frac{1}{p''}$ &c. be the roots of the equation

$$0 = w - H + \frac{H'}{w} - \text{&c.,}$$

we will have

$$\beta = -\frac{1}{p} = -\frac{u\,dx}{du}$$

hence

$$u = e^{px}, \quad u' = e^{p'x}, \quad u'' = e^{p''x}, \quad \text{&c.}$$

whence we will form

$$\overline{u} = e^{p'x}\left(\frac{p - p'}{p}\right)$$

$$\overline{\overline{u}} = e^{p''x} \cdot \left(\frac{p - p''}{p}\right) \cdot \left(\frac{p' - p''}{p'}\right)$$

$$\overline{\overline{\overline{u}}} = e^{p'''x}\left(\frac{p - p'''}{p}\right) \cdot \left(\frac{p' - p'''}{p'}\right) \cdot \left(\frac{p'' - p'''}{p''}\right)$$

$$\overline{u'} = e^{p''x}\left(\frac{p - p''}{p}\right) \cdot \overline{\overline{u'}} = e^{p'''x}\left(\frac{p - p'''}{p}\right) \cdot \left(\frac{p' - p'''}{p'}\right) \quad \text{&c.}$$

$$\overline{u''} = e^{p'''x}\left(\frac{p - p'''}{p}\right) \quad \text{&c.}$$

&c.

therefore

$$\frac{(n-1)}{\overline{u}} = e^{p^{n-1}x} \cdot \left(\frac{p - p^{n-1}}{p}\right) \cdot \left(\frac{p' - p^{n-1}}{p'}\right) \cdots \left(\frac{p^{n-2} - p^{n-1}}{p^{n-2}}\right)$$

hence

$$Z^{n-1} = \frac{-p \cdot p' \cdot p'' \cdots p^{n-1}}{(p - p^{n-1}) \cdot (p' - p^{n-1}) \cdot (p'' - p^{n-1}) \cdots (p^{n-2} - p^{n-1}) \cdot e^{p^{n-1}x}}$$

whence we will conclude

$$Z^{n-2} = \frac{-p \cdot p' \cdot p'' \cdots p^{n-1}}{(p - p^{n-2}) \cdot (p' - p^{n-2}) \cdots (p^{n-1} - p^{n-2}) \cdot e^{p^{n-2}x}}$$

$$\cdots$$

$$Z = \frac{-p \cdot p' \cdot p'' \cdots p^{n-1}}{(p' - p) \cdot (p'' - p) \cdots (p^{n-1} - p) \cdot e^{px}}$$

hence

$$y = \frac{p \cdot p' \cdot p'' \cdots p^{n-1} \cdot e^{px}}{(p'-p) \cdot (p''-p) \cdots (p^{n-1}-p)} \left(A - \int \frac{X dx}{e^{px}} \right)$$

$$+ \frac{p \cdot p' \cdot p'' \cdots p^{n-1} \cdot e^{p'x}}{(p-p') \cdot (p''-p') \cdot \&c.} \left(A' - \int \frac{X dx}{e^{p'x}} \right)$$

$$+ \&c.$$

we suppose further in equation (A), $H = Ax$, $H' = A'x^2$, $H'' = A''x^3$ &c. it is easy to perceive that by supposing in equation (D), $w = mx$, we will satisfy this equation, & we will have by dividing by, x,

$$0 = m - A + (1+m) \cdot \frac{A'}{m} - (1+m) \cdot (1+2m) \cdot \frac{A''}{mm}$$

$$+ (1+m)(1+2m)(1+3m) \cdot \frac{A'''}{m^3} - \&c.$$

By resolving this equation, we will have a number, n, of values for, m, & consequently a number, n, of particular integrals of equation (D). This case is known enough for us to dispense of entering into any detail in its regard. This method of integrating the equations of the form

$$X = y + H\frac{dy}{dx} + H'\frac{ddy}{dx^2} \cdots + H^{n-1} \cdot \frac{d^n y}{dx^n}$$

by seeking a number n, or $n-1$, of particular integrals of equation (D) embraces therefore all the known cases where the integration has succeeded to the present, & there is no doubt that we can by its means discover new, & much more extensive of them.

<div align="center">X.</div>

If we suppose that equation (A) ascends only to the second degree, equation (D) will become

$$0 = dx \cdot (H' - Hw + ww) + H'dw \qquad \text{(o)}$$

of which it will suffice to find a single particular integral, in order to integrate the equation

$$X = y + H\frac{dy}{dx} + H'\frac{ddy}{dx^2}.$$

for let, β, be this integral by substituting it into equation (E), we will have

$$X = T + \frac{dT}{dx} \cdot \frac{H'}{\beta}$$

whence we will conclude

$$T = e^{-\int \frac{\beta dx}{H'}} \cdot \left(A + \int \frac{\beta X dx}{H'} \cdot e^{\int \frac{\beta dx}{H'}} \right)$$

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therefore we will have by article IV.

$$T = e^{-\int \frac{dx}{\beta}} \cdot \left(C + \int \left(\frac{e^{-\int \left[\frac{\beta X dx}{H'} - \frac{dx}{\beta} \right]}}{\beta} \cdot \left(A + \int \frac{\beta X dx}{H'} \cdot e^{\int \frac{\beta dx}{H'}} \right) \right) \right)$$

if instead of knowing a value of, w, we would know a particular value of, y, in the equation

$$0 = y + H \frac{dy}{dx} + H' \frac{ddy}{dx^2}.$$

let, $y = A'u$, be this value, we will have $w = \beta = -\frac{u dx}{du}$, & we will integrate as, above,

$$X = y + H \frac{dy}{dx} + H' \frac{ddy}{dx^2}.$$

thence, & from the remark which we have made article VII. results the following two theorems.

The equation,

$$0 = dx(H' - Hw + ww) + H'dw \qquad \text{(o)}$$

is completely integrable, when we know a particular integral of it.

The equation of Riccati will be therefore integrable yet when we will know a particular integral, since this equation is contained in the preceding, & more generally still the equation

$$0 = dx(P + Qy + Ryy) + Sdy$$

P, Q, R, S, being any functions whatever of, x, is integrable, when we know a single particular value which satisfies for, y, in this equation; because by dividing by, R, we will have

$$0 = \frac{P}{R} dx + \frac{Q}{R} ydx + yydx + \frac{S}{R} dy$$

In order to restore this equation to formula (o) I multiply it by, $\frac{P}{S}$, & I make, $\frac{P}{S} dx = dz$, this which gives

$$0 = \frac{P}{R} dz + \frac{Q}{R} ydz + yydz + \frac{P}{R} dy$$

P, Q, R, becoming then some functions of z. Let there be $\frac{P}{R} = H'$, & $\frac{Q}{R} = -H$, we will have,

$$0 = dz(H' - Hy + yy) + H'dy;$$

an equation which is the same as equation (o).

The equation

$$X = y + H \frac{dy}{dx} + H' \cdot \frac{ddy}{dx^2} \qquad \text{(7)}$$

is integrable yet when $H = w + \frac{H'}{w} \left(1 + \frac{dw}{dx} \right)$.

If we suppose,

$$w = (m + nx + px^2 \cdots + hx^r)^q,$$

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the equation

$$X = y + \frac{dy}{dx}[(m + nx + px^2 \cdots + hx^r)^q$$
$$+ Q(1 + q(m + nx \cdots + hx^r)^{q-1}[n + 2px \cdots hrx^{r-1})]$$
$$+ \frac{ddy}{dx^2}[Q(m + nx + px^2 \cdots + hx^r)^q]$$

is integrable whatever be, Q. We can therefore by means of the preceding theorem, find in an infinite number of cases the integral of equation (7) whatever be, Q, namely by giving to w, an infinity of values.

<div align="center">XI.</div>

Equation (D) will become for the third order

$$0 = w - H + \left[1 + \frac{dw}{dx}\right] \cdot \left(\frac{H'}{w} - \left(1 + \frac{2dw}{dx}\right) \cdot \frac{H''}{ww}\right) + \frac{ddw}{dx^2} \cdot \frac{H''}{w} \qquad (P)$$

of which it suffices to find two particular integrals, in order to have the complete integral of this one

$$X = y + H\frac{dy}{dx} + H'\frac{ddy}{dx^2} + H'' \cdot \frac{d^3y}{dx^3}$$

it will be easy to find thus an infinity of equations of third order, & of superior orders integrable, & to form thus for each order of differentials, a class of equations very general, & integrable, but I content myself to have given the method.

<div align="center">XII.</div>

<div align="center">Application of the preceding method to the integral
calculus in finite differences.</div>

Although the integral calculus in the finite differences is the foundation of the entire theory of series, however this interesting branch of analysis is yet quite far from the point of perfection where we have carried the others. Mr. Euler has given in truth in his institutions many very good, & very ingenious methods in order to integrate a differential function in finite differences & in a single variable, but the integration of the differential equations is an absolutely new part, if we except from it one or two cases which the theory of recurrent series contains, & the excellent essay which Mr. the Marquis de Condorcet has given on this material in his integral calculus.[4] I myself am therefore here proposed to study it thoroughly, by applying the method of which I have made usage, above for the infinitely small differences; it has led me to find the general term of a quite extensive class of series, & of which the known series are only some particular cases, at the same time to many other remarks which have appeared to me

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[4]When I wrote this in the month of March 1771 there had appeared nothing more on this matter; since this time I have had occasion to see a quite beautiful memoir of Mr. the Marquis de Condorcet on the finite differences which will appear in the volume of memoirs of the Académie des Sciences de France, for the year 1770 but the researches of this illustrious geometer have nothing in common with mine. Except that he observes in the same manner that the theorem of Mr. de la Grange holds equally for the finite differences.

important to demonstrate, for example, that the beautiful theorem of Mr. de la Grange according to which the equation

$$X = y + H\frac{dy}{dx} + H' \cdot \frac{ddy}{dx^2} + \&c.$$

X, H, H' being some functions whatever of, x, is integrable in the same cases as this

$$0 = y + H\frac{dy}{dx} + H' \cdot \frac{ddy}{dx^2} + \&c.$$

holds equally for the finite differences, & to determine in a quite simple manner the integral of the first of these equations, when we know the integral of the second.

XIII.

In order for us to form a precise idea of the differential equations in the finite differences, we imagine a series

$$y' + y'' + y''' \cdots + y^x$$

formed according to a law such as we have constantly

$$X^x = M^x y^x + V^x \cdot \triangle \cdot y^x + p^x \cdot \triangle^x y^x \cdots + S^x \triangle^n \cdot y^x \qquad (A)$$

X^x, M^x, V^x &c. being some functions any whatever of the index, x, of which the difference is supposed constant & equal to 1, & the characteristic $\triangle$ designating in the manner of Mr. Euler the finite difference of a quantity. The preceding equation will be a differential equation in the finite differences which can generally represent the equations of this kind where the variable y^x, & its differences are under a linear form.

Although we can easily form other differential equations in which for example y^x & its differences would be multiplied by themselves, or the ones by the others, however those which are contained in equation (A) are the only ones which it is veritably interesting to know well, because they alone can serve in the theory of series; thus we will adhere to examine them with care.

XIV.

Many principles of the integral calculus in the infinitely small differences, hold equally for the finite differences, thus every function of x, for example which will satisfy for y^x, in equation (A) & which will contain a number n, of arbitrary constants, will be the complete integral of it. This principle which is of greatest usage in the integral calculus in the infinitely small differences, is not of a less extensive usage, in the calculus in the finite differences.

By a particular integral of a differential equation, I understand as previously every function of x, which substituted for y^x, in this equation makes vanish all the terms of it.

I will caution here that for the convenience of the calculus I will suppose that H, H', H'' &c. express some different quantities, & which can have no relation

among themselves, instead as this, H, H', H'' &c. express as ordinarily that which, H, becomes when the index increases successively by 1, 2, 3, &c., & the following, $H_{\prime}$, $H_{\prime\prime}$, $H_{\prime\prime\prime}$, &c. express that which this same quantity, H, becomes when the index diminishes successively by 1, 2, 3 &c. this put.

<div align="center">XV.</div>

As we have

$$\triangle \cdot y^x = y^{x+1} - y^x$$
$$\triangle^2 y^x = y^{x+2} - 2y^{x+1} + y^x$$
$$\triangle^3 y^x = y^{x+3} - 3y^{x+2} + 3y^{x+1} - y^x$$
<div align="center">&c.</div>

we can put equation (A) under this form

$$\begin{aligned} X^x =\,& y^x(M^x - N^x + P^x - \text{\&c.} \\ &+ y^{x+1}(N^x - 2p^x + \text{\&c.}) \\ &+ \text{\&c.} \\ &\cdots \\ &+ y^{x+n} \cdot S^x \end{aligned}$$

we can consequently give to it this form

$$X^x = y^x + H^x \cdot y^{x+1} + {}'H^x \cdot y^{x+2} + {}''H^x \cdot y^{x+3} \cdots + {}^{n-1}H^x \cdot y^{x+2} \qquad (B)$$

& this equation represents generally every equation linear in the finite differences. The equation

$$X^x = y^x + H^x y^{x+1}$$

is of the first order; this one

$$X^x = y^x + H^x \cdot y^{x+1} + {}'H^x \cdot y^{x+2}$$

is of the second order, & thus in sequence. We are going presently to resolve the following problems.

<div align="center">XVI.</div>
<div align="center">PROBLEM II.</div>

Let be proposed to integrate the equation of the first order

$$X^x = y^x + H^x y^{x+1}$$

Solution.

I put this equation under this form,

$$y^{x+1} = -\frac{1}{H^x} \cdot y^x + \frac{X^x}{H^x}$$

let $-\frac{1}{H^x} + R^x$, &, $\frac{X^x}{H^x} = Z^x$, & we will have,

$$y^{x+1} = R^x y^x + Z^x$$

an equation which it is necessary to integrate; now it gives

$$y' = Ry + Z$$
$$y'' = R'y' + Z'$$

by substituting into this second equation, in the place of, y', its value deduced from the first, we will have

$$y'' = R'Ry + R'Z$$
$$+ Z'$$

therefore,

$$y''' = R''R'y' + R''Z'$$
$$+ Z''$$

By substituting always in the place of, y', its value, we will have

$$y''' = R'' \cdot R' \cdot y + R'' \cdot R' \cdot Z$$
$$+ R''Z'$$
$$+ Z''$$

&

$$y^{iv} = R''' \cdot R'' \cdot R'y' + R''' \cdot R'' \cdot Z'$$
$$+ R''' \cdot Z''$$
$$+ Z'''$$

However

$$y^{iv} = R''' \cdot R'' \cdot R' \cdot Ry + R''' \cdot R'' \cdot R'Z$$
$$+ R''' \cdot R''Z'$$
$$+ R''' \cdot Z''$$
$$+ Z'''$$

we will have in the same manner

$$y^v = \mathbf{R}^{iv} \cdot R''' \cdot R'' \cdot R' \cdot Ry + R^{iv} \cdot R''' \cdot R'' \cdot R'Z$$
$$+ R^{iv} \cdot R''' \cdot R''Z'$$
$$+ R^{iv} \cdot R''' \cdot Z''$$
$$+ R^{iv}Z'''$$
$$+ R^{iv}$$

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& generally

$$y^x = \mathrm{R}^{x-1} \cdot R^{x-2} \cdot R^{x-3} \cdot R^{x-4} \cdots Ry + R^{x-1} \cdot R^{x-2} \cdots R'Z$$
$$+ R^{x-1} \cdot R^{x-2} \cdots + R''Z'$$
$$+ R^{x-1} \cdot R^{x-2} \cdots + R'''Z''$$
$$\cdots$$
$$+ R^{x-1}$$

Or by supposing, $y = A$

$$y^x = R \cdot R' \cdot R'' \cdots R^{x-1} \left(A + \frac{Z}{R} + \frac{Z'}{RR'} \cdots + \frac{Z^{x-1}}{R \cdot R' \cdots R^{x-1}} \right)$$

Therefore,

$$y^x = R \cdot R' \cdot R'' \cdots R^{x-1} \left(A + \sum \frac{Z^x}{R \cdot R' \cdots R^x} \right)$$

the characteristic, Σ, designating the integral in the finite differences. For more simplicity, I will denote by, ∇R^{x-1}, the product $R \cdot R' \cdots R^{x-1}$. This which gives,

$$y^x = \nabla \cdot R^{x-1} \left(A + \sum \frac{Z^x}{\nabla R^x} \right)$$

Whence we will conclude by substituting in the place of, Z^x, & R^{x-1}, their values

$$y^x = \frac{1}{\nabla(-H^{x-1})} \cdot \left(A + \sum [-X^x \triangle (-H^{x-1})] \right)$$

if, H^x, is constant, & equal to $-\frac{1}{p}$ we will have

$$y^x = p^{x-1} \left(A - \sum \cdot \left[\frac{X^x}{p^{x-1}} \right] \right)$$

XVII.
PROBLEM III.

Let it be proposed to integrate the differentio-differential equation

$$X^x = y^x + H^x \cdot y^{x+1} + {}'H^x \cdot y^{x+2} + {}''H^x \cdot y^{x+3} \cdots + {}^{n-1}H^x \cdot y^{x+n} \qquad (B)$$

Solution.

I suppose that we have,

$$w^x y^{x+1} + y^x = T^x \qquad (C)$$

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& we will form the following equations

$$w^{x+1}y^{x+2} + y^{x+1} = T^{x+1}$$
$$w^{x+2}y^{x+3} + y^{x+2} = T^{x+2}$$
$$w^{x+3}y^{x+4} + y^{x+3} = T^{x+3}$$
$$\cdots$$
$$w^{x+n-1}y^{x+n} + y^{x+n-1} = T^{x+n-1}$$

I multiply the first of these equations by, $'\beta$, the second by, $''\beta$, the third by, $'''\beta$, &c., & I add them with equation (C) this which gives

$$T^x + {}'\beta \cdot T^{x+1} + {}''\beta \cdot T^{x+2} + {}'''\beta \cdot T^{x+3} \cdots + {}^{n-1}\beta \cdot T^{x+n-1}$$
$$= y^x + (w^x + {}'\beta)y^{x+1} + ({}'\beta w^{x+1} + {}''\beta)y^{x+2}$$
$$+ ({}''\beta w^{x+2} + {}'''\beta)y^{x+3} + ({}'''\beta w^{x+3} + {}^{iv}\beta)y^{x+4} \cdots$$
$$+ ({}^{n-1}\beta w^{x+n-1})y^{x+n}$$

by comparing this equation with equation (B), we will have

1° $\quad X^x = T^x + {}'\beta \cdot T^{x+1} = {}''\beta \cdot T^{x+2} \cdots {}^{n-1}\beta \cdot T^{x+n-1}$

2° $\quad$ The following

$$w^x + {}'\beta = H^x$$
$$'\beta \cdot w^{x+1} + {}''\beta = {}'H^x$$
$$''\beta \cdot w^{x+2} + {}'''\beta = {}''H^x$$
$$\cdots$$
$${}^{n-1}\beta w^{x+n-1} = {}^{n-1}H^x$$

whence we will conclude

$$'\beta = H^x - w^x$$
$$''\beta = {}'H^x - H^x w^{x+1} + w^x \cdot w^{x+1}$$
$$'''\beta = {}''H^x - {}'H^x \cdot w^{x+2} + {}'H^x w^{x+1} \cdot w^{x+2} - w^x \cdot w^{x+1} \cdot w^{x+2} \text{ \&c.}$$
$$\cdots$$
$${}^{n-1}\beta = \pm \left(w^x \cdot w^{x+1} \cdots w^{x+n-2} - H^x \cdot w^{x+1} \cdots w^{x+n-2} \right.$$
$$\left. + {}'H^x \cdot w^{x+2} \cdots w^{x+n-2} - \text{\&c.} \right) = \frac{{}^{n-1}H^x}{w^x + w^{x+n-1}}$$

the sign, $+$, having place if, n, is odd, & the sign, $-$, if it is even. We will have therefore in order to resolve the problem, the following two equations.

$$X^x = T^x + T^{x+1}(H^x - w^x) + T^{x+2}({}'H^x - Hw^{x+1} + w^x \cdot w^{x+1})$$
$$+ \text{\&c.} \cdots + T^{x+n-1} \cdot \frac{{}^{n-1}H^x}{w^x + w^{x+n-1}} \tag{D}$$

$$0 = 1 - \frac{H^x}{w^x} + \frac{'H^x}{w^x \cdot w^{x+1}} - \frac{''H^x}{w^x \cdot w^{x+1} \cdot w^{x+2}} \cdots \pm \frac{{}^{n-1}H^x}{w^x \cdots w^{x+n-1}} \qquad (E)$$

<div align="center">XVIII.</div>

Equations (*D*) & (*E*) are of a degree inferior to the proposed, & equation (*D*) is of the same form; now it is not necessary to resolve generally these equations. It suffices to know a number, n, of values which satisfy for, w^x, in equation (*E*), because by substituting these values into equation (*D*), we will form in it a number, n, of equations of which it will suffice to find for each a particular integral. Let V^x, $'V^x$, $''V^x$ &c. be the particular values of, w^x, & L^x, $'L^x$, $''L^x$ &c. the corresponding values of, T^x, we will have

$$V^x y^{x+1} + y^x = L^x$$
$$'V^x y^{x+1} + y^x = 'L^x$$
$$''V^x y^{x+1} + y^x = ''L^x$$
<div align="center">&c.</div>

& the complete integral of equation (*B*) will be

$$y^x = \frac{1}{\nabla(-V^{x-1})}[A - \sum L^x \nabla(-V^{x-1})]$$
$$+ \frac{1}{\nabla(-'V^{x-1})} \cdot ['A - \sum 'L^x \nabla(-'V^{x-1})]$$
$$\cdots$$
$$+ \frac{1}{\nabla(-{}^{n-1}V^{x-1})} \cdot [{}^{n-1}A - \sum {}^{n-1}L^x \nabla(-{}^{n-1}V^{x-1})]$$

since this integral contains a number, n, of arbitrary constants.

<div align="center">XIX.</div>

I suppose that $X = 0$, & the complete integral of the equation

$$0 = y^x + H^x \cdot y^{x+1} + 'H^x \cdot y^{x+2} \cdots + {}^{n-1}H^x \cdot y^{x+n-2}$$

will be

$$y^x = \frac{A}{\nabla(-V^{x-1})} + \frac{'A}{\nabla(-'V^x y^{x-1})} \cdots \frac{{}^{n-1}A}{\nabla(-{}^{n-1}V^x y^{x-1})}$$

Presently if we substitute into equation (*D*), V^x, in the place of, w^x, & if we suppose next that the complete integral of this new equation which I call (*D'*), is when we suppose, $X^x = 0$,

$$T^x = CR^x + 'C'R^x + ''C''R^x \cdots + {}^{n-2}C^{n-2}R^x$$

It is easy to see that since we have,

$$V^x y^{x+1} + y^x = T^x$$

the complete integral of the equation

$$0 = y^x + H^x \cdot y^{x+1} \cdots + {}^{n-1}H^x \cdot y^{x+2}$$

will be

$$y^x = \frac{1}{\nabla(-V^{x-1})} \begin{pmatrix} \left(\begin{array}{l} A - C \cdot \sum [R^x \nabla(-V^{x-1})] \\ \quad - {}'C \cdot \sum [{}'R^x \nabla(-V^{x-1})] \end{array} \right) \\ \left(\begin{array}{l} \cdots \\ \quad - {}^{n-2}C \cdot \sum [{}^{n-2}R^x \nabla(-V^{x-1})] \end{array} \right) \end{pmatrix}$$

By comparing this last integral with the preceding, we will have

$$\frac{1}{\nabla(-V^{x-1})} \cdot \sum [R^x \nabla(-V^{x-1})] = \frac{1}{\nabla(-{}'V^{x-1})}$$

$$\frac{1}{\nabla(-V^{x-1})} \cdot \sum [{}'R^x \nabla(-V^{x-1})] = \frac{1}{\nabla(-{}''V^{x-1})}$$

&c.

Therefore

$$R^x = \nabla \frac{\left(\frac{\nabla(-V^{x-1})}{\nabla(-{}'V^{x-1})} \right)}{\nabla(-V^{x-1})} \qquad {}'R^x = \nabla \frac{\left(\frac{\nabla(-V^{x-1})}{\nabla(-{}''V^{x-1})} \right)}{\nabla(-V^{x-1})}$$

&c.

Therefore if we know how to resolve equation (B) by supposing $X^x = 0$ in it, we will know how to resolve equation (D') by supposing in the same manner $X^x = 0$ in it. Let then, u, ${}'u$, ${}''u$, &c. be the particular values of, y^x, in equation (B), so that its complete integral is

$$y^x = Au + {}'A'u + {}''A''u \cdots + + {}^{n-1}A^{n-1}u$$

we will have, $u = \frac{1}{\nabla(-V^{x-1})}$, and the integral of equation (D) by supposing $X^x = 0$ in it will be

$$T^x = Cu\nabla \left(\frac{{}'u}{u} \right) + {}'Cu\nabla \left(\frac{{}''u}{u} + {}''Cu\nabla \left(\frac{{}'''u}{u} \cdots + {}^{n-2}Cu\nabla \left(\frac{{}^{n-1}u}{u} \right) \right. \right.$$

Now if we know how to integrate equation (D') by supposing in it whatever X^x, we will know in the same manner how to integrate under the same supposition equation (B). Because let then, Z^x, be the complete integral of, T^x, in equation (D), we will have for the complete integral of, y^x, in equation (B)

$$y^x = \frac{1}{\nabla(-V^{x-1})} \cdot [A - \sum (Z^x \nabla(-V^{x-1})]$$

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since this integral contains a number, n, of arbitrary constants. Therefore the difficulty to integrate

$$X^x = y^x + {'}H^x \cdot y^{x+1} \cdots + {}^{n-1}H^x \cdot y^{x+n}$$

when we know how to integrate this one

$$0 = y^x + {'}H^x \cdot y^{x+1} \cdots + {}^{n-1}H^x \cdot y^{x+n}$$

is reduced to integrating the equation

$$
\begin{aligned}
X^x =&\, T^x + T^{x+1}(H^x - V^x) \\
&+ \text{\&c.} \\
&\cdots \\
&+ T^{x+n-1} \cdot \frac{{}^{n-1}H^x}{V^{x+n-1}}
\end{aligned}
\qquad (D')
$$

which is of degree, $n-1$, & since we know how to resolve by supposing in it $X^x = 0$; we will make in the same manner the resolution of this one to depend, on the resolution of an equation of degree, $n-2$, & thus in sequence; whence there results that the equation

$$X^x = y^x + H^x \cdot y^{x+1} \cdots + {}^{n-1}H^x \cdot y^{x+n}$$

is integrable in the same cases as this one

$$0 = y^x + H^x \cdot y^{x+1} \cdots + {}^{n-1}H^x \cdot y^{x+n}$$

This which is the beautiful theorem that Mr. de la Grange has found for the infinitely small differences, & that the preceding method gives us thus means to extend to the finite differences.

<div align="center">XX.</div>

This method leads us yet further, namely to find all at once the general expression of, y^x. Because, Z^x, being the complete integral of T^x in equation (D'), being as above, u, ${'}u$, ${''}u$ &c. the particular values of, y^x, in equation (B) when we suppose $X^x = 0$ because of $\frac{1}{\nabla(-V^{x-1})} = u$, we will have

$$y^x = u\left(A - \sum \frac{Z}{u}\right)$$

for the complete integral of equation (B), X^x, being any whatever. If we name $\overline{u}$, $\overline{{'}u}$, $\overline{{''}u}$, &c. the particular values of, T^x, in equation (D') when we suppose in it $X^x = 0$, we will have in the same manner

$$Z^x = \overline{u}\left({'}A - \sum \frac{{'}Z^x}{\overline{u}}\right)$$

<div align="center">28</div>

We will have similarly

$$' Z^x = \overline{\overline{u}} \left('' A - \sum \frac{'' Z^x}{\overline{\overline{u}}} \right)$$

&c.

until finally we arrive to this algebraic equation

$$^{n-1} Z^x = X^x$$

We will have therefore by substituting

$$y^x = u(A - \sum \left(\frac{\overline{u}}{u} \left(' A - \left[\left(\frac{\overline{\overline{u}}}{\overline{u}} \cdot \left('' A - \sum \left(\frac{\overline{\overline{\overline{u}}}}{\overline{\overline{u}}} \right. \right. \right. \right. \right.$$

$$(''' A \cdots - \sum \left\{ \frac{\frac{(n-1)}{\overline{u}}}{\frac{(n-2)}{\overline{u}}} \cdot \left(^{n-1} A - \sum \frac{X^x}{\frac{(n-1)}{\overline{u}}} \right) \right\} \tag{K}$$

It is necessary presently to determine, $\overline{u}, \overline{\overline{u}}$ &c. Now we have by article XVIII.

$$\overline{u} = u \triangle \left(\frac{'u}{u} \right), \quad \overline{'u} = u \triangle \left(\frac{''u}{u} \right), \quad \overline{''u} = u \triangle \left(\frac{'''u}{u} \right) \quad \text{&c.}$$

We will have likewise

$$\overline{\overline{u}} = \overline{u} \triangle \left(\frac{\overline{'u}}{\overline{u}} \right), \quad \overline{\overline{'u}} = \overline{u} \triangle \left(\frac{\overline{''u}}{\overline{u}} \right), \quad \overline{\overline{''u}} = \overline{u} \triangle \left(\frac{\overline{'''u}}{\overline{u}} \right) \quad \text{&c.}$$

$$\overline{\overline{\overline{u}}} = \overline{\overline{u}} \triangle \left(\frac{\overline{\overline{'u}}}{\overline{\overline{u}}} \right) \quad \text{&c.}$$

& formula (K) will become

$$y^x = u \left[A - \sum \left[\triangle \left(\frac{'u}{u} \right) \cdot \left[' A - \sum \left[\triangle \left(\frac{\overline{'u}}{\overline{u}} \right) \right. \right. \right. \right. \cdot$$

$$\left('' A \cdots - \sum \left[\triangle \left\{ \frac{\frac{(n-2)}{'u}}{\frac{(n-2)}{\overline{u}}} \cdot \left[^{n-1} A - \sum \frac{X^x}{\frac{(n-1)}{\overline{u}}} \right. \right. \right\} \right. \tag{o}$$

If we would know only a number, $n - 1$, of particular values in the equation

$$0 = y^x + H^x y^{x+1} + \text{&c.}$$

the integration would no longer have difficulty; because instead of arriving as previously to the algebraic equation, $^{n-1} z^x = X^x$, we would arrive to an equation of this form,

$$X^x = {}^{n-2} Z^x + S^x \cdot {}^{n-2} Z^{x+1}$$

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an equation which we know how to integrate by the second problem.

If instead of knowing how to resolve the equation,

$$0 = y^x + H^x y^{x+1} + \&\text{c.},$$

we would know a number, n, or, $n-1$, of values for, w^x, in equation (E), the preceding formulas would serve equally, because we have,

$$u = \frac{1}{\nabla(-V^{x-1})} \qquad {}'u = \frac{1}{\nabla(-{}'V^{x-1})} \qquad \&\text{c.}$$

<div align="center">XXI.</div>

We can further simplify formula (o), by putting it under this form

$$
\begin{aligned}
y^x =&\, Au + {}'A'u + {}''A''u \cdots + {}^{n-1}A^{n-1}u \\
&\pm u \sum \left(\triangle\left(\frac{'u}{u}\right) \cdot \sum\left(\triangle\left(\frac{\overline{'u}}{\overline{u}}\right) \cdots \sum \frac{X^x}{(n-1)}{\overline{u}} \right) \right)
\end{aligned}
\qquad (\sigma)
$$

the sign, $+$, having place, if, n, is even, & the sign, $-$, if it is odd: now we imagine first that the differential equation (B) is only of the second order, & we will have

$$y^x = Au + {}'A'u + \pm u \sum \left(\triangle\left(\frac{'u}{u}\right) \cdot \sum \frac{X^x}{\overline{u}} \right)$$

Now

$$\sum \left(\triangle\left(\frac{'u}{u}\right) \cdot \sum \frac{X^x}{\overline{u}} \right) = \frac{'u}{u}\sum \frac{X^{x-1}}{\overline{u}} - \sum \frac{X^{x-1}}{\overline{u}} \cdot \frac{'u}{u}$$

Therefore

$$
\begin{aligned}
y^x =&\, u\left(A - \sum \frac{X^{x-1}}{\overline{u}} \cdot \frac{'u}{u} \right) \\
&+ {}'u\left({}'A + \sum \frac{X^{x-1}}{\overline{u}_\prime} \right)
\end{aligned}
$$

But we have,

$$\overline{u} = u\triangle\left(\frac{'u}{u}\right) = u \cdot \frac{'u'}{u'} - {}'u,$$

By multiplying by, $\frac{u'}{'u'}$ we will have

$$\overline{u} \cdot \frac{u'}{'u'} = -\left({}'u \cdot \frac{u'}{'u'} - u \right).$$

<div align="center">30</div>

Hence if we make, $\bar{u} = {}'Z$ & if we call, Z, that which $\bar{u}$ becomes, when we change, ${}'u$, into u, & u into ${}'u$, we will have

$$y^x = u\left(A + \sum \frac{X^{x-1}}{Z}\right)$$
$$+ {}'u\left({}'A + \sum \frac{X^{x-1}}{{}'Z}\right)$$

&

$$Z = {}'u_{\prime}\triangle\left(\frac{u'}{{}'u_{\prime}}\right)$$

$${}'Z = u_{\prime}\triangle\left(\frac{{}'u_{\prime}}{u_{\prime}}\right)$$

I will observe however that by integrating differently, we would have a value of y^x, which would appear different, but of which it is easy to recognize the identity with the preceding. Because we will have

$$\sum\left(\triangle\left(\frac{{}'u}{u}\right)\cdot\sum\frac{X^x}{\bar{u}}\right) = \frac{{}'u}{u}\cdot\sum\frac{X^x}{\bar{u}} - \sum\frac{X^x}{\bar{u}}\cdot\frac{{}'u'}{u'}.$$

whence we will conclude

$$y^x = u\left(A + \sum\frac{X^x}{Z'}\right)$$
$$+ {}'u\left({}'A + \sum\frac{X^x}{{}'Z'}\right)$$

Now it is easy to see that this expression of y^x is the same as the preceding, that is to say, that we have

$$\left.\begin{array}{l} u\left(A + \sum\dfrac{X^x}{Z'}\right) \\[2mm] + {}'u\left({}'A + \sum\dfrac{X^x}{{}'Z'}\right) \end{array}\right\} = \left\{\begin{array}{l} u\left(A + \sum\dfrac{X^{x-1}}{Z}\right) \\[2mm] + {}'u\left({}'A + \sum\dfrac{X^{x-1}}{{}'Z}\right) \end{array}\right.$$

or that which reverts to the same, that

$$\frac{u}{Z} = \frac{-{}'u}{{}'Z};$$

or that

$$\frac{u}{{}'u_{\prime}}\triangle\left(\frac{{}'u_{\prime}}{u_{\prime}}\right) = \frac{-{}'u}{u_{\prime}}\triangle\left(\frac{u'}{{}'u_{\prime}}\right),$$

or finally that,

$$\frac{u'}{{}'u_{\prime}}\frac{-u}{u_{\prime}} = \frac{-u}{u_{\prime}} + \frac{{}'u}{{}'u_{\prime}}$$

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that which is clear.

If we suppose presently that equation (B) is of the third order, we will have,

$$y^x = Au + {'}A'u + {''}A''u - u\sum\left[\triangle\left(\frac{'u}{u}\right)\cdot\sum\left[\triangle\left(\frac{\overline{'u}}{\overline{u}}\right)\cdot\sum\frac{X^x}{\overline{\overline{u}}}\right)\right]$$

now

$$\triangle\left[\triangle\left(\frac{\overline{'u}}{\overline{u}}\right)\cdot\triangle\frac{X^x}{\overline{\overline{u}}}\right) = \frac{\overline{'u}}{\overline{u}}\triangle\frac{X^{x-1}}{\overline{\overline{u}}} - \sum\frac{X^{x-1}}{\overline{\overline{u}}_{/}}\cdot\frac{\overline{'u}}{\overline{u}}$$

$$= \frac{\triangle\left(\frac{''u}{u}\right)}{\triangle\left(\frac{'u}{u}\right)}\cdot\triangle\frac{X^x}{\overline{\overline{u}}}\cdot - \triangle\frac{X^{x-1}}{\overline{\overline{u}}_{/}}\cdot\frac{\overline{'u}}{\overline{u}}$$

Hence

$$\sum\left[\triangle\left(\frac{'u}{u}\right)\sum\left[\triangle\left(\frac{\overline{'u}}{\overline{u}}\right)\sum\frac{X^x}{\overline{\overline{u}}}\right.\right. = \frac{''u}{u}\sum\frac{X^{x-2}}{\overline{\overline{u}}_{//}} - \frac{'u}{u}\sum\frac{X^{x-2}}{\overline{\overline{u}}_{//}}\cdot\frac{\overline{'u}}{\overline{u}_{/}}$$

$$+ \sum\frac{X^{x-2}}{\overline{\overline{u}}_{//}}\left(\frac{\overline{'u}}{\overline{u}_{/}}\cdot\frac{\overline{'u}}{u} - \frac{''u}{u}\right)$$

Therefore

$$y^x = u\left(A - \sum\frac{X^{x-2}}{\overline{\overline{u}}_{//}}\cdot\left[\frac{\overline{'u}}{\overline{u}_{/}}\cdot\frac{'u}{u} - \frac{''u}{u}\right]\right)$$

$$+ {'}u\left({'}A - \sum\frac{X^{x-2}}{\overline{\overline{u}}_{//}}\cdot\frac{\overline{'u}_{/}}{\overline{u}_{/}}\right)$$

$$+ {''}u\left({''}A - \sum\frac{X^{x-2}}{\overline{\overline{u}}_{//}}\right)$$

by observing the same process, we will have the value of y^x, by supposing the differential equation (B) of such order as we will wish.

<div align="center">XXII.</div>

But here is a quite simple method in order to conclude this value of y^x. We resume for this the formula

$$y^x = u\left[A - \sum\left[\triangle\left(\frac{'u}{u}\right)\cdot\left[{'}A - \sum\left[\triangle\left(\frac{\overline{'u}}{\overline{u}}\right)\cdot\right.\right.\right.\right.$$

$$\left({''}A\cdots - \sum\left[\triangle\left\{\frac{\overset{(n-2)}{'u}}{\overset{(n-2)}{\overline{u}}}\cdot\left[{}^{n-1}A - \sum\frac{X^x}{\overset{(n-1)}{\overline{u}}}\right.\right.\right\}\right.\right)$$

By differentiating, we will have

$$-\triangle\left(\frac{y^x}{u}\right) = \triangle\left(\frac{'u}{u}\right)\cdot\left['A - \sum\left[\triangle\left(\frac{''u}{\overline{u}} - \&c.\right)\right.\right.$$

Whence we will conclude by dividing by, $\triangle\left(\frac{'u}{u}\right)$ & differentiating

$$\triangle\left\{\frac{\triangle\left(\frac{y^x}{u}\right)}{\triangle\left(\frac{'u}{u}\right)}\right\} = \triangle\left(\frac{''u}{\overline{u}}\right)['''A - \&c.$$

We will have therefore by continuing to differentiate thus.

$$^{n-1}A - \sum\frac{X^x}{\dfrac{(n-1)}{\overline{u}}} = \gamma^x\cdot y^x + '\gamma^x\cdot y^{x+1}\cdots + {}^{n-1}\gamma^x\cdot y^{x+n-1}$$

γ^x, $'\gamma^x$ &c. being some functions of u, $'u$, $''u$ &c.

If instead of u, we had considered, $'u$, as the first of the values of y^x, of equation (B) when we suppose in it $X^x = 0$, & u, as the second we would have had

$$^{n-1}A - \sum\frac{X^x}{\boxed{\dfrac{(n-1)}{\overline{u}}}} = \boxed{\gamma^x}\cdot y^x + \boxed{'\gamma^x}\cdot y^{x+1}\cdots + \boxed{^{n-1}\gamma^x}\cdot y^{x+n-1}$$

$\boxed{\dfrac{(n-1)}{\overline{u}}}$, & $\boxed{\gamma^x}$ &c. being that which $\dfrac{(n-1)}{\overline{u}}$, γ^x &c. become when we suppose in it, u, & reciprocally: now if we suppose $X^x = 0$, we will have

$$^{n-1}A = \gamma^x\cdot y^x + '\gamma^x\cdot y^{x+1}\cdots + {}^{n-1}\gamma^x\cdot y^{x+n-1}$$

$$^{n-1}A = \boxed{\gamma^x}\cdot y^x + \boxed{'\gamma^x}\cdot y^{x+1}\cdots + \boxed{^{n-1}\gamma^x}\cdot y^{x+n-1}$$

We will have therefore

$$\gamma^x\cdot y^x + '\gamma^x\cdot y^{x+1}\cdots + {}^{n-1}\gamma^x\cdot y^{x+n-1} = \boxed{\gamma^x}\cdot y^x\cdots + \boxed{^{n-1}\gamma^x}\cdot y^{x+n-1}$$

an equation which must be identical, because if it were not, then this equation being differential of the order, $n-1$, would have however for complete integral

$$y^x = Au + \cdots + {}^{n-1}A^{n-1}u$$

which contains, n, arbitrary constants, that which is absurd. It is necessary therefore that

$$^{n-1}A - \sum\frac{X^x}{\dfrac{(n-1)}{\overline{u}}} = {}^{n-1}A - \sum\frac{X^x}{\boxed{\dfrac{(n-1)}{\overline{u}}}}$$

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hence that, $\overset{(n-1)}{\overline{u}} = \boxed{\overset{(n-1)}{\overline{u}}}$ Thus the expression, $\overset{(n-1)}{\overline{u}}$ will remain always the same, granted we change, u, into, $'u$, & $'u$, into u. It would be easy to establish in the same manner that if in, $\overset{(n-1)}{\overline{u}}$, we change, u, into $''u$, & $''u$, into u, or, $'u$, into $''u$, & $''u$ into $'u$, or, $'''u$, into $'u$, & $'u$, into $'''u$, &c., & generally, $^k u$, into $^i u$, & $^i u$, into $^k u$, k, & i, being less than, $n-1$, the expression $\overset{(n-1)}{\overline{u}}$, will remain always the same, & that thus, whatever order that we give to the values, u, $'u$ &c. in order to form $\overset{(n-1)}{\overline{u}}$, this expression will remain constantly the same, provided that, ^{n-1}u, is always considered as the last of these values.

Let now, $\overset{(n-1)}{\overline{u}} = {}^{n-1}z$, & we imagine that after having treated, ^{n-1}u, as the last of the values u, $'u$, $''u$ &c. we regard, ^{n-2}u, as this last, let, ^{n-2}z, be that which ^{n-1}z becomes, when we change, ^{n-2}u, into ^{n-1}u, & reciprocally we will have

$$- {}^{n-2}A - \sum \frac{X^x}{{}^{n-1}z} = \underline{\gamma}^x \cdot y^x \cdots + {}^{n-1}\underline{\gamma}^x \cdot y^{x+n-1}$$

$\underline{\gamma}^x$ &c., being that which, γ^x, &c. become, when we change, ^{n-1}u, into ^{n-2}u, & reciprocally. I give to, ^{n-2}A, the sign, $-$, because formula (o) gives, by supposing in it $X^x = 0$

$$y^x = Au - {}'A'u \cdots \mp {}^{n-2}A^{n-2}u + {}^{n-1}A^{n-1}u$$

the sign, $+$, having place if, n, is odd, & the sign, $-$, if it is even. We will have likewise

$$^{n-3}A - \sum \frac{X^x}{{}^{n-3}z} = \underline{\underline{\gamma}}^x \cdot y^x \cdots + {}^{n-1}\underline{\underline{\gamma}}^x \cdot y^{x+n-1}$$

& thus in sequence; by arranging all these equations in the following order,

$$^{n-1}A - \sum \frac{X^x}{{}^{n-1}z} = \gamma^x \cdot y^x \cdots + {}^{n-1}\gamma^x \cdot y^{x+n-1} \tag{4}$$

$$- {}^{n-2}A - \sum \frac{X^x}{{}^{n-2}z} = \underline{\gamma}^x \cdot y^x \cdots + {}^{n-1}\underline{\gamma}^x \cdot y^{x+n-1} \tag{10}$$

$$^{n-3}A - \sum \frac{X^x}{{}^{n-3}z} = \underline{\underline{\gamma}}^x \cdot y^x \cdots + {}^{n-1}\underline{\underline{\gamma}}^x \cdot y^{x+n-1} \tag{7}$$

$$\cdots$$

$$\pm A - \sum \frac{X^x}{z} = \underset{(n-1)}{\gamma}^x \cdot y^x \cdots + {}^{n-1}\underset{(n-1)}{\gamma}^x \cdot y^{x+n-1} \tag{15}$$

& adding them together after having multiplied the first by ^{n-1}u, the second by ^{n-2}u,

&c., & the last by u, we will have an equation of this form

$$\lambda^x y^x \cdots {}^{n-1}\lambda^x \cdot y^{x+n-1} = u\left(A \pm \sum \frac{X^x}{z}\right)$$
$$+ \, 'u\left(-'A \pm \sum \frac{X^x}{'z}\right)$$
$$\cdots$$
$$+ \, {}^{n-1}u\left(\mp^{n-1}A \pm \sum \frac{X^x}{{}^{n-1}z}\right)$$

this which gives by supposing in it, $X^x = 0$,

$$\lambda^x y^x \cdots {}^{n-1}\lambda^x \cdot y^{x+n-1} = Au - 'A'u \cdots \mp {}^{n-1}A^{n-1}u$$

But we have under this same supposition

$$y^x = Au - 'A'u + ''A''u \cdots + {}^{n-1}A^{n-1}u$$

Hence

$$y^x = \lambda^x y^x \cdots {}^{n-1}\lambda^x \cdot y^{x+n-1}$$

Now this equation must be identical, otherwise although it is of order, $n - 1$, as we have

$$y^x = Au \cdots + {}^{n-1}A^{n-1}u$$

its integral would contain a number, n, of arbitrary constants, that which is absurd. We will have therefore for the complete integral of equation (B), by changing of sign, as this is permitted the arbitrary negative constants

$$y^x = u\left(A \pm \sum \frac{X^x}{z}\right)$$
$$+ \, 'u\left('A \pm \sum \frac{X^x}{'z}\right) \tag{n}$$
$$\cdots$$
$$+ \, {}^{n-1}u\left({}^{n-1}A \pm \sum \frac{X^x}{{}^{n-1}z}\right)$$

the sign, $+$, having place if, n, is odd, & the sign, $-$, if it is even; we resume now the equations (4), (10), (7), (15); they give

$${}^{n-1}A - \sum \frac{X^{x-1}}{{}^{n-1}z} = \gamma^{x-1} \cdots + {}^{n-1}\gamma^{x-1} \cdot y^{x+n-2}$$

$$\cdots$$

$$\pm A - \sum \frac{X^{x-1}}{z_{\prime}} = \frac{\gamma}{(n-1)}{}^{x-1} \cdot y^{x-1} \cdots + {}^{n-1}\frac{\gamma}{(n-1)}{}^{x-1} \cdot y^{x+n-2}$$

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multiplying the first of these equations by, ^{n-1}u, the second by ^{n-2}u, &c., & thus in sequence, we will have by adding them an equation of this form

$$\lambda^x y^{x-1} \cdots {}^{n-2}\lambda^x \cdot y^{x+n-2} = u\left(A \pm \sum \frac{X^{x-1}}{z_{\prime}}\right)$$
$$+ \,'u\left(-'A \pm \sum \frac{X^{x-1}}{'z_{\prime}}\right)$$
$$+ \,\&\text{c.}$$

We will have therefore by supposing, $X^x = 0$

$$\lambda^x y^{x-1} \cdots {}^{n-1}\lambda^x \cdot y^{x+n-2} = Au - \,'A'u + \&\text{c.}$$

Therefore

$$\lambda^x y^{x-1} \cdots {}^{n-1}\lambda^x \cdot y^{x+n-1} = y^x$$

an equation which must be identical. Hence we will have, by changing of sign the negative constants

$$y^x = u\left(A \pm \sum \frac{X^{x-1}}{z_{\prime}}\right)$$
$$+ \,'u\left('A \pm \sum \frac{X^{x-1}}{'z_{\prime}}\right)$$
$$+ \,\&\text{c.}$$

We will find in the same manner

$$y^x = u\left(A \pm \sum \frac{X^{x-2}}{z_{\prime\prime}}\right)$$
$$+ \,'u\left('A \pm \sum \frac{X^{x-2}}{'z_{\prime\prime}}\right)$$
$$+ \,\&\text{c.}$$

and thus in sequence, until we arrive to this last expression inclusively

$$y^x = u\left(A \pm \sum \frac{X^{x-n+1}}{z_{n-1}}\right)$$
$$+ \,'u\left('A \pm \sum \frac{X^{x-n+1}}{'z_{n-1}}\right)$$
$$+ \,\&\text{c.}$$

& all these expressions of, y^x, must be the same, as we have remarked above that this holds for the equations of the second order; by comparing together these expressions,

we will form the following equations

$$\frac{u}{z_\prime} + \frac{\prime u}{\prime z_\prime} + \frac{\prime\prime u}{\prime\prime z_\prime} \cdots + \frac{^{n-1}u}{^{n-1}z_\prime} = 0$$

$$\frac{u}{z_{\prime\prime}} + \frac{\prime u}{\prime z_{\prime\prime}} + \cdots + \frac{^{n-1}u}{^{n-1}z_{\prime\prime}} = 0$$

$$\cdots$$

$$\frac{u}{z_{n-1}} + \frac{\prime u}{\prime z_{n-1}} \cdots + \frac{^{n-1}u}{^{n-1}z_{n-1}} = 0$$

XXIII.

We resume now equation (E), which is

$$0 = 1 - \frac{H^x}{w^x} + \frac{\prime H^x}{w^x \cdot w^{x+1}} - \frac{\prime\prime H^x}{w^x \cdot w^{x+1} w^{x+2}} \cdots + \frac{^{n-1}H^x}{w^x \cdots w^{x+n-1}} \qquad (E)$$

We suppose

$$H^x = C \cdot \phi^x$$

$$\prime H^x = \prime C \cdot \phi^x \cdot \phi^{x+1}$$

$$\prime\prime H^x = \prime\prime C \cdot \phi^x \cdot \phi^{x+1} \cdot \phi^{x+2}$$

$$\&\text{c.}$$

C, $\prime C$, $\prime\prime C$ &c. being some constants any whatever, & ϕ^x, a function any whatever of, x, then the equation

$$X^x = y^x + C \cdot \phi^x \cdot y^{x+1} + \prime C \cdot \phi^x \cdot \phi^{x+1} \cdot y^{x+2}$$
$$+ \prime\prime C \cdot \phi^x \cdot \phi^{x+1} \cdot \phi^{x+2} \cdot y^{x+3} + \&\text{c.} \qquad (F)$$
$$+ {}^{n-1}C \cdot \phi^x \cdots \phi^{x+n-1} y^{x+n}$$

will be integrable; because if we suppose in equation (E),

$$w^x = a \cdot \phi^x,$$

a, being constant, it will become

$$0 = 1 - \frac{C}{a} + \frac{\prime C}{aa} - \frac{\prime\prime C}{a^3} \cdots \pm \frac{^{n-1}C}{a^n}$$

whence we will have a number, n, of values for, a, & consequently for, w^x, because $w^x = a \cdot \phi^x$. We can put formula (F) under this form

$$y^{x+n} = - \frac{^{n-2}C}{^{n-1}C} \frac{y^{x+n-1}}{\phi^{x+n-1}} - \frac{^{n-3}C}{^{n-1}C} \frac{y^{x+n-2}}{\phi^{x+n-1}\phi^{x+n-2}} + \&\text{c.}$$

$$+ \frac{X^x}{^{n-1}C \cdot \phi^x \cdots \phi^{x+n-1}}$$

& consequently under this one which is simpler

$$
\begin{aligned}
y^x =& A \cdot \phi^x \cdot y^{x-1} + {'A} \cdot \phi^x \cdot \phi^{x-1} \cdot y^{x-2} \\
& + {''A} \cdot \phi^x \cdot \phi^{x-1} \cdot \phi^{x-2} \cdot y^{x-3} + \&\text{c.} \qquad\qquad (H) \\
& + X^x
\end{aligned}
$$

this formula of which we can find the general term by the preceding article is much more extensive than any of those which the geometers have examined to the present; because if, $\phi^x, = 1$, that which is the simplest case, then it is changed into this one

$$
y^x = Ay^{x-1} + {'A} \cdot y^{x-2} + {''A} \cdot y^{x-3} \cdots + X^x
$$

this which is, as we know, the general expression of recurrent series.

XXIV.

Among the infinite number of series which formula (H) offers us, we will choose in first place those, in which we have $\phi^x = x$, & we will consider the series formed according to this law

$$
y^x = A \cdot y^{x-1} + {'A} \cdot x \cdot \overline{x-1} \cdot y^{x-2} \cdots + X^x
$$

by supposing in it first, $X^x = 0$; we will see in the series that the researches which we are going to make on this formula extend easily to the case, where we will have ϕ^x, equal to a function any whatever of x. We examine presently this equation when it rises only to the second order. It becomes then

$$
y^x = Ax \cdot y^{x-1} + {'A} \cdot x \cdot \overline{x-1} \cdot y^{x-2}
$$

in order to give an example of a series formed according to this law, we suppose $A = 2$, & $'A = 3$, & we will have,

$$
y^x = 2x \cdot y^{x-1} + 3x \cdot x \cdot \overline{x-1} \cdot y^{x-2},
$$

whence we will form the sequence,

$$
1, \ 4, \ 42, \ 480, \ 7320, \ 131040, \ \&\text{c.}
$$

we seek now the value of, y^x, in the differential equation

$$
y^x = Ax \cdot y^{x-1} + {'A} \cdot x \cdot \overline{x-1} \cdot y^{x-2}
$$

I put it under this form

$$
0 = y^x + \frac{A}{{'A} \cdot \overline{x+1}} y^{x+1} - \frac{y^{x+2}}{{'A} \cdot \overline{x+1} \cdot \overline{x+2}}.
$$

by comparing it with equation (B) of problem II; we will have

$$
X^x = 0, \quad H^x = \frac{A}{{'A} \cdot \overline{x+1}}, \quad {'H^x} = \frac{-1}{{'A} \cdot \overline{x+1} \cdot \overline{x+2}}
$$

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& equation (*E*) will give,

$$0 = 1 - \frac{A}{'A \cdot x + 1 \cdot w^x} - \frac{1}{'A \cdot x + 1 \cdot x + 2 \cdot w^x \cdot w^{x+1}}$$

I suppose $w^x = \frac{1}{a \cdot x + 1}$, & we will have, $0 = 1 - \frac{Aa}{'A} - \frac{a^2}{'A}$ or $'A = Aa + a^2$. Hence

$$a = -\frac{1}{2}A + \sqrt{'A + \frac{1}{4}A^2}$$

are expressed by, $-p$, & $-'p$ these two values of a; we will have therefore by that which precedes,

$$y^x = 1 \cdot 2 \cdot 3 \cdots x (Bp^x + 'B \cdot 'B^x)$$

this is the general term of this new kind of series, when the differential equation does not exceed the second order.

In order to determine B & $'B$, it is necessary to suppose that the first two terms of the series are given; let M & $'M$, be these two terms, & we will have

$$M = Bp + 'B'p$$
$$'M = {}^2Bp^2 + 2'B'p^2$$

Therefore

$$B = \frac{'M - 2M'p}{2p(p - 'p)}; \qquad 'B = \frac{'M - 2Mp}{2'p('p - p)}$$

Hence

$$y^x = 1 \cdot 2 \cdot 3 \cdots x \cdot \left(\frac{'M - 2M'p}{2p(p - 'p)}\right) \cdot p^x + \frac{'M - 2Mp}{2'p('p - p)} \cdot p^x \qquad (\lambda)$$

if we have $p = 'p$, that is to say, if the two roots of the equation $'A = Aa + a^2$, are equals, we will make $'p = p + dp$, whence we will conclude easily

$$y^x = 1 \cdot 2 \cdot 3 \cdots x \cdot p^{x-1} \left[\frac{4Mp - 'M}{2p} + \left(\frac{'M - 2Mp}{2\,p}\right)x\right] \qquad (\lambda)$$

in order to apply the preceding formulas to an example, let as above $A = 1$, $'A = 3$, we will have $A = -1 + 2$, therefore $p = -1$, & $'p = -3$. We suppose that the first two terms of the series are 1, & 4, so that $M = 1$ & $'M = 4$. We will have therefore $B = -\frac{1}{4}$, $'B = \frac{1}{4}$. Hence formula (λ) will give

$$y^x = 1 \cdot 2 \cdot 3 \cdots x \cdot \frac{1}{4}(3^x \pm 1),$$

the sign, $+$, having place if, x, is odd, & the sign, $-$, if it is even. If we wish to have for example the fifth term of the series, we will have,

$$y^x = 1 \cdot 2 \cdot 3 \cdots x \cdot \frac{1}{4}(3^5 + 1) = 7320$$

as previously.

XXV.

We suppose presently that we have to resolve the differential equation of the third order.

$$y^x = Ax \cdot y^{x-1} + {}'A \cdot x \cdot \overline{x-1} \cdot y^{x-2} + {}''A \cdot x \cdot \overline{x-1} \cdot \overline{x-2} \cdot y^{x-3}$$

I put it under this form

$$0 = y^x + \frac{{}'A}{{}''A \cdot \overline{x+1}} \cdot y^{x+1} + \frac{{}''A}{{}''A \cdot \overline{x+1} \cdot \overline{x+2}} y^{x+2}$$

$$- \frac{y^{x+3}}{{}''A \cdot \overline{x+1} \cdot \overline{x+2} \cdot \overline{x+3}}$$

by comparing with equation (*B*) of problem II we will have,

$$X^x = 0, \qquad\qquad H^x = \frac{{}'A}{{}''A \cdot \overline{x+1}},$$

$${}'H^x = \frac{A}{{}''A \cdot \overline{x+1} \cdot \overline{x+2}}, \qquad {}''H^x = \frac{-1}{{}''A \cdot \overline{x+1} \cdot \overline{x+2} \cdot \overline{x+3}}$$

& equation (*E*) will give

$$0 = 1 - \frac{{}'A}{{}''A \cdot \overline{x+1} \cdot w^x} + \frac{A}{{}''A \cdot \overline{x+1} \cdot \overline{x+2} \cdot w^x \cdot w^{x+1}}$$

$$+ \frac{1}{{}''A \cdot \overline{x+1} \cdot \overline{x+2} \cdot \overline{x+3} \cdot w^x \cdot w^{x+1} \cdot w^{x+2}}$$

I suppose

$$w^x = \frac{1}{a \cdot \overline{x+1}},$$

& we will have,

$$0 = {}''A - {}'Aa + Aa^2 + a^3$$

Let, $-p, \, -{}'p, \, -{}''p$, be the three values of, a, & we will have

$$y^x = 1 \cdot 2 \cdot 3 \cdots x(Bp^x + {}'B \cdot {}'p^x + {}''B \cdot {}''p^x)$$

Let $M, \, {}'M, \, {}''M$, be the three terms of the series, we will have in order to determine the constant $B, \, {}'B, \, {}''B$, the three equations

$$M = Bp + {}'B\,{}'p + {}''B\,{}''p$$

$$\frac{{}'M}{1 \cdot 2} = Bp^2 + {}'B\,{}'p^2 + {}''B\,{}''p^2$$

$$\frac{{}''M}{1 \cdot 2 \cdot 3} = Bp^3 + {}'B\,{}'p^3 + {}''B\,{}''p^3$$

Further we can raise ourselves to some more general considerations, because if we examine attentively the process of the preceding article, it is easy to see that if we have generally

$$y^x = A \cdot x \cdot y^{x-1} + {}'A \cdot x \cdot \overline{x-1} \cdot y^{x-2} + \&c. \cdots$$
$$+ {}^{n-1}Ax \cdots \overline{x-n+1} \cdot y^{x-n}$$

we will have

$$y^x = 1 \cdot 2 \cdot 3 \cdots x (Bp^x + {}'B\,{}'p^x \cdots + {}^{n-1}B\,{}^{n-1}p^x) \qquad (\gamma)$$

$-p, -{}'p, -{}''p$ &c. being the roots of, a, in the equation

$$0 = {}^{n-1}A - {}^{n-2}Aa + {}^{n-3}Aa^3 \cdots + Aa^{n-1} + a^n$$

the sign, $+$, having place if, n, is odd, & the sign, $-$, if it is even, or if M, $'M$, $''M$,&c. are the first terms of the series, we will have in order to determine the constants, B, $'B$, $''B$ &c. the number, n, of the following equations

$$M = Bp + {}'B\,{}'p + {}''B\,{}''p \cdots + {}^{n-1}B\,{}^{n-1}p$$

$$\frac{'M}{1 \cdot 2} = Bp^2 + {}'B\,{}'p^2 + {}''B\,{}''p^2 \cdots + {}^{n-1}B\,{}^{n-1}p^2$$

$$\frac{''M}{1 \cdot 2 \cdot 3} = Bp^3 + {}'B\,{}'p^3 + {}''B\,{}''p^3 \cdots + {}^{n-1}B\,{}^{n-1}p^3$$

$$\cdots$$

$$\frac{{}^{n-1}M}{1 \cdot 2 \cdot 3 \cdots n} = Bp^n + {}'B\,{}'p^n + {}''B\,{}''p^n \cdots + {}^{n-1}B\,{}^{n-1}p^n$$

in order to resolve these equations we can avail ourselves of the ordinary methods of elimination, but here is one which seems to me more convenient & simpler.

I multiply the first equation by, ${}^{n-1}p$, & I subtract it from the second, I multiply in the same manner the second by, ${}^{n-1}p$, & I subtract it from the third, & thus in sequence, this which produces the following equations

$$\frac{'M}{1 \cdot 2} - M \cdot {}^{n-1}p = Bp \cdot (p - {}^{n-1}p) + {}'B\,{}'p \cdot ({}'p - {}^{n-1}p)$$
$$\cdots + {}^{n-2}B\,{}^{n-2}p({}^{n-2}p - {}^{n-1}p)$$

$$\frac{''M}{1 \cdot 2 \cdot 3} - \frac{'M}{1 \cdot 2} \cdot {}^{n-1}p = Bp^2 \cdot (p - {}^{n-1}p) + {}'B\,{}'p^2 \cdot ({}'p - {}^{n-1}p)$$
$$\cdots + {}^{n-2}B \cdot {}^{n-2}p^2({}^{n-2}p - {}^{n-1}p)$$

$$\cdots$$

$$\frac{{}^{n-1}M}{1 \cdot 2 \cdots n} - \frac{{}^{n-2}M}{1 \cdot 2 \cdots n-1} \cdot {}^{n-1}p = Bp^{n-1} \cdot (p - {}^{n-1}p)$$
$$\cdots + {}^{n-2}B\,{}^{n-2}p^{n-1}({}^{n-2}p - {}^{n-1}p)$$

I multiply again the first of these equations by ^{n-2}p, & I subtract it from the second. I multiply similarly the second by ^{n-2}p, & I subtract it from the third, & thus in sequence, this which gives

$$\frac{''M}{1\cdot 2\cdot 3} - \frac{'M}{1\cdot 2}(^{n-1}p + {}^{n-2}p) + M\cdot {}^{n-1}p\cdot {}^{n-2}p$$
$$= Bp\cdot (p - {}^{n-2}p)\cdot (p - {}^{n-1}p)$$
$$+ {}'B'p\cdot ('p - {}^{n-2}p)\cdot ('p - {}^{n-1}p)$$
$$\cdots$$
$$+ {}^{n-3}B\cdot {}^{n-3}p\cdot (^{n-3}p - {}^{n-2}p)\cdot (^{n-3}p - {}^{n-1}p)$$

$$\frac{'''M}{1\cdot 2\cdot 3\cdot 4} - \frac{''M}{1\cdot 2\cdot 3}(^{n-1}p + {}^{n-2}p) + \frac{'M}{1\cdot 2}\cdot {}^{n-1}p\cdot {}^{n-2}p$$
$$= Bp^2(p - {}^{n-2}p)\cdot (p - {}^{n-1}p)$$
$$+ {}'B'p^2('p - {}^{n-2}p)\cdot ('p - {}^{n-1}p)$$
$$+ {}^{n-3}B\cdot {}^{n-3}p(^{n-3}p - {}^{n-2}p)\cdot (^{n-3}p - {}^{n-1}p)$$
$$\&c.$$

by operating on these last equations as on the preceding, we will have

$$\frac{'''M}{1\cdot 2\cdot 3\cdot 4} - \frac{''M}{1\cdot 2\cdot 3}(^{n-1}p + {}^{n-2}p + {}^{n-3}p)$$
$$+ \frac{'M}{1\cdot 2}[(^{n-2}p + {}^{n-1}p)\cdot {}^{n-3}p + {}^{n-1}p\cdot {}^{n-2}p]$$
$$- M\cdot {}^{n-1}p\cdot {}^{n-2}p\cdot {}^{n-3}p$$
$$= Bp(p - {}^{n-1}p)\cdot (p - {}^{n-2}p)\cdot (p - {}^{n-3}p)$$
$$+ \&c.$$

whence it is easy to see that if we name,
f, the sum of all the roots, $'p$, $''p$ &c. with the exception of p,
h, the sum of their products two by two,
l, the sum of their products three by three,
q, the sum of their products four by four,
&c.
$'f$, the sum of all the roots p, $''p$ &c. with the exception of $'p$,
$'h$, the sum of their products two by two,
&c.
$''f$, the sum of all the roots, p, $'p$, $'''p$ &c. with the exception of $''p$,
&c.

we will have

$$B = \frac{^{n-1}M - nf^{n-2}M + n \cdot \overline{n-1}h \cdot {}^{n-3}M - n\,\overline{n-1} \cdot \overline{n-2}\,l^{n-4}M + \&c.}{1 \cdot 2 \cdot 3 \cdots n \cdot p \cdot (p - {}'p)(p - {}''p)(p - {}'''p) \cdot \&c.}$$

$${}'B = \frac{^{n-1}M - n'f^{n-2}M + n \cdot \overline{n-1}\,'h \cdot {}^{n-3}M - n \cdot \overline{n-1} \cdot \overline{n-2}\,'l^{n-4}M + \&c.}{1 \cdot 2 \cdot 3 \cdots n \cdot {}'p \cdot ({}'p - p)({}'p - {}''p)({}'p - {}'''p) \cdot \&c.}$$

&c.

Let for brevity,

$$\frac{^{n-1}M - nf^{n-2}M + n \cdot \overline{n-1}h \cdot {}^{n-3}M - \&c.}{1 \cdot 2 \cdot 3 \cdots n.} = N$$

$$\frac{^{n-1}M - n'f \cdot {}^{n-2}M + \&c.}{1 \cdot 2 \cdot 3 \cdots n.} = {}'N$$

&c.

We will have therefore

$$y^x = 1 \cdot 2 \cdot 3 \cdots x \left\{ \begin{array}{l} \dfrac{N}{(p - {}'p)(p - {}''p)(p - {}'''p) \cdot \&c.} \cdot p^{x-1} \\[2mm] + \dfrac{{}'N}{({}'p - p)({}'p - {}''p)({}'p - {}'''p) \cdot \&c.} \cdot {}'p^{x-1} \\[2mm] + \&c. \end{array} \right\} \qquad (\widetilde{\omega})$$

the quantities f, h, l, q, can be determined quite easily in the following manner.

Let the equation of this article be resumed

$$0 = a^n + A \cdot a^{n-1} - {}'A \cdot a^{n-2} + {}''A \cdot a^{n-3} \cdots \pm {}^{n-1}A$$

I divide it by $a + p$ that which gives

$$\begin{aligned} 0 = {}& a^{n-1} + a^{n-2}(A - p) \\ & - a^{n-3}({}'A + Ap - pp) \\ & + a^{n-4}({}''A + {}'Ap + Ap^2 - p^3) \\ & - \&c. \end{aligned}$$

whence it is easy to conclude

$$\begin{aligned} f &= A - p, \\ h &= -({}'A + Ap - pp) \\ l &= {}''A + {}'Ap + Ap^2 - p^3 \\ & \quad \&c. \\ {}'f &= A - {}'p \\ {}'h &= -({}'A + A'p - 'p^2) \\ & \quad \&c. \end{aligned}$$

If we have $p = {}'p$ we will have $N = {}'N$; let then ${}'p = p + dp$, &

$$\frac{1}{(p - {}''p)(p - {}'''p) \cdot \text{\&c.}} = Q,$$

we will have

$$\frac{{}'N}{({}'p - p)({}'p - {}''p) \,\text{\&c.}} = {}'NQ + {}'Ndp \cdot \frac{dQ}{dp}$$

Now it is easy to see by the formation of N, & ${}'N$, that $N = {}'N + \frac{d\,'N}{dp}\,dp$. Hence ${}'N = N - \frac{dN}{dp}\,dp$. Therefore

$$\frac{{}'N}{({}'p - {}''p)({}'p - {}'''p) \,\text{\&c.}} = dp\left(\frac{NQ}{dp} + \frac{NdQ}{dp} - \frac{QdN}{dp}\right),$$

&

$$\frac{{}'N \cdot {}'p^{x-1}}{({}'p - {}''p)({}'p - {}'''p) \,\text{\&c.}} = dp\left(\frac{NQ}{dp} + \frac{NdQ}{dp} - \frac{QdN}{dp} + (x-1) \cdot \frac{NQ}{p}\right) \cdot p^{x-1}$$

Hence because $p - {}'p = -dp$, & ${}'p - p = dp$, we will have

$$\frac{N \cdot p^{x-1}}{(p - {}'p)(p - {}''p) \,\text{\&c.}} + \frac{{}'N \cdot {}'p^{x-1}}{({}'p - p)({}'p - {}''p) \cdot \text{\&c.}}$$

$$= p^{x-1}\left(\frac{NdQ}{dp} - \frac{QdN}{dp} + NQ \cdot \frac{\overline{x-1}}{p}\right)$$

$$= N^2 \cdot p^{x-2}\left(\overline{x-1} \cdot R + p\frac{dR}{dp}\right)$$

by supposing $R = \frac{Q}{N}$. We will have

$$y^x = 1 \cdot 2 \cdot 3 \cdots x \left\{ \begin{array}{l} N^2 \cdot p^{x-2}\left(\overline{x-1} \cdot R + \dfrac{pdR}{dp}\right) \\[2ex] + \dfrac{{}''N}{({}''p - p)^2 \cdot ({}''p - {}'''p) \,\text{\&c.}} \cdot {}''p^{x-1} \\[2ex] + \text{\&c.} \end{array} \right. \qquad (\widetilde{\omega})$$

If moreover we have in this last formula $p = {}''p$, we will make yet ${}''p = p + dp$, & thus in sequence it will be superfluous to pause ourselves further in these different cases.

If we consider with attention the process of the preceding articles, we must notice that it gives us the general term of the series formed according to this equation

$$y^x = A \cdot \phi^x \cdot y^{x-1} + {}'A \cdot \phi^x \cdot \phi^{x-1} \cdot y^{x-2}$$
$$+ {}''A \cdot \phi^x \cdot \phi^{x-1} \cdot \phi^{x-2} \cdot y^{x-3} + \&\text{c.}$$

because the whole subsisting as in these articles with this sole difference that we must make here

$$N = \frac{{}^{n-1}M - nf \cdot {}^{n-2}M + \&\text{c.}}{\phi' \cdot \phi'' \cdots \phi^n}$$

$${}'N = \frac{{}^{n-1}M - n\,{}'f \cdot {}^{n-2}M + \&\text{c.}}{\phi' \cdot \phi'' \cdots \phi^n}$$

$$\&\text{c.}$$

the formulas $(\widetilde{\omega})$, & $\overset{(m)}{\sim}$ will become

$$y^x = \phi \cdot \phi' \cdot \phi'' \cdots \phi^x \left\{ \begin{array}{l} \dfrac{N}{(p - {}'p)^2 \cdot (p - {}'p)\&\text{c.}} \cdot p^{x-1} \\[2ex] + \dfrac{{}'N}{({}'p - p)({}'p - {}''p) \cdot \&\text{c.}} \cdot {}'p^{x-1} \\[2ex] + \&\text{c.} \end{array} \right. \qquad ({}'\widetilde{\omega})$$

$$y^x = \phi \cdot \phi' \cdot \phi'' \cdots \phi^x \left\{ \begin{array}{l} N^2 \cdot p^{x-2} \left(\overline{x-1} \cdot R + p\dfrac{dR}{dp} \right) \\[2ex] + \dfrac{{}''N}{({}''p - p)^2({}''p - {}'''p) \cdot \&\text{c.}} \cdot {}''p^{x-1} \\[2ex] + \&\text{c.} \end{array} \right. \qquad \left({}'\overset{m}{\sim}\right)$$

If $\phi^x = 1$, the preceding series are changed into recurrent series, & we will have thus in a very direct manner, the general term of recurrent series: thence results this theorem.

If we name T^x, the general term of a recurrent series such that

$$y^x = A \cdot y^{x-1} + {}'A \cdot y^{x-2} + {}''A \cdot y^{x-3} \cdots + {}^{n-1}A \cdot y^{x-n},$$

the general term of a series such that we have

$$y^x = A\phi^x \cdot y^{x-1} + {}'A \cdot \phi^x \cdot \phi^{x-1}y^{x-2} \cdots$$
$$+ {}^{n-1}A \cdot \phi^x \cdots \phi^{x+1-n}y^{x-n}$$

& in which the first n terms which are arbitraries, are the same as in the preceding, will be

$$y^x = \phi^{n+1} \cdot \phi^{n+2} \cdots \phi^x T^x$$

This is that of which it is easy to assure ourselves besides; because if we substitute this value of y^x, into the equation

$$y^x = A\phi^x \cdot y^{x-1} \cdots {}^{n-1}A \cdot \phi^x \cdots \phi^{x+1-n}y^{x-n}$$

we will have

$$\phi^{n+1} \cdots \phi^x T^x = A \cdot \phi^{n+1} \cdot \phi^{n+2} \cdots \phi^x T^{x-1} \cdots$$
$$+ {}^{n-1}A \cdot \phi^{n+1} \cdots \phi^x y^{x-n}$$

or

$$T^x = A \cdot T^{x-1} \cdots {}^{n-1}A \cdot T^{x-n}$$

Now this last equation holds, since T^x, is the general term of the recurrent series formed by the equation

$$y^x = A \cdot y^{x-1} \cdots + {}^{n-1}A \cdot y^{x-n}$$

<h2 style="text-align:center">XXIX.</h2>

We have supposed until here in formula (H) of article XXII. $X^x = 0$ & this was indispensable before integrating it. By supposing X^x whatever, we are going presently to examine this formula under this hypothesis. Now if in the equation

$$y^x = A\phi^x \cdot y^{x-1} \cdots {}^{n-1}A \cdot \phi^x \cdots \phi^{x-n+1}y^{x-n} + X^x,$$

we suppose $X^x = 0$, we will have as we have seen previously

$$y^x = \phi' \cdot \phi'' \cdots \phi^x \cdot (Bp^x + {}'B'p^x \cdots + {}^{n-1}B^{n-1}p^x)$$

whence we will form by the articles XVIII & following

$$u = \phi' \cdot \phi'' \cdots \phi^x p^x$$
$${}'u = \phi' \cdot \phi'' \cdots \phi^x {}'p^x$$
$$\cdots$$
$${}^{n-1}u = \phi' \cdot \phi'' \cdots \phi^x \cdot {}^{n-1}p^x$$

We form presently the expressions of ${}^{n-1}z$, ${}^{n-2}z$ &c. in order to substitute them into formula (n) of article XXI. We will have

$$\overline{u} = u\triangle\left(\frac{{}'u}{u}\right) = \phi' \cdot \phi'' \cdots \phi^x \cdot p^x \cdot \triangle\left(\frac{{}'p^x}{p^x}\right)$$
$$= \phi' \cdot \phi'' \cdots \phi^x \cdot \left(\frac{{}'p - p}{p}\right) \cdot {}'p^x$$
$$\overline{{}'u} = \phi' \cdot \phi'' \cdots \phi^x \cdot \left(\frac{{}''p - p}{p}\right) \cdot {}''p^x$$
$$\overline{{}''u} = \phi' \cdot \phi'' \cdots \phi^x \cdot \left(\frac{{}'''p - p}{p}\right) \cdot {}'''p^x$$
&c.

Hence

$$\bar{\bar{u}} = \overline{'u} \triangle \left(\frac{\overline{'u}}{\bar{u}}\right) = \phi' \cdot \phi'' \cdots \phi^x \left(\frac{''p - p}{p}\right)\left(\frac{''p - 'p}{'p}\right) \cdot {}''p^x$$

$$\overline{\overline{'u}} = \phi' \cdot \phi'' \cdots \phi^x \cdot \left(\frac{'''p - p}{p}\right)\left(\frac{'''p - 'p}{'p}\right) \cdot {}'''p^x$$

&c.

$$\bar{\bar{\bar{u}}} = \bar{\bar{u}} \triangle \left(\frac{\overline{\overline{'u}}}{\bar{\bar{u}}}\right) = \phi' \cdot \phi'' \cdots \phi^x \left(\frac{'''p - p}{p}\right)\left(\frac{'''p - 'p}{'p}\right)\left(\frac{'''p - ''p}{''p}\right)$$

&c.

& thus in sequence, we will have therefore

$$z = \phi' \cdot \phi'' \cdots \phi^x \cdot p^x \cdot \left(\frac{p - 'p}{'p}\right)\left(\frac{p - ''p}{''p}\right)\left(\frac{p - '''p}{'''p}\right) \cdots \left(\frac{p - {}^{n-1}p}{{}^{n-1}p}\right)$$

$$'z = \phi' \cdot \phi'' \cdots \phi^{x+n-1} \cdot {}'p^x \cdot \left(\frac{'p - p}{p}\right)\left(\frac{'p - ''p}{''p}\right) \text{ \&c.}$$

whence we will conclude by substituting these values into formula (*n*)

$$y^x = \frac{\phi' \cdot \phi'' \cdots \phi^x \cdot p^x}{\left(\frac{p - 'p}{'p}\right)\left(\frac{p - ''p}{''p}\right) \cdot \text{\&c.}} \cdot \left(A \pm \sum \frac{X^x}{\phi' \cdot \phi'' \cdots \phi^x \cdot p^x}\right)$$

$$+ \frac{\phi' \cdot \phi'' \cdots \phi^x \cdot {}'p^x}{\left(\frac{'p - p}{p}\right)\left(\frac{'p - ''p}{''p}\right) \text{\&c.}} \cdot \left('A \pm \sum \frac{X^x}{\phi' \cdot \phi'' \cdots \phi^x \cdot {}'p^x}\right)$$

$$+ \text{\&c.}$$

<div align="center">XXX.</div>

If $p = {}'p$; we will make $'p = p + dp$. Let

$$N = \frac{''p \cdot '''p \cdots {}^{n-1}p}{(p - ''p)(p - '''p) \text{ \&c.}}$$

we will have therefore

$$y^x = \frac{-N'}{dp} \cdot p\,\phi' \cdot \phi'' \cdots \phi^x \cdot p^x \cdot \left(A \pm \sum \frac{X^x}{\phi' \cdot \phi'' \cdots \phi^x \cdot p^x}\right)$$

$$+ \left(\frac{Np}{dp} + \frac{pdN}{dp}\right) \cdot \phi' \cdot \phi'' \cdots \phi^x (p + dp)^x \cdot$$

$$\left('A \pm \sum \frac{X^x}{\phi' \cdot \phi'' \cdots \phi^x (p + dp)}\right)$$

$$+ \text{\&c.}$$

this which gives all reductions made

$$y^x = \phi' \cdot \phi'' \cdots \phi^x \cdot p^x \left\{ \begin{array}{l} \left(\dfrac{N^x + pdN}{dp} \right) \cdot \left({}'A \pm \sum \dfrac{X^x}{\phi' \cdot \phi'' \cdots \phi^x \cdot p^x} \right) \\[2ex] - N \left(A \pm \sum \dfrac{X^x \cdot (x+1)}{\phi' \cdot \phi'' \cdots \phi^x \cdot p^x} \right) \\[2ex] + \&\text{c.} \end{array} \right\}$$

If moreover we have $p = {}''p$, we will make in this last expression $''p = p + dp$, & thus in sequence; it suffices to suppose $\phi^x = 1$, in order to have the general term of the recurrent series.

<div align="center">XXXI.</div>

We are going presently to examine some other cases, in which the proposed equation (B) is integrable. But we suppose that it rises only to the second order. Equation (E) becomes then

$$0 = 1 - \frac{H^x}{w^x} + \frac{{}'H^x}{w^x \cdot w^{x+1}}$$

of which it suffices to find a particular integral; every time therefore that we will have

$$H^x = w^x + \frac{{}'H^x}{w^{x+1}},$$

the equation

$$X^x = y^x + H^x y^{x+1} + {}'H^x y^{x+2}$$

will be integrable.

The equation

$$X^x = y^x + \left(w^x + \frac{{}'H^x}{w^{x+1}} \right) y^{x+1} + {}'H^x y^{x+2},$$

is integrable w^x, & $'H^x$, being some functions any whatever of x. If w^x, is constant, & equal $a \cdot m$, the equation

$$X^x = y^x + \left(m + \frac{{}'H^x}{m} \right) y^{x+1} + {}'H^x \cdot y^{x+2}$$

will be integrable. The case of the recurrent series is contained in this one; because if we wish to integrate

$$X^x = y^x + A y^{x+1} + {}'A \cdot y^{x+2},$$

we will make

$$'H^x = A$$

$$m + \frac{{}'H^x}{m} = {}'A$$

this which gives two values for m, & consequently for w^x. By giving to w^x some other values, we will have some other integrable equations. We see that this method is extended in the same manner to the differential equations of all orders, but it is useless to enter into a greater detail in this regard.

I will not quit this material without indicating a remarkable usage of the integral calculus to the finite differences in the formation of series: some examples make it better to understand all that we could say. Let x be the *sinus* of an angle z, & y its *cosinus*, we have generally

$$\sin nz = y \cdot \sin(n-1)z - \sin(n-2)z$$

therefore

$$\sin z = x$$
$$\sin 2z = 2xy$$
$$\sin 3z = 4xy^2 - x$$
$$\sin 4z = 8xy^3 - 4xy$$
$$\sin 5z = 16xy^4 - 8xy^2 + x$$
$$\&c.$$

it is easy to find, this put, the general expression of $\sin nz$; it is easy to notice. $1\,^\circ$ that the preceding expressions are the product of x by a rational function, & the whole of y, $2\,^\circ$ that in this function the highest exponent of y, is smaller by one unit than the number which multiplies the angle z, $3\,^\circ$ that the exponents of y, diminish according to an arithmetic progression, of which 2 is the difference, & that these exponents are always positive. It is easy to see that this must hold for the *sinus* of any multiple whatever of z; the expression of $\sin nz$, will be therefore the following form

$$\sin nz = x(A \cdot y^{n-1} + B \cdot y^{n-3} + C \cdot y^{n-5} + D \cdot y^{n-7} + \&c.)$$

It is necessary presently to determine the coefficients, A, B, C, D, &c. we have

$$\sin \overline{n-1} \cdot z = x(A_{\prime}y^{n-2} + B_{\prime}y^{n-4} + C_{\prime}y^{n-6} + D_{\prime}y^{n-8} + \&c.)$$

&

$$\sin \overline{n+1} \cdot z = x(A'y^n + B'y^{n-2} + C'y^{n-4} + D'y^{n-6} + \&c.)$$

$$= 2y \cdot \sin nz - \sin \overline{n-1} \cdot z = x \left\{ \begin{array}{l} 2A \cdot y^n + 2B \cdot y^{n-2} + 2C \cdot y^{n-4} \\ + 2D \cdot y^{n-6} + \&c. - A_{\prime}y^{n-2} + B_{\prime}y^{n-4} \\ - C_{\prime}y^{n-6} + \&c. \end{array} \right\}$$

We will have therefore by comparing

$1\,^\circ$ $A' = 2A$. Therefore by problem II, $A = 2^n \cdot H$, (H, being a constant). Now n being, 1, $A = 1$. Therefore $H = \frac{1}{2}$ Hence $A = 2^{n-1}$.

$2\,^\circ$ $B' = 2B - A$. Now $B' = 2B - 2^{n-2}$. Whence we will have by integrating,

$$B = 2^{n-1} \sum \frac{-2^{n-1}}{2^n} = -2^{n-3} \sum 1 = -2^{n-3}(n+H);$$

as y cannot have a negative exponent, in the expression of $\sin nz$, it is necessary that B, be zero, when $n = 2$. Hence $B = -2^{n-3}(n-2)$.

$3°$ $C' = 2C - B, = 2C + 2^{n-4}(n-3)$. Therefore $C = 2^{n-5}\left(\frac{nn-7n}{2} + H\right)$. We will determine H by this condition that C be zero, when $n = 4$. Therefore $H = 6$. Hence

$$C = 2^{n-5} \cdot \frac{\overline{n-3} \cdot \overline{n-4}}{1 \cdot 2}$$

$4°$ $D' = 2D - C$, whence we will conclude easily

$$D = -\frac{2^{n-7} \cdot \overline{n-4} \cdot \overline{n-5} \cdot \overline{n-6}}{1 \cdot 2 \cdot 3} \text{ \&c.}$$

Therefore

$$\sin nz = x \left\{ \begin{array}{l} 2^{n-1} \cdot y^{n-1} - \dfrac{n-2}{1} \cdot 2^{n-3} y^{n-3} \\[2mm] + \dfrac{\overline{n-3} \cdot \overline{n-4}}{1 \cdot 2} \cdot 2^{n-5} y^{n-5} \\[2mm] - \dfrac{\overline{n-4} \cdot \overline{n-5} \cdot \overline{n-6}}{1 \cdot 2 \cdot 3} \cdot 2^{n-7} y^{n-7} + \text{\&c.} \end{array} \right\}$$

Let next $z =$ ang. $\sin x$. By differentiating, we will have

$$\frac{dz}{dx} = \frac{1}{\sqrt{1-xx}}$$

Presently we wish to have the general expression of $\frac{d^n z}{dx^n}$; dx being supposed constant.
Let $y = \frac{1}{\sqrt{1-xx}}$, we will have

$$\frac{dy}{dx} = \frac{x}{(1-xx)^{\frac{3}{2}}}$$

$$\frac{ddy}{dx^2} = \frac{2x^2+1}{(1-xx)^{\frac{5}{2}}}$$

$$\frac{d^3 y}{dx^3} = \frac{6x^3+9x}{(1-xx)^{\frac{7}{2}}}$$

$$\frac{d^4 y}{dx^4} = \frac{24x^4+72x^2+9}{(1-xx)^{\frac{9}{2}}}$$

If in the least we consider with attention these expressions of $\frac{dy}{dx}$, $\frac{d^2 y}{dx^2}$ &c. it is easy to perceive that the general expression of $\frac{d^n y}{dx^n}$ must have the following form

$$\frac{d^n y}{dx^n} = \frac{Ax^n + B \cdot x^{n-2} + Cx^{n-4} + D \cdot x^{n-6} + F \cdot x^{n-8} + \text{\&c.}}{(1-xx)^{n+\frac{1}{2}}}$$

Hence

$$\frac{d^{n+1} y}{dx^{n+1}} = \frac{A'x^{n+1} + B' \cdot x^{n-1} + C' \cdot x^{n-3} + D' \cdot x^{n-5} + F' \cdot x^{n-7} + \text{\&c.}}{(1-xx)^{n+\frac{3}{2}}}$$

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By differentiating $\frac{d^n y}{dx^n}$ we will have

$$\frac{d^{n+1}y}{dx^{n+1}} = \frac{\begin{array}{ll}(2n+1)A \cdot x^{n+1} & + (2n+1)B \cdot x^{n-1} \\ & + (2n+1)C \cdot x^{n-3} + \&\text{c.} \\ -nA & + nA + (n-2)B \\ & - (n-2)B - (n-4)C\end{array}}{(1-xx)^{n+\frac{3}{2}}}$$

We will have by comparing these two expressions of, $\frac{d^{n+1}y}{dx^{n+1}}$, & integrating appropriately the values of A, B, C, D &c. & we will find after having made the calculus as previously

$$d^{n-1}y \cdot dx = d^n \cdot \text{ang. } \sin x = \frac{1 \cdot 2 \cdot 3 \cdot \overline{n-1} \cdot dx^n}{(1-xx)^{n-\frac{1}{2}}}$$

$$\left\{\begin{array}{l} x^{n-1} + \dfrac{1}{2} \cdot \dfrac{\overline{n-1} \cdot \overline{n-2}}{1 \cdot 2} \cdot x^{n-3} \\[2mm] + \dfrac{1 \cdot 3}{2 \cdot 4} \cdot \dfrac{\overline{n-1} \cdot \overline{n-2} \cdot \overline{n-3} \cdot \overline{n-4}}{1 \cdot 2 \cdot 3 \cdot 4} \cdot x^{n-5} \\[2mm] + \dfrac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \dfrac{\overline{n-1} \cdot \overline{n-2} \cdots \overline{n-6}}{1 \cdot 2 \cdot 3 \cdots 6} \cdot x^{n-7} \\[2mm] + \&\text{c.} \end{array}\right\}$$

This method, as we see, is that of the indeterminates, with this difference that the undetermined coefficients instead of being constants, are here variables, & given by as many of the equation in the finite differences.

<div align="center">

XXXIII.

Application of the preceding method
to the partial differences.

</div>

In order to integrate the equation in the partial differences $\frac{dy}{dx} = \frac{a\,dy}{dt}$, we will observe that $\frac{dy}{dx}dx + \frac{dy}{dt}dt = dy$, whence it is easy to conclude, $dy = \frac{dy}{dt}(dt + a\,dx)$. Hence $y = \phi(t + ax) \cdot \phi$, serving to designate a function any whatever. This put.

<div align="center">

PROBLEM II.

</div>

We propose to integrate the differential equation

$$X = \frac{d^n y}{dx^n} + a' \cdot \frac{d^n y}{dx^{n-1} \cdot dt} + a'' \cdot \frac{d^n y}{dx^{n-2} dt^2} \cdots + \frac{a^n \cdot d^n y}{dt^n} \qquad (A)$$

a', a'' &c., being some constants any whatever, & X, a function any whatever of x.

Let $y = z + u$, z, being a function of x alone; equation (*A*) gives

$$X = \frac{d^n z}{dx^n} + \frac{d^n u}{dx^n} + a' \cdot \frac{d^n u}{dx^{n-1} \cdot dt} + \&\text{c.}$$

I make $X = \frac{d^n z}{dx^n}$. Hence, we will have

$$0 = \frac{d^n u}{dx^n} + a' \cdot \frac{d^n u}{dx^{n-1} \cdot dt} + a'' \cdot \frac{d^n u}{dx^{n-2} dt^2} + \&\text{c.} \qquad (A')$$

Let, $\frac{du}{dx} = w \frac{du}{dt}$, w being constant. If we differentiate this equation by making vary $1°$, x, $2°$ t, we will have the following two

$$\frac{ddu}{dx^2} = w \frac{ddu}{dx\,dt}$$
$$\frac{ddu}{dx\,dt} = w \frac{ddu}{dt^2}$$

If we differentiate in the same manner these two equations by making vary $1°$ x, $2°$ t, we will have the following three

$$\frac{d^3 u}{dx^3} = w \frac{d^3 u}{dx^2 dt}$$
$$\frac{d^3 u}{dx^2 dt} = w \frac{d^3 u}{dx\,dt^2}$$
$$\frac{d^3 u}{dx\,dt^2} = w \frac{d^3 u}{dt^3}$$

By differentiating thus successively, we will find

$$\frac{d^n u}{dx^n} = w \cdot \frac{d^n u}{dx^{n-1} dt}$$
$$\frac{d^n u}{dx^{n-1} dt} = w \cdot \frac{d^n u}{dx^{n-2}\,dt^2}$$
$$\cdots$$
$$\frac{d^n u}{dx\,dt^{n-1}} = w \cdot \frac{d^n u}{dt^n}$$

If we multiply the second of these equations by w', the third by w'', the fourth by w''' &c., & if we add them together with the first, we will have

$$0 = \frac{d^n u}{dx^n} + (w' - w) \cdot \frac{d^n u}{dx^{n-1}\,dt} + (w'' - ww') \cdot \frac{d^n u}{dx^{n-2}\,dt^2}$$
$$\cdots + (w^{n-1} - ww^{n-2}) \cdot \frac{d^n u}{dx\,dt^{n-1}} - ww^{n-1} \cdot \frac{d^n u}{dt^n}$$

By comparing this equation with equation (A'), we will form the following

$$w' - w = a'$$
$$w'' - ww' = a''$$
$$w''' - ww'' = a'''$$
$$\cdots$$
$$w^{n-1} - ww^{n-1} = a^{n-1}$$
$$-w\,w^{n-1} = a^n$$

Whence we will conclude,

$$w' = a' - w$$
$$w'' = a'' - a'w + w^2$$
$$w''' = a''' - a''w + a'w^2 + w^3$$
$$\cdots$$
$$w^{n-1} = a^{n-1} - a^{n-2}w + a^{n-3}w^2 \cdots + w^{n-1}$$

Hence, we will have

$$0 = w^n + a' \cdot w^{n-1} + w'' \cdot w^{n-2} \cdots + a^n$$

Let p, p', p'' be the values of w, in this equation. We will have therefore the following equations,

$$\frac{du}{dx} = p \cdot \frac{du}{dt}$$
$$\frac{du}{dx} = p' \cdot \frac{du}{dt}$$
$$\&c.$$

Hence the complete integral of equation (A') will be

$$u = \phi(t + px) + \phi'(t + p'x) + \phi''(t + p''x)$$
$$\cdots + \phi^{n-1}(t + p^{n-1}x)$$

I must observe that Mr. D'Alembert has integrated this same equation in a very elegant manner, in the fourth volume of his *Opuscules*: also I have integrated it here only in order to show how my method is applied to the calculus of partial differences.

<div align="center">XXXIV.</div>

Here are presently many theorems on the integral calculus, which have not at all yet been, that I know, noticed by any geometer, & which have appeared to me to be of some utility in the infinitesimal analysis.

Let $Mdx = 0$ be a differential equation of order n, M being a finite function, & homogeneous in x, y, & in their first, second ... & n^{th} differences. I suppose besides

this function that we can make at will, dx, or, dy, constant, or variable. Let now, $dy = pdx$, $dp = qdx$, $dq = rdx$, $dr = \int dx$ &c. Mr. Euler has shown in his *Institutions* of the differential calculus, that, M, will be in this case a function of x, y, p, q, r, s, &c. Now it is clear that, p, is of null dimension; q, of the dimension, -1, r, of the dimension -2 &c. Since therefore, M, is a homogeneous function, by naming, h, its dimension, we will have

$$M = x^h \cdot \text{function} \left(\frac{y}{x}, \ p, \ qx, \ rx^2, \ \int x^3 \ \&c. \right)$$

Hence

$$0 = \text{function} \left(\frac{y}{x}, \ p, \ qx, \ rx^2, \ \int x^3 \ \&c. \right) \qquad (A)$$

I make presently,

$$\frac{y}{x} = u, \quad \& \ \frac{dx}{x} = \frac{du}{t},$$

this which gives

$$\frac{dy}{dx} = u + t = p.$$

Next,

$$q = \frac{d(u+t)}{dx}, \quad \& \ qx = \frac{x}{dx}.$$

$$d(u+t) = \frac{t \cdot d(u+t)}{du}$$

Differentiating, we will have,

$$\frac{dq}{dx} = \frac{d\left(\frac{t \cdot d(u+t)}{du} \right)}{x \cdot dx} - \frac{q}{x} = r$$

Hence

$$rx^2 = \frac{t \cdot d\left(\frac{t[d(u+t)]}{du} \right)}{du} = \frac{t \cdot d(u+t)}{du}$$

I name this quantity γ for brevity. Differentiating anew the previous equation, we will have

$$\frac{dr}{dx} = s = \frac{d\gamma}{x^2 \, dx} - \frac{2r}{x}$$

Hence,

$$sx^3 = d\gamma \cdot \frac{x}{dx} - 2rx^2 = \frac{d\gamma \cdot t}{du} - 2\gamma,$$

& thus in sequence. This put. Substituting in the place of, $\frac{y}{x}$, p, qx, rx^2 &c. these values into the equation (A) it will become

$$0 = \text{function} \left\{ u, \ u + t, \ t \cdot \frac{d(u+t)}{du}, \ \frac{t \cdot d\left(\frac{t \, d(u+t)}{du} \right)}{du}, \ \frac{t \cdot d\left(\frac{t \cdot d(u+t)}{du} \right)}{du} \ \&c. \right\} \qquad (B)$$

which equation contains no more than some differences of order $n-1$.

We can therefore thus lower a homogeneous equation of order, n, to another of order $n - 1$.

It can happen that the equation lowered to another of an inferior order remains yet homogeneous. In this case we will treat it anew, as the preceding.

In order to give an example, let

$$0 = dx\left(\gamma + A\frac{dy}{dx} + A' \cdot x \cdot \frac{ddy}{dx^2}\right)$$

dx being constant. It will be changed into the following

$$0 = y + Ap + A'q^x,$$

whence we will conclude

$$0 = u + A(u + t) + A' \cdot \frac{t \cdot d(u + t)}{du}$$

a homogeneous equation, & of the first degree.

Generally the homogeneous equation $Mdx = 0$, lowered to another of an inferior degree will remain yet homogeneous, if M is homogeneous with respect to, x, & to its differences, & if it is moreover with respect to, y, & its differences. Because it is clear that under this supposition equation (B) is homogeneous.

If in the equation, $Mdx = 0$, the dimension of x, & of its differences, less the dimension of y, & of its differences makes for each term a constant quantity, we will render this equation homogeneous by making $x = \frac{1}{z}$.

Let as previously the equation, $Mdx = 0$, of order, n, M, being a finite function, & homogeneous of x, y, & of their first, second ...& n^{th} differences, in which we can make at will, dx, or dy, constant, or variable. I suppose moreover that this function is homogeneous with respect to y, & to its differences, such that for each term the number of the dimensions of this variable, & of its differences, is the same; I name it h. This put. Let, $dy = pdx$, $dp = qdx$, $dq = rdx$ &c. M, will be a function of x, y, p, q, r, s &c., & p, q, r, s &c. will be of the dimension 1 with respect to y: I make,

$$\frac{dy}{dx} = ty = p,$$

therefore

$$\frac{dy}{y} = tdx$$

Next

$$\frac{dp}{dx} = q = y\frac{dt}{dx} + \frac{t\,dy}{dx} = y\left(\frac{dt}{dx} + t^2\right);$$

therefore,

$$\frac{dq}{dx} = y \cdot \frac{d\left(\frac{dt}{dx} + t^2\right)}{dx} + \frac{dy}{dx}\left(\frac{dt}{dx} + t^2\right)$$

$$= y\left\{\frac{d\left(\frac{dt}{dx} + t^2\right)}{dx} + t \cdot \left(\frac{dt}{dx} + t^2\right)\right\} = r, \text{ \&c.}$$

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& by substituting in the place of p, q, r, &c. these values into M, we will have

$$M = y^h \cdot \text{function} \left\{ \begin{array}{l} x,\ t,\ \dfrac{dt}{dx},\ \dfrac{d \cdot \left(\frac{dt}{dx} \right)}{dx}, \\[4ex] \dfrac{d \cdot \left\{ \frac{d\left(\frac{dt}{dx} \right)}{dx} \right\}}{dx} \quad \text{\&c.} \end{array} \right\}$$

Hence

$$0 = \text{function} \left\{ x,\ t,\ \frac{dt}{dx},\ d \cdot \frac{\left(\frac{dt}{dx} \right)}{dx}\ \text{\&c.} \right\}$$

an equation which contains no more than some differences of order $n - 1$.